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TENSOR PRODUCTS OF HOMOTOPY GERSTENHABER ALGEBRAS

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Abstract

On the tensor product of two homotopy Gerstenhaber algebras we construct a Hirsch algebra structure which extends the canonical dg algebra structure. Our result applies more generally to tensor products of "level 3 Hirsch algebras" and also to the Mayer–Vietoris double complex.

1. Introduction

Let R be a commutative unital ring and A an augmented associative differential graded (dg) algebra over R. A *Hirsch algebra* structure on A is a (possibly non-associative) multiplication in the normalized bar construction $\bar{B}A$ of A which is a morphism of coalgebras and has the counit $\mathbf{1} \in \bar{B}A$ as a unit. It is uniquely determined by its associated twisting cochain

$$E\colon \bar{B}A \otimes \bar{B}A \to A.$$

Because the map $a_1 \otimes b_1 \mapsto E([a_1], [b_1])$ is essentially a $\cup_1$ product for A (without strict Hirsch formulas), the product of a Hirsch algebra is always commutative up to homotopy in the naive sense.

Let $a = [a_1|\cdots|a_k] \in \bar{B}_k A$ and $b = [b_1|\cdots|b_l] \in \bar{B}_l A$. A Hirsch algebra satisfying $E(a, b) = 0$ for all $k > 1$ is called a *level 3 Hirsch algebra* in [**6**]. It is a *homotopy Gerstenhaber algebra* (or "homotopy G-algebra") if in addition the resulting multiplication is associative. Important examples of homotopy Gerstenhaber algebras are the cochain complex of a simplicial set or topological space [**1**], the Hochschild cochains of an associative algebra [**5**], [**4**, Sec. 5.1], [**9**] and the cobar construction of a dg bialgebra over $\mathbb{Z}_2$ [**6**].

Let A' and A'' be two Hirsch algebras. Then $A' \otimes A''$ is a dg algebra, again commutative up to homotopy in the naive sense. In this paper we address the question of whether such a homotopy is part of a system of higher homotopies. We obtain the following result:

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Theorem 1.1. *Let A' and A'' be two level 3 Hirsch algebras. Then $A' \otimes A''$ is a Hirsch algebra in a natural way, and the shuffle map $\bar{B}A' \otimes \bar{B}A'' \to \bar{B}(A' \otimes A'')$ is multiplicative.*

As shown in Remark 5.4, one cannot generally hope for the tensor product of two level 3 Hirsch algebras to be again of the same type. Whether Hirsch algebras are closed under tensor products remains open, see Question 5.5.

The paper is organized as follows: In Section 2 we introduce the notation needed for the later parts. The Hirsch algebra structure of $A = A' \otimes A''$ is constructed in Section 3. Example 3.1 shows how our twisting cochain $E\colon \bar{B}A \otimes \bar{B}A \to A$ looks like in small degrees, and Example 3.2 illustrates a general recipe for computing it explicitly. Section 4 contains the proof that E is well-defined and that the shuffle map is multiplicative. We conclude by reformulating our result in an operadic language and applying it to the Mayer–Vietoris double complex in Section 5.

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2. Notation

We work in a cohomological setting, so that differentials are of degree $+1$. We denote the desuspension of a complex C by $s^{-1}C$, and the canonical chain map $s^{-1}C \to C$ of degree 1 by σ. Anticipating the definition of the bar construction, we also write $\sigma^{-1}(c) = [c]$ for $c \in C$. The differential on $s^{-1}C$ is given by $d[c] = -[dc]$.

Let A be an augmented, unital associative dg algebra over R with multiplication map $\mu_A\colon A \otimes A \to A$ and augmentation $\varepsilon_A\colon A \to R$. Denote the augmentation ideal of A by $\bar{A}$, so that $A = R \oplus \bar{A}$ canonically.

Note that there are canonical isomorphisms of complexes

$$\begin{aligned} s^{-1}A' \otimes A'' &\to s^{-1}(A' \otimes A''), &\qquad [a'] \otimes a'' &\mapsto [a' \otimes a''], \\ A' \otimes s^{-1}A'' &\to s^{-1}(A' \otimes A''), &\qquad a' \otimes [a''] &\mapsto (-1)^{|a'|}[a' \otimes a'']. \end{aligned} \tag{1}$$

Although we are mostly interested in the normalized bar construction $\bar{B}A$ of A, it will be convenient to consider the unnormalized bar construction BA as well. This is the tensor coalgebra of the desuspension of A (instead of $\bar{A}$),

$$BA = T(s^{-1}A) = \bigoplus_{k \geqslant 0} (s^{-1}A)^{\otimes k}.$$

We write $B_kA = (s^{-1}A)^{\otimes k}$ and for elements $[a_1|\cdots|a_k] \in B_kA$. The differential on BA is the sum of the tensor product differential $d_\otimes$ and the differential

$$\partial = \sum_{i=1}^{k-1} 1^{\otimes i-1} \otimes \tilde{\mu} \otimes 1^{\otimes k-i-1}\colon B_kA \to B_{k-1}A.$$

Here $\tilde{\mu}$ denotes the desuspension of μ,

$$\tilde{\mu} = \sigma^{-1}\mu(\sigma \otimes \sigma)\colon s^{-1}A \otimes s^{-1}A \to s^{-1}A.$$

We write $\mathbf{1} \in B_0A$ for the counit of BA and α for the canonical twisting cochain

$$\alpha\colon BA \to B_1A = s^{-1}A \xrightarrow{\sigma} A.$$

Let M be a right dg-A-module and N a left dg-A-module with structure maps $\mu_M\colon M \otimes A \to M$ and $\mu_N\colon A \otimes N \to N$, respectively. The two-sided bar construction of the triple (M, A, N) is

$$B(M, A, N) = M \otimes BA \otimes N$$

with differential $d_{B(M,A,N)} = d_{M\otimes BA\otimes N} + \partial'$, where

$$\partial' = (\mu_M(1 \otimes \alpha) \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1) - (1 \otimes 1 \otimes \mu_N(\alpha \otimes 1))(1 \otimes \Delta \otimes 1), \qquad (2)$$

and with augmentation

$$\varepsilon_{B(M,A,N)}\colon B(M, A, N) \to M \otimes_A N,$$
$$m[a_1|\dots|a_k]n \mapsto \begin{cases} m \otimes n & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We write repeated (co)associative maps in the form

$$\mu^{(k)}\colon A^{\otimes k} \to A,$$
$$\Delta^{(k)}\colon T(s^{-1}A) \to T(s^{-1}A)^{\otimes k},$$

for instance, and we agree that $\mu^{(0)}$ is the unit map $\iota\colon R \to A$.

We will also need the concatenation operator

$$\nabla\colon BA \otimes BA \to BA, \quad [a_1|\cdots|a_k] \otimes [b_1|\cdots|b_l] \mapsto [a_1|\cdots|a_k|b_1|\cdots|b_l],$$

which satisfies

$$d(\nabla) = \nabla^{(3)}(1 \otimes \tilde{\mu} \otimes 1)(1 \otimes \alpha \otimes \alpha \otimes 1)(\Delta \otimes \Delta) \qquad (3)$$

and

$$\begin{aligned}(\alpha \otimes 1)\Delta\nabla &= (\alpha \otimes \nabla)(\Delta \otimes 1) + \varepsilon_{BA} \otimes (\alpha \otimes 1)\Delta \\ &= (1 \otimes \nabla)\big((\alpha \otimes 1)\Delta \otimes 1\big) + \varepsilon_{BA} \otimes (\alpha \otimes 1)\Delta, \qquad (4a) \\ (1 \otimes \alpha)\Delta\nabla &= (\nabla \otimes \alpha)(1 \otimes \Delta) + (1 \otimes \alpha)\Delta \otimes \varepsilon_{BA} \\ &= (\nabla \otimes 1)\big(1 \otimes (1 \otimes \alpha)\Delta\big) + (1 \otimes \alpha)\Delta \otimes \varepsilon_{BA}. \qquad (4b)\end{aligned}$$

On both the unnormalized and the normalized bar construction, we will only consider multiplications which are coalgebra maps and have the counit $\mathbf{1}$ as a (two-sided) unit. We do not require the multiplication to be associative.

Any such multiplication $f\colon BA \otimes BA \to BA$ is uniquely determined by its twisting cochain $E = \alpha f$, which satisfies

$$d(E) = E \cup E,$$
$$E(\mathbf{1}, -) = E(-, \mathbf{1}) = \alpha.$$

We will only consider twisting cochains E satisfying both conditions.

Any multiplication on the normalized bar construction $\bar{B}A \subset BA$ can be extended to BA in a canonical way: Define $E([1], \mathbf{1}) = E(\mathbf{1}, [1]) = 1$ and, for $a = [a_1|\cdots|a_k]$,

$b = [b_1|\cdots|b_l] \in BA$, set $E(a,b) = 0$ if $k+l > 1$ and some $a_i = 1$ or some $b_j = 1$. Then $E(a,b) \in \bar{A}$ whenever $k+l > 1$. We call a twisting cochain having these additional properties *normalized*. Any normalized twisting cochain $E\colon BA \otimes BA \to A$ comes from a unique multiplication on $\bar{B}A$.

For a map $E\colon BA \otimes BA \to A$ and $a \in BA$ we define

$$E_a\colon BA \to A, \quad b \mapsto E(a,b).$$

In this notation, the properties of a multiplication on BA become

$$d(E_a) = -E_{da} + \sum_{i=0}^{k} (-1)^{|[a_1|\cdots|a_i]|} \mu\big(E_{[a_1|\cdots|a_i]} \otimes E_{[a_{i+1}|\cdots|a_k]}\big)\Delta, \tag{5a}$$

$$E_{\mathbf{1}}(b) = \alpha(b), \tag{5b}$$

$$E_a(\mathbf{1}) = \alpha(a), \tag{5c}$$

for $a = [a_1|\cdots|a_k]$ and $b = [b_1|\cdots|b_l] \in BA$. If E is normalized, then one additionally has

$$E_a(b) = 0 \quad \text{if } k+l > 1 \text{ and some } a_i = 1 \text{ or some } b_j = 1, \tag{6a}$$

$$\varepsilon(E_a(b)) = 0 \quad \text{if } k+l > 1. \tag{6b}$$

If E is of level 3, then condition (5a) is equivalent to the two identities

$$d(E_{[a_1]}) = -E_{d[a_1]} + \mu\big(\alpha \otimes E_{[a_1]} + (-1)^{|a_1|-1} E_{[a_1]} \otimes \alpha\big)\Delta, \tag{7a}$$

$$E_{[a_1 a_2]} = (-1)^{|a_1|-1} \mu(E_{[a_1]} \otimes E_{[a_2]})\Delta. \tag{7b}$$

3. Construction of the twisting cochain

Let A' and A'' be two level 3 Hirsch algebras with twisting cochains E' and E'', respectively. Set $A = A' \otimes A''$. We are going to inductively define maps $G_a\colon BA \to B(A,A,A)$ of degree $|a|+1$ for $a \in BA$ and then set $E_a = \varepsilon_{B(A,A,A)} G_a$. In Section 4 we will show that this defines a twisting cochain $E\colon BA \otimes BA \to A$, hence a multiplication in BA. Moreover, if both E and E'' are normalized, then so is E.

For the construction as well as for the proof, it is convenient to identify $B(A,A,A)$ with $A \otimes BA \otimes A$. This is an isomorphism of graded R-modules; the difference between the two differentials is given by (2). We write $a = [a_1|\cdots|a_k] \in BA$ with $a_i = a_i' \otimes a_i''$.

For $k = 0$ we set $E_{\mathbf{1}} = \alpha$ as required by (5b). We define for $k = 1$

$$G_{[a_1]} = \big((E'_{[a_1']} \otimes \mu_{A''}) \otimes 1 \otimes (\mu_{A'} \otimes E''_{[a_1'']})\big)\Delta^{(3)}$$

and for $k > 1$

$$G_a = M(E'_{[a_1']}, E''_{[a_1'']}, G_{[a_2|\cdots|a_k]}).$$

Here we have used the abbreviation

$$M(\tilde{E}', \tilde{E}'', \tilde{G}) = (1 \otimes 1 \otimes \mu_A)\big((\tilde{E}' \otimes \mu_{A''}) \otimes 1 \otimes (\mu_{A'} \otimes \tilde{E}'') \otimes 1\big) \\ (1 \otimes \Delta\nabla^{(3)} \otimes 1)(1 \otimes 1 \otimes (\sigma^{-1} \otimes 1 \otimes 1)\tilde{G})\Delta^{(3)} \tag{8}$$

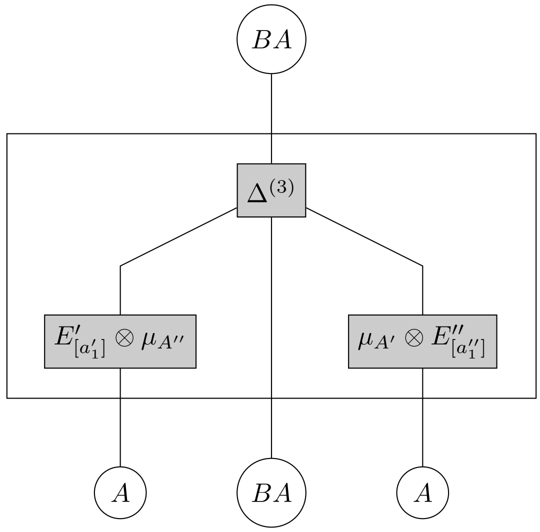

Figure 1: “Electronic diagram” for $G_{[a_1]}$

for maps $\tilde{E}'\colon BA' \to A'$, $\tilde{E}''\colon BA'' \to A''$ and $\tilde{G}\colon BA \to A \otimes BA \otimes A$. Moreover, by $\tilde{E}' \otimes \mu_{A''}\colon BA \to A$ we mean the map

$$[b_1|\cdots|b_k] \mapsto \Bigl(\prod_{i>j}(-1)^{(|b_i'|-1)|b_j''|}\Bigr)\tilde{E}'([b_1'|\cdots|b_k']) \otimes \mu_{A''}(b_1'' \otimes \cdots \otimes b_k''),$$

and similarly by $\mu_{A'} \otimes \tilde{E}''\colon BA \to A$

$$[b_1|\cdots|b_k] \mapsto \Bigl(\prod_{i>j}(-1)^{|b_i'|(|b_j''|-1)}\Bigr)\mu_{A'}(b_1' \otimes \cdots \otimes b_k') \otimes \tilde{E}''([b_1''|\cdots|b_k'']).$$

By identities (1), the differentials of these maps are

$$d(\tilde{E}' \otimes \mu_{A''}) = d(\tilde{E}') \otimes \mu_{A''}, \qquad d(\mu_{A'} \otimes \tilde{E}'') = \mu_{A'} \otimes d(\tilde{E}'').$$

Figures 1 and 2 visualize the definitions of $G_{[a_1]}$ and of $M(\tilde{E}', \tilde{E}'', \tilde{G})$.

Example 3.1. The following list shows $E(a,b)$ for $a \in B_kA$ and $b \in B_lA$ with $k \leqslant 2$ and $l \leqslant 2$. We are ignoring signs here.

$$E([a_1],[b_1]) = a_1'b_1' \otimes E''([a_1''],[b_1'']) + E'([a_1'],[b_1']) \otimes b_1''a_1'', \tag{9}$$

$$\begin{aligned} E([a_1],[b_1|b_2]) &= a_1'b_1'b_2' \otimes E''([a_1''],[b_1''|b_2'']) \\ &\quad + E'([a_1'],[b_1'])b_2' \otimes b_1''E''([a_1''],[b_2'']) \\ &\quad + E'([a_1'],[b_1'|b_2']) \otimes b_1''b_2''a_2'', \end{aligned} \tag{10}$$

$$E([a_1|a_2],[b_1]) = a_1'E'([a_2'],[b_1']) \otimes E''([a_1''],[b_1''])a_2'', \tag{11}$$

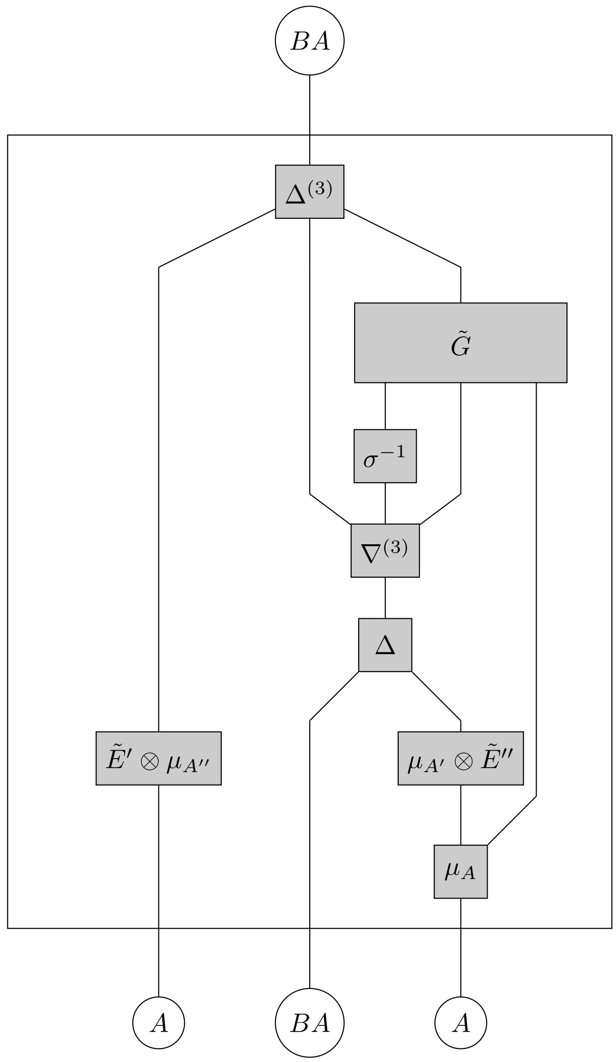

Figure 2: "Electronic diagram" for $M(\tilde{E}', \tilde{E}'', \tilde{G})$

$$\begin{aligned} E([a_1|a_2],[b_1|b_2]) &= a_1' E'([a_2'],[b_1'])b_2' \otimes E''([a_1''],[b_1''])E''([a_2''],[b_2'']) \\ &+ a_1' E'([a_2'],[b_1'])b_2' \otimes E''([a_1''],[b_1''|b_2''])a_2'' \\ &+ a_1' E'([a_2'],[b_1'|b_2']) \otimes E''([a_1''],[b_1''b_2''])a_2'' \\ &+ a_1' b_1' E'([a_2'],[b_2']) \otimes E''([a_1''],[b_1''|b_2''])a_2'' \\ &+ E'([a_1'],[b_1'])E'([a_2'],[b_2']) \otimes b_1'' E''([a_1''],[b_2''])a_2''. \end{aligned} \tag{12}$$

Example 3.2. We give a general recipe for computing $E(a,b)$ as in Example 3.1. To show all features of the algorithm, we illustrate it with $a = \left[a_1' \otimes a_1'' \;\middle|\; a_2' \otimes a_2''\right]$ and $b = \left[b_1' \otimes b_1'' \;\middle|\; \cdots \;\middle|\; b_5' \otimes b_5''\right]$. We are going to explain how to obtain the terms $c' \otimes c'' \in A' \otimes A''$ appearing in $E(a,b)$, again ignoring signs for simplicity.

We start by looking at the component $c' \in A'$. Take $[b_1'|\cdots|b_l']$ and cut it into $2k$ pieces such that the pieces at positions 3, 5, …, $2k-1$ have length at least 1. In our example, one such decomposition is

$$[b_1'] \otimes [b_2'|b_3'] \otimes [b_4'|b_5'] \otimes \mathbf{1}.$$

(The last piece has length 0.) Now apply $E'_{[a_i']}$ to the $(2i-1)$-th group and then multiply everything together:

$$E'_{[a_1']}([b_1']) \cdot b_2' b_3' \cdot E'_{[a_2']}([b_4'|b_5']) \cdot 1 = E'([a_1'],[b_1'])b_2'b_3'E'([a_2'],[b_4'|b_5']) = c'.$$

These are the possible factors $c' \in A'$ of the terms $c' \otimes c''$ appearing in $E(a,b)$.

For each such factor, we now describe which factors $c'' \in A''$ appear: Switch from primed to doubly primed variables and multiply the components within the odd-numbered groups together to obtain

$$\left[b_1'' \;\middle|\; b_2'' \;\middle|\; b_3'' \;\middle|\; b_4''b_5''\right].$$

Take the first factor of the tensor product (in the example, $[b_1'']$) apart. Cut the rest

$$\left[b_2'' \;\middle|\; b_3'' \;\middle|\; b_4''b_5''\right]$$

into k pieces. Only cuts satisfying the following condition are allowed: If some b_j' appears as argument to $E'_{[a_i']}$, then the corresponding element b_j'' can only appear in the $(i-1)$-th piece or earlier. In our example, this forces the second piece to be empty, hence the first piece is everything. Now plug the i-th piece into $E''_{[a_i'']}$ and multiply everything together, including the first factor we have put apart earlier:

$$b_1'' \cdot E''_{[a_1'']}([b_2''|b_3''|b_4''b_5'']) \cdot E''_{[a_1'']}(\mathbf{1}) = b_1'' E''([a_1''],[b_2''|b_3''|b_4''b_5''])a_1'' = c''.$$

Summing up,

$$E'([a_1'],[b_1'])b_2'b_3'E'([a_2'],[b_4'|b_5']) \otimes b_1''E''([a_1''],[b_2''|b_3''|b_4''b_5''])a_1''$$

is one term appearing in $E(a,b)$. (There are 70 terms altogether.)

The reason for the length condition imposed in the first step is the following: The recursive definition of G_a together with the assignment $E_a = \varepsilon G_a$ force everything that "runs through" $E'_{[a_i']} \otimes \mu$, $i > 1$, to "go through" some $\mu \otimes E''_{[a_j'']}$ with $j < i$ as well. Because $(E'_{[a_i']} \otimes \mu)(\mathbf{1}) = a_i' \otimes 1$ and $E''_{[a_j'']}(1) = 0$, the length of the argument of $E'_{[a_i']}$ must therefore be at least 1 if $i > 1$.

Remark 3.3. The multiplication in $\bar{B}(A' \otimes A'')$ is not associative in general, not even if it is so in $\bar{B}A'$ and $\bar{B}A''$ (which means that A' and A'' are homotopy Gerstenhaber algebras). In the latter case one has

$$\big([a] \cdot [b]\big) \cdot [c] + [a] \cdot \big([b] \cdot [c]\big) = d(h)([a], [b], [c])$$

for $a = a' \otimes a''$, $b = b' \otimes b''$, $c = c' \otimes c'' \in A' \otimes A''$ and

$$\begin{aligned} h([a], [b], [c]) = &\big[a' E([b'], [c']) \otimes E([a''], [c''|b''])\big] \\ &+ \big[E([a'], [b'|c']) \otimes E([b''], [c''])\, a''\big]. \end{aligned}$$

(We are again ignoring signs here.)

Question 3.4. Is $\bar{B}(A' \otimes A'')$ an A_∞-algebra if A' and A'' are homotopy Gerstenhaber algebras?

4. Proof of the main result

In Section 3 we constructed a map $G_a \colon BA \to B(A, A, A)$ for each $a \in BA$. They can be assembled into a map $G \colon BA \otimes BA \to B(A, A, A)$. We now study its differential.

Denote the left and right action of A on $B(A, A, A)$ by μ_L and μ_R, respectively, and let β be the twisting cochain

$$\beta = \varepsilon_{BA} \otimes \alpha_{BA} \colon BA \otimes BA \to R \otimes A = A.$$

Proposition 4.1. *The differential of G is*

$$d(G) = \mu_L(\beta \otimes G)\Delta_{BA \otimes BA} + \mu_R(G \otimes (E - \beta))\Delta_{BA \otimes BA}.$$

Proof. We again identify $B(A, A, A)$ with $A \otimes BA \otimes A$. Taking equation (2) into account, we have to show

$$\begin{aligned} d(G_a) = &-G_{da} \\ &+ (\mu \otimes 1 \otimes 1)(\alpha \otimes G_a)\Delta_{BA} \\ &+ \sum_{i=1}^{k} (-1)^{|[a_1|\cdots|a_i]|} (1 \otimes 1 \otimes \mu)(G_{[a_1|\cdots|a_i]} \otimes E_{[a_{i+1}|\cdots|a_k]})\Delta_{BA} \\ &- (\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G_a \\ &+ (1 \otimes 1 \otimes \mu)(1 \otimes 1 \otimes \alpha \otimes 1)(1 \otimes \Delta \otimes 1)G_a \end{aligned} \tag{13}$$

for all $a = [a_1|\cdots|a_k] \in BA$ We proceed by induction on k. Write $\tilde{E}' = E'_{[a'_1]}$ and $\tilde{E}'' = E''_{[a''_1]}$. Recall that we have

$$\big|E'_{[a'_1]}\big| = |a'_1|, \qquad \big|E''_{[a''_1]}\big| = |a''_1|, \qquad \big|G_a\big| = |a| + 1.$$

For $k = 1$, *i.e.*, $a = [a'_1 \otimes a''_1] \in s^{-1}A$, we have

$$\begin{aligned} d(G_a) = &\big((d(E'_{[a'_1]}) \otimes \mu) \otimes 1 \otimes (\mu \otimes E''_{[a''_1]})\big)\Delta^{(3)} \\ &+ (-1)^{|a'_1|}\big((E'_{[a'_1]} \otimes \mu) \otimes 1 \otimes (\mu \otimes d(E''_{[a''_1]}))\big)\Delta^{(3)} \end{aligned} \tag{14}$$

using formula (7a)

$$\begin{aligned}
&= -\big((E'_{d[a'_1]} \otimes \mu) \otimes 1 \otimes (\mu \otimes E''_{[a''_1]})\big)\Delta^{(3)} \\
&\quad - (-1)^{|a'_1|}\big((E'_{[a'_1]} \otimes \mu) \otimes 1 \otimes (\mu \otimes E''_{d[a''_1]})\big)\Delta^{(3)} \\
&\quad + (\mu \otimes 1 \otimes 1)\big(\alpha \otimes (E'_{[a'_1]} \otimes \mu) \otimes 1 \otimes (\mu \otimes E''_{[a''_1]})\big)\Delta^{(4)} \\
&\quad + (-1)^{|a'_1|-1}(\mu \otimes 1 \otimes 1)\big((E'_{[a'_1]} \otimes \mu) \otimes \alpha \otimes 1 \otimes (\mu \otimes E''_{[a''_1]})\big)\Delta^{(4)} \\
&\quad + (-1)^{|a'_1|}(1 \otimes 1 \otimes \mu)\big((E'_{[a'_1]} \otimes \mu) \otimes 1 \otimes \alpha \otimes (\mu \otimes E''_{[a''_1]})\big)\Delta^{(4)} \\
&\quad + (-1)^{|a'_1|+|a''_1|-1}(1 \otimes 1 \otimes \mu) \\
&\qquad \big((E'_{[a'_1]} \otimes \mu) \otimes 1 \otimes (\mu \otimes E''_{[a''_1]}) \otimes \alpha\big)\Delta^{(4)} \qquad (15) \\
&= -G_{da} \\
&\quad + (\mu \otimes 1 \otimes 1)(\alpha \otimes G_{[a_1]})\Delta \\
&\quad + (-1)^{|[a_1]|}(1 \otimes 1 \otimes \mu)(G_{[a_1]} \otimes E_{\mathbf{1}})\Delta \\
&\quad - (\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G_a \\
&\quad + (1 \otimes 1 \otimes \mu)(1 \otimes 1 \otimes \alpha \otimes 1)(1 \otimes \Delta \otimes 1)G_a. \qquad (16)
\end{aligned}$$

For $k > 1$, we write $\tilde{a} = [a_2|\cdots|a_k]$ and $\tilde{G} = G_{\tilde{a}}$. Then, using definition (8),

$$\begin{aligned}
d(G_a) &= d\big(M(\tilde{E}', \tilde{E}'', \tilde{G})\big) \qquad (17) \\
&= M(d(\tilde{E}'), \tilde{E}'', \tilde{G}) + (-1)^{|a'_1|}M(\tilde{E}', d(\tilde{E}''), \tilde{G}) \\
&\quad + (-1)^{|a'_1|+|a''_1|}(1 \otimes 1 \otimes \mu_A)\big((\tilde{E}' \otimes \mu_{A''}) \otimes 1 \otimes (\mu_{A'} \otimes \tilde{E}'') \otimes 1\big) \\
&\qquad (1 \otimes \Delta d(\nabla^{(3)}) \otimes 1)(1 \otimes 1 \otimes \sigma^{-1} \otimes 1 \otimes 1)(1 \otimes 1 \otimes \tilde{G})\Delta^{(3)} \\
&\quad + (-1)^{|a'_1|+|a''_1|-1}M(\tilde{E}', \tilde{E}'', d(\tilde{G})) \qquad (18)
\end{aligned}$$

using (3), $\tilde{\mu}(1 \otimes \sigma^{-1}) = \sigma^{-1}\mu(\sigma \otimes 1)$ and $\tilde{\mu}(\sigma^{-1} \otimes 1) = -\sigma^{-1}\mu(1 \otimes \sigma)$

$$\begin{aligned}
&= M(d(\tilde{E}'), \tilde{E}'', \tilde{G}) + (-1)^{|a'_1|}M(\tilde{E}', d(\tilde{E}''), \tilde{G}) \\
&\quad + (-1)^{|a_1|}M\big(\tilde{E}', \tilde{E}'', (\mu \otimes 1 \otimes 1)(\alpha \otimes \tilde{G})\Delta\big) \\
&\quad + (-1)^{|a_1|-1}M\big(\tilde{E}', \tilde{E}'', (\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)\tilde{G}\big) \\
&\quad + (-1)^{|a_1|-1}M(\tilde{E}', \tilde{E}'', d(\tilde{G}))\ ; \qquad (19)
\end{aligned}$$

$$\begin{aligned}
G_{d_\otimes a} = M(E'_{d[a'_1]}, \tilde{E}'', \tilde{G}) &+ (-1)^{|a'_1|}M(\tilde{E}', E''_{d[a''_1]}, \tilde{G}) \\
&+ (-1)^{|a_1|-1}M(\tilde{E}', \tilde{E}'', G_{d_\otimes \tilde{a}})\ ; \qquad (20)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=2}^{k} M(\tilde{E}', \tilde{E}'', (1 \otimes 1 \otimes \mu)(G_{[a_2|\cdots|a_i]} \otimes E_{[a_{i+1}|\cdots|a_k]})\Delta) \\
&\qquad = \sum_{i=2}^{k}(1 \otimes 1 \otimes \mu)(G_{[a_1|\cdots|a_i]} \otimes E_{[a_i|\cdots|a_k]})\Delta\ ; \qquad (21)
\end{aligned}$$

$$M(\tilde{E}', \tilde{E}'', G_{\partial\tilde{a}}) = \sum_{i=2}^{k-1} (-1)^{|[a_2|\cdots|a_i]|} G_{[a_1|\cdots|a_i a_{i+1}|\cdots|a_k]} \;; \tag{22}$$

$$M(\mu(\alpha \otimes \tilde{E}')\Delta, \tilde{E}'', \tilde{G}) = (\mu \otimes 1 \otimes 1)(\alpha \otimes G)\Delta \;; \tag{23}$$

$$M(\tilde{E}', \mu(\alpha \otimes \tilde{E}'')\Delta, \tilde{G}) = (-1)^{|a_1'|}(1 \otimes 1 \otimes \mu)(1 \otimes 1 \otimes \alpha \otimes 1)(1 \otimes \Delta \otimes 1)G \;; \tag{24}$$

and

$$\begin{aligned} &M(\mu(\tilde{E}' \otimes \alpha)\Delta, \tilde{E}'', \tilde{G}) \\ &\quad = (-1)^{|a_1''|}(\mu \otimes 1 \otimes \mu)\big((\tilde{E}' \otimes \mu) \otimes 1 \otimes 1 \otimes (\mu \otimes \tilde{E}'') \otimes 1\big) \\ &\qquad (1 \otimes 1 \otimes \Delta\nabla^{(3)} \otimes 1)(1 \otimes (\alpha \otimes 1)\Delta \otimes (\sigma^{-1} \otimes 1 \otimes 1)\tilde{G})\Delta^{(3)} \end{aligned} \tag{25}$$

using (4a) and the fact that $\tilde{G}$ maps to $A \otimes BA \otimes A$

$$\begin{aligned} &= (-1)^{|a_1''|}(\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)M(\tilde{E}', \tilde{E}'', \tilde{G}) \\ &\quad + (-1)^{|a_1''|}(1 \otimes 1 \otimes \mu) \\ &\qquad \big(1 \otimes 1 \otimes (\mu \otimes \tilde{E}'') \otimes 1\big)(\mu \otimes \Delta \otimes 1)\big((\tilde{E}' \otimes \mu) \otimes \tilde{G}\big)\Delta. \end{aligned} \tag{26}$$

We consider the case $k = 2$ first.

$$\begin{aligned} &= (-1)^{|a_1''|}(\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)M(\tilde{E}', \tilde{E}'', \tilde{G}) \\ &\quad + (-1)^{|a_1''|}(1 \otimes 1 \otimes \mu) \\ &\qquad \big(1 \otimes 1 \otimes (\mu \otimes \tilde{E}'') \otimes 1\big)(\mu \otimes \Delta \otimes 1)\big((\tilde{E}' \otimes \mu) \otimes G_{[a_2]}\big)\Delta \end{aligned} \tag{27}$$

using (7b) in the form $\mu\big((E'_{[a_1']} \otimes \mu) \otimes (E'_{[a_2']} \otimes \mu)\big)\Delta = (-1)^{|a_1'|-1} E'_{[a_1' a_2']} \otimes \mu$

$$\begin{aligned} &= (-1)^{|a_1''|}(\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G \\ &\quad + (-1)^{|a_1|+|a_1'||a_1''|-1}(1 \otimes 1 \otimes \mu)\big(1 \otimes 1 \otimes (\mu \otimes E''_{[a_1'']}) \otimes 1\big) \\ &\qquad (1 \otimes \Delta \otimes 1)G_{[a_1' a_2' \otimes a_2'']}\Delta \end{aligned} \tag{28}$$

using (7b) in the form $\mu\big((\mu \otimes E''_{[a_1'']}) \otimes (\mu \otimes E''_{[a_2'']})\big)\Delta = (-1)^{|a_1''|-1} \mu \otimes E''_{[a_1'' a_2'']}$

$$\begin{aligned} &= (-1)^{|a_1''|}(\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G \\ &\quad + (-1)^{|a_1'|+|a_1''||a_2'|} G_{[a_1' a_2' \otimes a_1'' a_2'']} \end{aligned} \tag{29}$$

using $a_1 a_2 = (-1)^{|a_1''||a_2'|} a_1' a_2' \otimes a_1'' a_2''$

$$\begin{aligned} &= (-1)^{|a_1''|}(\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G \\ &\quad + (-1)^{|a_1'|} G_{[a_1 a_2]}. \end{aligned} \tag{30}$$

Continuing at (26) for $k > 2$ and using the same identities as before,

$$\begin{aligned}
&= (-1)^{|a_1''|}(\mu\otimes 1\otimes 1)(1\otimes\alpha\otimes 1\otimes 1)(1\otimes\Delta\otimes 1)G \\
&\quad + (-1)^{|a_1|+|a_1'||a_1''|-1}(1\otimes 1\otimes\mu)\big(1\otimes 1\otimes(\mu\otimes E''_{[a_1'']})\otimes 1\big) \\
&\qquad (1\otimes\Delta\otimes 1)M(E'_{[a_1' a_2']}, E''_{[a_2'']}, G_{[a_3|\cdots|a_k]}) \qquad (31)\\
&= (-1)^{|a_1''|}(\mu\otimes 1\otimes 1)(1\otimes\alpha\otimes 1\otimes 1)(1\otimes\Delta\otimes 1)G \\
&\quad + (-1)^{|a_1'|+|a_1''||a_2'|}M(E'_{[a_1' a_2']}, E''_{[a_1'' a_2'']}, G_{[a_3|\cdots|a_k]}) \qquad (32)\\
&= (-1)^{|a_1''|}(\mu\otimes 1\otimes 1)(1\otimes\alpha\otimes 1\otimes 1)(1\otimes\Delta\otimes 1)G \\
&\quad + (-1)^{|a_1'|}G_{[a_1 a_2|a_3|\cdots|a_k]}. \qquad (33)
\end{aligned}$$

So the result is the same for all $k \geqslant 2$.

$$\begin{aligned}
&M(\tilde{E}', \mu(\tilde{E}''\otimes\alpha)\Delta, \tilde{G}) \\
&\quad = (1\otimes 1\otimes\mu(\mu\otimes 1))\big((\tilde{E}'\otimes\mu)\otimes 1\otimes(\mu\otimes\tilde{E}'')\otimes\alpha\otimes 1\big) \\
&\qquad (1\otimes\Delta^{(3)}\nabla^{(3)}\otimes 1)(1\otimes 1\otimes\sigma^{-1}\otimes 1\otimes 1)(1\otimes 1\otimes\tilde{G})\Delta^{(3)} \qquad (34)
\end{aligned}$$

using (4b)

$$\begin{aligned}
&= -M\big(\tilde{E}', \tilde{E}'', (1\otimes 1\otimes\mu)(1\otimes 1\otimes\alpha\otimes 1)(1\otimes\Delta\otimes 1)\tilde{G}\big) \\
&\quad + (1\otimes 1\otimes\mu(1\otimes\mu))\big((\tilde{E}'\otimes\mu)\otimes 1\otimes(\mu\otimes\tilde{E}'')\otimes 1\otimes 1\big) \\
&\qquad (1\otimes\Delta\otimes 1\otimes\varepsilon\otimes 1)(1\otimes 1\otimes\tilde{G})\Delta^{(3)} \qquad (35)\\
&= -M\big(\tilde{E}', \tilde{E}'', (1\otimes 1\otimes\mu)(1\otimes 1\otimes\alpha\otimes 1)(1\otimes\Delta\otimes 1)\tilde{G}\big) \\
&\quad + (1\otimes 1\otimes\mu)(G_{[a_1]}\otimes E_{\tilde{a}})\Delta. \qquad (36)
\end{aligned}$$

Putting all terms together finishes the proof. □

Proposition 4.2. *The map $E\colon BA\otimes BA\to A$ is a twisting cochain. Moreover, if E' and E'' are normalized, then so is E.*

Proof. To verify (5a), we compute:

$$\begin{aligned}
d(E_a) &= \varepsilon\, d(G_a) \\
&= -\varepsilon\, G_{da} + \mu(\alpha\otimes E_a)\Delta \\
&\quad + \sum_{i=1}^{k}(-1)^{|[a_1|\cdots|a_i]|}\mu(E_{[a_1|\cdots|a_i]}\otimes E_{[a_{i+1}|\cdots|a_k]})\Delta \\
&= -E_{da} + \sum_{i=0}^{k}(-1)^{|[a_1|\cdots|a_i]|}\mu\big(E_{[a_1|\cdots|a_i]}\otimes E_{[a_{i+1}|\cdots|a_k]}\big)\Delta.
\end{aligned}$$

Condition (5b) holds by definition. Condition (5c) holds for $k = 1$ because $G_{[a_1]}(\mathbf{1}) = (a_1'\otimes 1)\otimes\mathbf{1}\otimes(1\otimes a_1'')$. For $k > 1$, one similarly has $G_a(\mathbf{1}) \in (A'\otimes 1)\otimes BA\otimes A$, hence $\varepsilon(G_a(\mathbf{1})) = 0$ by condition (5c) for E''. (This is related to the length condition in Example 3.2.)

Assume now that E' and E'' are normalized. For the proof of (6a) one inductively shows $G_a(b) \in \bigoplus_{m\geqslant 1} A\otimes B_m A\otimes A$ if some $a_i = 1$ or some $b_j = 1$. For (6b), notice that the image of $E'_{[a'_1]} \otimes \mu$ lies in $\bar{A}' \otimes A'' \subset \bar{A}$ if $a'_1 \in \bar{A}'$, and analogously for $\mu \otimes E''_{[a''_1]}$. Hence, $\varepsilon(G_a(b)) \in \bar{A}$ if $k \geqslant 1$ and $a_1 \in \bar{A}$. □

We now turn to the shuffle maps

$$\begin{aligned} \nabla\colon BA' \otimes BA'' &\to B(A' \otimes A''), \\ \nabla\colon \bar{B}A' \otimes \bar{B}A'' &\to \bar{B}(A' \otimes A''), \end{aligned} \tag{37}$$

cf. [**8**, Sec. 7.1].

Proposition 4.3. *The shuffle maps* (37) *are multiplicative.*

Proof. It suffices to consider the unnormalized bar construction. We have to show that the diagram

$$\begin{array}{ccc} (BA' \otimes BA'') \otimes (BA' \otimes BA'') & \xrightarrow{\nabla \otimes \nabla} & B(A' \otimes A'') \otimes B(A' \otimes A'') \\ \downarrow & & \\ (BA' \otimes BA') \otimes (BA'' \otimes BA'') & & \downarrow \mu \\ \downarrow \mu \otimes \mu & & \\ BA' \otimes BA'' & \xrightarrow{\nabla} & B(A' \otimes A'') \end{array}$$

commutes. Because all maps are morphisms of coalgebras, it is enough to verify that the two associated twisting cochains coincide.

Take two elements $a = a' \otimes a'' \in B_{p'}A' \otimes B_{p''}A''$ and $b = b' \otimes b'' \in B_{q'}A' \otimes B_{q''}A''$. The twisting cochain of the composition via $BA' \otimes BA''$ vanishes unless $p' = q' = 0$ or $p'' = q'' = 0$. Consider now the twisting cochain of the other composition. It follows from properties (5b) and (5c) and the inductive definition of G_a that for $p' > 0$ this twisting cochain vanishes if $p'' > 0$ or $q'' > 0$. The case $p'' > 0$ is analogous. It is therefore enough to check the two cases $a = a' \otimes \mathbf{1}$, $b = b' \otimes \mathbf{1}$ and $a = \mathbf{1} \otimes a''$, $b = \mathbf{1} \otimes b''$. That both twisting cochains agree follows again inductively from the definition of G_a. □

5. Operadic reformulation

It is useful to translate Theorem 1.1 into the language of operads. Let $\mathcal{A}ss$ be the operad of associative augmented unital R-algebras. We write $\mu \in \mathcal{A}ss(2)$ for the multiplication, $\varepsilon \in \mathcal{A}ss(1)$ for the augmentation and $\iota \in \mathcal{A}ss(0)$ for the unit. An operad under $\mathcal{A}ss$ is a morphism of operads $\mathcal{A}ss \to \mathcal{P}$.

We define the *Hirsch operad* $\mathcal{H}$ to be the dg operad under $\mathcal{A}ss$ generated by operations $E_{kl} \in \mathcal{H}(k+l)_{1-k-l}$ subject to the relations (5) and (6) (modulo the desuspension) plus the generators and relations for $\mathcal{A}ss$. A Hirsch algebra then is the same as an algebra over $\mathcal{H}$.

Let $\mathcal{H}_3$ be the dg operad under $\mathcal{A}ss$ describing level 3 Hirsch algebras. It is the quotient of $\mathcal{H}$ by the relations $E_{kl} = 0$ for $k > 1$. Equivalently, it is generated by operations $E_{1k} \in \mathcal{H}_3(1+k)_{-k}$ and E_{01} subject to the relations (5) and (6) with (5a) replaced by (7), and of course again plus the generators and relations for $\mathcal{A}ss$.

Theorem 5.1. *The construction in Section 3 defines a morphism* $f\colon \mathcal{H} \to \mathcal{H}_3 \otimes \mathcal{H}_3$ *of dg operads under* $\mathcal{A}ss$.

Proof. Let $\mathcal{P}$ be the free dg operad under $\mathcal{A}ss$ generated by the operations E_{kl}. It is clear that our construction defines a morphism of dg operads under $\mathcal{A}ss$

$$\mathcal{P} \to \mathcal{H}_3 \otimes \mathcal{H}_3.$$

Moreover, we know that the relations for $\mathcal{H}$ hold whenever $\mathcal{H}_3 \otimes \mathcal{H}_3$ acts on a tensor product of two $\mathcal{H}_3$-algebras A' and A''. More precisely, we have proven that the composed map

$$\mathcal{P} \to \mathcal{H}_3 \otimes \mathcal{H}_3 \to \mathcal{E}nd(A') \otimes \mathcal{E}nd(A'')$$

factors through $\mathcal{H}$. Because A' and A'' can be free $\mathcal{H}_3$-algebras (*cf.* [**7**, Sec. I.1.4]), this implies that the necessary relations hold already in $\mathcal{H}_3 \otimes \mathcal{H}_3$. □

Example 5.2. The homotopy Gerstenhaber algebra structure on the cochain complex $C^*(X)$ of a simplicial set X is constructed by dualizing a "homotopy Gerstenhaber coalgebra" structure on the chain complex $C(X)$. Therefore, for simplicial sets X and Y there is a natural action of $\mathcal{H}_3 \otimes \mathcal{H}_3$ on the complex dual to $C(X) \otimes C(Y)$, and the canonical map

$$C^*(X) \otimes C^*(Y) \to \big(C(X) \otimes C(Y)\big)^*$$

is a morphism of $\mathcal{H}_3 \otimes \mathcal{H}_3$-algebras, hence of $\mathcal{H}$-algebras. Note however that the dual of the shuffle map

$$C^*(X \times Y) \xrightarrow{\nabla^*} \big(C(X) \otimes C(Y)\big)^*$$

is *not* a morphism of Hirsch algebras. (∇^* already fails to commute with the operation (9).)

An analogous remark applies to Hochschild cochains.

Example 5.3. Let A be a cosemisimplicial $\mathcal{H}_3$-algebra. By this we mean a collection A^q, $q \geqslant 0$, of $\mathcal{H}_3$-algebras together with morphisms $d_i\colon A^q \to A^{q+1}$, $0 \leqslant i \leqslant q+1$, satisfying the usual coface relations, *cf.* [**8**, Def. 8.40]. Then the associated total complex $\operatorname{Tot} A^*$ is an algebra over $\mathcal{H}_3 \otimes \mathcal{H}_3$ in the following way: Let $E \otimes E' \in \mathcal{H}_3(m)_n \otimes \mathcal{H}_3(m)_{n'}$, and $a_i \in A^{q_i}$ for $1 \leqslant i \leqslant m$. Set $q = \sum_i q_i - n'$. Via the coface operators, E' determines morphisms $\phi_i\colon A^{q_i} \to A^q$ in the same way as it acts on the unnormalized cochains of a simplicial set. We can therefore set

$$(E \otimes E')(a_1, \dots, a_m) = E\big(\phi_1(a_1), \dots, \phi_m(a_m)\big) \in A^q.$$

(If $q < q_i$ for some i, we define the result to be 0.)

An important special case of this is the Mayer–Vietoris double complex

$$C^{pq}(\mathcal{U}) = \prod_{i_0 < \dots < i_q} C^p(U_{i_0} \cap \dots \cap U_{i_q}; R)$$

associated to an ordered cover $\mathcal{U} = (U_i)_{i \in I}$ of a simplicial set, *cf.* [**2**, §§8, 14] for instance. In this case Theorem 5.1 says that $\operatorname{Tot} C^{**}(\mathcal{U})$ has the structure of a Hirsch algebra which extends the familiar dg algebra structure. Note also that the canonical inclusion map

$$C^*(X;R) \to \operatorname{Tot} C^{**}(\mathcal{U}), \quad \alpha \mapsto \bigl(\alpha|_{U_i}\bigr)_{i\in I} \in C^{*0}(\mathcal{U};R)$$

is a morphism of Hirsch algebras because for $n' > 0$ the maps $\phi_1, \dots, \phi_m$ described above vanish on the image of the inclusion map, and for $n' = 0$ they must all be the identity map.

Remark 5.4. Assume $R = \mathbb{Z}_2$ and let $\tau = (12) \in S_2$. Note that μ is basis of $\mathcal{H}_3(2)_0$ over $R[S_2]$, and E_{11} is one for $\mathcal{H}_3(2)_{-1}$. A direct computation shows that up to applying τ and transposing the factors, $h = \mu \otimes E_{11} + E_{11} \otimes \tau\mu \in (\mathcal{H}_3 \otimes \mathcal{H}_3)(2)_{-1}$ is the only solution to $d(h) = \mu \otimes \mu + \tau\mu \otimes \tau\mu$. Hence, our definition (9) of $f(E_{11})$ is essentially the only possible choice. Together with $d(f(E_{21})) \neq 0$, this also proves that one cannot hope for a morphism $\mathcal{H}_3 \to \mathcal{H}_3 \otimes \mathcal{H}_3$ of dg operads under $\mathcal{A}ss$ because condition (7b) never holds.

But of course one may ask:

Question 5.5. Is $\mathcal{H}$ a dg Hopf operad under $\mathcal{A}ss$? In other words, is the tensor product of two Hirsch algebras again a Hirsch algebra?

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