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AN ALGORITHMIC APPROACH TO DOLD-PUPPE COMPLEXES

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Abstract

A Dold-Puppe complex is the image $NF\Gamma(C.)$ of a chain complex $C.$ under the composition of the functors Γ, F and N, where Γ and N are given by the Dold-Kan correspondence and F is a not necessarily linear functor between two abelian categories. The first half of this paper gives an algorithm that streamlines the calculation of $\Gamma(C.)$. The second half gives an algorithm that allows the explicit calculation of the Dold-Puppe complex $NF\Gamma(C.)$ in terms of the cross-effect functors of F.

Introduction

Let R and S be rings. The construction of the left derived functors $L_kF\colon R$-**mod** $\to S$-**mod** of any covariant right exact functor $F\colon R$-**mod** $\to S$-**mod** is achieved by applying three functors. The first functor constructs a projective resolution $P.$ of the R-module M of which we wish to calculate the derived functor. Then the functor F is applied to the resolution $P.$ giving the chain complex $F(P.)$. Lastly, $L_kF(M)$ is defined to be $H_k(F(P.))$, the k^{th} homology of the chain complex $F(P.)$. However, for a given module M the projective resolution of M is unique only up to chain-homotopy equivalence, so this construction crucially depends on the fact that F preserves chain homotopies. In general this fact does not hold when F is a non-linear functor such as the l^{th} symmetric power functor, Sym^l, or the l^{th} exterior power functor, Λ^l. In the paper [**DP**] Dold and Puppe overcome this problem and define the derived functors of non-linear functors by passing to the category of simplicial complexes using the Dold-Kan correspondence.

The Dold-Kan correspondence gives a pair of functors Γ and N that provide an equivalence between the category of bounded chain complexes and the category of simplicial complexes; under this correspondence chain homotopies correspond to simplicial homotopies. Furthermore, in the simplicial world all functors preserve simplicial homotopy (not just linear functors). Because of this the above definition of the derived functors of F becomes well defined for any functor when $F(P.)$ is replaced by the complex $NF\Gamma(P.)$. We call chain complexes of the form $NF\Gamma(C.)$ Dold-Puppe complexes, for any bounded chain complex $C.$.

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Let R be a ring and let I be an ideal in R that is locally generated by a non-zero divisor. If $P.$ is a length-one R-projective resolution of a projective R/I-module V, then the homology of the Dold-Puppe complex $N \operatorname{Sym}^k \Gamma(P.)$, $k \geqslant 1$, has been explicitly computed in [**Kö**]. These computations yield a very natural and new proof of the classical Adams-Riemann-Roch theorem for regular closed immersions and hence a new approach to the seminal Grothendieck-Riemann-Roch theorem avoiding the comparatively involved deformation to the normal cone; see [**Kö**].

If $C.$ is a chain complex of length bigger than 1, then the calculation of the Dold-Puppe complex $NF\Gamma(C.)$ is normally too complicated to be performed on a couple of pieces of paper, and the nature of the calculation means that errors easily creep in. In this paper we analyse and elucidate its combinatorial structure, and exploiting this structure that we have revealed, we develop an algorithm that computes this Dold-Puppe complex. We hope that this explicit description of the Dold-Puppe complex will help later work in calculating its homology, particularly in concrete example situations. Moreover, we expect that it will be useful in computing maps between the homology of different Dold-Puppe complexes, such as the plus and diagonal maps occurring in [**Kö**]: for such calculations one often has to find representatives on the complex level for elements of the homology.

We now describe the contents of each section in more detail.

In Section 1 we introduce an ordering on the set $\operatorname{Mor}([n],[k])$ of order-preserving maps between $[n] := \{0 < 1 < \cdots < n\}$ and $[k] := \{0 < 1 < \cdots < k\}$ (see Definition 1.9). Basically the entire paper is based on this crucial definition. We show at the end of Section 1 that composition with the face maps $\delta_i\colon [n-1] \to [n]$ and degeneracy maps $\sigma_i\colon [n] \to [n-1]$ is "well-behaved" with respect to this ordering (see Theorem 1.13).

The simplicial complex $\Gamma(C.)$ is defined by

$$\Gamma(C.)_n = \bigoplus_{k=0}^{n} \bigoplus_{\mu \in \operatorname{Sur}([n],[k])} C_k,$$

so we have a copy of the direct summand C_k for each surjective order-preserving map $\mu\colon [n] \to [k]$. The face and degeneracy operators in the simplicial complex $\Gamma(C.)$ are defined in terms of composition of μ with the maps δ_i and σ_i. In Section 2 we show how the results in Section 1 can be used to streamline the calculation of the face and degeneracy operators in the simplicial complex $\Gamma(C.)$ (see Theorem 2.2 and Example 2.3).

In Section 3 we summarize the results on cross-effect functors that are needed for the final section.

The Dold-Puppe complex $NF\Gamma(C.)$ is constructed by modding out the images of the degeneracy operators in $F\Gamma(C.)$. To calculate this we apply the theory of cross-effect functors to decompose both the numerator and denominator into the direct sum of cross-effect modules, the non-degenerate modules corresponding to the terms that appear in the numerator but not in the denominator. However, the decomposition produces many, many terms and seeing which are non-degenerate is far from obvious. In Section 4 we give a criterion that identifies the non-degenerate terms (see Proposition 4.4). Using the ordering we introduced in Section 1, we later give an algorithm that constructs all relevant non-degenerate terms, thus avoiding the need to check each of the many terms one by one. We finally illustrate the methods developed in this paper in the case when $C.$ is a chain complex of modules over a commutative ring of length 2 and F is the symmetric-square functor (see Example 4.13).

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Notation

Let Δ be the category whose objects are the non-empty finite totally ordered sets $[n] := \{0 < 1 < \cdots < n\}$, $n \in \mathbb{N}$, and the set of morphisms, $\mathrm{Mor}([n],[k])$, between $[n]$ and $[k]$ consists of all the order-preserving maps between them. Recall that for each $i \in \{0,\ldots,n\}$ the face map $\delta_i\colon [n-1] \to [n]$ is the unique injective order-preserving map with $\delta_i^{-1}(i) = \emptyset$, and for each $i \in \{0,\ldots,n-1\}$ the degeneracy map $\sigma_i\colon [n] \to [n-1]$ is the unique order-preserving surjective map with $\sigma_i^{-1}(i) = \{i, i+1\}$. For a category $\mathcal{A}$, a simplicial object A in $\mathcal{A}$ is a contravariant functor $A\colon \Delta \to \mathcal{A}$. We write A_n for $A([n])$, d_i for the *face operator* $A(\delta_i)\colon A_n \to A_{n-1}$, s_i for the *degeneracy operator* $A(\sigma_i)\colon A_{n-1} \to A_n$ and $\mathrm{Sur}([n],[k])$ for the set of surjective morphisms between $[n]$ and $[k]$.

1. Partitions and composition with face/degeneracy maps in Δ

For the whole of this section let us fix the natural numbers n and k. In this section we introduce an ordering on $\mathrm{Mor}([n],[k])$, investigate the maps $\mu \mapsto \mu\delta_i$ and $\nu \mapsto \nu\sigma_i$ between $\mathrm{Mor}([n],[k])$ and $\mathrm{Mor}([n-1],[k])$ and show that these maps behave in a nice way with respect to the introduced ordering.

This ordering will be used throughout this paper. In Section 2 it will allow us to describe algorithms that streamline the calculation of the face and degeneracy operators in the simplicial complex $\Gamma(C.)$ (for any bounded chain complex $C.$). In Section 4 the ordering will help us to give an algorithmic description of the Dold-Puppe complex $NF\Gamma(C.)$.

Definition 1.1. For an n-tuple $x := (x_1,\ldots,x_n) \in \mathbb{N}^n$ we write $|x|$ for $\sum_{l=1}^n x_l$, and we call x *a partition of m of length n* if $|x| = m$. If each $x_i \neq 0$, then we call x a *proper partition*, otherwise we call x an *improper partition*. We write x_i for the i^{th} entry of x.

A function $\mu\colon [n] \to [k]$ is determined by $\mu^{-1}(0)$, $\mu^{-1}(1),\ldots,\mu^{-1}(k)$. If μ is a monotonically increasing function, then the sets $\mu^{-1}(0)$, $\mu^{-1}(1),\ldots,\mu^{-1}(k)$ consist of consecutive elements of $[n]$. Because of this it is sufficient to know the sizes of these sets; hence, we can think of a morphism $\mu\colon [n] \to [k]$ as a partition of $n+1$ of length $k+1$. A surjective morphism would correspond to a proper partition and a non-surjective morphism would correspond to an improper partition.

Notation 1.2. Let $\mu \in \mathrm{Mor}([n],[k])$. The partition $(|\mu^{-1}(0)|, \dots, |\mu^{-1}(n)|)$ is denoted by μ^*. Note that $\mu_i^* = |\mu^{-1}(i-1)|$.

Lemma 1.3. *The cardinality of the set of surjective order-preserving morphisms between the sets* $[n]$ *and* $[k]$ *is given by the binomial coefficient* $\binom{n}{k}$*:*

$$|\operatorname{Sur}([n],[k])| = \binom{n}{k}.$$

Proof. If $\mu\colon [n] \to [k]$ is a surjective morphism, then the sets $\mu^{-1}(i)$, $i = 0, 1, \dots, k$ are non-empty, disjoint, their union is $[n]$ and they consist of consecutive elements of $[n]$. So if we know the smallest elements of $\mu^{-1}(1)$, $\mu^{-1}(2), \dots, \mu^{-1}(k)$, then we have determined μ. Since we know $0 = \mu(0)$ the smallest elements are in the set $\{1, \dots, n\}$. So there are as many elements of $\operatorname{Sur}([n],[k])$ as there are ways of choosing k elements from a set of size n. □

Notation 1.4. For $i \in \{0, \dots, n\}$ define $\overline{\delta}_i\colon \mathrm{Mor}([n],[k]) \to \mathrm{Mor}([n-1],[k])$ by $\mu \mapsto \mu\delta_i$, and for $i \in \{0, \dots, n-1\}$ define $\overline{\sigma}_i\colon \mathrm{Mor}([n-1],[k]) \to \mathrm{Mor}([n],[k])$ by $\nu \mapsto \nu\sigma_i$. By abuse of notation we write $\operatorname{Im} \overline{\sigma}_i$ for $\overline{\sigma}_i(\operatorname{Sur}([n-1],[k]))$.

Lemma 1.5. *For all* $i \in \{0, \dots, n-1\}$ *we have* $\overline{\delta}_i\overline{\sigma}_i = \mathrm{id}$*, and hence* $\overline{\sigma}_i$ *is injective and* $\overline{\delta}_i$ *is surjective; also* $\overline{\delta}_n$ *is surjective.*

Proof. The result follows directly from $\sigma_i\delta_i = \mathrm{id}$ for $i \in \{0, \dots, n-1\}$ and from $\sigma_{n-1}\delta_n = \mathrm{id}$. □

Definition 1.6. Let a be a partition of length k and x a partition of length $l \leqslant k$. Then we call x an *initial partition of* a if $x_i = a_i$ for $1 \leqslant i \leqslant l$. We write $a = (x, y)$ where y is the partition of length $k - l$ defined by $y_i = a_{i+l}$ for $1 \leqslant i \leqslant k - l$. (Note that we may allow either x or y to be the empty partition.)

Since knowing the effects of $\overline{\delta}_i$ and $\overline{\sigma}_i$ are essential in calculating d_i and s_i it is useful to have a quick way of working out the partitions $(\mu\delta_i)^*$ and $(\mu\sigma_i)^*$ from the partition μ^*.

Lemma 1.7.

(a) *Let* $\mu \in \mathrm{Mor}([n-1],[k])$ *and* $i \in \{0, \dots, n-1\}$*. We write* $\mu^* = (x, d, y)$ *with partitions* x, y *and a positive integer* d *such that* $|x| < i + 1 \leqslant |x| + d$*. Then the partition* $(\mu\sigma_i)^*$ *is equal to* $(x, d+1, y)$*.*

(b) *Let* $\mu \in \mathrm{Mor}([n],[k])$ *and* $i \in \{0, \dots, n\}$*. As above we write* $\mu^* = (x, d, y)$ *so that* $|x| < i + 1 \leqslant |x| + d$*. Then the partition* $(\mu\delta_i)^*$ *is equal to* $(x, d-1, y)$*.*

Proof. It is clear that we can write μ^* in the stated way. Note that $d \neq 0$, so $d - 1$ is non-negative.

By definition, for every μ in $\mathrm{Mor}([n-1],[k])$ we have $\mu_l^* = |\mu^{-1}(l-1)|$, and we also have $(\mu\sigma_i)^{-1}(l-1) = \sigma_i^{-1}\mu^{-1}(l-1)$. Recalling that σ_i is the unique surjective map $[n] \to [n-1]$ with $\sigma_i^{-1}(i) = \{i, i+1\}$, we see $|(\mu\sigma_i)^{-1}(l-1)| = |\mu^{-1}(l-1)|$ if and only if $i \notin \mu^{-1}(l-1)$, and $|(\mu\sigma_i)^{-1}(l-1)| = |\mu^{-1}(l-1)| + 1$ if and only if $i \in \mu^{-1}(l-1)$; i.e., $\mu_l^* = (\mu\sigma_i)_l^*$ if and only $i \notin \mu^{-1}(l-1)$, and $(\mu\sigma_i)_l^* = \mu_l^* + 1$ if and only if $i \in \mu^{-1}(l-1)$.

Let L be the length of x. Remembering that i is the $(i+1)^{\text{th}}$ element of $[n]$ we find that $i \in \mu^{-1}(L)$ and so, by the last sentence of the previous paragraph, we find $(\mu\sigma_i)^* = (x, d+1, y)$.

Similarly, we get our result for δ_i. $\square$

Lemma 1.8. *Let* $\mu \in \mathrm{Sur}([n],[k])$, *and let* $i \in \{0,\ldots,n\}$. *Then the morphism* $\overline{\delta}_i(\mu) = \mu\delta_i$ *is not surjective if and only if the partition* μ^* *is of the form* $(x,1,y)$, *where* x *is a partition of* i. *In this case we have the commutative diagram*

$$\begin{tikzcd}
& {[n]} \arrow[dr, "\mu"] & \\
{[n-1]} \arrow[ur, "\delta_i"] \arrow[dr, "\hat{\mu}"'] & & {[k]}, \\
& {[k-1]} \arrow[ur, "\delta_j"'] &
\end{tikzcd}$$

where $\hat{\mu}$ *is the surjection with* $\hat{\mu}^* = (x,y)$ *and* j *is the length of* x*; in particular* $i = 0$ *if and only if* $j = 0$.

Proof. The equivalence follows directly from Lemma 1.7(b). The additional statements are easy to check. $\square$

If a and b are both partitions of the same number over the same number of places and x is an initial partition of both, then we call x a *common initial partition of* a *and* b. Because a and b are of finite length there must be some longest common initial partition (even if it is of length 0, or it is equal to a).

Definition 1.9. If x is the longest common initial partition of $a = (x,y)$ and $b = (x,z)$, then we say $a < b$ if and only if $y_1 < z_1$. This gives the lexicographic ordering on the set of partitions and finally, via the bijection $\mu \mapsto \mu^*$, a total order on $\mathrm{Mor}([n],[k])$.

Notation 1.10. For $i \in \{0,\ldots,n\}$ let

$$S_i^{n,k} := \{\mu \in \mathrm{Sur}([n],[k]) \mid \mu^* \text{ is of the form } (x,y) \text{ where } |x| = i+1\}$$

and let

$$\widetilde{S_i^{n,k}} := \{\mu \in \mathrm{Sur}([n],[k]) \mid \mu^* \text{ is of the form } (x,1,y) \text{ where } |x| = i\}.$$

Note that $\widetilde{S_i^{n,k}} \subset S_i^{n,k}$ and Lemma 1.8 tells us that the set $\widetilde{S_i^{n,k}}$ coincides with the set $\{\mu \in \mathrm{Sur}([n],[k]) \mid \overline{\delta}_i(\mu) \text{ is not a surjection}\}$.

Lemma 1.11. *For each* $i \in \{0,\ldots,n-1\}$ *we have* $|S_i^{n,k}| = \binom{n-1}{k-1}$. *Furthermore, for each* $i \in \{1,\ldots,n-1\}$ *we have* $|\widetilde{S_i^{n,k}}| = \binom{n-2}{k-2}$ *and finally* $|\widetilde{S_n^{n,k}}| = \binom{n-1}{k-1}$.

Note in the statement above, if the lower entry of a binomial coefficient is negative, then the binomial coefficient is meant to be 0.

Proof. If $\mu \in S_i^{n,k}$, then for some l we have that i is the maximal element of $\mu^{-1}(l)$. Furthermore, we know that n is the maximal element of $\mu^{-1}(k)$. Therefore, choosing an element μ of $S_i^{n,k}$ amounts to the same as choosing the maximal elements for all but one of the sets $\mu^{-1}(0), \dots, \mu^{-1}(k-1)$ from the $n-1$ elements of $[n] \setminus \{i, n\}$; hence, $|S_i^{n,k}| = \binom{n-1}{k-1}$.

For $i \in \{1, \dots, n-1\}$, if $\mu \in \widetilde{S_i^{n,k}}$, then for some l we have that $i-1$ is the maximal element of $\mu^{-1}(l)$, and also i is the maximal element of $\mu^{-1}(l+1)$, i.e., choosing an element μ of $\widetilde{S_i^{n,k}}$ amounts to the same as choosing the maximal elements for all but two of the sets $\mu^{-1}(0), \dots, \mu^{-1}(k-1)$ from the $n-2$ elements of $[n] \setminus \{i-1, i, n\}$; hence, $|\widetilde{S_i^{n,k}}| = \binom{n-2}{k-2}$.

For the last statement we merely observe that $\widetilde{S_n^{n,k}} = S_{n-1}^{n,k}$ and use the first result. $\square$

Proposition 1.12. *For each $i \in \{0, \dots, n-1\}$, the set* $\mathrm{Sur}([n],[k])$ *is the disjoint union of $S_i^{n,k}$ and* $\operatorname{Im} \overline{\sigma}_i$*:*

$$\mathrm{Sur}([n],[k]) = S_i^{n,k} \amalg \operatorname{Im} \overline{\sigma}_i.$$

Note that $S_n^{n,k} = \mathrm{Sur}([n],[k])$ and there is no map $\overline{\sigma}_n$.

Proof. First we prove $S_i^{n,k}$ and $\operatorname{Im} \overline{\sigma}_i$ are disjoint. Let $\mu \in \mathrm{Sur}([n],[k])$. The partition μ^* has an initial partition of $i+1$ if and only if there is some l such that i is the maximal element of $\mu^{-1}(l)$ (remember i is the $(i+1)^{\text{th}}$ element of $[n]$). If i is the maximal element of $\mu^{-1}(l)$, then $\mu(i) \neq \mu(i+1)$. But $\mu \in \operatorname{Im} \overline{\sigma}_i$ means that for some $\nu \in \mathrm{Sur}([n-1],[k])$ we have $\mu = \nu\sigma_i$. So $\mu(i) = \nu\sigma_i(i) = \nu(i) = \nu\sigma_i(i+1) = \mu(i+1)$. Therefore μ cannot be both in $S_i^{n,k}$ and $\operatorname{Im} \overline{\sigma}_i$.

Now we prove that the union of $S_i^{n,k}$ and $\operatorname{Im} \overline{\sigma}_i$ form the whole of $\mathrm{Sur}([n],[k])$ by using a counting argument. We know that $S_i^{n,k} \cap \operatorname{Im} \overline{\sigma}_i = \emptyset$ so $|S_i^{n,k} \cup \operatorname{Im} \overline{\sigma}_i| = |S_i^{n,k}| + |\operatorname{Im} \overline{\sigma}_i|$. Lemma 1.5 tells us that $\overline{\sigma}_i$ is injective. From this we see that

$$|S_i^{n,k}| + |\operatorname{Im} \overline{\sigma}_i| = |S_i^{n,k}| + |\mathrm{Sur}([n-1],[k])|$$

and using Lemmas 1.3 and 1.11 we obtain

$$|S_i^{n,k}| + |\mathrm{Sur}([n-1],[k])| = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} = |\mathrm{Sur}([n],[k])|,$$

as desired. $\square$

Theorem 1.13.

(a) The map $\overline{\sigma}_i\colon$ $\mathrm{Mor}([n-1],[k]) \to \mathrm{Mor}([n],[k])$ is strictly order-preserving for each $i \in \{0, \dots, n-1\}$.

(b) The map $\overline{\delta}_i\colon$ $\mathrm{Sur}([n],[k]) \to \mathrm{Mor}([n-1],[k])$ is strictly order-preserving on both $\operatorname{Im} \overline{\sigma}_i$ *and $S_i^{n,k}$ for each $i \in \{0, \dots, n-1\}$, and $\overline{\delta}_n$ is strictly order-preserving on* $\mathrm{Sur}([n],[k]) = S_n^{n,k}$.

Note that while $\overline{\delta}_i$ is order-preserving on these two complementary subsets of $\mathrm{Sur}([n],[k])$ it is *not* order-preserving on the whole of $\mathrm{Sur}([n],[k])$; for an illustration of this look at the calculation at the end of Section 2.

Proof. (a) Suppose $\mu, \nu \in \mathrm{Mor}([n-1],[k])$ and $\mu < \nu$. As in Lemma 1.7 we write the partition μ^* in the form (x, d, y) where $|x| < i+1 \leqslant |x| + d$. Let a be the longest common partition of μ^* and ν^*, so $\mu^* = (a, b)$ and $\nu^* = (a, c)$ for appropriate partitions b and c with $b_1 < c_1$. We will show the desired inequality $(\mu\sigma_i)^* < (\nu\sigma_i)^*$ by distinguishing three cases: (i) a is longer than x, (ii) a has the same length as x and (iii) a is shorter than x.

(i) If a is longer than x, then we can write a in the form (x, d, w) for some (possibly empty) partition w. Then $\mu^* = (x, d, w, b)$ and $\nu^* = (x, d, w, c)$ and hence by Lemma 1.7 we see that $(\mu\sigma_i)^* = (x, d+1, w, b)$ and $(\nu\sigma_i)^* = (x, d+1, w, c)$. So the longest common initial partition of $(\mu\sigma_i)^*$ and $(\nu\sigma_i)^*$ is $(x, d+1, w)$, and since $b_1 < c_1$, we see that $(\mu\sigma_i)^* < (\nu\sigma_i)^*$.

(ii) If a has the same length as x (i.e., if $a = x$), then $d = b_1$ and since we have $b_1 < c_1$ we see $i + 1 \leqslant |x| + d < |x| + c_1$. Using Lemma 1.7 we obtain $(\mu\sigma_i)^* = (x, b_1 + 1, y)$ and $(\nu\sigma_i)^* = (x, c_1 + 1, z)$ for appropriate partitions y and z. So the longest common initial partition of $(\mu\sigma_i)^*$ and $(\nu\sigma_i)^*$ is x, and since $b_1 + 1 < c_1 + 1$, we see that $(\mu\sigma_i)^* < (\nu\sigma_i)^*$.

(iii) If x is longer than a we write $x = (a, x')$ for some non-empty partition x'. Then $\mu^* = (x, d, y) = (a, x', d, y)$. As in Lemma 1.7 we write $\nu^* = (w, d', z)$ where $|w| < i + 1 \leqslant |w| + d'$. We know that $|a| \leqslant |x| < i + 1$ and a is an initial partition of ν^* so $w = (a, w')$ for some possibly empty partition w'. We now show the desired inequality $(\mu\sigma_i)^* < (\nu\sigma_i)^*$ by distinguishing two subcases: (α) w' is non-empty, (β) w' is empty.

(α) If w' is not empty, then $\mu^* = (a, x', d, y)$ and $\nu^* = (a, w', d', z)$. Since $\mu^* < \nu^*$ we find that $x'_1 < w'_1$. Applying Lemma 1.7 we find that $(\mu\sigma_i)^* = (a, x', d+1, y)$ and $(\nu\sigma_i)^* = (a, w', d'+1, z)$. So the longest common initial partition of $(\mu\sigma_i)^*$ and $(\nu\sigma_i)^*$ is a, and since $x'_1 < w'_1$, we see that $(\mu\sigma_i)^* < (\nu\sigma_i)^*$.

(β) If w' is empty, then $\mu^* = (a, x', d, y)$ and $\nu^* = (a, d', z)$ where $|a| < i + 1 \leqslant |a| + d'$. Since $\mu^* < \nu^*$ we see that $x'_1 < d'$. Applying Lemma 1.7 we find that $(\mu\sigma_i)^* = (a, x', d+1, y)$ and $(\nu\sigma_i)^* = (a, d'+1, z)$. So the longest common initial partition of $(\mu\sigma_i)^*$ and $(\nu\sigma_i)^*$ is a, and since $x_1 < d' < d' + 1$, we see that $(\mu\sigma_i)^* < (\nu\sigma_i)^*$.

(b) That $\overline{\delta}_i$ is order-preserving on $\mathrm{Im}\,\overline{\sigma}_i$ follows directly from Lemma 1.5 and part (a). Although (the first half of) the proof that $\overline{\delta}_i$ is strictly order-preserving on $S_i^{n,k}$ is pretty similar to (the first half of) the proof of part (a) we include all details for the reader's convenience.

Suppose $\mu, \nu \in S_i^{n,k}$ with $\mu < \nu$. As in Lemma 1.7 we write the partition μ^* in the form (x, d, y) where $|x| < i + 1 \leqslant |x| + d$. Let a be the longest common partition of μ^* and ν^*, so $\mu^* = (a, b)$ and $\nu^* = (a, c)$ for appropriate partitions b and c with $b_1 < c_1$. We will now show the desired inequality $(\mu\delta_i)^* < (\nu\delta_i)^*$ by distinguishing three cases: (i) a is longer than x, (ii) a has the same length as x and (iii) a is shorter than x. Only case (iii) will make use of the assumption that $\mu, \nu \in S_i^{n,k}$.

(i) If a is longer than x, then we can write a in the form (x, d, w) for some (possibly empty) partition w. Then $\mu^* = (x, d, w, b)$ and $\nu^* = (x, d, w, c)$ and hence by Lemma 1.7 we see that $(\mu\delta_i)^* = (x, d-1, w, b)$ and $(\nu\delta_i)^* = (x, d-1, w, c)$. So the longest common initial partition of $(\mu\delta_i)^*$ and $(\nu\delta_i)^*$ is $(x, d-1, w)$, and since $b_1 < c_1$, we see that $(\mu\delta_i)^* < (\nu\delta_i)^*$.

(ii) If a has the same length as x (i.e., if $a = x$), then $d = b_1$ and since we have $b_1 < c_1$ we see $i+1 \leqslant |x| + d < |x| + c_1$. Using Lemma 1.7 we obtain

$$(\mu\delta_i)^* = (x, b_1 - 1, y) \quad \text{and} \quad (\nu\delta_i)^* = (x, c_1 - 1, z)$$

for appropriate partitions y and z. So the longest common initial partition of $(\mu\delta_i)^*$ and $(\nu\delta_i)^*$ is x, and since $b_1 - 1 < c_1 - 1$, we see that $(\mu\delta_i)^* < (\nu\delta_i)^*$.

(iii) If x is longer than a, then we write $x = (a, x')$ for some non-empty partition x'. Then $\mu^* = (x, d, y) = (a, x', d, y)$. As in Lemma 1.7 we write $\nu^* = (w, d', z)$ with $|w| < i+1 \leqslant |w| + d'$. We know that $|a| \leqslant |x| < i+1$ and a is an initial partition of ν^* so $w = (a, w')$ for some possibly empty partition w'. We now show the desired inequality $(\mu\sigma_i)^* < (\nu\sigma_i)^*$ by distinguishing two subcases: (α) w' is non-empty, (β) w' is empty.

(α) If w' is not empty, then $\mu^* = (a, x', d, y)$ and $\nu^* = (a, w', d', z)$. Since $\mu^* < \nu^*$ we find that $x'_1 < w'_1$. Applying Lemma 1.7 we find that $(\mu\delta_i)^* = (a, x', d-1, y)$ and $(\nu\delta_i)^* = (a, w', d'-1, z)$. So the longest common initial partition of $(\mu\delta_i)^*$ and $(\nu\delta_i)^*$ is a, and since $x'_1 < w'_1$, we see that $(\mu\delta_i)^* < (\nu\delta_i)^*$.

(β) If w' is empty, then $\mu^* = (a, x', d, y)$ and $\nu^* = (a, d', z)$ where $|a| < i+1 \leqslant |a| + d'$. Since $\mu^* < \nu^*$ we see that $x'_1 < d'$. Applying Lemma 1.7 we find that $(\mu\delta_i)^* = (a, x', d-1, y)$ and $(\nu\delta_i)^* = (a, d'-1, z)$. As $x'_1 < d'$ we have either $x'_1 < d'-1$ or $x'_1 = d'-1$.

If $x'_1 < d'-1$, then the longest common initial partition of $(\mu\delta_i)^*$ and $(\nu\delta_i)^*$ is a, and since $x'_1 < d'-1$, we have $(\mu\delta_i)^* < (\nu\delta_i)^*$.

If $x'_1 = d'-1$, then we observe the following: we have written ν^* as (a, d', z) so that $|a| < i+1 \leqslant |a| + d'$, but $\nu \in S_i^{n,k}$ so ν^* begins with a partition of $i+1$; hence, $|a| + d' = i+1$. Now $i+1 = |a| + d' = |a| + x'_1 + 1$, so $|a| + x'_1 = i$, i.e., the partition (a, x'_1) (which is an initial partition of μ^*) is a partition of i. But $\mu \in S_i^{n,k}$ so μ begins with a partition of $i+1$ and μ^* is a proper partition, so μ^* begins with the partition $(a, x'_1, 1)$, i.e., $x = (a, x'_1)$ and $d = 1$. So $\mu^* = (a, x'_1, 1, y)$ and $\nu^* = (a, d', z)$.

By Lemma 1.7 we find that

$$(\mu\delta_i)^* = (a, x'_1, 0, y) \quad \text{and} \quad (\nu\delta_i)^* = (a, d'-1, z) = (a, x'_1, z).$$

Since all the entries of ν^* are positive we have $z_1 > 0$. So the longest common initial partition of $(\mu\delta_i)^*$ and $(\nu\delta_i)^*$ is (a, x'_1), and since $0 < z_1$, we find that

$$(\mu\delta_i)^* < (\nu\delta_i)^*. \qquad \square$$

2. The face and degeneracy operators in the simplicial object $\Gamma(C.)$

For an abelian category $\mathcal{A}$ the Dold-Kan correspondence gives two mutually inverse functors Γ and N between the category $\mathrm{Ch}_{\geqslant 0}(\mathcal{A})$ of bounded chain complexes and the category $\mathcal{SA}$ of simplicial objects in $\mathcal{A}$. For a chain complex $C. \in \mathrm{Ch}_{\geqslant 0}(\mathcal{A})$ the functor $\Gamma(C.)$ is usually defined by $\Gamma(C.)_n = \bigoplus_{k=0}^n \bigoplus_{\sigma \in \mathrm{Sur}([n],[k])} C_k$. So $\Gamma(C.)$ contains $|\mathrm{Sur}([n],[k])|$ copies of C_k and these copies are indexed by elements of $\mathrm{Sur}([n],[k])$. We write $\Gamma(C.)_{n,k}$ to denote $\bigoplus_{\sigma \in \mathrm{Sur}([n],[k])} C_k$ considered as a sub-sum of $\Gamma(C.)_n$.

The effect of the degeneracy operator $s_i \colon \Gamma(C.)_{n-1} \to \Gamma(C.)_n$ on the copy of C_k indexed by $\mu \in \mathrm{Sur}([n-1],[k])$ is to identify it with the copy of $C_k \in \Gamma(C.)_n$ indexed by $\overline{\sigma}_i(\mu)$ (cf. Notation 1.4).

The effect of the face operator $d_i \colon \Gamma(C.)_n \to \Gamma(C.)_{n-1}$ on the copy of C_k indexed by $\mu \in \mathrm{Sur}([n],[k])$ depends on the nature of $\overline{\delta}_i(\mu)$ (cf. Notation 1.4):

- If $\overline{\delta}_i(\mu)$ is surjective, then C_k is identified with the copy of C_k indexed by $\overline{\delta}_i(\mu)$;
- If $\overline{\delta}_i(\mu)$ is not surjective, and $\overline{\delta}_i(\mu) = \delta_0 \hat{\mu}$ for some $\hat{\mu} \in \mathrm{Sur}([n-1],[k-1])$ (cf. Lemma 1.8), then d_i maps the copy of C_k indexed by μ to the copy of C_{k-1} indexed by $\hat{\mu}$ with the same action as the differential of $C.$;
- If $\overline{\delta}_i(\mu)$ is not surjective, and $\overline{\delta}_i(\mu) = \delta_j \hat{\mu}$ for some $\hat{\mu} \in \mathrm{Sur}([n-1],[k-1])$ and for some $j \neq 0$ (cf. Lemma 1.8), then C_k is mapped to 0.

This can be expressed more concisely in symbols than in words. For $\mu \in \mathrm{Sur}([n],[k])$ we write $C_{k,\mu}$ to denote the copy of C_k in $\bigoplus_{\sigma \in \mathrm{Sur}([n],[k])} C_k$ that is contributed by μ and also, for $m \in C_k$, we write (m,μ) to denote $m \in C_{k,\mu}$. The face and degeneracy maps in $\Gamma(C.)$ are defined as follows:

$$s_i(m,\mu) := (m, \overline{\sigma}_i(\mu)),$$
$$d_i(m,\mu) := \begin{cases} (m, \overline{\delta}_i(\mu)) & \text{if } \overline{\delta}_i(\mu) \text{ is surjective} \\ (\partial(m), \hat{\mu}) & \text{if } \overline{\delta}_i(\mu) = \delta_0 \hat{\mu} \text{ with } \hat{\mu} \in \mathrm{Sur}([n-1],[k-1]) \\ 0 & \text{if } \overline{\delta}_i(\mu) = \delta_j \hat{\mu} \text{ with } \hat{\mu} \in \mathrm{Sur}([n-1],[k-1]) \text{ and } j \neq 0. \end{cases}$$

The object of this section is to rewrite these expressions using results from the previous section and to thereby make the calculation of the face and degeneracy operators simpler.

Lemma 1.3 tells us that for natural numbers n and k,

$$\Gamma(C.)_n = \Gamma(C.)_{n,0} \oplus \Gamma(C.)_{n,1} \oplus \cdots \oplus \Gamma(C.)_{n,n} = C_0^{\binom{n}{0}} \oplus C_1^{\binom{n}{1}} \oplus \cdots \oplus C_n^{\binom{n}{n}};$$

again each copy of C_k is indexed by the element of $\mathrm{Sur}([n],[k])$ that contributes it. But now we can use the ordering on $\mathrm{Sur}([n],[k])$ that we defined in Section 1 to order the copies of C_k. Because of this we will tend to use the ordinal associated to $\mu \in \mathrm{Sur}([n],[k])$ instead of μ to index a copy of C_k, i.e., if μ is the m^{th} element of $\mathrm{Sur}([n],[k])$ we will usually write $C_{k,m}$ instead of $C_{k,\mu}$.

Combining various results from the previous section we get the following proposition. For $n,k \in \mathbb{N}$ and $A \subset \mathrm{Sur}([n],[k])$ we write A^C for the complement of A in the set $\mathrm{Sur}([n],[k])$.

Proposition 2.1. *Let $n > 0$ and $k \in \{0,\dots,n\}$.*

(a) (i) For each $i \in \{0,\dots,n-1\}$ the sets $\mathrm{Sur}([n-1],[k])$ and $(S_i^{n,k})^C$ have the same cardinality.

(ii) The sets $S_0^{n,k}$ and $\mathrm{Sur}([n-1],[k-1])$ have the same cardinality.

(iii) For each $i \in \{1, \dots, n\}$ *the sets* $S_{i-1}^{n-1,k}$ *and* $S_i^{n,k} \setminus \widetilde{S_i^{n,k}}$ *have the same cardinality.*

(b) For each $i \in \{0, \dots, n-1\}$ *the map* $\overline{\sigma}_i \colon \mathrm{Sur}([n-1],[k]) \to \mathrm{Sur}([n],[k])$ *sends the* l^{th} *element of* $\mathrm{Sur}([n-1],[k])$ *to the* l^{th} *element of* $(S_i^{n,k})^C$.

(c) (i) If $\mu \in S_0^{n,k}$*, then for some* $\hat{\mu} \in \mathrm{Sur}([n-1],[k-1])$ *we have* $\overline{\delta}_0(\mu) = \delta_0\hat{\mu}$*. Moreover, the map* $\mu \mapsto \hat{\mu}$ *acts on* $S_0^{n,k}$ *by sending the* l^{th} *element of* $S_0^{n,k}$ *to the* l^{th} *element of* $\mathrm{Sur}([n-1],[k-1])$.

(ii) For each $i \in \{0, \dots, n-1\}$ *the map* $\overline{\delta}_i \colon \mathrm{Sur}([n],[k]) \to \mathrm{Mor}([n-1],[k])$ *acts on the set* $(S_i^{n,k})^C$ *by sending the* l^{th} *element of* $(S_i^{n,k})^C$ *to the* l^{th} *element of* $\mathrm{Sur}([n-1],[k])$.

(iii) For each $i \in \{1, \dots, n\}$ *the map* $\overline{\delta}_i \colon \mathrm{Sur}([n],[k]) \to \mathrm{Mor}([n-1],[k])$ *acts on the set* $S_i^{n,k} \setminus \widetilde{S_i^{n,k}}$ *by sending the* l^{th} *element of* $S_i^{n,k} \setminus \widetilde{S_i^{n,k}}$ *to the* l^{th} *element of* $S_{i-1}^{n-1,k}$.

Part (a) of this proposition ensures that the later statements are well defined.

Note for $i \neq 0$ we do not describe the action of $\overline{\delta}_i$ on $\widetilde{S_i^{n,k}}$ because from Lemma 1.8 we know for $\mu \in \widetilde{S_i^{n,k}}$ the map $\overline{\delta}_i(\mu)$ will be a non-surjection equal to $\delta_j\hat{\mu}$ where $j \neq 0$; hence, the action of d_i on $C_{k,\mu}$ will just be the zero map (see the definition of Γ at the beginning of this section).

Proof. Part (a)(i) follows from Proposition 1.12 and the injectivity of $\overline{\sigma}_i$ (Lemma 1.5). Part (a)(ii) follows from Lemmas 1.3 and 1.11. Lemma 1.11 furthermore tells us that for $i \in \{1, \dots, n-1\}$ we have $|S_i^{n,k}| = \binom{n-1}{k-1}$ and that $|\widetilde{S_i^{n,k}}| = \binom{n-2}{k-2}$, and therefore $|S_i^{n,k} \setminus \widetilde{S_i^{n,k}}| = \binom{n-1}{k-1} - \binom{n-2}{k-2} = \binom{n-2}{k-1} = |S_{i-1}^{n-1,k}|$ (the final step is given by Lemma 1.11 again). Furthermore, $S_n^{n,k} = \mathrm{Sur}([n],[k])$ and by using Lemma 1.11 twice we see that $|\widetilde{S_n^{n,k}}| = \binom{n-1}{k-1}$, so $|S_n^{n,k} \setminus \widetilde{S_n^{n,k}}| = \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k} = |S_{n-1}^{n-1,k}|$. Thus, we have shown part (a)(iii) of this theorem for all $i \in \{1, \dots, n\}$.

Part (b) is seen by applying Proposition 1.12 and Theorem 1.13(a) to part (a)(i).

If $\mu \in S_0^{n,k}$, then μ^* is of the form $(1, y)$ for an appropriate partition y. Applying Lemma 1.8 gives us the first sentence of part (c)(i), and also tells us that $\hat{\mu}^*$ is the partition y. Clearly the map that sends $(1, y)$ to y is order-preserving. Now using (a)(ii) we get (c)(i). By applying Theorem 1.13(b) to part (a)(i) we get part (c)(ii). Finally, part (c)(iii) follows by applying Theorem 1.13(b) to part (a)(iii) of this statement. Note that $\overline{\delta}_i(S_i^{n,k} \setminus \widetilde{S_i^{n,k}}) \subseteq S_{i-1}^{n-1,k}$. □

Theorem 2.2. *Let* $n > 0$.

(a) Let $i \in \{0, \dots, n-1\}$*, fix* $k \in \{0, \dots, n\}$ *and let* $\underline{c} \in \Gamma(C.)_{n-1,k}$*; then we have* $s_i(\underline{c}) \in \Gamma(C.)_{n,k}$*. More precisely, write*

$$\underline{c} = (c_1, \dots, c_{\binom{n-1}{k}}) \quad \text{and} \quad s_i(\underline{c}) = (b_1, \dots, b_{\binom{n}{k}});$$

then $s_i(\underline{c})$ *is given by the following relations:*

(i) If the l^{th} *element of* $\mathrm{Sur}([n],[k])$ *is an element of* $S_i^{n,k}$*, then* $b_l = 0$.

(ii) If the l^{th} *element of* $\mathrm{Sur}([n],[k])$ *is the* m^{th} *element of* $(S_i^{n,k})^C$*, then* $b_l = c_m$.

(b) Let $\underline{c} = (c_{k,l})_{k=0,\dots,n;l=1,\dots,\binom{n}{k}} \in \Gamma(C.)_n$. *Then*

$$d_0(\underline{c}) = (b_{k,l})_{k=0,\dots,n-1;l=1,\dots,\binom{n-1}{k}} \in \Gamma(C.)_{n-1}$$

is given by the following relation: $b_{k,l} = \partial(c_{k+1,l}) + c_{k,\binom{n-1}{k-1}+l}$.

(c) Let $i \in \{1,\dots,n-1\}$, *fix* $k \in \{0,\dots,n\}$ *and let* $\underline{c} \in \Gamma(C.)_{n,k}$; *then we have* $d_i(\underline{c}) \in \Gamma(C.)_{n-1,k}$. *More precisely, write*

$$\underline{c} = (c_1,\dots,c_{\binom{n}{k}}) \text{ and } \quad d_i(\underline{c}) = (b_1,\dots,b_{\binom{n-1}{k}});$$

then $d_i(\underline{c})$ *is given by the following relations:*

(i) If the l^{th} *element of* $\mathrm{Sur}([n-1],[k])$ *is an element of* $(S_{i-1}^{n-1,k})^C$, *then* $b_l = c_{\alpha(l)}$, *where* $\alpha(l)$ *is the ordinal associated with the* l^{th} *element of* $(S_i^{n,k})^C$.

(ii) If the l^{th} *element of* $\mathrm{Sur}([n-1],[k])$ *is the* m^{th} *element of* $S_{i-1}^{n-1,k}$, *then* $b_l = c_{\alpha(l)} + c_{\beta(m)}$ *where* $\alpha(l)$ *is the ordinal associated to the* l^{th} *element of* $(S_i^{n,k})^C$ *and* $\beta(m)$ *is the ordinal associated to the* m^{th} *element of* $S_i^{n,k} \setminus \widetilde{S_i^{n,k}}$.

(d) Fix $k \in \{0,\dots,n\}$ *and let* $\underline{c} \in \Gamma(C.)_{n,k}$; *then we have* $d_n(\underline{c}) \in \Gamma(C.)_{n-1,k}$. *More precisely, write* $\underline{c} = (c_1,\dots,c_{\binom{n}{k}})$ *and* $d_n(\underline{c}) = (b_1,\dots,b_{\binom{n-1}{k}})$; *then* $d_n(\underline{c})$ *is given by the following relation: Let* $\beta(l)$ *denote the ordinal associated with the* l^{th} *element of* $S_n^{n,k} \setminus \widetilde{S_n^{n,k}} = \mathrm{Sur}([n],[k]) \setminus S_{n-1}^{n,k}$; *then* $b_l = c_{\beta(l)}$.

Proof. Part (a) follows from Proposition 2.1(b). To prove part (b) we first observe that $S_0^{n,k} = \{\mu \in \mathrm{Sur}([n],[k]) \mid \mu^* = (1,x) \text{ where } |x| = n)\}$; so $S_0^{n,k}$ consists of the first $\binom{n-1}{k-1}$ elements of $\mathrm{Sur}([n],[k])$. Now part (b) follows from Proposition 2.1(c)(i) and (c)(ii). Part (c) follows from Lemma 1.8 and Proposition 2.1(c)(ii) and (c)(iii). Finally part (d) follows from Lemma 1.8 and Proposition 2.1(c)(iii). □

In Example 2.3 below we look at the case when the chain complex $C.$ is of length 2, to help elucidate the previous results. But first we give some general instructions on how to read that example.

While part (b) of the previous theorem is a very explicit formula which allows us to instantly describe the action of the face operator d_0, we first need to calculate the sets $S_i^{n,k}$ (and $\widetilde{S_i^{n,k}}$) to be able to use the other parts for describing the degeneracy operators and the other face operators.

For each n that we are concerned with (the position in the simplicial complex $\Gamma(C.)$) and each $k \in \{1,\dots,\min(n,l)\}$ (where l stands for the length of the chain complex $C.$), we draw a table to help us determine these sets. We label the columns of the table by the possible values of i (0 through to n). We label the rows of the table with both the partition and the ordinal associated with the elements of $\mathrm{Sur}([n],[k])$. If a cell in the table has its column labelled by i and its row is labelled by a partition μ^* that has an initial partition of $i+1$, then we mark the cell with a $\times$ mark. If that initial partition ends with a 1, then we also mark the cell with a *. So if a cell is marked with a $\times$ mark, then the corresponding surjection μ is an element of the set $S_i^{n,k}$. If the cell is also marked with a *, then μ is an element of the set $\widetilde{S_i^{n,k}}$. We do not draw any tables for $k = 0$ because all face and degeneracy operators act just as the identity on the single copy of C_0 in $\Gamma(C.)_n$.

We now explain how to use the tables we have made to calculate the degeneracy operators. For this paragraph we fix $i \in \{0, \dots, n-1\}$ and $k \in \{0, \dots, n\}$, let $\underline{c} \in \Gamma(C.)_{n-1,k}$ and write $\underline{c} = (c_1, \dots, c_{\binom{n-1}{k}})$. The vector $s_i(\underline{c}) \in \Gamma(C.)_{n,k}$ is an $\binom{n}{k}$-tuple. By Theorem 2.2(a) the entries of $s_i(\underline{c})$ are either 0 or one of $c_1, \dots, c_{\binom{n-1}{k}}$; more specifically, $c_1, \dots, c_{\binom{n-1}{k}}$ each occur once in $s_i(\underline{c})$ and occur *in order*, with zeroes in all the other entries. We find where the zeroes are in $s_i(\underline{c})$ by looking at the column labelled i in the table we made for (n, k); if there is an $\times$ in the l^{th} row of this column, then (by Theorem 2.2(a)(i)) the l^{th} entry of $s_i(\underline{c})$ is zero.

We now explain how to calculate the face operator d_n. For this paragraph we fix $k \in \{0, \dots, n\}$, let $\underline{c} \in \Gamma(C.)_{n,k}$ and write $\underline{c} = (c_1, \dots, c_{\binom{n}{k}})$. If $k = n$, then $\Gamma(C.)_{n-1,k}$ is just the zero module, so $d_n(\underline{c}) = 0$. In general, the vector $d_n(\underline{c}) \in \Gamma(C.)_{n-1,k}$ is an $\binom{n-1}{k}$-tuple. By Theorem 2.2(d) each entry of $d_i(\underline{c})$ is one of $c_1, \dots, c_{\binom{n}{k}}$; more specifically, $\binom{n-1}{k}$ elements of $c_1, \dots, c_{\binom{n}{k}}$ occur in $d_n(\underline{c})$; they occur once and they occur *in order*. To determine which entries do not occur in $d_n(\underline{c})$ we look at the n^{th} column of the table we drew for (n, k). If a * occurs in the l^{th} row, then (by Theorem 2.2(d)) c_l does not occur in $d_n(\underline{c})$.

We finally explain how to calculate the face operators other than d_0 and d_n. For this and the next paragraph we fix $i \in \{1, \dots, n-1\}$ and $k \in \{0, \dots, n\}$; let $\underline{c} \in \Gamma(C.)_{n,k}$ and write $\underline{c} = (c_1, \dots, c_{\binom{n}{k}})$. If $k = n$, then $\Gamma(C.)_{n-1,k}$ is just the zero module, so $d_i(\underline{c}) = 0$. In general, the vector $d_i(\underline{c}) \in \Gamma(C.)_{n-1,k}$ is an $\binom{n-1}{k}$-tuple. By Theorem 2.2(c) each entry of $d_i(\underline{c})$ is either one of $c_1, \dots, c_{\binom{n}{k}}$ or the sum of two of them; more specifically, each of $c_1, \dots, c_{\binom{n}{k}}$ occur at most once in $d_i(\underline{c})$, either by itself or as part of a sum, but might not occur at all.

We now proceed in three steps. In the first step we determine those entries of $d_i(\underline{c})$ that consist of the sum of two entries of $\underline{c}$ (but not yet the summands). To do so we look at the column labelled $i - 1$ in the table we have drawn for $(n - 1, k)$; if the l^{th} row of that column has a $\times$ mark in it, then the l^{th} entry of $d_i(\underline{c})$ is the sum of two entries of $\underline{c}$ (by Theorem 2.2(c)(ii)). For the second and third step we look at the column labelled i in the table we have made for (n, k). In this column there are as many rows with no $\times$ mark as there are entries of $d_i(\underline{c})$ (by Proposition 2.1(a)(i)). The second step now is to write the entries of $\underline{c}$ indexed by the ordinals of these rows into $d_i(\underline{c})$ *in order*. Still in the same column of the same table, there are as many rows that are marked with a $\times$ but not with a * as there are entries of $d_i(\underline{c})$ that contain a sum (by Proposition 2.1(a)(iii)). The final, third step is to write the entries of $\underline{c}$ indexed by the ordinals of these rows *in order* into those entries of $d_i(\underline{c})$ we have identified in the first step to contain a sum and join them by a plus sign with the entries we have already made in the second step. This accomplishes calculating $d_i(\underline{c})$ by Theorem 2.2(c). Finally, it may be worth mentioning that if the l^{th} row (still in the same column of the same table) contains both a $\times$ mark and a * mark, then c_l does not occur in $d_i(\underline{c})$.

Example 2.3. Let $C \to B \to A$ be a chain complex of length 2, placed in degrees 0, 1 and 2, which has differential ∂. For $n \geqslant 0$ let $\Gamma_n := \Gamma(C \to B \to A)_n$. For each $n \in \{1, 2, 3, 4, 5\}$ we calculate all the degeneracy operator $s_i \colon \Gamma_{n-1} \to \Gamma_n$ and all the face operators $d_i \colon \Gamma_n \to \Gamma_{n-1}$. But first we write write down the tables as introduced above.

Table for $(n, k) = (1, 1)$:

	0	1
1 $(1,1)$	$\times^*$	$\times^*$

Tables for $(n, k) = (2, 1)$ and $(n, k) = (2, 2)$:

	0	1	2
1 $(1,2)$	$\times^*$		$\times$
2 $(2,1)$		$\times$	$\times^*$

	0	1	2
1 $(1,1,1)$	$\times^*$	$\times^*$	$\times^*$

Tables for $(n, k) = (3, 1)$ and $(n, k) = (3, 2)$:

	0	1	2	3
1 $(1,3)$	$\times^*$			$\times$
2 $(2,2)$		$\times$		$\times$
3 $(3,1)$			$\times$	$\times^*$

	0	1	2	3
1 $(1,1,2)$	$\times^*$	$\times^*$		$\times$
2 $(1,2,1)$	$\times^*$		$\times$	$\times^*$
3 $(2,1,1)$		$\times$	$\times^*$	$\times^*$

Tables for $(n, k) = (4, 1)$ and $(n, k) = (4, 2)$:

	0	1	2	3	4
1 $(1,4)$	$\times^*$				$\times$
2 $(2,3)$		$\times$			$\times$
3 $(3,2)$			$\times$		$\times$
4 $(4,1)$				$\times$	$\times^*$

	0	1	2	3	4
1 $(1,1,3)$	$\times^*$	$\times^*$			$\times$
2 $(1,2,2)$	$\times^*$		$\times$		$\times$
3 $(1,3,1)$	$\times^*$			$\times$	$\times^*$
4 $(2,1,2)$		$\times$	$\times^*$		$\times$
5 $(2,2,1)$		$\times$		$\times$	$\times^*$
6 $(3,1,1)$			$\times$	$\times^*$	$\times^*$

Tables for $(n, k) = (5, 1)$ and $(n, k) = (5, 2)$:

	0	1	2	3	4	5
1 $(1,5)$	$\times^*$					$\times$
2 $(2,4)$		$\times$				$\times$
3 $(3,3)$			$\times$			$\times$
4 $(4,2)$				$\times$		$\times$
5 $(5,1)$					$\times$	$\times^*$

	0	1	2	3	4	5
1 (1,1,4)	$\times^*$	$\times^*$				$\times$
2 (1,2,3)	$\times^*$		$\times$			$\times$
3 (1,3,2)	$\times^*$			$\times$		$\times$
4 (1,4,1)	$\times^*$				$\times$	$\times^*$
5 (2,1,3)		$\times$	$\times^*$			$\times$
6 (2,2,2)		$\times$		$\times$		$\times$
7 (2,3,1)		$\times$			$\times$	$\times^*$
8 (3,1,2)			$\times$	$\times^*$		$\times$
9 (3,2,1)			$\times$		$\times$	$\times^*$
10 (4,1,1)				$\times$	$\times^*$	$\times^*$

The face and degeneracy operators between $\Gamma_0 = A$ and $\Gamma_1 = B \oplus A$ act as follows:

$$d_i((b;a)) = \begin{cases} \partial(b) + a & \text{for } i = 0 \\ a & \text{for } i = 1, \end{cases}$$
$$s_0(a) = (0;a).$$

The face and degeneracy operators between $\Gamma_1 = B \oplus A$ and $\Gamma_2 = C \oplus B^2 \oplus A$ act as follows:

$$d_i((c;,b_1,b_2;a)) = \begin{cases} \partial(c) + b_2; \partial(b_1) + a) & \text{for } i = 0 \\ (b_1 + b_2; a) & \text{for } i = 1 \\ (b_1;a) & \text{for } i = 2, \end{cases}$$
$$s_i((b;a)) = \begin{cases} (0,b;a) & \text{for } i = 0 \\ (b,0;a) & \text{for } i = 1. \end{cases}$$

The face and degeneracy operators between

$$\Gamma_2 = C \oplus B^2 \oplus A \quad \text{and} \quad \Gamma_3 = C^3 \oplus B^3 \oplus A$$

act as follows:

$$d_i((c_1,c_2,c_3;b_1,b_2,b_3;a)) = \begin{cases} (c_3, \partial(c_1) + b_2, \partial(c_2) + b_3; \partial(b_1) + a) & \text{for } i = 0 \\ (c_2 + c_3; b_1 + b_2, b_3; a) & \text{for } i = 1 \\ (c_1 + c_2; b_1, b_2 + b_3; a) & \text{for } i = 2 \\ (c_1; b_1, b_2; a) & \text{for } i = 3, \end{cases}$$
$$s_i((c;b_1,b_2;a)) = \begin{cases} (0,0,c;0,b_1,b_2;a) & \text{for } i = 0 \\ (0,c,0;_1,0,b_2;a) & \text{for } i = 1 \\ (c,0,0;b_1,b_2,0;a) & \text{for } i = 2. \end{cases}$$

The face and degeneracy operators between

$$\Gamma_3 = C^3 \oplus B^3 \oplus A \quad \text{and} \quad \Gamma_4 = C^6 \oplus B^4 \oplus A$$

act as follows:

$$d_i((c_1, c_2, c_3, c_4, c_5, c_6; b_1, b_2, b_3, b_4; , a))$$
$$= \begin{cases} (c_4, c_5, c_6; \partial(c_1) + b_2, \partial(c_2) + b_3, \partial(c_3) + b_4; \partial(b_1) + a) & \text{for } i = 0 \\ (c_2 + c_4, c_3 + c_5, c_6; b_1 + b_2, b_3, b_4; a) & \text{for } i = 1 \\ (c_1 + c_2, c_3, c_5 + c_6; b_1, b_2 + b_3, b_4; a) & \text{for } i = 2 \\ (c_1, c_2 + c_3, c_4 + c_5; b_1, b_2, b_3 + b_4; a) & \text{for } i = 3 \\ (c_1, c_2, c_4; b_1, b_2, b_3; a) & \text{for } i = 4, \end{cases}$$

$$s_i((c_1, c_2, c_3; b_1, b_2, b_3; a))$$
$$= \begin{cases} (0, 0, 0, c_1, c_2, c_3; 0, b_1, b_2, b_3; a) & \text{for } i = 0 \\ (0, c_1, c_2, 0, 0, c_3; b_1, 0, b_2, b_3; a) & \text{for } i = 1 \\ (c_1, 0, c_2, 0, c_3, 0; b_1, b_2, 0, b_3; a) & \text{for } i = 2 \\ (c_1, c_2, 0, c_3, 0, 0; , b_1, b_2, b_3, 0; a) & \text{for } i = 3. \end{cases}$$

The face and degeneracy operators between

$$\Gamma_4 = C^6 \oplus B^4 \oplus A \quad \text{and} \quad \Gamma_5 = C^{10} \oplus B^5 \oplus A$$

act as follows:

$$d_i((c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}; b_1, b_2, b_3, b_4, b_5; a))$$
$$= \begin{cases} (c_5, c_6, c_7, c_8, c_9, c_{10}; \partial(c_1) + b_2, \partial(c_2) + b_3, \partial(c_3) + b_4, \partial(c_4) + b_5; \partial(b_1) + a) \\ (c_2 + c_5, c_3 + c_6, c_4 + c_7, c_8, c_9, c_{10}; b_1 + b_2, b_3, b_4, b_5; a) \\ (c_1 + c_2, c_3, c_4, c_6 + c_8, c_7 + c_9, c_{10}; b_1, b_2 + b_3, b_4, b_5; a) \\ (c_1, c_2 + c_3, c_4, c_5 + c_6, c_7, c_9 + c_{10}; b_1, b_2, b_3 + b_4, b_5; a) \\ (c_1, c_2, c_3 + c_4, c_5, c_6 + c_7, c_8 + c_9; b_1, b_2, b_3, b_4 + b_5; a) \\ (c_1, c_2, c_3, c_5, c_6, c_8; b_1, b_2, b_3, b_4; a), \end{cases}$$

$$s_i((c_1, c_2, c_3, c_4, c_5, c_6; b_1, b_2, b_3, b_4; a))$$
$$= \begin{cases} (0, 0, 0, 0, c_1, c_2, c_3, c_4, c_5, c_6; 0, b_1, b_2, b_3, b_4; a) & \text{for } i = 0 \\ (0, c_1, c_2, c_3, 0, 0, 0, c_4, c_5, c_6; b_1, 0, b_2, b_3, b_4; a) & \text{for } i = 1 \\ (c_1, 0, c_2, c_3, 0, c_4, c_5, 0, 0, c_6; b_1, b_2, 0, b_3, b_4; a) & \text{for } i = 2 \\ (c_1, c_2, 0, c_3, c_4, 0, c_5, 0, c_6, 0; b_1, b_2, b_3, 0, b_4; a) & \text{for } i = 3 \\ (c_1, c_2, c_3, 0, c_4, c_5, 0, c_6, 0, 0; b_1, b_2, b_3, b_4, 0; a) & \text{for } i = 4. \end{cases}$$

Looking for instance at the case $i = 1$ in the previous table for d_i one sees that c_5 appears before c_3 and that c_6 appears before c_4. This reflects that $\overline{\delta}_1$ is not order-preserving on the whole of $\mathrm{Sur}([5], [2])$, as stated after Theorem 1.13.

3. Cross-effect functors

In this section we summarize some definitions and results about cross-effect functors that are relevant to our work; see [**EM**] for proofs and more details.

Recall a functor $G\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is called linear if, for any sequence $A_1, \ldots, A_n$ of objects in $\mathcal{A}$, we have the relation $G(\oplus_{i=1}^n A_i) = \oplus_{i=1}^n G(A_i)$

in $\mathcal{B}$. The main result of the theory of cross-effect functors (Theorem 3.4) gives us an analogous decomposition for any *non-linear* functor $F\colon \mathcal{A} \to \mathcal{B}$ with the property that $F(0_{\mathcal{A}}) = 0_{\mathcal{B}}$. This decomposition we get in $\mathcal{B}$ has a term for each subsum of the original sum in $\mathcal{A}$ (rather than for each summand as with a linear functor). The terms of this sum in $\mathcal{B}$ are given by cross-effect functors of F.

For the rest of this section we let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between an additive category $\mathcal{A}$ and an abelian category $\mathcal{B}$ with $F(0_{\mathcal{A}}) = 0_{\mathcal{B}}$. The condition $F(0_{\mathcal{A}}) = 0_{\mathcal{B}}$ is equivalent to the condition that the image of any zero homomorphism in $\mathcal{A}$ under F is a zero homomorphism in $\mathcal{B}$.

Definition 3.1. For $f_1, \dots, f_n \in \operatorname{Hom}(A, B)$ we define the morphism

$$F(f_1 \top \cdots \top f_n) \in \operatorname{Hom}(F(A), F(B))$$

by the following equation:

$$F(f_1 \top \cdots \top f_n) = \sum_{k=1}^{n} \sum_{j_1 < \cdots < j_k} (-1)^{n-k} F(f_{j_1} + \cdots + f_{j_k}).$$

The function $F(- \top \cdots \top -)$ has the following properties. For each permutation π of $\{1, \dots, n\}$ we have $F(f_1 \top \cdots \top f_n) = F(f_{\pi(1)} \top \cdots \top f_{\pi(n)})$. Whenever any of the functions f_i are zero we get $F(f_1 \top \cdots \top f_n) = 0$. By rearranging the definition we get the relation $F(f_1 + \cdots + f_n) = \sum_{k=1}^{n} \sum_{j_1 < \cdots < j_k} F(f_{j_1} \top \cdots \top f_{j_k})$.

Notation 3.2. Let $A = A_1 \oplus \cdots \oplus A_n$ be a direct sum in the additive category $\mathcal{A}$. For each non-empty subset $\alpha = \{j_1 < \cdots < j_k\}$ of $\{1, \dots, n\}$ and each $j \in \alpha$, we write A^α for $\bigoplus_{l \in \alpha} A_l$, i^α for the canonical injection $A^\alpha \to A$, p^α for the canonical projection $A \to A^\alpha$, ψ_j^α for the map $A^\alpha \to A^\alpha$, $(a_{j_1}, \dots, a_{j_k}) \mapsto (0, \dots, 0, a_j, 0, \dots, 0)$ and just ψ_j if $\alpha = \{1, \dots, n\}$. We also write $(A_j, j \in \alpha)$ for the tuple $(A_{j_1}, \dots, A_{j_k})$.

Definition 3.3. The n^{th} cross-effect of F is a functor $\mathcal{A}^n \to \mathcal{B}$. It acts on objects by

$$\operatorname{cr}_n(F)(A_1, \dots, A_n) = F(\psi_1 \top \cdots \top \psi_n) F(A_1 \oplus \cdots \oplus A_n).$$

For the collection of morphisms $f_l\colon A_l \to B_l$, $1 \leqslant l \leqslant n$, the morphism

$$\operatorname{cr}_n(F)(f_1, \dots, f_n)\colon \operatorname{cr}_n(F)(A_1, \dots, A_n) \to \operatorname{cr}_n(F)(B_1, \dots, B_n)$$

is induced by $F(f_1 \oplus \cdots \oplus f_n)\colon F(A_1 \oplus \cdots \oplus A_n) \to F(B_1 \oplus \cdots \oplus B_n)$.

Definition 3.3 is a technical definition of cross-effect functors that does not really give much intuition about how one should think of them. It is better to think of cross-effect functors as the terms of a direct-sum decomposition as given in Theorem 3.4 below; Theorem 3.6 gives us the justification of this mental picture. In a sense, Theorem 3.6 is a converse of Theorem 3.4, because it says that if we have an appropriate collection of functors which give a decomposition of $G(\bigoplus_{i=1}^n A)$, then they are (up to isomorphism) the cross-effect functors of G.

Theorem 3.4. *Let $A_1, \dots, A_n \in \mathcal{A}$. The maps*

$$\mathrm{cr}_{|\alpha|}(F)(A_j, j \in \alpha) \subseteq F(\oplus_{j\in\alpha} A_j) \xrightarrow{F(i^\alpha)} F(A_1 \oplus \cdots \oplus A_n), \quad \alpha \subseteq \{1, \dots, n\},$$

induce the following direct-sum decomposition of $F(A_1 \oplus \cdots \oplus A_n)$:

$$\bigoplus_{\alpha \subseteq \{1,\dots,n\}} \mathrm{cr}_{|\alpha|}(F)(A_j, j \in \alpha) \cong F(A_1 \oplus \cdots \oplus A_n);$$

here, for each subset $\alpha = \{j_1 < \cdots < j_{|\alpha|}\}$ of $\{1, \dots, n\}$, the direct summand of the left-hand side that is indexed by α corresponds to the sub-object

$$F(\psi_{j_1} \top \cdots \top \psi_{j_{|\alpha|}})F(A_1 \oplus \cdots \oplus A_n)$$

of the right-hand side. In particular, given any subset β of $\{1, \dots, n\}$, the subsum $\oplus_{\alpha\subset\beta} \mathrm{cr}_{|\alpha|}(F)(A_j, j \in \alpha)$ of the left-hand side corresponds to the image of

$$F(i^\beta)\colon F(\oplus_{j\in\beta} A_j) \to F(A_1 \oplus \cdots \oplus A_n)$$

on the right-hand side.

Cross-effect functors also have the following properties. Whenever any of the objects A_j for $j \in \{1, \dots, n\}$ is the zero object, then $\mathrm{cr}_n(F)(A_1, \cdots, A_n)$ is also the zero object. For each permutation π of $\{1, \dots, n\}$ we get a natural isomorphism

$$\mathrm{cr}_n(F)(A_1, \dots, A_n) \cong \mathrm{cr}_n(F)(A_{\pi(1)}, \dots, A_{\pi(n)}).$$

Definition 3.5. If $\mathrm{cr}_n(F)$ is the zero functor, then we say that F *is a functor of degree less than* n. In this case F is also of degree less than m for any $m > n$. Because of this F has a well-defined *degree*. The degree of F is either a non-negative integer or infinity.

The following theorem gives us a characterization of the cross-effect functors of F by their appearance in a direct-sum decomposition as in Theorem 3.4.

Theorem 3.6. *For each subset α of $\{1, \dots, n\}$ let E_α be a covariant functor between $\mathcal{A}^{|\alpha|}$ and $\mathcal{B}$, which is zero when any of its arguments is zero. If we have a natural isomorphism*

$$h\colon \bigoplus_{\alpha \subset \{1,\dots,n\}} E_\alpha(A_j, j \in \alpha) \cong F(A_1 \oplus \cdots \oplus A_n),$$

then h induces an isomorphism

$$E_\alpha(A_j, j \in \alpha) \cong F(\psi_{j_1} \top \cdots \top \psi_{j_{|\alpha|}})F(A_1 \oplus \cdots \oplus A_n).$$

In particular, we get a natural isomorphism $E_\alpha \cong \mathrm{cr}_{|\alpha|}(F)$.

Example 3.7. Let R be a commutative ring and let Sym^2 denote the symmetric-square functor from the abelian category of R-modules to itself. For any R-modules

$M_1, \ldots, M_n$ we have a natural isomorphism

$$\mathrm{Sym}^2(M_1 \oplus \cdots \oplus M_n) \cong \left(\bigoplus_{i=1}^{n} \mathrm{Sym}^2(M_i) \right) \bigoplus \left(\bigoplus_{1 \leqslant i < j \leqslant n} M_i \otimes M_j \right),$$

which easily follows from the well-known case $n = 2$ by induction on n. From Theorem 3.6 we therefore obtain

$$\mathrm{cr}_i(\mathrm{Sym}^2)(M_1, \ldots, M_i) = \begin{cases} \mathrm{Sym}^2(M_1) & \text{if } i = 1 \\ M_1 \otimes M_2 & \text{if } i = 2 \\ 0 & \text{if } i > 2. \end{cases}$$

Similarly, we obtain

$$\mathrm{cr}_i(\mathrm{Sym}^3)(M_1, \ldots, M_i) = \begin{cases} \mathrm{Sym}^3(M_1) & \text{if } i = 1 \\ \mathrm{Sym}^2(M_1) \otimes M_2 \oplus M_1 \otimes \mathrm{Sym}^2(M_2) & \text{if } i = 2 \\ M_1 \otimes M_2 \otimes M_3 & \text{if } i = 3 \\ 0 & \text{if } i \geqslant 3. \end{cases}$$

In particular, Sym^2 is of degree 2, Sym^3 is of degree 3, and more generally, Sym^n is of degree n.

4. Expressing Dold-Puppe complexes in terms of cross-effect modules

Let $\mathcal{A}$ be an abelian category. Previously we have worked with the functor Γ; now we introduce its inverse $N \colon \mathcal{SA} \to \mathrm{Ch}_{\geqslant 0}\, \mathcal{A}$. Let $X.$ be a simplicial object in $\mathcal{A}$. The normalized chain complex $N(X.)$ of $X.$ is given by

$$N(X.)_n := X_n \Big/ \sum_{i=0}^{n-1} \mathrm{Im}\, s_i,$$

with its differential induced by the alternating sum of the face maps of $X.$:

$$\partial = \sum_{i=0}^{n} (-1)^i d_i \colon X_n \to X_{n-1}$$

(for $n \geqslant 0$). An important application of the Dold-Kan correspondence is the construction of Dold-Puppe complexes, i.e., complexes of the form $NF\Gamma(C.)$ where $C.$ is a chain complex and $F \colon \mathcal{A} \to \mathcal{B}$ is a functor between abelian categories (that has been extended to the category $\mathcal{SA}$ in the obvious way).

In [**Kö**] the first-named author uses cross-effect functors to give a description of the Dold-Puppe complex of a chain complex $C. = (P \to Q)$ of length one (i.e., $C_n = 0$ when $n > 1$) in the category $\mathrm{Ch}_{\geqslant 0}(\mathcal{A})$. Lemma 2.2 of [**Kö**] proves that

$$NF\Gamma(P \to Q)_n \cong \mathrm{cr}_n(F)(P, \ldots, P) \oplus \mathrm{cr}_{n+1}(F)(Q, P, \ldots, P)$$

and gives an explicit description of the differential. The aim of this section is to generalize this result and give a similar description of Dold-Puppe complexes in terms of cross-effect functors when the original complex is longer.

For the rest of this section we fix a functor $F\colon \mathcal{A} \to \mathcal{B}$ from an additive category $\mathcal{A}$ to an abelian category $\mathcal{B}$ with the property that $F(0_{\mathcal{A}}) = 0_{\mathcal{B}}$, we fix a chain complex $C.$ in $\mathcal{A}$ and we fix a positive integer n.

The following definition introduces another way of denoting elements of $\mathrm{Sur}([n],[k])$, which will make it easier to deal with the problems in this section.

Definition 4.1. Let $\mathcal{P}_n$ denote the set of subsets of $\{0,1,\dots,n-1\}$. We define a bijective map $^{\triangle}$ as follows:

$$\begin{aligned} ^{\triangle}\colon\ &\amalg_{k=0}^{n} \mathrm{Sur}([n],[k]) \to \mathcal{P}_n \\ &\mu \in \mathrm{Sur}([n],[k]) \mapsto \mu^{\triangle} := \{\max \mu^{-1}(0), \dots, \max \mu^{-1}(k-1)\}, \end{aligned}$$

where max is the function that gives the maximum element of a set. For each $k \in \{0,\dots,n\}$, we use the symbol $^{\triangle}$ also for the induced bijection between $\mathrm{Sur}([n],[k])$ and the set $\mathcal{P}_n^k$ of subsets of $\{0,\dots,n-1\}$ of cardinality k.

Note that we have omitted $\max \mu^{-1}(k)$ in the list of elements of $\mu^{\triangle}$ because $\max \mu^{-1}(k)$ is always equal to n. For every $0 \leqslant i \leqslant n-2$, the partition μ^* obviously begins with a partition of $i+1$ (in the sense of Definition 1.6) if and only if $i \in \mu^{\triangle}$. We will be using this observation extensively when we refer to results of Section 2.

Definition 4.2. We say that a subset α of the disjoint union $\amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$ is *honourable* if $\cup_{\mu\in\alpha}\, \mu^{\triangle} = \{0,1,\dots,n-1\}$.

Notation 4.3. Let $\alpha \subset \amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$. For each $k \in \{0,\dots,n\}$ we write α_k for the intersection $\alpha \cap \mathrm{Sur}([n],[k])$. For $C_0,\dots,C_n \in \mathcal{A}$ we write $(C_{0,\alpha_0},\dots,C_{n,\alpha_n})$ for the following $|\alpha|$-tuple:

$$(\underbrace{C_0,\dots,C_0}_{|\alpha_0|\text{ times}},\dots,\underbrace{C_n,\dots,C_n}_{|\alpha_n|\text{ times}}).$$

Proposition 4.4. *We have a canonical isomorphism*

$$NF\Gamma(C.)_n \cong \bigoplus_{\substack{\alpha\subset \amalg_{k=0}^{n} \mathrm{Sur}([n],[k]) \\ \alpha \text{ is honourable}}} \mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\dots,C_{n,\alpha_n}).$$

Proof. Using the definitions of N and Γ we see that

$$NF\Gamma(C.)_n = F\Big(\bigoplus_{k=0}^{n} \bigoplus_{\mu\in\mathrm{Sur}([n],[k])} C_k\Big) \Big/ \sum_{i=0}^{n-1} \mathrm{Im}\, F(s_i).$$

Expanding the numerator in terms of cross effects according to Theorem 3.4 we get the formula

$$F\Big(\bigoplus_{k=0}^{n} \bigoplus_{\mu\in\mathrm{Sur}([n],[k])} C_k\Big) = \bigoplus_{\alpha\subseteq \amalg_{k=0}^{n} \mathrm{Sur}([n],[k])} \mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\dots,C_{n,\alpha_n}).$$

Theorem 2.2(a) tells us that s_i maps the direct sum $\bigoplus_{k=0}^{n} \bigoplus_{\mu\in\mathrm{Sur}([n-1],[k])} C_k$ isomorphically to the subsum $\bigoplus_{k=0}^{n} \bigoplus_{\mu\in(S_i^{n,k})^C} C_k$ of $\bigoplus_{k=0}^{n} \bigoplus_{\mu\in\mathrm{Sur}([n],[k])} C_k$. Applying

Theorem 3.4 again we see that

$$\operatorname{Im} F(s_i) = \bigoplus_{\alpha} \operatorname{cr}_{|\alpha|}(F)(C_{0,\alpha_0}, \dots, C_{n,\alpha_n}),$$

where the last sum ranges over all subsets $\alpha \subset \amalg_{k=0}^n \operatorname{Sur}([n],[k])$ where $i \notin \cup_{\mu \in \alpha} \mu^{\triangle}$. From this we see that $\operatorname{cr}_{|\alpha|}(F)(C_{0,\alpha_0}, \dots, C_{n,\alpha_n})$ is not a direct summand of $\operatorname{Im} F(s_i)$ if and only if $i \in \cup_{\mu \in \alpha} \mu^{\triangle}$. A module is a direct summand of $NF\Gamma(C.)_n$ if and only if it is not a direct summand of $\sum_{i=0}^{n-1} \operatorname{Im} F(s_i)$, and hence we see the desired result. □

Although the expression for $NF\Gamma(C.)_n$ given in the previous proposition is quite compact, it still contains many vanishing terms: whenever $|\alpha|$ is bigger than the degree of F or α_k is non-empty for k bigger than the length of $C.$, the cross-effect module $\operatorname{cr}_{|\alpha|}(F)(C_{0,\alpha_0}, \dots, C_{n,\alpha_n})$ vanishes. The rest of this section is devoted to the problem of quickly finding those honourable subsets α for which $\operatorname{cr}_{|\alpha|}(F)(C_{0,\alpha_0}, \dots, C_{n,\alpha_n})$ does not vanish. A first (still rather rough) result in this direction is Corollary 4.6 below. Later we will describe an algorithm that produces the relevant honourable subsets fairly quickly.

Proposition 4.5.

(a) Let α be an honourable subset of $\amalg_{k=0}^n \operatorname{Sur}([n],[k])$. Then we have the inequality $\sum_{k=0}^n k|\alpha_k| \geqslant n$.

(b) Conversely, let $(a_0, \dots, a_n) \in \mathbb{N}^{n+1}$ with $a_k \leqslant \binom{n}{k}$ for each $k \in \{0, \dots, n\}$. If $\sum_{k=0}^n k a_k \geqslant n$, then there is some honourable subset α of $\amalg_{k=0}^n \operatorname{Sur}([n],[k])$ with $|\alpha_k| = a_k$ for each $k \in \{0, \dots, n\}$.

Proof. First we prove part (a). We know α is honourable, so by definition

$$\cup_{k=0}^n \cup_{\mu \in \alpha_k} \mu^{\triangle} = \{0, 1, \dots, n-1\},$$

hence,

$$\sum_{k=0}^n k|\alpha_k| = \sum_{k=0}^n \sum_{\mu \in \alpha_k} |\mu^{\triangle}| \geqslant |\{0, 1, \dots, n-1\}| = n.$$

Now we prove part (b). Because $|\{0, \dots, n-1\}| = n \leqslant \sum_{k=0}^n k a_k$ and $a_k \leqslant \binom{n}{k}$ we can cover the set $\{0, \dots, n-1\}$ using a_1 subsets of cardinality 1, a_2 subsets of cardinality $2, \dots, a_{n-1}$ subsets of cardinality $n-1$ and a_n subsets of cardinality n. Take such a covering β and define α to be the preimage of β under the map ${}^{\triangle}\colon \amalg_{k=0}^n \operatorname{Sur}([n],[k]) \to \mathcal{P}_n$ introduced in Definition 4.1. Then α has the desired properties. □

Corollary 4.6. *The length of the Dold-Puppe complex $NF\Gamma(C.)$ is less than or equal to the product ld of the length l of $C.$ and the degree d of F. Equality is achieved if the module $\operatorname{cr}_d(F)(C_l, \dots, C_l)$ is not the zero module.*

Proof. Proposition 4.4 tells us that

$$NF\Gamma(C.)_n \cong \bigoplus_{\substack{\alpha \subset \amalg_{k=0}^n \operatorname{Sur}([n],[k]) \\ \alpha \text{ is honourable}}} \operatorname{cr}_{|\alpha|}(F)(C_{0,\alpha_0}, \dots, C_{n,\alpha_n}).$$

If $|\alpha| > d$, then $\operatorname{cr}_{|\alpha|}(F)(C_{0,\alpha_0}, \dots, C_{n,\alpha_n})$ vanishes. Also the properties of cross-effects tell us if any of the modules are zero, then cross-effect modules involving them will also vanish, in particular any which involve any copies of $C_{l'}$ where $l' > l$ vanish. So the only non-zero cross-effect modules in $NF\Gamma(C.)_n$ are those which correspond to subsets of $\amalg_{k=0}^{\min\{n,l\}} \mathrm{Sur}([n],[k])$ that are honourable and of cardinality d or less.

It therefore suffices to show that, if $n > ld$, there does not exist any honourable subset α of $\amalg_{k=0}^{\min\{n,l\}} \mathrm{Sur}([n],[k])$ that satisfies $|\alpha| \leqslant d$. Suppose α is such a subset. As $|\alpha_k| = 0$ for $k > \min\{n,l\} = l$ (we may assume $d \geqslant 1$), we obtain

$$\sum_{k=0}^{n} |\alpha_k| k = \sum_{k=0}^{l} |\alpha_k| k \leqslant \sum_{k=0}^{l} |\alpha_k| l = l|\alpha| \leqslant ld < n.$$

This contradicts Proposition 4.5(a).

To prove equality is achieved if $\mathrm{cr}_d(F)(C_l,\ldots,C_l)$ is not the zero module, we set $n = dl$, $a_l = d$ and $a_k = 0$ if $k \neq l$. Proposition 4.5(b) tells us that there is some honourable set $\alpha \subset \amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$ with $|\alpha_k| = a_k$ for each $k \in \{0,\ldots,n\}$. This condition tells us that $\alpha \subset \mathrm{Sur}([n],[l])$. So $\mathrm{cr}_{|\alpha|}(F)(C_{0,\alpha_0},\ldots,C_{n,\alpha_n}) = \mathrm{cr}_d(F)(C_{l,\alpha_l})$. This is non-zero by assumption and a direct summand of $NF\Gamma(C.)_n$ because of our choice of α. □

The following definition will be useful in describing the algorithm mentioned above.

Definition 4.7.

(a) We define a total order on the powerset $\mathcal{P}_n$ of $\{0,1,\ldots,n-1\}$ as follows. Let $x = \{i_1 < \cdots < i_k\}$ and $y = \{j_1 < \cdots < j_{k'}\}$ be sets in $\mathcal{P}_n$. Then $x \leqslant y$ if and only if $k' < k$ or ($k' = k$ and $(i_1,\ldots,i_k) \leqslant (j_1,\ldots,j_k)$ in the lexicographic ordering).

(b) Let T be a subset of $\mathcal{P}_n$ and let x be a set in T. We say that x is *superfluous* in T if $\cup_{y\in T}\, y = \cup_{y\in T\setminus\{x\}}\, y$.

(c) We say that an honourable subset α of $\amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$ is *minimal* if $\alpha^{\triangle}$ does not contain any superfluous sets.

Recall that we have introduced a total order on $\mathrm{Sur}([n],[k])$ in Definition 1.9 for each $k \in \{0,1,\ldots,n\}$. It is easy to see that the bijection ${}^{\triangle}\colon \mathrm{Sur}([n],[k]) \to \mathcal{P}_n^k$ is order preserving. The following easy procedure is an efficient way for checking whether a subset T of $\mathcal{P}_n$ contains superfluous sets, particularly in the context of the algorithm described later.

Procedure 4.8. Let T be a subset of $\mathcal{P}_n$. We first order the sets in T using the ordering introduced in Definition 4.7(a), say $T = \{x_1 < \cdots < x_m\}$. For each $r = 2,\ldots,m$ and for each $i \in x_r$ we then check whether $i \in x_1 \cup \cdots \cup x_{r-1}$. If so, we underline i in each of the sets $x_1,\ldots,x_r$ where it occurs. There are two ways for this procedure to stop: (1) we perform the check (and if necessary the underlining) described above for each $r \in \{2,\ldots,m\}$ and each $i \in x_r$ and at each stage we find that no set in T has all of its elements underlined; (2) at some point we find some set x in T with each of its elements underlined. In case (1) no superfluous sets are contained in T; in case (2) the set x is superfluous in T.

Example 4.9. Let $n = 4$.

(a) Applying Procedure 4.8 to $T = \{\{0,1\},\{0,3\},\{0\}\}$, we first obtain $\{\underline{0},1\} < \{\underline{0},3\}$ and then $\{\underline{0},1\} < \{\underline{0},3\} < \{\underline{0}\}$; hence, the last set $\{\underline{0}\}$ is superfluous.

(b) Applying Procedure 4.8 to $T = \{\{0,1\},\{1,2\},\{2,3\}\}$, we first obtain $\{0,\underline{1}\} < \{\underline{1},2\}$ and then $\{0,\underline{1}\} < \{\underline{1},\underline{2}\} < \{\underline{2},3\}$; hence, the second set $\{1,2\}$ is superfluous.

(c) Applying Procedure 4.8 to $T = \{\{0,1,2\},\{1,3\}\}$, we obtain $\{0,\underline{1},2\} < \{1,3\}$; hence, none of the sets in T is superfluous.

(d) Procedure 4.8 applied to $T = \{\{0,1\},\{1,2\},\{1\},\{2\},\{3\}\}$ stops at $\{0,\underline{1}\} < \{\underline{1},2\} < \{\underline{1}\}$.

We now describe an algorithm which finds all *minimal* honourable subsets of the set $\amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$ in an efficient way. Via the bijection ${}^{\triangle}\colon \amalg_{k=0}^{n} \mathrm{Sur}([n],[k]) \to \mathcal{P}_n$ (see Definition 4.1) this amounts to finding all subsets T of $\mathcal{P}_n$ such that $\cup_{x\in T}\, x = \{0,1,\ldots,n-1\}$ and such that T does not contain any superfluous sets. Below we first inductively define a finite list $T_1, T_2, \ldots$ of subsets of $\mathcal{P}_n$. From the construction it will be immediately clear that $T_1, T_2, \ldots$ is the list of all subsets of $\mathcal{P}_n$ which do not contain any superfluous sets. We finally just discard those subsets from the list which are not honourable.

Definition 4.10. We inductively define a finite list $T_1, T_2, \ldots$ of distinct subsets of $\mathcal{P}_n$ containing no superfluous sets as follows. Let $T_1 := \{\{0,1,\ldots,n-1\}\}$ and suppose $T_1,\ldots,T_m$ have already been defined. We write T_m in the form $\{x_1 < \cdots < x_r\}$ with some sets $x_1,\ldots,x_r$ in $\mathcal{P}_n$. If $r = 1$ and $x_1 = \{n-1\}$, i.e., if x_1 is the maximal set in $\mathcal{P}_n \setminus \{\emptyset\}$, then the list $T_1,\ldots,T_m$ is complete. We now assume this is not the case. If T_m is not honourable, then (since by construction T_m contains no superfluous set) there exists a set y in $\mathcal{P}_n$ bigger than x_r such that $\{x_1 < \cdots < x_r < y\}$ does not contain any superfluous set; we choose y to be minimal with this property and define $T_{m+1} := \{x_1 < \cdots < x_r < y\}$. If T_m is honourable, then there exists an index $s \in \{1,\ldots,r\}$ and a set y in $\mathcal{P}_n$ bigger than x_s such that $\{x_1 < \cdots < x_{s-1} < y\}$ does not contain any superfluous set. We choose $s \in \{1,\ldots,r\}$ to be maximal and $y \in \mathcal{P}_n$ to be minimal with this property and define $T_{m+1} := \{x_1 < \cdots < x_{s-1} < y\}$.

Example 4.11. For $n = 3$ the previous definition gives the following list $T_1, T_2, \ldots$ of subsets of $\mathcal{P}_3$. Following the convention introduced in Procedure 4.8 we underline certain elements to be able to easily detect superfluous sets.

$$\begin{aligned}
&T_1 = \{\{0,1,2\}\}, T_2 = \{\{0,1\}\}, T_3 = \{\{\underline{0},1\} < \{\underline{0},2\}\}, T_4 = \{\{0,\underline{1}\} < \{\underline{1},2\}\},\\
&T_5 = \{\{0,1\} < \{2\}\}, T_6 = \{\{0,2\}\}, T_7 = \{\{0,\underline{2}\} < \{1,\underline{2}\}\}, T_8 = \{\{0,2\} < \{1\}\},\\
&T_9 = \{\{1,2\}\}, T_{10} = \{\{1,2\} < \{0\}\}, T_{11} = \{\{0\}\}, T_{12} = \{\{0\} < \{1\}\},\\
&T_{13} = \{\{0\} < \{1\} < \{2\}\}, T_{14} = \{\{0\} < \{2\}\}, T_{15} = \{\{1\}\}, T_{16} = \{\{1\} < \{2\}\},\\
&T_{17} = \{\{2\}\}.
\end{aligned}$$

The subsets T_1, T_3, T_4, T_5, T_7, T_8, T_{10} and T_{13} correspond to minimal honourable subsets of $\amalg_{k=0}^{3} \mathrm{Sur}([n],[k])$.

As explained earlier, in order to calculate the direct-sum decomposition in Proposition 4.4 there is no need to find those honourable subsets α of $\amalg_{k=0}^{n} \mathrm{Sur}([n],[k])$ for which α_k is non-empty for k bigger than the length l of $C.$. In other words, rather than starting the inductive procedure in Definition 4.10 at the smallest set $\{0, 1, \dots, n-1\}$ in $\mathcal{P}_n$, it suffices to begin at $\{0, 1, \dots, \min\{n, l\} - 1\}$.

Example 4.12. In this example we apply Definition 4.10 in the case $n = 4$. We begin the induction only at $\{\{0, 1\}\}$ rather than at $T_1 = \{\{0, 1, 2, 3\}\}$, i.e., we assume $l = 2$. For simplicity, we omit the external brackets for T_i; we in fact omit the name T_i as well (but keep the order of the list of course), and we moreover write down only subsets of $\mathcal{P}_n$ which correspond to minimal *honourable* subsets. The result is as follows:

$$
\begin{aligned}
&\{0,1\} < \{0,2\} < \{0,3\}, \{0,1\} < \{0,2\} < \{3\}, \{0,1\} < \{0,3\} < \{2\},\\
&\{0,1\} < \{1,2\} < \{1,3\}, \{0,1\} < \{1,2\} < \{3\}, \{0,1\} < \{1,3\} < \{2\}, \{0,1\} < \{2,3\},\\
&\{0,1\} < \{2\} < \{3\}, \{0,2\} < \{0,3\} < \{1\}, \{0,2\} < \{1,2\} < \{2,3\},\\
&\{0,2\} < \{1,2\} < \{3\}, \{0,2\} < \{1,3\}, \{0,2\} < \{2,3\} < \{1\}, \{0,2\} < \{1\} < \{3\},\\
&\{0,3\} < \{1,2\}, \{0,3\} < \{1,3\} < \{2,3\}, \{0,3\} < \{1,3\} < \{2\}, \{0,3\} < \{2,3\} < \{1\},\\
&\{0,3\} < \{1\} < \{2\}, \{1,2\} < \{1,3\} < \{0\}, \{1,2\} < \{2,3\} < \{0\}, \{1,2\} < \{0\} < \{3\},\\
&\{1,3\} < \{2,3\} < \{0\}, \{1,3\} < \{0\} < \{2\}, \{2,3\} < \{0\} < \{1\},\\
&\{0\} < \{1\} < \{2\} < \{3\}.
\end{aligned}
$$

The object of the following example is to illustrate the methods developed earlier in this paper.

Example 4.13. Let R be a commutative ring and let $C \xrightarrow{\partial} B \xrightarrow{\partial} A$ be a chain complex of R-modules of length 2 (sitting in degrees 0, 1 and 2). The goal of this example is to explicitly write down the Dold-Pupppe complex $Q. := N\,\mathrm{Sym}^2\,\Gamma(C \to B \to A)$. We proceed in two steps. In the first step we write down the object Q_n for $n = 0, 1, \dots$ (using the method developed in this section), and in the second step we write down the differential $\Delta\colon Q_n \to Q_{n-1}$ for $n = 1, 2, \dots$ (using the calculations made at the end of Section 2).

By Corollary 4.6 the chain complex $Q.$ is of length 4. From Proposition 4.4 we immediately get $D_0 = \mathrm{Sym}^2(A)$. To calculate D_n for $n = 1, 2, 3, 4$ we first find all honourable subsets of $\amalg_{k=0}^{n} \mathrm{Sur}([n], [k])$. The subsets of $\mathcal{P}_n$ listed below correspond to minimal honourable subsets of $\amalg_{k=0}^{n} \mathrm{Sur}([n], [k])$. As explained earlier, before Example 4.12, we write down only those subsets T of $\mathcal{P}_n$ whose sets contain at most two elements. Furthermore, we write down only those subsets T of $\mathcal{P}_n$ which contain at most two sets (because the degree of Sym^2 is 2). As in Example 4.12 we omit the exterior brackets. For $n = 3$ and $n = 4$ we use Examples 4.11 and 4.12, respectively:

$$
\begin{aligned}
n = 1:&\quad \{0\}\\
n = 2:&\quad \{0,1\}, \{0\} < \{1\}\\
n = 3:&\quad \{0,1\} < \{0,2\}, \{0,1\} < \{1,2\}, \{0,1\} < \{2\}, \{0,2\} < \{1,2\},\\
&\quad \{0,2\} < \{1\}, \{1,2\} < \{0\}\\
n = 4:&\quad \{0,1\} < \{2,3\}, \{0,2\} < \{1,3\}, \{0,3\} < \{1,2\}.
\end{aligned}
$$

We finally add to these lists those subsets T of $\mathcal{P}_n$ which correspond to non-minimal honourable subsets. As above we are only interested in subsets T of $\mathcal{P}_n$ of cardinality at most 2; hence, the lists for $n = 3$ and $n = 4$ do not change. For $n = 1$ and $n = 2$ the completed lists are as follows:

$$\begin{aligned}
n = 1:&\quad \{0\}, \{0\} < \emptyset \\
n = 2:&\quad \{0,1\}, \{0,1\} < \{0\}, \{0,1\} < \{1\}, \{0,1\} < \emptyset, \{0\} < \{1\}.
\end{aligned}$$

(By the way, this also illustrates that it is more efficient to first find the minimal honourable subsets and then to add the relevant non-minimal honourable subsets than to immediately go for all honourable subsets.) Hence, the objects $Q_0, \ldots, Q_4$ are as follows:

$$\begin{aligned}
Q_0 &= \mathrm{Sym}^2(A) \\
Q_1 &= \mathrm{Sym}^2(B_1) \oplus B_1 \otimes A \\
Q_2 &= \mathrm{Sym}^2(C_1) \oplus C_1 \otimes B_1 \oplus C_1 \otimes B_2 \oplus C_1 \otimes A \oplus B_1 \otimes B_2 \\
Q_3 &= C_1 \otimes C_2 \oplus C_1 \otimes C_3 \oplus C_1 \otimes B_3 \oplus C_2 \otimes C_3 \oplus C_2 \otimes B_2 \oplus C_3 \otimes B_1 \\
Q_4 &= C_1 \otimes C_6 \oplus C_2 \otimes C_5 \oplus C_3 \otimes C_4.
\end{aligned}$$

Here, for instance, the module C_5 in Q_4 refers to the fifth copy of the module C in $\Gamma(C \to B \to A)_4 = C^6 \oplus B^4 \oplus A$, using the ordering of copies of C introduced in Section 2.

We finally turn to the differential $\Delta\colon Q_n \to Q_{n-1}$ for $n = 1, 2, 3, 4$. It is induced by $\sum_{i=0}^{n} (-1)^i d_i$ (see Section 3). Here, d_i denotes the i^{th} face operator in $\mathrm{Sym}^2 \Gamma(C \to B \to A)$; i.e., d_i is the symmetric square of the i^{th} face operator in $\Gamma(C \to B \to A)$. Using the calculation given in Example 2.3 and some elementary facts about the cross-effects of Sym^2, we obtain the following action of d_i on each direct summand of Q_n for $n = 1, 2, 3, 4$:

$$\begin{aligned}
n = 1\colon\ d_0\colon&\ \mathrm{Sym}^2(B_1) \to \mathrm{Sym}^2(A), \quad bb' \mapsto \partial(b)\partial(b') \\
&\ B_1 \otimes A \to \mathrm{Sym}^2(A), \quad b \otimes a \mapsto \partial(b)a \\
d_1\colon&\ \text{acts as the zero map on } Q_1
\end{aligned}$$

$$\begin{aligned}
n = 2\colon\ d_0\colon&\ \mathrm{Sym}^2(C_1) \to \mathrm{Sym}^2(B_1), \quad cc' \mapsto \partial(c)\partial(c') \\
&\ C_1 \otimes B_1 \to B_1 \otimes A, \quad c \otimes b \mapsto \partial(c) \otimes \partial(b) \\
&\ C_1 \otimes B_2 \to \mathrm{Sym}^2(B_1), \quad c \otimes b \mapsto \partial(c)b \\
&\ C_1 \otimes A \to B_1 \otimes A, \quad c \otimes a \mapsto \partial(c) \otimes a \\
&\ B_1 \otimes B_2 \to B_1 \otimes A, \quad b \otimes b' \mapsto b' \otimes \partial(b) \\
d_1\colon&\ \text{acts as the zero map on the first four direct summands of } Q_2 \\
&\ B_1 \otimes B_2 \to \mathrm{Sym}^2(B_1), \quad b \otimes b' \mapsto bb' \\
d_2\colon&\ \text{acts as the zero map on } Q_2
\end{aligned}$$

$$\begin{aligned}
n = 3\colon\ d_0\colon&\ C_1 \otimes C_2 \to B_1 \otimes B_2, \quad c \otimes c' \mapsto \partial(c) \otimes \partial(c') \\
&\ C_1 \otimes C_3 \to C_1 \otimes B_1, \quad c \otimes c' \mapsto c' \otimes \partial(c) \\
&\ C_1 \otimes B_3 \to B_1 \otimes B_2, \quad c \otimes b \mapsto \partial(c) \otimes b \\
&\ C_2 \otimes C_3 \to C_1 \otimes B_2, \quad c \otimes c' \mapsto c' \otimes \partial(c)
\end{aligned}$$

$C_2 \otimes B_2 \to B_1 \otimes B_2, \quad c \otimes b \mapsto b \otimes \partial(c)$
$C_3 \otimes B_1 \to C_1 \otimes A, \quad c \otimes b \mapsto c \otimes \partial(b)$

d_1: acts as the zero map on the first three direct summands of Q_3
$C_2 \otimes C_3 \to \mathrm{Sym}^2(C_1), \quad c \otimes c' \mapsto cc'$
$C_2 \otimes B_2 \to C_1 \otimes B_1, \quad c \otimes b \mapsto c \otimes b$
$C_3 \otimes B_1 \to C_1 \otimes B_1, \quad c \otimes b \mapsto c \otimes b$

d_2: acts as the zero map on the 2^{nd}, 4^{th} and 6^{th} direct summand of Q_3
$C_1 \otimes C_2 \to \mathrm{Sym}^2(C_1), \quad c \otimes c' \mapsto cc'$
$C_1 \otimes B_3 \to C_1 \otimes B_2, \quad c \otimes b \mapsto c \otimes b$
$C_2 \otimes B_2 \to C_1 \otimes B_2, \quad c \otimes b \mapsto c \otimes b$

d_3: acts as the zero map on Q_3

$n = 4$: d_0: $C_1 \otimes C_6 \to C_3 \otimes B_1, \quad c \otimes c' \mapsto c' \otimes \partial(c)$
$C_2 \otimes C_5 \to C_2 \otimes B_2, \quad c \otimes c' \mapsto c' \otimes \partial(c)$
$C_3 \otimes C_4 \to C_1 \otimes B_3, \quad c \otimes c' \mapsto c' \otimes \partial(c)$

d_1: acts as the zero map on the first direct summand of Q_4
$C_2 \otimes C_5 \to C_1 \otimes C_2, \quad c \otimes c' \mapsto c \otimes c'$
$C_3 \otimes C_4 \to C_1 \otimes C_2, \quad c \otimes c' \mapsto c' \otimes c$

d_2: acts as the zero map on the last direct summand of Q_4
$C_1 \otimes C_6 \to C_1 \otimes C_3, \quad c \otimes c' \mapsto c \otimes c'$
$C_2 \otimes C_5 \to C_1 \otimes C_3, \quad c \otimes c' \mapsto c \otimes c'$

d_3: acts as the zero map on the first direct summand of Q_4
$C_2 \otimes C_5 \to C_2 \otimes C_3, \quad c \otimes c' \mapsto c \otimes c'$
$C_3 \otimes C_4 \to C_2 \otimes C_3, \quad c \otimes c' \mapsto c \otimes c'$

d_4: acts as the zero map on Q_4

Note that in the case $n = 4$ the only i for which the image of d_i intersected with $C_3 \otimes B_1$, $C_2 \otimes B_2$ or $C_1 \otimes B_3$ does not vanish is 0; hence, $\sum_{i=0}^{4}(-1)^i d_i$ is injective if d_0 is injective. The latter holds for instance if $\partial\colon C \to B$ is injective and the R-module C is projective. This reproves the case $k = 4$ of Theorem 6.4 in [**Kö**], which states that $H_4 N \,\mathrm{Sym}^2 \Gamma(P.(V)) \cong 0$ for a certain projective resolution $P.(V)$.

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