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ALGEBRAIC COBORDISM AND GROTHENDIECK GROUPS OVER SINGULAR SCHEMES

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(*communicated by J. F. Jardine*)

Abstract

A theorem of Levine-Morel states that algebraic cobordism groups are isomorphic to (multiplicative) Grothendieck groups over smooth schemes. We extend this theorem to singular schemes. As a consequence, we provide a new proof of the singular Riemann-Roch theorem of Baum-Fulton-MacPherson and a new type of Riemann-Roch theorem with respect to pullbacks of locally complete morphisms.

1. Introduction

Let k be a field. Let us denote by Sch_k the category of separated k-schemes of finite type and by qSch_k (resp. Sm_k) its full subcategory of quasi-projective (resp. smooth) k-schemes. By a smooth morphism in Sch_k, we will always mean a smooth and quasi-projective morphism. In particular, a smooth k-scheme will always be assumed to be quasi-projective over k.

We recall that an *oriented cohomology theory* A^* on Sm_k is a contravariant functor $X \mapsto A^*(X)$ sending $X \in \mathrm{Sm}_k$ to the category of graded commutative rings equipped with functorial push-forwards for projective morphisms, satisfying certain properties such as the projective bundle formula and homotopy. Please refer to [**6**, Def. 1.1.2] for full details.

An important feature of oriented cohomology theories is that they have a formal group law structure that describes how the first Chern classes behave with respect to the tensor product of line bundles. An oriented cohomology theory is called *additive*, *multiplicative*, and *periodic* if its formal group law is additive, multiplicative, and periodic respectively.

In [**6**], Levine and Morel construct a universal oriented cohomology theory on Sm_k, called *algebraic cobordism* and written as Ω^*, which is the algebro-geometric version of Quillen's complex cobordism. They show that Ω^* has the universal formal group law. That is to say, given a formal group law (F_R, R), there is a unique homomorphism $\Omega^*(k) \to R$ sending F_Ω to F_R, which allows one to construct the universal theory with formal group law (F_R, R) as $\Omega^*_F(X) := \Omega(X) \otimes_{\Omega(k)} R$. It is of particular interest when $R = \mathbb{Z}[\beta, \beta^{-1}]$. Let us use $\Omega^*_\times$ to denote $\Omega^* \otimes_{\Omega(k)} \mathbb{Z}[\beta, \beta^{-1}]$. It turns out that

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$\Omega^*_\times$ is the universal multiplicative periodic theory on Sm_k. The homological notation for $\Omega^*_\times$ will be denoted by $\Omega^\times_*$.

Remark 1.1. For the construction of Ω_* (or equivalently, Ω^*), please refer to [**6**, §2.4]. However, for the purpose of this paper it suffices to know that $\Omega_*(X)$ (and $\Omega^\times_*(X)$ resp.) is essentially generated by $[f\colon Y \to X]$ (and $[f\colon Y \to X]\beta^n$ resp.), called cobordism cycles, with Y being a smooth scheme and the morphism f being projective.

We recall the following universal property of K-theory from [**7**]:

Theorem 1.2 (Levine-Morel)**.** *Let A^* be a multiplicative periodic oriented cohomology theory on Sm_k. Then there exists one, and only one, morphism of oriented cohomology theories $\mathrm{ch}_A\colon K_0[\beta,\beta^{-1}] \to A^*$, where $K_0[\beta,\beta^{-1}] = K_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta,\beta^{-1}]$.*

By the universality of $\Omega^*_\times$ on Sm_k, this yields:

Corollary 1.3 (Levine-Morel)**.** *Suppose that k has characteristic zero. Then the canonical transformation $\Omega^* \to K_0[\beta,\beta^{-1}]$ descends to an isomorphism of multiplicative oriented cohomology theories $\Omega^*_\times \to K_0[\beta,\beta^{-1}]$ on Sm_k.*

It is natural to ask if this natural isomorphism over Sm_k can be extended to one over Sch_k. For this purpose, it is necessary to replace *oriented cohomology theories* on Sm_k by *oriented Borel-Moore homology theories* on Sch_k.

An *oriented Borel-Moore homology theory* A_* on Sch_k is a functor $X \mapsto A_*(X)$ sending X in Sch_k to the category of graded abelian groups with functorial push-forward for projective morphisms, and pullback maps for locally complete intersection (l.c.i.) morphisms, satisfying some natural axioms. See Definition 5.1.3 of [**6**] for details.

Note that on Sch_k, K-theory shall be replaced by G-theory. Let us abbreviate the phrase *Oriented Borel-Moore* to OBM.

Remark 1.4. From Theorem 7.1.3 and Remark 4.1.12 of [**6**], Ω_* (and $\Omega^\times_*$ resp.) is the universal OBM homology theory (and the universal multiplicative OBM homology theory resp.) on Sch_k.

We are able to prove the following main result of this paper:

Theorem 1.5. *Let k be a field of characteristic zero. Then $G_0[\beta,\beta^{-1}]$ is the universal multiplicative* OBM *homology theory on Sch_k. That is to say, for any multiplicative* OBM *homology theory A_* on Sch_k, there is a unique natural transformation of* OBM *homology theories $\tau\colon G_0[\beta,\beta^{-1}] \to A_*$.*

In fact, the canonical natural transformation $\theta_G\colon \Omega_* \to G_0[\beta,\beta^{-1}]$ descends to a natural transformation of OBM homology theories on Sch_k,

$$\theta^\times_G\colon \Omega^\times_* \to G_0[\beta,\beta^{-1}], \tag{1}$$

where for a scheme X the map $\theta^\times_G$ is defined by the following:

$$[f\colon Y \to X]\beta^n \mapsto f_*[\mathcal{O}_Y]\beta^{n+\dim_k Y}.$$

Remark 1.6. The transformation θ_G is natural by the universality of Ω_*. As being natural only concerns commutativity with push-forwards but not the factor β^n, $\theta_G^\times$ is thus natural. On the other hand, the universality of $\Omega_*^\times$ implies that $\theta_G^\times$ is actually the unique natural transformation between the two theories, which is compatible with l. c. i. pullbacks and the first Chern class operators (i.e., a morphism of OBM homology theories).

We prove directly that the map (1) is an isomorphism on Sch_k, which yields Theorem 1.5 via the universality of $\Omega_*^\times$.

We apply the main theorem to two situations. The first (Corollary 1.7) gives a new version of singular Riemann-Roch with respect to pullbacks by locally complete morphisms, and the second (Corollary 1.8) provides a new proof of the singular Riemann-Roch theorem of Baum-Fulton-MacPherson.

Corollary 1.7 (l. c. i. Riemann-Roch)**.** *Let $f\colon Y \to X \in \mathrm{qSch}_k$ be an* l. c. i. *morphsim of relative degree d. Then we have the following commutative diagram:*

$$\begin{array}{ccc} G_0(X) & \xrightarrow{\quad f^* \quad} & G_0(Y) \\ {\scriptstyle \tau_0}\downarrow & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{CH}(X)_{\mathbb{Q}} & \xrightarrow{\tilde{\mathrm{td}}(T_f)\circ f^*} & \mathrm{CH}(Y)_{\mathbb{Q}}, \end{array}$$

*where, for a vector bundle $E \to Y$ over Y, $\tilde{\mathrm{td}}(E)\colon \mathrm{CH}_*(Y)_{\mathbb{Q}} \to \mathrm{CH}_*(Y)_{\mathbb{Q}}$ sending $a \mapsto \mathrm{td}(E) \cap a$ by the cap-product map $\mathrm{CH}^*(Y)_{\mathbb{Q}} \otimes \mathrm{CH}_*(Y)_{\mathbb{Q}} \xrightarrow{\cap} \mathrm{CH}_*(Y)_{\mathbb{Q}}$ defined in [**3**].*

Corollary 1.8 (Singular Riemann-Roch)**.** *Let $f\colon X \to Y$ be a projective morphism in qSch_k. Then the following diagram is commutative:*

$$\begin{array}{ccc} G_0(X) & \xrightarrow{\ f_*\ } & G_0(Y) \\ {\scriptstyle \tau_0}\downarrow & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{CH}(X)_{\mathbb{Q}} & \xrightarrow[f_*]{} & \mathrm{CH}(Y)_{\mathbb{Q}}, \end{array}$$

where τ_0 is the restriction to degree zero of the natural transformation

$$\tau\colon G_0[\beta, \beta^{-1}] \to \mathrm{CH} \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(\mathrm{td})}.$$

*Moreover, τ_0 coincides with the local Chern class morphism in [**1**].*

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2. Several lemmas

This section provides several preliminary results needed for the proof of the main theorem.

Let us recall the following two localization theorems.

Theorem 2.1 (Quillen [**8**]). *Let X be a noetherian scheme, $i\colon Z \to X$ a closed immersion, and $j\colon U \to X$ the open complement of Z. Then there is a natural long exact sequence*

$$\begin{aligned} \ldots \to G_n(Z) \xrightarrow{i_*} G_n(X) \xrightarrow{j^*} G_n(U) & \\ \xrightarrow{\delta} G_{n-1}(Z) \to \ldots \to G_1(U) & \\ \xrightarrow{\delta} G_0(Z) \xrightarrow{i_*} G_0(X) \xrightarrow{j^*} G_0(U) \to 0. & \end{aligned}$$

Theorem 2.2 (Levine-Morel [**6**]). *Let X be in Sch_k. Let $i\colon Z \to X$ be a closed immersion and $j\colon U \to X$ the open complement. Then the sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0$$

is exact.

As the tensor product is right exact, we have:

Corollary 2.3. *Let X be in Sch_k, $i\colon Z \to X$ a closed immersion and $j\colon U \to X$ the open complement. Then the sequence $\Omega_*^{\times}(Z) \xrightarrow{i_*} \Omega_*^{\times}(X) \xrightarrow{j^*} \Omega_*^{\times}(U) \to 0$ is exact.*

Throughout this section we assume that k admits resolution of singularities, and we abbreviate $G_0(X)[\beta, \beta^{-1}]$ to $G_0(X)_\beta$.

Lemma 2.4. *Take X in Sch_k. Let $i\colon X_{\mathrm{red}} \to X$ be the reduction of X. Then the maps*

$$\begin{aligned} i_*\colon \Omega_*^{\times}(X_{\mathrm{red}}) \to \Omega_*^{\times}(X), \\ i_*\colon G_0(X_{\mathrm{red}})_\beta \to G_0(X)_\beta \end{aligned}$$

are isomorphisms.

Proof. The result for G_0 follows from Theorem 2.1 applied to $i\colon X_{\mathrm{red}} \to X$, since the complement is empty.

For $\Omega_*^{\times}$, this follows from the same result for Ω_*, which then follows directly from the definition. □

Lemma 2.5. *For $X \in \mathrm{Sch}_k$, the map $\theta_G^{\times}(X)\colon \Omega_*^{\times}(X) \to G_0(X)_\beta$ is surjective.*

Proof. If X is in Sm_k, then we may use Theorem 1.2 and the fact that $K_0[\beta, \beta^{-1}] = G_0[\beta, \beta^{-1}]$ on Sm_k.

In general, we may assume that X is reduced. Then X admits a filtration by reduced closed subschemes with $U_l := X_l \setminus X_{l-1}$ in Sm_k. In particular, X_0 is in Sm_k and the result is thus proven for X_0.

We have the commutative diagram

$$\begin{array}{ccccccc}
\Omega^{\times}_*(X_{l-1}) & \xrightarrow{i_*} & \Omega^{\times}_*(X_l) & \xrightarrow{j^*} & \Omega^{\times}_*(U_l) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} & & \\
G_0(X_{l-1})_\beta & \xrightarrow[i_*]{} & G_0(X_l)_\beta & \xrightarrow[j^*]{} & G_0(U_l)_\beta & \longrightarrow & 0.
\end{array}$$

The rows are exact by Theorem 2.1 and Corollary 2.3. The result follows by induction on l and a diagram chase. □

Lemma 2.6. *Let $p\colon V \to X$ be a vector bundle of rank $n+1$ in Sch_k, and $q\colon P = P(V) \to X$ the associated projective bundle. Then $q_*\colon \Omega_*(P) \to \Omega_*(X)$ is surjective.*

Proof of the special case. Let us first prove the case where $V = X \times_k \mathbb{A}^{n+1}$; thus $P = X \times_k \mathbb{P}^n$. There is a closed immersion $i\colon X \to X \times_k \mathbb{P}^n$ such that $q \circ i = \mathrm{id}_X$. The composition of the induced morphisms

$$q_* \circ i_*\colon \Omega_*(X) \to \Omega_*(X \times_k \mathbb{P}^n) \to \Omega_*(X)$$

is the identity on $\Omega_*(X)$. It follows that q_* is surjective. The lemma holds for this case.

Proof of the general case. Now let $V \to X$ be a general vector bundle of rank $n+1$. Let Z be a proper closed subscheme of X such that the restriction of P to $U := X \setminus Z$, the complement of Z in X, is $U \times_k \mathbb{P}^n$. We denote by $P^{'}$ the restriction of P to Z. We have the following commutative diagram of morphisms of localization sequences:

$$\begin{array}{ccccccc}
\Omega_*(P^{'}) & \longrightarrow & \Omega_*(P) & \longrightarrow & \Omega_*(U \times_k \mathbb{P}^n) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle q_*} & & \downarrow & & \\
\Omega_*(Z) & \longrightarrow & \Omega_*(X) & \longrightarrow & \Omega_*(U) & \longrightarrow & 0.
\end{array}$$

The vertical map on the left is surjective by induction on dimension of X, and the vertical map on the right is surjective as shown in the special case; we thus conclude that the map q_* is surjective by the 5-lemma. □

Lemma 2.7. *Let M be in Sm_k and $Z \subset M$ a reduced closed subscheme.*

Consider the following commutative diagram:

$$\begin{array}{ccc}
D & \hookrightarrow & M' \\
\downarrow & & \downarrow{\scriptstyle p} \\
Z & \hookrightarrow & M,
\end{array}$$

where p is a sequence of blowups along smooth centers lying over Z, $D = p^{-1}(Z)$;

then both vertical maps in the following commutative diagram are surjective:

$$\begin{array}{ccc} \Omega_*^{\times}(D) & \twoheadrightarrow & G_0(D)_\beta \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \Omega_*^{\times}(Z) & \twoheadrightarrow & G_o(Z)_\beta. \end{array}$$

Proof. Since p is a sequence of blowups along smooth centers lying over Z, it suffices to show that the lemma holds for the case where M' is the blowup of M along some smooth subscheme F of Z, as displayed in the following diagram:

$$\begin{array}{ccccc} E & \hookrightarrow & D & \hookrightarrow & M_F \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ F & \hookrightarrow & Z & \hookrightarrow & M. \end{array}$$

Let U denote the complement of F in Z, which is the same as the complement of E in D. We then have the following commutative diagram, with the rows being the respective exact localization sequences:

$$\begin{array}{ccccccc} \Omega_*(E) & \longrightarrow & \Omega_*(D) & \longrightarrow & \Omega_*(U) & \longrightarrow & 0 \\ {\scriptstyle p_*}\downarrow & & {\scriptstyle p_*}\vdots & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \Omega_*(F) & \longrightarrow & \Omega_*(Z) & \longrightarrow & \Omega_*(U) & \longrightarrow & 0. \end{array}$$

The map p_* on the left is surjective by Lemma 2.6 as $p\colon E \to F$ is a projective bundle over F. The surjectivity of the dashed map p_* then follows by the 5-lemma.

The surjectivity of $G_0(D)_\beta \to G_0(Z)_\beta$ follows from the commutativity of the diagram. □

Lemma 2.8. *Let D be a reduced finite type k-scheme, D_2 an irreducible component of D, and D_1 the union of the remaining irreducible components of D, so $D = D_1 \cup D_2$. Let $D_{12} = D_1 \cap D_2$ with inclusions $i_j\colon D_{12} \to D_j$, $\phi_j\colon D_j \to D$ for $j = 1, 2$. If we write $i_*^- = (i_{1*}, -i_{2*})$ and $\phi = \phi_{1*} + \phi_{2*}$, we have:*

1. *The sequence*

$$G_0(D_{12})_\beta \xrightarrow{i_*^-} G_0(D_1)_\beta \oplus G_0(D_2)_\beta \xrightarrow{\phi} G_0(D)_\beta \to 0$$

   is exact.

2. *The map $\phi\colon \Omega_*^{\times}(D_1) \oplus \Omega_*^{\times}(D_2) \to \Omega_*^{\times}(D)$ is surjective.*

Proof of (1). Consider the morphism $p\colon D_1 \amalg D_2 \to D_1 \cup D_2$ induced by closed embeddings $D_j \to D_1 \cup D_2$ for $j = 1, 2$. Let $U_j := D_j \setminus D_{12}$ with open immersions $\sigma_j\colon U_j \to D_j$ for $j = 1, 2$, and let $i\colon D_{12} \to D$ be the inclusion. We denote by $\sigma_j^{'}$ the open immersions $U_j \to D$ for $j = 1, 2$. Let $\sigma_* := \sigma_{1*} \oplus \sigma_{2*}$ and $\sigma_*^{'} := (\sigma_{1*}^{'}, \sigma_{2*}^{'})$.

Since

$$D_1 \amalg D_2 \setminus D_{12} \amalg D_{12} = D \setminus D_{12} = U_1 \amalg U_2,$$

we have the following morphism of localization sequences:

$$\begin{CD}
G_1(U_1) \oplus G_1(U_2) @>{\mathrm{id}}>> G_1(U_1) \oplus G_1(U_2) \\
@V{\partial_1 \oplus \partial_2}VV @VV{\partial_1 + \partial_2}V \\
G_0(D_{12}) \oplus G_0(D_{12}) @>{\Sigma}>> G_0(D_{12}) \\
@VV{i_{1*} \oplus i_{2*}}V @VV{i_*}V \\
G_0(D_1) \oplus G_0(D_2) @>{p_*}>> G_0(D) \\
@VV{\sigma^*}V @VV{{\sigma'}^*}V \\
G_0(U_1) \oplus G_0(U_2) @>{\mathrm{id}}>> G_0(U_1) \oplus G_0(U_2) \\
@VVV @VVV \\
0 @. 0,
\end{CD} \tag{2}$$

where Σ is the sum map.

We note that

$$\ker(p_*) \subset \ker(\sigma'^* \circ p_*) = \ker(\sigma^*) = \operatorname{im}(i_{1*} \oplus i_{2*}).$$

Thus, if $y = y_1 \oplus y_2$ is in $\ker(p_*)$, then there are elements $x_i \in G_0(D_{12})$ with $y_1 = i_{1*}(x_1)$, $y_2 = i_{2*}(x_2)$. Since $p_*(i_{1*}(x_1) \oplus i_{2*}(x_2)) = 0$, we have $i_*(x_1 + x_2) = 0$; hence there are elements $\alpha_i \in G_1(U_i)$ with $\partial_1(\alpha_1) + \partial_2(\alpha_2) = x_1 + x_2$. Replacing x_i with $x_i - \partial_i(\alpha_i)$, we may assume that $x_1 = -x_2$ in $G_0(D_{12})$; i.e., there is an $x \in G_0(D_{12})$ with

$$y_1 = i_{1*}(x), \; y_2 = -i_{2*}(x)$$

which proves the exactness of our sequence (1) at $G_0(D_1)_\beta \oplus G_0(D_2)_\beta$. The surjectivity of ϕ in (1) follows from diagram (2) and the 5-lemma, noting that the maps Σ and id are surjective.

Proof of (2). Using the right exact localization sequence of $\Omega_*^\times$, the same argument as for the surjectivity in (1) applies to prove the surjectivity of ϕ. □

Lemma 2.9. *Let D be a strict normal crossing divisor on a scheme $M \in \mathrm{Sm}_k$. Then $\Omega_*^\times(D) \xrightarrow{\sim} G_0(D)_\beta$.*

Proof. We may assume that D is reduced.

Let us write $D = D_1 \cup D_2$, where D_2 is an irreducible component of D. We proceed by induction on the number of irreducible components of D as well as on the dimension of D. As in the previous Lemma 2.8, we write $D_{12} = D_1 \cap D_2$, and use $i_j \colon D_{12} \to D_j$ and $\phi_j \colon D_j \to D$ for $j = 1, 2$ to denote the inclusions.

We have the following commutative diagram:

$$\begin{array}{ccccccc}
\Omega_*^\times(D_{12}) & \xrightarrow{i_*^-} & \Omega_*^\times(D_1) \oplus \Omega_*^\times(D_2) & \xrightarrow{\phi} & \Omega_*^\times(D) & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \\
G_0(D_{12})_\beta & \xrightarrow[i_*^-]{} & G_0(D_1)_\beta \oplus G_0(D_2)_\beta & \xrightarrow[\phi]{} & G_0(D)_\beta & \longrightarrow & 0,
\end{array}$$

where $i_*^- = (i_{1*}, -i_{2*})$ and $\phi = \phi_{1*} + \phi_{2*}$. The first two of the three vertical maps are isomorphisms by induction, while the third one is surjective. Clearly the top row is a complex; in addition, the bottom row is exact by Lemma 2.8(1) and the top map ϕ is surjective by Lemma 2.8(2).

We fill $K := \operatorname{coker}(i_*)$ into the following diagram:

$$\begin{array}{ccccccc}
\Omega_*^\times(D_{12}) & \xrightarrow{i_*^-} & \Omega_*^\times(D_1) \oplus \Omega_*^\times(D_2) & \xrightarrow{\phi} & \Omega_*^\times(D) & \longrightarrow & 0 \\
& & & \searrow \quad & \nearrow & & \\
\downarrow \wr & & \downarrow \wr & K & \downarrow & & \\
& & & \quad \searrow^{\psi} & & & \\
G_0(D_{12})_\beta & \xrightarrow[i_*^-]{} & G_0(D_1)_\beta \oplus G_0(D_2)_\beta & \longrightarrow & G_0(D)_\beta & \longrightarrow & 0
\end{array}$$

with the sequence

$$\Omega_*^\times(D_{12}) \to \Omega_*^\times(D_1) \oplus \Omega_*^\times(D_2) \to K \to 0$$

being exact. Since $\phi \circ i_* = 0$, we have a surjective map $K \to \Omega_*^\times(D)$. By the 5-lemma $\psi\colon K \to G_0(D)_\beta$ is an isomorphism; hence the surjection $\Omega_*^\times(D) \to G_0(D)_\beta$ is an isomorphism. $\square$

Lemma 2.10. *Let M be in Sm_k and let $Z \subset M$ a reduced closed subscheme. Let $F \subset M$ be a smooth closed subscheme contained in Z. We denote by M_F the blowup of M along F with the canonical projective morphism $p\colon M_F \to M$. Then the sequence*

$$0 \to \ker(p_*) \to G_n(M_F) \xrightarrow{p_*} G_n(M) \to 0$$

is split exact.

Proof. It suffices to show that $p_* \circ p^* = \mathrm{id}$ on $G_n(M) = K_n(M)$. We have the projection formula

$$p_*(a \cdot p^*(b)) = p_*(a) \cdot b$$

for all $a \in K_0(M_F)$ and $b \in G_n(M)$. Thus, for any $x \in G_n(M)$, we have

$$p_*(p^*(x)) = p_*([\mathcal{O}_{M_F}] \cdot p^*(x)) = p_*([\mathcal{O}_{M_F}]) \cdot x.$$

However, $R^q p_*(\mathcal{O}_{M_F}) = 0$ for $q > 0$, and $p_*(\mathcal{O}_{M_F}) = \mathcal{O}_M$; so $p_*([\mathcal{O}_{M_F}]) = [\mathcal{O}_M]$, and $p_*(p^*(x)) = [\mathcal{O}_M] \cdot x = x$. $\square$

Lemma 2.11. *Borrowing notation from the preceding Lemma 2.10, we denote by* D *the exceptional divisor* $p^{-1}(F)$.

Let K *be the kernel of*

$$p_*\colon G_0(M_F) \to G_0(M)$$

and K' *be the kernel of*

$$p_*\colon G_0(D) \to G_0(Z).$$

Then the inclusion $i\colon D \to M_F$ *induces an isomorphism* $K' \simeq K$.

Proof. Let us look at the following diagram:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & K' & \overset{}{\underset{i_*}{\dashrightarrow}} & K & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
G_1(M_F) & \longrightarrow & G_1(U) & \longrightarrow & G_0(D) & \longrightarrow & G_0(M_F) & \longrightarrow & G_0(U) & \longrightarrow & 0 \\
{\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
G_1(M) & \longrightarrow & G_1(U) & \longrightarrow & G_0(Z) & \longrightarrow & G_0(M) & \longrightarrow & G_0(U) & \longrightarrow & 0, \\
\downarrow & & & & \downarrow & & \downarrow & & & & \\
0 & & & & 0 & & 0 & & & &
\end{array}
$$

where i_* is the natural map induced by $i\colon D \to M_F$ and the rows are the respective localization sequences.

Surjectivity of i_*: To see this, we pick an element a in K. It goes to 0 in $G_0(M)$ and thus goes to 0 in $G_0(U)$ as well by the commutativity of the diagram. Exactness implies that there is an element of b of $G_0(D)$ whose image in $G_0(M_F)$ is a. Let c be the image of b in $G_0(Z)$. Then c goes to 0 in $G_0(M)$, so it comes from an element d in $G_1(U)$. Let e be the image of d in $G_0(D)$. Then $b - e$ belongs to K' and its image in K is a.

Injectivity of i_*: Let x be such an element that $i_*(x) = 0$. Then it is the image of some element y in $G_1(U)$, which goes to 0 in $G_0(Z)$ by commutativity. Therefore, y is the image of some element z in $G_1(M)$. Since p_* is split surjective by Lemma 2.10, we can lift z to an element $\tilde{z}$ in $G_1(M_F)$, whose image in $G_1(U)$ is y. Therefore, x is the image of $\tilde{z}$ in $G_0(D)$, which is then 0. We conclude that i_* is injective. □

Remark 2.12. Let us consider the following localization commutative diagrams:

$$\begin{array}{ccccccc}
\Omega_*^\times(D) & \longrightarrow & \Omega_*^\times(M_F) & \longrightarrow & \Omega_*^\times(U) & \longrightarrow & 0 \\
\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
G_0(D)_\beta & \longrightarrow & G_0(M_F)_\beta & \longrightarrow & G_0(U)_\beta & \longrightarrow & 0,
\end{array}$$

$$\begin{array}{ccccccc}
\Omega_*^\times(Z) & \longrightarrow & \Omega_*^\times(M) & \longrightarrow & \Omega_*^\times(U) & \longrightarrow & 0 \\
\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
G_0(Z)_\beta & \longrightarrow & G_0(M)_\beta & \longrightarrow & G_0(U)_\beta & \longrightarrow & 0.
\end{array}$$

From Lemma 2.11 it is easy to deduce that

$$\ker(\Omega_*^\times(D) \to \Omega_*^\times(Z)) \to \ker(\Omega_*^\times(M_F) \to \Omega_*^\times(M))$$

is surjective. This is because Lemma 2.11 still holds if we replace Z by F, and D by $E := p^{-1}(F)$; i.e., the map

$$\ker(G_0(E)_\beta \to G_0(F)_\beta) \to \ker(G_0(M_F)_\beta \to G_0(M)_\beta)$$

is an isomorphism. We can replace the $G_0[\beta, \beta^{-1}]$ by $K_0[\beta, \beta^{-1}]$ since everything is smooth; similarly, $K_0[\beta, \beta^{-1}]$ is isomorphic to theory $\Omega_*^\times$ by Corollary 1.3. Thus

$$\ker(\Omega_*^\times(E) \to \Omega_*^\times(F)) \to \ker(\Omega_*^\times(M_F) \to \Omega_*^\times(M))$$

is an isomorphism. Since the map

$$\ker(\Omega_*^\times(E) \to \Omega_*^\times(F)) \to \ker(\Omega_*^\times(M_F) \to \Omega_*^\times(M))$$

factors through

$$\ker(\Omega_*^\times(D) \to \Omega_*^\times(Z)),$$

the surjectivity of

$$\ker(\Omega_*^\times(D) \to \Omega_*^\times(Z)) \to \ker(\Omega_*^\times(M_F) \to \Omega_*^\times(M))$$

follows.

3. Main theorem

Let Z be a k-scheme which admits an embedding into some smooth k-scheme M. By Hironaka [**4**], there is a sequence of blowups of M, $p\colon M^{'} \to M$, along smooth centers lying over Z such that $D := p^{-1}(Z)$ is a strict normal crossing divisor of $M^{'}$.

To be more precise, we have the following diagram of blowups:

$$\begin{array}{ccccccccccc}
M^{'} = & M_r & \xrightarrow{p_r} & \cdots & \longrightarrow & M_1 & \xrightarrow{p_1} & M_0 & \xrightarrow{p_0} & M \\
& \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
D = & D_r & \longrightarrow & \cdots & \longrightarrow & D_1 & \longrightarrow & D_0 & \longrightarrow & Z,
\end{array}$$

where

- $p_{i+1}\colon M_{i+1} \to M_i$ is the blowup of M_i along some smooth $F_i \subset D_i$ for $i = 0, \dots, r-1$,
- $D_{i+1} = p_{i+1}^{-1}(D_i)$ for $i = 0, \dots, r-1$,
- $p = p_0 \circ \dots \circ p_r$.

Lemma 3.1. *In the commutative diagram of short exact sequences,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i^{''} & \longrightarrow & \Omega_*^{\times}(M_i) & \longrightarrow & \Omega_*^{\times}(M) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K_i^{'} & \longrightarrow & \Omega_*^{\times}(D_i) & \longrightarrow & \Omega_*^{\times}(Z) & \longrightarrow & 0, \end{array}$$

the map $K_i^{'} \to K_i^{''}$ is surjective for all $i = 0, \dots, r$.

In particular, $K_r^{'} \to K_r^{''}$ is surjective.

Proof. We proceed by induction.

For $i = 0$, p_0 is only a single blowup, and the claim follows from Remark 2.12. Let us assume the claim for $i \geqslant 0$. We must show that the claim holds for $i+1$.

Note that $K_{i+1}^{'} \to K_i^{'}$ is surjective by Lemma 2.7 applied to p_{i+1}, and $K_{i+1}^{''} \to K_i^{''}$ is surjective since $p_{i+1*} : G_0(M_{i+1}) \to G_0(M_i)$ is (split) surjective and $G_0(M_j)_\beta = \Omega_*^{\times}(M_j)$ as M_j is smooth. Letting

$$N' := \ker(\Omega_*^{\times}(D_{i+1}) \to \Omega_*^{\times}(D_i)) \quad \text{and} \quad M' := \ker(\Omega_*^{\times}(M_{i+1}) \to \Omega_*^{\times}(M_i)),$$

then we have the natural morphism of short exact sequences as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & K_{i+1}^{'} & \xrightarrow{p_{i+1*}} & K_i^{'} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & M' & \longrightarrow & K_{i+1}^{''} & \xrightarrow[p_{i+1*}]{} & K_i^{''} & \longrightarrow & 0. \end{array}$$

We see that f' is surjective because p_{i+1} is a single blowup, and that $f^{''}$ is surjective by induction. The lemma thus follows by the 5-lemma. $\square$

Lemma 3.2. *In the commutative diagram of short exact sequences,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_i & \longrightarrow & G_0(M_i)_\beta & \longrightarrow & G_0(M)_\beta & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K_i & \longrightarrow & G_0(D_i)_\beta & \longrightarrow & G_0(Z)_\beta & \longrightarrow & 0, \end{array}$$

the map $K_i \to L_i$ is an isomorphism for all $i = 0, \dots, r$.

In particular $K_r \to L_r$ is an isomorphism.

Proof. The same argument applies as in the preceding lemma using the isomorphism of Lemma 2.11 instead of the surjection of Remark 2.12. $\square$

Theorem 3.3. $\theta_G^{\times}(Z)\colon \Omega_*^{\times}(Z) \to G_0(Z)_\beta$ *is an isomorphism.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_r' & \longrightarrow & \Omega_*^{\times}(D) & \longrightarrow & \Omega_*^{\times}(Z) & \longrightarrow & 0 \\ & & \downarrow & & \cong\downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_r & \longrightarrow & G_0(D)_\beta & \longrightarrow & G_0(Z)_\beta & \longrightarrow & 0, \end{array}$$

where the middle map is an isomorphism by Lemma 2.9. It follows that $K_r' \to K_r$ is injective.

By Claims 3.1 and 3.2, we have the isomorphism $K_r \simeq L_r$ and the epimorphism $K_r' \twoheadrightarrow K_r''$. Moreover, $K_r'' \simeq L_r$ because M and M' are both smooth.

We conclude that $K_r' \to K_r$ is surjective in view of the following commutative diagram:

$$\begin{array}{ccc} K_r' & \twoheadrightarrow & K_r'' \\ \downarrow & & \simeq\downarrow \\ K_r & \xrightarrow{\simeq} & L_r. \end{array}$$

Therefore, $K_r' \simeq K_r$ which implies that $\Omega_*^{\times}(Z) \simeq G_0(Z)_\beta$. This completes the proof that the natural transformation (1) is an isomorphism. As we have already remarked, this proves Theorem 1.5. □

Remark 3.4. From the proof of the theorem, it is easy to see that the isomorphism $\Omega_*^{\times}(Z) \simeq G_0(Z)_\beta$ does not depend on the choice of embeddings $Z \hookrightarrow M$, nor does it depend on the choice of the resolution blowup sequences. This is because what we have proved is actually only the injectivity of the canonical surjective map $\Omega_*^{\times}(Z) \twoheadrightarrow G_0(Z)_\beta$.

4. Applications: Riemann-Roch

4.1. l.c.i. R.R.

Let A_* be an OBM homology theory. We recall briefly how to twist A_* into a new OBM theory. Please refer to §8.2 of [**5**] and §10.5 of [**7**] for details.

Let $\tau = (\tau_i) \in \prod_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. Following Levine and Morel, one can twist A_* by τ as follows:

The groups and push-forward maps are unchanged:

$$A_*^{(\tau)}(X) := A_*(X), f_*^{(\tau)} = f_*.$$

To define the twisting of the pullback for an l. c. i. morphism $f\colon X \to Y$, let us choose a factorization of f as $f = qi$, with $i\colon Y \to P$ a regular embedding and $q\colon P \to X$ a smooth morphism. We have the *relative tangent bundle* $T_q \to P$, defined as the vector bundle whose dual has sheaf of sections the relative differentials $\Omega^1_{Y/X}$. Letting $\mathcal{I}$ be the ideal sheaf of Y in P, we let $N_i \to Y$ be the bundle whose dual has sheaf of sections $\mathcal{I}/\mathcal{I}^2$. We let $[N_f] \in K^0(Y)$ be the class $[N_i] - [i^*T_q]$. We call $[N_f]$

the virtual normal bundle of $f\colon Y \to X$. It is easy to see that $[N_f]$ is independent of the choice of the factorization of f.

We define

$$f^*_{(\tau)} := \tilde{c}_\tau(N_f) \circ f^*,$$

and for any line bundle L over X, we set

$$\tilde{c}_1^{(\tau)}(L) := \tilde{c}_\tau(L) \circ \tilde{c}_1(L),$$

where for a vector bundle $E \to X$, the construction $\tilde{c}_\tau(E)$ is given by Lemma 8.1 of [**5**].

Remark 4.1. This does define a new oriented Borel-Moore homology theory on Sch_k, denoted by $A_*^{(\tau)}$. The definition of $\tilde{c}_1^{(\tau)}(L)$ can be rewritten as $\tilde{c}_1^{(\tau)}(L) = \lambda_{(\tau)}(\tilde{c}_1(L))$, where $\lambda_{(\tau)}(u) = \sum_{i\geqslant 0} \tau_i \cdot u^{i+1} \in A_*(k)[[u]]$. It is then clear that to give a twisting is equivalent to giving a formal series $\lambda_{(\tau)}(u)$ with leading term u.

Example 4.2. The Chow theory CH_* has the structure of OBM homology theory on Sch_k. We give $\mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]$ the structure of OBM homology theory on Sch_k by taking the $\mathbb{Q}[\beta, \beta^{-1}]$-linear extension; i.e.,

$$f^*_{\mathrm{CH}_* \otimes \mathbb{Q}[\beta,\beta^{-1}]} := f^*_{\mathrm{CH}} \otimes \mathrm{id}$$

and similarly for all other structures.

We can produce a new theory on Sch_k, denoted by $\mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(\mathrm{td})}$, by applying our twisting for the family τ given by

$$\tau = \lambda_{(\tau)}(u) = (1 - e^{-\beta u})/\beta.$$

In effect, the presence of the exponential term $e^{-\beta u}$ converts the additive OBM homology theory $\mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]$ into a multiplicative one $\mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(\mathrm{td})}$ on Sch_k with the multiplicative formal group law

$$F^{(\mathrm{td})}_{\mathrm{CH}} = u + v - \beta uv.$$

Corollary 4.3. *Suppose that k admits resolution of singularities. Then there is a unique natural transformation of* OBM *homology theories on* Sch_k

$$\tau\colon G_0[\beta, \beta^{-1}] \to \mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(\mathrm{td})}.$$

Proof. By Theorem 1.5 $G_0[\beta, \beta^{-1}]$ is the universal periodic multiplicative OBM homology theory on Sch_k. Thus, for any oriented OBM homology theory A_* on Sch_k with periodic multiplicative formal group law, there exists a unique natural transformation $\tau\colon G_0[\beta, \beta^{-1}] \to A_*$.

By construction, $\mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(\mathrm{td})}$ is an OBM theory on Sch_k with multiplicative periodic formal group law.

Thus we have a unique natural transformation of OBM homology theories

$$\tau\colon G_0[\beta, \beta^{-1}] \to \mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(\mathrm{td})}. \qquad \square$$

Remark 4.4. For a vector bundle E on X, we have the degree 0 endomorphism $\tilde{c}_{(\mathrm{td})^{-1}}(E)$ on $\mathrm{CH}_*(X)[\beta, \beta^{-1}]$. We can identify $\mathrm{CH}_*(X)$ with the degree 0 part of $\mathrm{CH}_*(X)[\beta, \beta^{-1}]$ by sending $x \in \mathrm{CH}_p(X)$ to $x\beta^{-p}$. We denote the restriction of $\tilde{c}_{(\mathrm{td})^{-1}}(E)$ to $\mathrm{CH}_*(X)$ by $\tilde{\mathrm{td}}E$. It follows that $\tilde{\mathrm{td}}(E)$ agrees with the classical Todd class automorphism of $\mathrm{CH}_*(X)$ as defined in [**2**].

Corollary 4.5 (l. c. i. Riemann-Roch)**.** *Let* $f\colon Y \to X \in \mathrm{qSch}_k$ *be an* l. c. i. *morphsim of relative degree d. Then we have the following commutative diagram:*

$$\begin{array}{ccc} G_0(X) & \xrightarrow{\ f^*\ } & G_0(Y) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{CH}(X)_{\mathbb{Q}} & \xrightarrow{\ \tilde{\mathrm{td}}(T_f)\circ f^*\ } & \mathrm{CH}(Y)_{\mathbb{Q}}, \end{array}$$

where, for a vector bundle $E \to Y$ *over* Y, $\tilde{\mathrm{td}}(E)\colon\ \mathrm{CH}_*(Y)_{\mathbb{Q}} \to \mathrm{CH}_*(Y)_{\mathbb{Q}}$ *sending* $a \mapsto \mathrm{td}(E) \cap a$ *by cap-product map* $\mathrm{CH}^*(Y)_{\mathbb{Q}} \otimes \mathrm{CH}_*(Y)_{\mathbb{Q}} \xrightarrow{\cap} \mathrm{CH}_*(Y)_{\mathbb{Q}}$ *defined in [**3**].*

Proof. By Corollary 4.3, there is a natural transformation of OBM homology theories

$$\tau\colon G_0[\beta, \beta^{-1}] \to \mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(\mathrm{td})}.$$

By restricting τ to degree zero, denoted by τ_0, the naturality of τ gives us the following commutative diagram for an l. c. i. morphism $f\colon Y \to X \in \mathrm{qSch}_k$:

$$\begin{array}{ccc} G_0(X) & \xrightarrow{\ f^*\ } & G_0(Y) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{CH}(X)_{\mathbb{Q}} & \xrightarrow[\ f^*_{(\mathrm{td})}\ ]{} & \mathrm{CH}(Y)_{\mathbb{Q}}. \end{array}$$

To finish the proof, it remains to verify that

$$f^*_{(\mathrm{td})} = \tilde{\mathrm{td}}(T_f) \circ f^*.$$

By definition,

$$f^*_{(\mathrm{td})} := \tilde{c}_{\mathrm{td}}(N_f) \circ f^*.$$

Since $N_f = -T_f$ in $K_0(Y)$, and since $\tilde{\mathrm{td}}(T_f)$ is the restriction of $\tilde{c}_{(\mathrm{td})^{-1}}(T_f)$ to the degree zero portion, it suffices to show that

$$\tilde{c}_{(\mathrm{td})}(-T_f) = \tilde{c}_{(\mathrm{td})^{-1}}(T_f).$$

But by definition of $(\tau)^{-1}$ and the multiplicative properties of $\tilde{c}_\tau$, we have

$$\tilde{c}_{(\tau)^{-1}}(E) = \tilde{c}_\tau(E)^{-1}$$

for all τ and E. Since $\tilde{c}_\tau(E)$ is multiplicative in E, we thus have

$$\tilde{c}_{(\tau)^{-1}}(E) = \tilde{c}_\tau(E)^{-1} = \tilde{c}_\tau(-E).$$

□

4.2. Singular R.R.

Corollary 4.6 (Singular R.R.)**.** *Let $f\colon X \to Y$ be a projective morphism in qSch_k. Then the following diagram is commutative:*

$$\begin{array}{ccc} G_0(X) & \xrightarrow{f_*} & G_0(Y) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{CH}(X)_{\mathbb{Q}} & \xrightarrow[f_*]{} & \mathrm{CH}(Y)_{\mathbb{Q}}, \end{array}$$

where τ_0 is the restriction to degree zero of the natural transformation

$$\tau\colon G_0[\beta,\beta^{-1}] \to \mathrm{CH}\otimes\mathbb{Q}[\beta,\beta^{-1}]^{(\mathrm{td})}.$$

*Moreover, τ_0 coincides with the local Chern class morphism in [**1**].*

Proof. The commutativity of the diagram is clear by restricting the natural transformation τ to degree zero, noting that τ is a transformation of OBM homology theories, and that the twisting construction does not alter the pushforward maps.

We claim that if P is a projective space $\mathbb{P}^n$, the term of degree n in $\tau_0([\mathcal{O}_p])$ is the fundamental class in $\mathrm{CH}_n(P)$, $[P]$. For this, we have canonical natural transformations

$$\Omega_* \xrightarrow{\theta_\times} \Omega_*^\times \xrightarrow{\theta_G^\times} G_0[\beta,\beta^{-1}] \xrightarrow{\tau} \mathrm{CH}_*[\beta,\beta^{-1}]^{(\mathrm{td})}.$$

Thus the composition

$$\tau\circ\theta_G^\times\circ\theta_\times\colon \Omega_* \to \mathrm{CH}_*[\beta,\beta^{-1}]^{(\mathrm{td})}$$

is the canonical natural transformation $\theta_{\mathrm{CH}^{(\mathrm{td})}}$ given by the universality of Ω_*. Similarly, the composition

$$\theta_G^\times\circ\theta_\times\colon \Omega_* \to G_0[\beta,\beta^{-1}]$$

is the canonical natural transformation $\theta_G\colon \Omega_* \to G_0[\beta,\beta^{-1}]$.

If A_* is a OBM homology theory on Sch_k , then for a cobordism cycle $[f\colon Y \to X]$, the canonical natural transformation $\theta_A\colon \Omega_* \to A_*$ has

$$\theta_A([f\colon Y\to X]) = f_*^A(1_Y^A).$$

Here $1_Y^A = p_Y^*(1)$, where $p\colon Y \to \mathrm{Spec}(k)$ is the structure morphism and $1 \in A_0(k)$ is the unit (note that by definition of a cobordism cycle, Y is irreducible and in Sm_k, and f is projective). We use the notation f_*^A to indicate the pushforward for the theory A.

For $A = G_0[\beta,\beta^{-1}]$, this gives $1_Y = [\mathcal{O}_Y]\beta^{\dim_k Y}$ and

$$\theta_G([\mathrm{id}\colon P\to P]) = \mathrm{id}_*(1_P) = [\mathcal{O}_P]\beta^n.$$

For $A = \mathrm{CH}_*[\beta,\beta^{-1}]^{(\mathrm{td})}$ we have

$$1_Y = (p_Y)^*_{(\mathrm{td})}(1_k) = \tilde{c}_{\mathrm{td}}(N_{p_Y})(p_Y^*(1_k)) = c_{\mathrm{td}}(N_{p_Y}) = c_{\mathrm{td}}(-T_Y);$$

hence

$$\theta_{\mathrm{CH}^{(\mathrm{td})}}([\mathrm{id}\colon P\to P]) = c_{\mathrm{td}}(-T_P)$$

and thus

$$\tau([\mathcal{O}_P]) = c_{\mathrm{td}}(-T_P)\beta^{-n} = c_{\mathrm{td}}(-T_P) \quad \text{(see Remark 4.4)}.$$

In degree 0, this is just the classical total Todd class of T_P, which written in $\mathrm{CH}^*(P)$ is:

$$\tau_0([\mathcal{O}_P]) = \mathrm{td}(T_P) = \mathrm{td}(\mathcal{O}_P(1))^{n+1} = \left[\frac{H}{1-e^{-H}}\right]^{n+1} = (1 + \frac{1}{2}H + \cdots)^{n+1},$$

where $H \in \mathrm{CH}^1(P)$ is the class of a hyperplane, and $1 \in \mathrm{CH}^0(P)$ is the usual fundamental class.

We conclude that τ_0 coincides with the localized Chern class map of [**1**] by the following uniqueness theorem of Baum-Fulton-MacPherson.

Theorem 4.7 (Baum-Fulton-MacPherson)**.** *There is only one additive natural transformation $\phi\colon G_0 \to \mathrm{CH}_. \otimes \mathbb{Q}$ with the property that if P is a projective space, the top dimensional cycle in $\phi(\mathcal{O}_P)$ is $[P]$.*

Remark 4.8. The transformation ϕ in the above theorem being natural means it commutes with push-forwards. That is, for a projective morphism $f\colon X \to Y$, the following diagram:

$$\begin{array}{ccc} G_0(X) & \xrightarrow{f_*} & G_0(Y) \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_0} \\ \mathrm{CH}(X)_. \otimes \mathbb{Q} & \xrightarrow[f_*]{} & \mathrm{CH}(Y)_. \otimes \mathbb{Q} \end{array}$$

commutes.

The proof is complete. □

Corollary 4.9 (Module)**.** *Let X be in Sch_k. Then for any $a \in K_0(X)$ and $b \in G_0(X)$, we have*

$$\tau_0(a \cdot b) = \tilde{\mathrm{ch}}(a)(\tau_0(b)).$$

Proof. By linearity, it suffices to prove it for the case where $a = [E]$ and $b = [\mathcal{F}]$ for $E \to X$ a vector bundle and $\mathcal{F}$ a coherent sheaf on X. By the splitting principle, it is further reduced to the case where E is a line bundle L, with projection $p\colon L \to X$. Let $\mathcal{L}$ denote the associated sheaf of sections of L.

We have the first Chern class operator map

$$\tilde{c}_1(L)\colon G_0(X)_\beta \to G_0(X)_\beta$$

defined as

$$\tilde{c}_1(L)(x) := s^* s_*(x)\beta^{-1},$$

where $s\colon X \to L$ is the zero section.

We resolve $\mathcal{O}_{s(X)}$, regarded as an $\mathcal{O}_L$-module, as follows

$$0 \to p^*(\mathcal{L}^\vee) \to \mathcal{O}_L \to \mathcal{O}_{s(X)} \to 0.$$

Using the fact that the pullback map p^* is flat we get the exact sequence

$$0 \to p^*(\mathcal{L}^\vee \otimes \mathcal{F}) \to p^*(\mathcal{F}) \to s_*(\mathcal{F}) \to 0.$$

Since s is a closed immersion, the higher direct images of s_* vanish. Thus in $G_0(L)$, we have

$$s_*([\mathcal{F}]) = [s_*\mathcal{F}] = [p^*(\mathcal{F})] - [p^*(\mathcal{L}^\vee \otimes \mathcal{F})].$$

Since p is flat and $s^*p^* = \mathrm{id}$,

$$s^*([p^*(\mathcal{F})]) = [\mathcal{F}];\ s^*([p^*(\mathcal{L}^\vee \otimes \mathcal{F})]) = [\mathcal{L}^\vee \otimes \mathcal{F}],$$

and we then have

$$\tilde{c}_1(L)([\mathcal{F}]) = s^*s_*([\mathcal{F}])\beta^{-1} = ([\mathcal{F}] - [\mathcal{L}^\vee \otimes \mathcal{F}])\beta^{-1}.$$

The naturality of the canonical transformation

$$\tau \colon G_0(X)_\beta \to \mathrm{CH}_*(X)[\beta, \beta^{-1}]^{(\mathrm{td})}_{\mathbb{Q}},$$

gives us

$$\tau(\tilde{c}_1(L)([\mathcal{F}]) = \tilde{c}_1^{(\mathrm{td})}(L)(\tau([\mathcal{F}])).$$

Thus,

$$\tau([\mathcal{F}]\beta^{-1} - [\mathcal{L}^\vee][\mathcal{F}]\beta^{-1}) = (\beta^{-1} - \beta^{-1}e^{-\beta\tilde{c}_1(L)})\tau([\mathcal{F}]).$$

We easily deduce that, at degree 0,

$$\tau_0([\mathcal{L}^\vee][\mathcal{F}]) = e^{-\beta\tilde{c}_1(L)}\tau_0([\mathcal{F}]) = \mathrm{ch}(L^\vee) \cap \tau_0([\mathcal{F}]).$$

One should notice that the presence of β in $\mathrm{ch}(L^\vee)$ is due to the introduction of β in the twisting of CH_*-theory. Under the identification of sending $x \in \mathrm{CH}_p(X)$ to $x \cdot \beta^{-p}$, $\mathrm{ch}(L^\vee)$ becomes the classical Chern character of $L^\vee$, $e^{\tilde{c}_1(L^\vee)}$, which is equal to $e^{-\tilde{c}_1(L)}$.

The proof is then completed by replacing $L^\vee$ (resp. $\mathcal{L}^\vee$) by L (resp. $\mathcal{L}$). $\square$

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