Homology, Homotopy and Applications, vol. 15(2), 2013, pp.1–7

THE GEOMETRIC REALIZATION OF MONOMIAL IDEAL RINGS AND A THEOREM OF TREVISAN

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(*communicated by Donald M. Davis*)

Abstract

A direct proof is presented of a form of Alvise Trevisan's theorem [**7**], that every monomial ideal ring is represented by the cohomology of a topological space. Certain of these rings are shown to be realized by polyhedral products indexed by simplicial complexes.

1. Introduction

In the paper [**7**], Alvise Trevisan showed that every ring which is a quotient of an integral polynomial ring with two dimensional generators by an ideal of monomial relations, can be realized as the integral cohomology ring of a topological space. Moreover, he showed that the rings could be all realized with spaces which are generalized Davis-Januszkiewicz spaces. These spaces are colimits over *multicomplexes* which are generalizations of simplicial complexes.

Here is presented a direct proof of the "realization" part of Trevisan's theorem. It uses a result of Fröberg from [**5**] which asserts that a map known as "polarization" produces, in a natural way, a regular sequence of degree-two elements. This allows for the realization of any monomial ideal ring by a certain pullback.

It is noted also that certain families of monomial ideal rings, beyond Stanley-Reisner rings, can be realized as generalized Davis-Januszkiewicz spaces based on ordinary simplicial complexes. Of course, as Trevisan shows, multicomplexes are needed in general.

Through the paper, all cohomology is taken with *integral* coefficients.

2. The main result

Let $\mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial ring on generators of degree two and

$$M = \{m_j\}_{j=1}^{r}, \qquad m_j = x_1^{t_{1j}} x_2^{t_{2j}} \cdots x_n^{t_{nj}} \tag{1}$$

A. B. was supported in part by a Rider University Summer Research Fellowship and grant number 210386 from the Simons Foundation; F. R. C. was supported partially by DARPA grant number 2006-06918-01.

Received September 19, 2012, revised April 18, 2013; published on July 31, 2013.

2000 Mathematics Subject Classification: Primary: 13F55, Secondary: 55T20, 57T35.

Key words and phrases: monomial ideal ring, Stanley-Reisner ring, Davis-Januszkiewicz space, polarization, polyhedral product.

Article available at `http://intlpress.com/HHA/v15/n2/a1` and `doi:10.4310/HHA.2013.v15.n2.a1`

be a set of minimal monomials, that is, no monomial divides another. Here, the exponent t_{ij} might be equal to zero but every x_i must appear in some m_j. Notice that the set M is determined by the $n \times r$ matrix (t_{ij}). Denote by $I(M)$ the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ generated by the minimal monomials m_j and set

$$A = A(M) = \mathbb{Z}[x_1, \ldots, x_n]/I(M) \tag{2}$$

a *monomial ideal ring*. From this is defined a second monomial ideal ring $A(\overline{M})$ with monomial ideal generated by square free monomials. For each $i = 1, 2, \ldots, n$ set

$$t_i = \max\{t_{i1}, t_{i2}, \ldots, t_{ir}\}, \tag{3}$$

the largest entry in the i-th row of (t_{ij}). Next, introduce new variables of degree two $y_{i1}, y_{i2}, \ldots, y_{it_i}$ for each $i = 1, 2, \ldots, n$. For each monomial $m_j = x_1^{t_{1j}} x_2^{t_{2j}} \cdots x_n^{t_{nj}}$, set

$$\overline{m}_j = (y_{11}y_{12} \cdots y_{1t_{1j}})(y_{21}y_{22} \cdots y_{2t_{2j}}) \cdots (y_{n1}y_{n2} \cdots y_{nt_{nj}}). \tag{4}$$

Let $\overline{M} = \{\overline{m}_j\}_{j=1}^{r}$ and define an algebra $B = B(\overline{M})$ by

$$B = \mathbb{Z}[y_{11}, y_{12}, \ldots, y_{1t_1}, y_{21}, y_{22}, \ldots, y_{2t_2}, \ldots, y_{n1}, y_{n2}, \ldots, y_{nt_n}]/I(\overline{M}). \tag{5}$$

The monomials here are square-free so B is a Stanley-Reisner algebra which determines a simplicial complex $K(\overline{M})$. (This process which constructs B from A is known in the literature as *polarization*.) Associated to this simplicial complex is a fibration

$$Z\big(K(\overline{M}); (D^2, S^1)\big) \longrightarrow \mathcal{DJ}(K(\overline{M})) \longrightarrow BT^{d(\overline{M})},$$

where $d(\overline{M}) = \sum_{1=1}^{n} t_i$, with t_i as in (3), $\mathcal{DJ}(K(\overline{M}))$ is the Davis-Januszkiewicz space of the simplicial complex $K(\overline{M})$, and $Z\big(K(\overline{M}); (D^2, S^1)\big)$ is the moment-angle complex corresponding to $K(\overline{M})$, [**3**]. Recall that the Davis-Januszkiewicz space has the property that

$$H^*\big(\mathcal{DJ}(K(\overline{M}))\big) \cong B. \tag{6}$$

Define next a diagonal map $\Delta\colon T^n \longrightarrow T^{d(\overline{M})}$ by

$$\Delta(x_1, x_2, \ldots, x_l) = \big(\Delta_{t_1}(x_1), \Delta_{t_2}(x_2), \ldots, \Delta_{t_n}(x_l)\big), \tag{7}$$

where $\Delta_{t_i}(x_i) = (x_i, x_i, \ldots, x_i) \in T^{t_i}$. In the diagram below, let $W(A)$ be defined as the pullback of the fibration.

$$\begin{array}{ccc}
Z\big(K(\overline{M}); (D^2, S^1)\big) & \xrightarrow[=]{} & Z\big(K(\overline{M}); (D^2, S^1)\big) \\
\big\downarrow & & \big\downarrow \\
W(A) & \xrightarrow[\widetilde{\Delta}]{} & \mathcal{DJ}(K(\overline{M})) \\
\big\downarrow & & \big\downarrow \\
BT^n & \xrightarrow{B\Delta} & BT^{d(\overline{M})}
\end{array} \tag{8}$$

The diagram (8) extends to a larger diagram

$$\begin{array}{ccccccc}
\ast & \longrightarrow & Z\big(K(\overline{M});(D^2,S^1)\big) & \underset{=}{\longrightarrow} & Z\big(K(\overline{M});(D^2,S^1)\big) & \longrightarrow & \ast \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
T^{d(\overline{M}-n)} & \longrightarrow & W(A) & \underset{\widetilde{\Delta}}{\longrightarrow} & \mathcal{D}\mathcal{J}(K(\overline{M})) & \underset{p}{\longrightarrow} & BT^{d(\overline{M}-n)} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & =\big\downarrow \\
T^{d(\overline{M}-n)} & \longrightarrow & BT^n & \overset{B\Delta}{\longrightarrow} & BT^{d(\overline{M})} & \longrightarrow & BT^{d(\overline{M}-n)}
\end{array}$$

where the fact that $W(A)$ is a pullback implies that

$$T^{d(\overline{M})-n} \xrightarrow{p} W(A) \xrightarrow{\widetilde{\Delta}} \mathcal{D}\mathcal{J}(K(\overline{M})) \tag{9}$$

is a fibration too. A long exact homotopy sequence argument comparing $W(A)$ to the homotopy fibre of p shows that

$$W(A) \xrightarrow{\widetilde{\Delta}} \mathcal{D}\mathcal{J}(K(\overline{M})) \xrightarrow{p} BT^{d(\overline{M})-n} \tag{10}$$

is a homotopy fibration. Recall that $d(\overline{M}) = \sum_{1=1}^{n} t_i$ and choose generators

$$H^*(BT^{d(\overline{M})-n}) \cong \mathbb{Z}[u_{12},\ldots,u_{1t_1},u_{22},\ldots,u_{2t_2},\ldots,u_{n2},\ldots,u_{nt_n}],$$

so that

$$p^*(u_{ik_i}) = y_{i1} - y_{ik_i} \quad i = 1,2,\ldots,n, \quad k_i = 2,3,\ldots,t_i.$$

This choice is possible because of the commutativity of the bottom right square in the large diagram above and the description of $H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big)$ given in (5) and (6). Set $\theta_{ik_i} := p^*(u_{ik_i})$. The proposition following is a basic result about the diagonal map Δ (the *polarization* map); a proof may be found in [**5**, page 30].

Proposition 2.1 (Fröberg)**.** *Over any field k, the sequence $\{\theta_{ik_i}\}$ is a regular sequence of degree-two elements in the ring $H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}));k\big)$.*

This result allows for a direct proof of the realization theorem.

Theorem 2.2. *There is an isomorphism of rings*

$$H^*\big(W(A);\mathbb{Z}\big) \longrightarrow A(M).$$

Proof. Working over a field k and following Masuda-Panov, [**6**, Lemma 2.1], we use the Eilenberg-Moore spectral sequence associated to the fibration (10). It has

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{H^*(BT^{d(\overline{M})-n})}\big(H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M})),k\big).$$

Now $H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big)$ is free as an $H^*(BT^{d(\overline{M})-n})$-module by Proposition 2.1, so

$$\begin{aligned}
\mathrm{Tor}^{*,*}_{H^*(BT^{d(\overline{M})-n})}\big(H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big),k\big) &= \mathrm{Tor}^{0,*}_{H^*(BT^{d(\overline{M})-n})}\big(H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big),k\big) \\
&= H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big) \otimes_{H^*(BT^{d(\overline{M})-n})} k \\
&= H^*\big(\mathcal{D}\mathcal{J}(K(\overline{M}))\big)\big/p^*(H^{>0}(BT^{d(\overline{M})-n})).
\end{aligned}$$

It follows that the Eilenberg-Moore spectral sequence collapses at the E_2 term and hence, as groups,

$$H^*\big(W(A)\big) = H^*\big(\mathcal{DJ}(K(\overline{M}))\big)/p^*(H^{>0}(BT^{d(\overline{M})-n})),$$

from which we conclude that $H^*\big(W(A);k\big)$ is concentrated in even degrees. Taking $k = \mathbb{Q}$ gives the result that in odd degree, $H^*\big(W(A);\mathbb{Z}\big)$ consists of torsion only. Unless this torsion is zero, the argument above with $k = \mathbb{F}_p$ for an appropriate p implies a contradiction. It follows that $H^*\big(W(A);\mathbb{Z}\big)$ is concentrated in even degrees.

Lemma 2.3. *The integral Serre spectral sequence of the fibration* (10) *collapses.*

Proof. The spaces in the fibration have integral cohomology concentrated in even degrees. □

The E_2-term of the Serre spectral sequence is

$$H^*\big(W(A);\mathbb{Z}\big) \otimes H^*\big(BT^{d(\overline{M})-n};\mathbb{Z}\big).$$

It follows that, as a ring, $H^*\big(W(A);\mathbb{Z}\big)$ is the quotient of $H^*\big(\mathcal{DJ}(K(\overline{M}))\big)$ by the two-sided ideal L generated by the image of p^*. So there is an isomorphism of graded rings,

$$H^*\big(W(A);\mathbb{Z}\big) \longrightarrow H^*\big(\mathcal{DJ}(K(\overline{M}))\big)/L \cong A(\overline{M})/L \cong A(M),$$

completing the proof of Theorem 2.2. □

Remark 2.4. The Eilenberg-Moore spectral sequence of the fibration

$$Z\big(K(\overline{M});(D^2,S^1)\big) \longrightarrow W(A) \longrightarrow BT^n$$

collapses and so it can be used to compute the cohomology of $Z\big(K(\overline{M});(D^2,S^1)\big)$, the two-connected covering of $W(A)$.

3. On the geometric realization of certain monomial ideal rings by ordinary polyhedral products

In this section, polyhedral products, [**1**], involving finite and infinite complex projective spaces are used to realize certain classes of monomial ideal rings. As noted earlier, generalizations of the Davis-Januszkiewicz spaces to the realm of multicomplexes are required in order to realize all monomial ideal rings; see Trevisan [**7**].

The class which can be realized by ordinary polyhedral products is restricted to those monomials

$$M = \{m_j\}_{j=1}^{r}, \qquad m_j = x_1^{t_{1j}} x_2^{t_{2j}} \cdots x_n^{t_{nj}}$$

of (1), which satisfy the condition:

⋇ t_{ij} is constant over all monomials m_j which have t_{ij} and *at least one other exponent* both non-zero.

In particular, a monomial ring of the form

$$\mathbb{Z}[x_1, x_2, x_3]/\langle x_1^2x_2, x_1^2x_3^4, x_3^5\rangle \tag{11}$$

can be realized by an ordinary polyhedral product. As usual, let $(\underline{X}, \underline{A})$ denote a family of CW pairs

$$\{(X_1, A_1), (X_2, A_2), \ldots, (X_n, A_n)\}.$$

Given a monomial ring $A(M)$ of the form (2), satisfying the condition ⋇ above, a simplicial complex K and a family of pairs $(\underline{X}, \underline{A})$ will be specified so that

$$H^*\big(Z(K; (\underline{X}, \underline{A})); \mathbb{Z}\big) = A(M),$$

where $Z(K; (\underline{X}, \underline{A})$ represents a polyhedral product as defined in [**1**].

Construction 3.1. Let K be the simplicial complex on n vertices $\{v_1, v_2, \ldots, v_n\}$ which has a minimal non-face corresponding to each m_i having *at least two* non-zero exponents. If m_i has non-zero exponents

$$t_{j_1 i}, t_{j_2 i}, \ldots, t_{j_t i},$$

then K will have a corresponding minimal non-face $\{v_{j_1}, v_{j_2}, \ldots, v_{j_t}\}$. Moreover, these will be the only minimal non-faces of K.

For example, the ring (11) above will have associated to it the simplicial complex K on vertices $\{v_1, v_2, v_3\}$ and will have minimal non-faces $\{v_1, v_2\}$ and $\{v_1, v_3\}$. So, K will be the disjoint union of a point and a one-simplex.

For the set of monomials M satisfying condition ⋇, the cases following are distinguished in terms of (1) for fixed $i \in \{1, 2, \ldots, n\}$.

1. For certain j, $t_{ij} = 1$, $t_{i'j} \neq 0$ for some $i' \neq i$ and $t_{ik} = 0$ otherwise.
2. For certain j, $t_{ij} = q_i > 1$, $t_{i'j} \neq 0$ for some $i' \neq i$ and $t_{ik} = 0$ otherwise.
3. $m_j = x_i^{s_i}$ for some j and $t_{ik} = 0$ for $k \neq j$.
4. $m_j = x_i^{s_i}$ for some j and if $t_{ik} \neq 0$ for $k \neq j$, then $t_{ik} = q_i < s_i$.

With this classification in mind, define a family of CW-pairs

$$(\underline{X}, \underline{A}) = \{(X_i, A_i) \colon i = 1, \ldots, n\}$$

by

$$(X_i, A_i) \quad = \quad \begin{cases} (\mathbb{C}P^\infty, *) & \text{if } i \text{ satisfies } (1), \\ (\mathbb{C}P^\infty, \mathbb{C}P^{q_i - 1}) & \text{if } i \text{ satisfies } (2), \\ (\mathbb{C}P^{s_i - 1}, *) & \text{if } i \text{ satisfies } (3), \\ (\mathbb{C}P^{s_i - 1}, \mathbb{C}P^{q_i - 1}) & \text{if } i \text{ satisfies } (4). \end{cases} \tag{12}$$

The next theorem describes the polyhedral products which have cohomology realizing the monomial ideal rings satisfying condition ⋇.

Theorem 3.2. *Let $A(M)$ be a monomial ring of the form* (2)*, satisfying the condition ⋇ and K, the simplicial complex defined by Construction 3.1, then*

$$H^*\big(Z(K; (\underline{X}, \underline{A})); \mathbb{Z}\big) = A(M)$$

where (X, A) is the pair specified by (12)*.*

Remark 3.3. The improvement here over [**2**, Theorem 10.5] consists of the inclusion of cases (3) and (4) above. The polyhedral products which realize the monomial ideal rings discussed in [**1**] have $X_i = \mathbb{C}P^\infty$ for all $i = 1, 2, \dots, n$.

Proof of Theorem 3.2. Set $Q = (q_1, q_2, \dots, q_n)$ with $q_i \geqslant 1$ for all i and write the spaces A_i of (12) as $\mathbb{C}P^{q_i-1}$ where $q_i = 1$ if $A_i = *$, a point. Write

$$(\underline{X}, \underline{A}) = (\underline{X}, \underline{\mathbb{C}P}^{Q-1}) = \{(X_i, \mathbb{C}P^{q_i-1}) \colon i = 1, 2, \dots, n\}$$

and consider the commutative diagram

$$\begin{array}{ccc} H^*(\prod_{i=1}^n X_i) & \xleftarrow[p^*]{} & H^*(\prod_{i=1}^n \mathbb{C}P^\infty) \\ \Big\downarrow i^* & & k^* \Big\downarrow \\ H^*\big(Z(K; (\underline{X}, \underline{\mathbb{C}P}^{Q-1}))\big) & \xleftarrow{h^*} & H^*\big(Z(K; (\underline{\mathbb{C}P}^\infty, \underline{\mathbb{C}P}^{Q-1}))\big) \end{array} \tag{13}$$

induced by the various inclusion maps. According to [**2**, Theorem 10.5], there is an isomorphism of rings

$$H^*\big(Z(K; (\underline{\mathbb{C}P}^\infty, \underline{\mathbb{C}P}^{Q-1}))\big) \longrightarrow \mathbb{Z}[x_1, \dots, x_n]/I(M^Q),$$

where $I(M^Q)$ is the ideal generated by all monomials $x_{i_1}^{q_{i_1}}, x_{i_2}^{q_{i_2}}, \dots, x_{i_k}^{q_{i_k}}$ corresponding to the minimal non-faces $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ of K. Moreover, the proof of [**2**, Lemma 10.3] shows that the composition i^*p^* is a surjection. The commutativity of diagram (13) implies that these relations all hold in $H^*\big(Z(K; (\underline{X}, \underline{\mathbb{C}P}^{Q-1}))\big)$. In addition to these, the relation $x_i^{s_i} = 0$ is included for each i satisfying $X_i = \mathbb{C}P^{s_i-1}$. These relations account for all the relations determined by $I(M)$. The remainder of the argument shows that $I(M)$ determines all relations in $H^*\big(Z(K; (\underline{X}, \underline{A})); \mathbb{Z}\big)$. Consider now the space

$$W_k = \mathbb{C}P^{q_1-1} \times \cdots \times \mathbb{C}P^{q_{k-1}-1} \times X_k \times \mathbb{C}P^{q_{k+1}-1} \times \cdots \times \mathbb{C}P^{q_n-1}$$

corresponding to the simplex $\{v_k\} \in K$, consisting of a single vertex. The composition

$$W_k \longrightarrow Z(K; (\underline{X}, \underline{\mathbb{C}P}^{Q-1})) \longrightarrow \textstyle\prod_{i=1}^n X_i$$

factors the natural inclusion $W_k \longrightarrow \prod_{i=1}^n X_i$. From this observation follows the fact that no other monomial relations occur in $H^*\big(Z(K; (\underline{X}, \underline{\mathbb{C}P}^{Q-1}))\big)$ other than those determined by $I(M)$. Suppose next that there is a linear relationship of the form

$$a\omega = \sum_{i=1}^k a_i \omega_i, \tag{14}$$

where $a, a_i \in \mathbb{Z}$ and ω, ω_i are monomials in the $x_i, i = 1, 2, \dots, n$. Without loss of generality, ω and ω_i can be assumed to be not divisible by any of the monomials in M. Suppose $\omega = x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_l}^{\lambda_l}$, then $\sigma = \{v_{j_1}, v_{j_2}, \dots, v_{j_l}\} \in K$ is a simplex and so is a full subcomplex of K. (The corresponding polyhedral product $Z(\sigma; (\underline{X}, \underline{\mathbb{C}P}^{Q-1}))$ is a product of finite and infinite complex projective spaces.) This implies, by [**4**, Lemma 2.2.3], that $H^*\big(Z(\sigma; (\underline{X}, \underline{\mathbb{C}P}^{Q-1}))\big)$ must be a direct summand in $H^*\big(Z(K; (\underline{X}, \underline{\mathbb{C}P}^{Q-1}))\big)$ contradicting the relation (14). $\square$

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