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INVERSE LIMITS OF FINITE TOPOLOGICAL SPACES

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Abstract

Extending a result of McCord, we prove that every finite simplicial complex is homotopy equivalent to the inverse limit of a sequence of finite spaces. In addition to generalizing McCord's theorem, this provides it with a more geometric motivation, demonstrating a sense in which the simplicial complex is successively better approximated by its finite models.

1. Introduction

McCord proved in [**4**] that every finite simplicial complex is weakly homotopy equivalent to a finite topological space, that is, a space with a finite number of points. There has recently been renewed interest in finite topological spaces; for example, in [**2**], Barmak and Minian present an approach to simple homotopy theory which is based on McCord's correspondence. In particular, they introduce the notion of a collapse of finite spaces and prove that it corresponds under this association to simplicial collapse, while in [**1**], they introduce a broader class of spaces than simplicial complexes, namely the so-called "h-regular CW complexes", to which McCord's analysis and their extension apply.

Expanding further upon McCord's work, Hardie and Vermeulen introduce in [**3**] a notion of barycentric subdivision of finite spaces, which, when applied to the finite model of a simplicial complex K, yields a sequence of finite spaces all weakly homotopy equivalent to K. By studying a homotopy category of finite T_0 spaces and proving its equivalence to the homotopy category of compact polyhedra, they prove a bijection between the homotopy set $[|K|, |L|]$ for finite simplicial complexes K, L and the direct limit of a sequence of homotopy sets between finite spaces.

The main result of this paper is the following:

Theorem 1.1. *Any finite simplicial complex is homotopy equivalent to the inverse limit of a sequence of finite spaces.*

Specifically, we show that if one considers the barycentrically subdivided finite models of a simplicial complex, as defined by Hardie and Vermeulen, but imposes them with the opposite topology, then one in fact obtains a homotopy equivalence

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(not merely a weak equivalence) between the simplicial complex and the inverse limit of its finite models.

2. Construction of the Finite Models

Let K be a finite simplicial complex. To construct its finite models, we begin by letting X_0 be the finite space whose points are in one-to-one correspondence with the faces of simplices of K, just as in McCord's definition of $\mathcal{X}(K)$. Also analogously to McCord, we make X_0 into a poset by declaring that if $x, y \in X_0$ correspond to the faces σ_x and σ_y of K, then $x \leqslant y$ if and only if $\sigma_x \subseteq \sigma_y$. However, we endow this space with the opposite topology to the topology on $\mathcal{X}(K)$; namely, the topology on X_0 is generated by the sets

$$B_x = \{y \in X_0 \mid x \leqslant y\}$$

for $x \in X_0$. The reason for this distinction involves the continuity of the maps p_n defined below.

For each $n \geqslant 0$, let K_n denote the n^{th} barycentric subdivision of K, and let X_n be the finite space whose points are in one-to-one correspondence with the faces of simplices of K_n. Using an analogous partial order on the points of X_n, we can endow each X_n with the topology generated by the sets B_x as above. (It should be noted that in the terminology of [**3**], the space X_n is precisely $(\mathcal{X}(K)^{(n)})^{op}$, the n^{th} barycentric subdivision of the finite space $\mathcal{X}(K)$ with the opposite topology.)

There is a natural map $p_n\colon |K| \to X_n$ for each n, since every point in K is contained in the interior of exactly one face of the n^{th} barycentric subdivision of K. Moreover, there is a unique projection map $q_n\colon X_n \to X_{n-1}$ making the following diagram commute:

$$\begin{array}{ccc} & |K| & \\ {\scriptstyle p_n}\swarrow & & \searrow{\scriptstyle p_{n-1}} \\ X_n & \xrightarrow{\quad q_n \quad} & X_{n-1}. \end{array}$$

In light of the correspondence between points in X_n and faces of simplices in K_n, we will typically denote the simplex corresponding to $x \in X_n$ by σ_x^n. It is straightforward to check that for each $n \geqslant 0$ and each $x \in X_n$, one has

$$p_n^{-1}(B_x) = \mathrm{st}_n(\sigma_x^n),$$

where $\mathrm{st}_n(\sigma_x^n)$ is the open star of σ_x^n in K_n. This implies in particular that the maps p_n are all continuous. They are also open maps, as is easily checked, so by the commutativity of the above diagram, this implies that each q_n is continuous.

3. Proof of Theorem 1.1

We now have an inverse system:

$$X_0 \xleftarrow{q_1} X_1 \xleftarrow{q_2} X_2 \xleftarrow{q_3} X_3 \xleftarrow{q_4} \cdots,$$

and we can define $\tilde{X}$ to be its inverse limit. The main work of the proof of Theorem 1.1 will be in showing that $|K|$ is homeomorphic to a quotient space of $\tilde{X}$.

Before doing so, however, it should be noted that the maps $p_n \colon |K| \to X_n$ are all quasifibrations with contractible fibers, and hence are still weak homotopy equivalences. To prove this, recall that for any basis element $B_x \subset X_n$, the set $p_n^{-1}(B_x) = \mathrm{st}_n(\sigma_x^n)$ is contractible. And, using the fact that B_x is the smallest open set containing x, it is readily verified that each B_x is also contractible. Hence the restriction $p_n|_{p_n^{-1}(B_x)} \colon p_n^{-1}(B_x) \to B_x$ is a weak homotopy equivalence for each basis element B_x, and by Theorem 6 of [**4**] this is sufficient to conclude that p_n is a weak homotopy equivalence.

Lemma 3.1. *If K is a finite simplicial complex and the finite spaces X_n are defined as above, then $|K|$ is homeomorphic to a quotient space of $\varprojlim X_n$.*

Proof. Given $x = (x_0, x_1, x_2, \ldots) \in \tilde{X}$, we can associate to x a sequence of points in $|K|$ by choosing an arbitrary element $a_n \in p_n^{-1}(x_n)$ for each $n \geqslant 0$. Because these points lie in nested simplices of increasingly fine barycentric subdivisions of K, any sequence obtained in this way converges to the same point.

We have thus established that there is a well-defined map

$$G \colon \tilde{X} \to |K|$$

given by sending $(x_0, x_1, x_2, \ldots)$ to the limit of any sequence $\{a_n\} \subset |K|$ where $p_n(a_n) = x_n$ for all n. To prove that G is continuous, let $U \subset |K|$ be any open set, and let $\overline{x} = (x_0, x_1, x_2, \ldots) \in G^{-1}(U)$. First, observe that

$$G^{-1}(U) \subset \prod_{n=0}^{\infty} p_n(U),$$

so we may as well assume that the sequence $\{a_n\}$ has $a_n \in U$ for all n. Since U is open and $\{a_n\}$ converges to a point in U, there is an open set V such that each $a_n \in V$ and such that $\overline{V} \subset U$. Now, the set $\prod p_n(V)$ is open since the p_n are open maps. Moreover, if $\overline{y} = (y_0, y_1, y_2, \ldots) \in \prod p_n(V)$, then $y_n \in p_n(V)$ for all n, so we can choose a sequence $\{b_n\} \subset V$ such that $p_n(b_n) = y_n$ for all n. Denote the limit of $\{b_n\}$ by b, so that $b = G(\overline{y})$. Then, since $\{b_n\} \subset V$, we have $b \in \overline{V} \subset U$, so $\overline{y} \in G^{-1}(U)$. Thus,

$$\overline{x} \in \prod_{n=0}^{\infty} p_n(V) \subset G^{-1}(U),$$

and hence $G^{-1}(U)$ is open.

Define an equivalence relation on $\tilde{X}$ by $x \sim y$ if and only if $G(x) = G(y)$, and denote by Y the corresponding quotient space of $\tilde{X}$. (In fact, one can check that this equivalence relation is simply the T_1 relation, wherein $x \sim y$ if and only if either every open set containing x also contains y or vice versa, since any open set in $\tilde{X}$ containing $(p_0(z), p_1(z), p_2(z), \ldots)$ necessarily contains every x such that $G(x) = z$. Thus, we might say that Y is the "T_1-ification" of $\tilde{X}$.)

We get an induced map $\tilde{G} \colon Y \to |K|$, which is by construction both well-defined and injective. Since $\tilde{G}([(p_0(x), p_1(x), p_2(x), \ldots)]) = x$ for any $x \in |K|$, it is also clearly surjective, and it is continuous because $G = \tilde{G} \circ \pi$, where $\pi\colon \tilde{X} \to Y$ is the quotient map. The inverse of $\tilde{G}$ is the continuous map $\tilde{G}^{-1}\colon |K| \to Y$ defined by

$$x \mapsto [(p_0(x), p_1(x), p_2(x), \ldots)],$$

so $\tilde{G}$ is a homeomorphism. □

All that remains, now, is to show that in fact Y is homotopy equivalent to $\tilde{X}$. This will be achieved by way of the following lemma:

Lemma 3.2. *The quotient space Y is homeomorphic to a deformation retract of $\tilde{X}$.*

Proof. First, observe that if $\tilde{x} \in \tilde{X}$ and $G(\tilde{x}) = y$, then every neighborhood of the point $(p_0(y), p_1(y), p_2(y), \ldots) \in \tilde{X}$ contains $\tilde{x}$.

Let E be any equivalence class under $\sim$, wherein every element defines a sequence converging to $x \in |K|$. Define a homotopy $h_E\colon E \times [0,1] \to E$ by

$$h_E(y,t) = \begin{cases} y & \text{if } t \in [0,1) \\ (p_0(x), p_1(x), \ldots) & \text{if } t = 1. \end{cases}$$

This map is easily seen to be continuous, and hence proves that every equivalence class is contractible.

Combining all of these homotopies on the various equivalence classes, we obtain a map $F\colon \tilde{X} \times [0,1] \to \tilde{X}$, which we claim is also continuous. To verify this, let $U \subset \tilde{X}$ be open, and define a subset $U^{BC} \subset U$ as follows:

$$U^{BC} = \{x \in U \mid (p_0(G(x)), p_1(G(x)), \ldots) \notin U\}.$$

These are the "boundary-convergent" points in U, those that we view as sequences of points in the open set U converging to a point that is not in U. The set U^{BC} is closed in U, and

$$F^{-1}(U) = (U \times [0,1]) \setminus (U^{BC} \times \{1\}),$$

so $F^{-1}(U)$ is open. Therefore, F is continuous, as claimed.

We have thus defined a deformation retraction of $\tilde{X}$ onto a subspace Z that contains exactly one element from each equivalence class. It is clear that if $i\colon Z \hookrightarrow \tilde{X}$ is the inclusion map and $\pi\colon \tilde{X} \to Y$ is the quotient map as above, then the map

$$f = \pi \circ i\colon Z \to Y,$$

is a bijection. Indeed, this map is a homeomorphism; for if $U \subset Z$ is open, then $U = V \cap Z$ for some open subset $V \subset \tilde{X}$, and $\pi(V) = f(U)$. And by the definition of Z, the set V is forced to contain every point in each equivalence class it intersects, so $\pi^{-1}(\pi(V)) = V$. In particular, $\pi(V)$ is open, so f is an open map. Conversely, if $U \subset Y$ is open, then $\pi^{-1}(V) \subset \tilde{X}$ is open, so $\pi^{-1}(V) \cap Z = f^{-1}(Z)$ is open in Z. Hence f is continuous.

Therefore, by the composition of the deformation retraction $\tilde{X} \to Z$ and the homeomorphism $f\colon Z \to Y$, we obtain the claim. □

The proof of Theorem 1.1 is now immediate:

Proof of Theorem 1.1. Composing the homeomorphism from Lemma 3.1 and the homotopy equivalence from Lemma 3.2, we obtain the desired result. □

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