Homology, Homotopy and Applications, vol. 11(1), 2009, pp.171–184

A HOMOTOPICAL ALGEBRA OF GRAPHS RELATED TO ZETA SERIES

TERRENCE BISSON AND ARISTIDE TSEMO

(*communicated by J. Daniel Christensen*)

Abstract

The purpose of this paper is to develop a homotopical algebra for graphs, relevant to the zeta series and the spectra of finite graphs. More precisely, we define a Quillen model structure in a category of graphs (directed and possibly infinite, with loops and multiple arcs allowed). The weak equivalences for this model structure are the Acyclics (graph morphisms which preserve cycles). The cofibrations and fibrations for the model are determined from the class of Whiskerings (graph morphisms produced by grafting trees). Our model structure seems to fit well with the importance of acyclic directed graphs in many applications.

1. Introduction

In this paper we develop a notion of homotopy within graphs, and demonstrate its relevance to the study of the zeta series and the spectrum of a finite graph. We will work throughout with a particular category of graphs, described in Section 2 below. Our graphs will be directed and possibly infinite, with loops and multiple arcs allowed.

Let us explain what we mean by homotopy here.

We are not concerned with the geometric realization of graphs as one-dimensional topological spaces. Since one-dimensional CW complexes are homotopic to disjoint unions of joins of circles, the usual invariants from algebraic topology cannot see much of the structure of a graph in this way. In any case, *directed* graphs are definitely not just part of topology (they are perhaps more related to new areas of *directed topology*, as in Fajstrup and Rosický [**5**]).

Homotopy originally referred to topological deformation of structure. But Quillen's remarkable notes on homotopical algebra [**14**] gave abstract axioms for working with concepts of homotopy in rather general categories. When these axioms are satisfied in a category, we say that we have given a "model structure" there. Quillen's axioms have led to new insights and developments in settings such as chain complexes and homological algebra, simplicial sets, topos theory, and small categories (including

Received September 21, 2008, revised January 29, 2009; published on May 19, 2009.
2000 Mathematics Subject Classification: 05C20, 18G55, 55U35.
Key words and phrases: category of directed graphs, topos, Quillen model structure, weak factorization system, cycles, zeta function.
This article is available at `http://intlpress.com/HHA/v11/n1/a8`

monoids, groups, groupoids, and posets). References include Cisinski [**1**], Dwyer and Spalinski [**3**], Hovey [**6**], Joyal and Tierney [**7**], Thomason [**17**], and many others. Also, recent proofs of the Bloch-Kato and Milnor conjectures are based upon development of a homotopical algebra for schemes; see Morel and Voevodsky [**13**].

A central part of giving a model structure in a category is the specification of which morphisms in the category are to be called "weak equivalences". In most applications, the weak equivalences are defined to be those morphisms which preserve some interesting invariant, such as homotopy type for topology, homology for chain complexes, geometric realization for simplicial sets, and nerve or topos of presheafs for small categories.

In our model structure we use the cycle structure of directed graphs to determine our weak equivalences. More precisely, we take as our weak equivalences the "acyclic" graph morphisms, which neither create nor destroy cycles. We hope that our model structure fits well with the role that acyclic directed graphs play in applications such as computer algorithms, analysis of the internet, random walks and Markov chains, and representations of quivers.

In Section 2 we set up our category Gph of graphs. In Section 3 we give background on weak factorization systems in general, and establish an example with classes of graph morphisms which we call Whiskerings and Surjectings. In Section 4 we give axioms for model structures in general, and define our model structure on Gph. In Section 5 we associate to each finite directed graph X a zeta series $Z_X(u)$. We show that if $f\colon X \to Y$ is an acyclic graph morphism, then $Z_X(u) = Z_Y(u)$ and the eigenvalues of the adjacency matrices of X and Y agree "up to zero eigenvalues". The paper ends with an appendix on the history of zeta series.

Acknowledgements

The first author would like to thank André Joyal and Jiří Rosický for encouragement, York University for hospitality during fall 2005, and his students in the "Geometry and Physics on Graphs" Research Experience for Undergraduates at Canisius College. We thank the Fields Institute for work space, and the referee for comments and suggestions.

2. A category of graphs

Let us establish precisely the objects and morphisms for our category Gph of graphs. For our purposes, a *graph* is a data structure $X = (X_0, X_1, s, t)$ with a set X_0 of *nodes*, a set X_1 of *arcs*, and a pair of functions $s, t\colon X_1 \to X_0$ which specify the *source* and *target* nodes of each arc. We may say that $a \in X_1$ is an arc *from node* $s(a)$ *to node* $t(a)$. A *graph morphism* $f\colon X \to Y$ is a pair of functions $f_1\colon X_1 \to Y_1$ and $f_0\colon X_0 \to Y_0$ such that $s \circ f_1 = f_0 \circ s$ and $t \circ f_1 = f_0 \circ t$.

Let **D** denote the graph with one node and no arcs. Let **A** denote the graph with one arc and two nodes (its source and target). Then the set of nodes of a graph X can be identified with the set of graph morphisms from **D** to X, and the set of arcs of X can be identified with the set of graph morphisms from **A** to X.

There is a stimulating discussion in Lawvere [**10**] of Gph as the category of presheafs on the small category with objects 0 and 1 and two non-identity morphisms from 0 to 1. It follows that Gph is a topos, and thus a category with many nice geometric and algebraic and logical properties; see Mac Lane and Moerdijk [**12**], for instance. We just call attention to a few aspects here.

The category Gph has all products, and all coproducts (sums); it also has pull-backs (fiber products) and pushouts. Also, products distribute over coproducts, etc. As in any presheaf category, these categorical constructions are performed "elementwise", where a graph has two types of elements, the nodes and the arcs. For instance, the empty product (terminal object) 1 is the graph with one node and one arc (which is a loop), and the empty coproduct (initial object) 0 is the graph with no nodes and no arcs.

The category Gph has geometric aspects; for instance, it is a cartesian closed category like a good category of "spaces". The category Gph also has logical aspects; for instance, there is a graph Ω which acts as generalized truth-values for graphs, in that graph morphisms $X \to \Omega$ classify sub-graphs of X (see Session 32 in Lawvere and Schanuel [**11**]).

3. Two classes of graph morphisms, and a weak factorization system

The *path graph* $\mathbf{P}_n$ has nodes $\{i : 0 \leqslant i \leqslant n\}$ and arcs $\{(i-1, i) : 1 \leqslant i \leqslant n\}$, with $s((i-1,i)) = i-1$ and $t((i-1,i)) = i$. Note that $\mathbf{P}_0 = \mathbf{D}$ and $\mathbf{P}_1 = \mathbf{A}$. A *path* in a graph X is just a graph morphism $\alpha\colon \mathbf{P}_n \to X$ for some non-negative integer n; we define $s(\alpha) = \alpha(0)$ and $t(\alpha) = \alpha(n)$. If a is an arc in X such that $t(\alpha) = s(a)$, then we define $\alpha a\colon \mathbf{P}_{n+1} \to X$, the *concatenation of* α *and* a.

Let us introduce some useful shorthand for arcs in a graph X. For any node x in a graph X, let $X(x,*)$ denote the set of those arcs in X which have source x, and let $X(*,x)$ denote the set of arcs with target x. Note that a graph morphism $f\colon X \to Y$ induces a function $f\colon X(x,*) \to Y(f(x),*)$, etc.

Here is our first class of graph morphisms.

Definition 3.1. A graph morphism $f\colon X \to Y$ is *Surjecting* if

$$f\colon X(x,*) \to Y(f(x),*)$$

is a surjective function for all $x \in X_0$.

A *discrete graph* is one with no arcs. We say that a node x is a *root* of the graph X if $X(*,x)$ is empty. Let $R(X)$ denote the set of roots in X, viewed as a discrete subgraph of X. A *rooted tree* is a graph T with one root r such that, for each node x in T, there is a unique (directed) path in T from r to x. For example, the path graph $\mathbf{P}_n$ is a rooted tree; and so is the infinite path $\mathbf{P}_\infty$ whose nodes are the set of non-negative integers and whose arcs are the set of ordered-pairs $(i-1,i)$ of non-negative integers, with $s((i-1,i)) = i-1$ and $t((i-1,i)) = i$.

For any node x in a graph X, we can define a rooted tree T_xX, the *tree of paths in* X *leaving* x. The nodes in T_xX are the finite paths in X with source x (note that x is considered as a path of length 0 in X); the arcs in T_xX are the triples $(\alpha, a, \alpha a)$

where αa is the concatenation of path α and arc a in X; and $s(\alpha, a, \alpha a) = \alpha$ and $t(\alpha, a, \alpha a) = \alpha a$. There are natural graph morphisms $T_x X \to X$ given by $\alpha \mapsto t(\alpha)$ and $(\alpha, a, \alpha a) \mapsto a$.

A *rooted forest* F is a coproduct of rooted trees. If F is a forest with roots $R(F)$, then we may form a new graph X_F as the pushout of the graph morphisms $r\colon R(F) \to F$ and $f\colon R(F) \to X$; we say that X_F is *formed by attaching the forest* F *to* X *along* R. For instance, if r is a root in a tree T and x is any node in a graph X, then X_T is formed by attaching the tree T to X at the node x (along the graph morphisms $r\colon \mathbf{D} \to T$ and $x\colon \mathbf{D} \to X$).

Definition 3.2. A graph morphism $f\colon X \to Y$ is a *Whiskering* if Y is formed by attaching some rooted forest to X.

For example, the graph morphism $\mathbf{s}\colon \mathbf{D} \to \mathbf{A}$ is a Whiskering, where $\mathbf{s}$ exhibits the "dot" graph $\mathbf{D}$ as the source subgraph of the "arrow" graph $\mathbf{A}$. Also, every isomorphism is a Whiskering, and $R(F) \to F$ is a Whiskering if F is a rooted forest.

Our goal in this section is to demonstrate some remarkable factorization properties of Surjectings and Whiskerings. Here is the conceptual background.

Definition 3.3. Let $\ell\colon X \to Y$ and $r\colon A \to B$ be morphisms in a category $\mathcal{S}$. We say that $\ell \dagger r$ when, for all f and g, if

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{g} & B \end{array}$$

commutes, then

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle \ell}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{g} & B \end{array}$$

commutes for some h. We say that h is a filler for the commutative diagram. We may also say that h *lifts* g along r, or that h *drops* ("extends") f along ℓ. Given two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms, we say $\mathcal{L} \dagger \mathcal{R}$ when we have $\ell \dagger r$ for every $\ell \in \mathcal{L}$ and every $r \in \mathcal{R}$. Given a class $\mathcal{F}$ of morphisms we may define

$$\mathcal{F}^{\dagger} = \{r : f \dagger r, \forall f \in \mathcal{F}\} \quad \text{and} \quad {}^{\dagger}\mathcal{F} = \{\ell : \ell \dagger f, \forall f \in \mathcal{F}\}.$$

Definition 3.4. A *weak factorization system* in $\mathcal{S}$ is given by two classes $\mathcal{L}$ and $\mathcal{R}$ such that $\mathcal{L}^{\dagger} = \mathcal{R}$ and $\mathcal{L} = {}^{\dagger}\mathcal{R}$ and such that, for any morphism c in $\mathcal{S}$, there exist $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ with $c = r \circ \ell$.

The notion of a weak factorization system has become a part of homotopical algebra; see Section 4.

The following three propositions combine to show that Surjectings and Whiskerings give a weak factorization system in Gph. Our Proposition 3.5 was inspired by an argument in Enochs and Herzog [**4**].

Proposition 3.5. *Any graph morphism $f\colon X \to Y$ factors as a Whiskering followed by a Surjecting.*

Proof. Recall that for each node y in Y we have the tree $T_y Y$ of paths in Y leaving y. From f we construct the rooted forest $F = \sum_{x \in X_0} T_{f(x)} Y$, with roots $R = X_0$ considered as a discrete subgraph of X. The pushout of $R \to F$ along the subgraph inclusion $R \to X$ defines a Whiskering $w\colon X \to X_F$. We have $g\colon F \to Y$ as a coproduct of the morphisms $T_{f(x)}Y \to Y$, and since $f\colon X \to Y$ and $g\colon F \to Y$ agree on R, they determine a unique graph morphism $p\colon X_F \to Y$. Note that $f = p \circ w$.

Let us show that p is a Surjecting. For any node z in X_F we must show that $p\colon X_F(z,*) \to Y(p(z),*)$ is surjective. But z is either a node x or a path in Y with source $f(x)$, for some $x \in X_0$.

In the first case, we have $p\colon X_F(x,*) \to Y(f(x),*)$ with

$$X_F(x,*) = X(x,*) \cup T_{f(x)}Y(f(x),*),$$

and $T_{f(x)}Y(f(x),*) \to Y(f(x),*)$ is a bijection.

In the second case, we have $p\colon X_F(\alpha,*) \to Y(p(\alpha),*)$ with

$$X_F(\alpha,*) = T_{f(x)}Y(\alpha,*) = Y(t(\alpha),*) = Y(p(\alpha),*).$$

In either case, $p\colon X_F(z,*) \to Y(p(z),*)$ is surjective. ☐

Proposition 3.6. Whiskering † Surjecting.

Proof. Let $f\colon Z \to Y$ be Surjecting. First we show lifting of rooted trees. More precisely, if T is a rooted tree with root x and we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{z} & Z \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & Y, \end{array}$$

then there is a filler $h\colon T \to Z$. This follows by induction on the length of path from the root to nodes of T, as follows. Suppose that we have extended h to paths of length n and let αa be a path of length $n+1$ in T. Let $x' = t(\alpha)$. Then $f(h(x')) = g(x')$, $g(a) \in Y(g(x'),*)$ and $f\colon Z(h(x'),*) \to Y(g(x'),*)$ is a surjective function, so there exists an arc $a' \in Z(h(x'),*)$ so that $f(a') = g(a)$. We extend h to αa by $h(a) = a'$.

More generally, consider any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g'} & Z \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle f} \\ X_F & \xrightarrow{g} & Y \end{array}$$

with w a Whiskering given by

$$\begin{array}{ccc} R(F) & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle w} \\ F & \xrightarrow{j} & X_F. \end{array}$$

We want to define a filler $h\colon X_F \to Z$, by extending g' along every tree T in F. This is possible since the square is commutative and f is Surjecting. $\square$

Proposition 3.7. *(Whiskering, Surjecting) is a weak factorization system.*

Proof. By the preceding proposition we have

$$\text{Whiskering} \subseteq {}^{\dagger}\text{Surjecting} \quad \text{and} \quad \text{Surjecting} \subseteq \text{Whiskering}^{\dagger}.$$

If $f\colon X \to Y$ is not in Surjecting, then there exists some $x \in X_0$ and some $a \in Y(f(x), *)$ which is not in the image of $X(x,*) \to Y(f(x),*)$. Consider the Whiskering $\mathbf{s}\colon \mathbf{D} \to \mathbf{A}$ and the commutative diagram

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{x} & X \\ \Big\downarrow{\scriptstyle \mathbf{D}} & & \Big\downarrow{\scriptstyle f} \\ \mathbf{A} & \xrightarrow{a} & Y \end{array}$$

for which there is no filler. This shows that $f \notin$ Surjecting implies $f \notin$ Whiskering†. It follows that Whiskering† = Surjecting. We will show that $f \notin$ Whiskering implies $f \notin {}^{\dagger}$Surjecting, by factoring. Suppose that $f\colon X \to Y$ with $f \notin$ Whiskering; then $f = p \circ w$ with some Whiskering $w\colon X \to X_F$ and some Surjecting $p\colon X_F \to Y$. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{w} & X_F \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{\text{id}} & Y. \end{array}$$

If this had a filler $h\colon Y \to X_F$, then we would have $p \circ h = \text{id}$. This would exhibit f as a "morphism retract" of w (a retract in the morphism category). But we show in the next lemma that this would give us the desired contradiction, finishing our proof. $\square$

Lemma 3.8. *Whiskerings are stable with respect to morphism retract.*

Proof. First we show that any retract of a rooted tree is a rooted tree. If T is a rooted tree with root x_0 and we have the following commutative diagram with $r \circ s = \text{id}_{T'}$

$$\begin{array}{ccccc} \mathbf{D} & \xrightarrow{\text{id}} & \mathbf{D} & \xrightarrow{\text{id}} & \mathbf{D} \\ \Big\downarrow{\scriptstyle x} & & \Big\downarrow{\scriptstyle x_0} & & \Big\downarrow{\scriptstyle x} \\ T' & \xrightarrow{s} & T & \xrightarrow{r} & T', \end{array}$$

then T' is a rooted tree with root x. This is clear since, for any node x' in T', the unique path α from x_0 to $s(x')$ gives $r \circ \alpha$ a path from x to $r(s(x')) = x'$, and there can be no other path in T' from x to x'.

More generally, consider any commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{s'} & Z & \xrightarrow{r'} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{s} & Z_F & \xrightarrow{r} & Y \end{array}$$

such that w is a Whiskering, $r \circ s = \mathrm{id}_Y$ and $r' \circ s' = \mathrm{id}_X$. The fact that $r \circ s = \mathrm{id}_Y$ implies that s is injective on nodes and arcs. Also w is injective on nodes and arcs since it is a Whiskering. This implies that f is injective on nodes and arcs, since the first square is commutative.

We will describe a rooted forest F' with roots R' a discrete subgraph of X, such that $Y = X_{F'}$.

The Whiskering w is given by a rooted forest F whose roots R form a discrete subgraph of Z. Let $R' = R \cap X$ as subgraphs of Z. Then R' is a discrete subgraph of X. For each $x \in R'$, consider the tree T in the forest F with root $x_0 = s(x)$. Then $r(T)$ is a retract of T, so $r(T)$ is a tree with root $x = r(x_0)$. Let $F' = \sum_{x \in R'} r(T)$. Then $Y = X_{F'}$, and we are done. □

4. A Quillen model structure on the category of graphs

Suppose that $\mathcal{S}$ is a category with finite limits and finite colimits. Then Quillen's notion [**14**] of "model category" can be expressed via the following axioms (which we learned from Section 7 of Joyal and Tierney [**8**]).

Definition 4.1. A *model structure* on $\mathcal{S}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of morphisms in $\mathcal{S}$ that satisfies:

(1) "three for two": if two of the three morphisms $a, b, a \circ b$ belong to $\mathcal{W}$, then so does the third;
(2) the pair $(\underline{\mathcal{C}}, \mathcal{F})$ is a weak factorization system (where $\underline{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}$);
(3) the pair $(\mathcal{C}, \underline{\mathcal{F}})$ is a weak factorization system (where $\underline{\mathcal{F}} = \mathcal{W} \cap \mathcal{F}$).

The morphisms in $\mathcal{W}$ are called *weak equivalences*. The morphisms in $\mathcal{C}$ are called *cofibrations*, and the morphisms in $\underline{\mathcal{C}}$ are called *acyclic cofibrations*. The morphisms in $\mathcal{F}$ are called *fibrations*, and the morphisms in $\underline{\mathcal{F}}$ are called *acyclic fibrations*.

Note that, according to Hovey [**6**, p. 28], "It tends to be quite difficult to prove that a category admits a model structure. The axioms are always hard to check."

Recall from Section 3 that the path graph $\mathbf{P}_n$ has nodes $\{0, \ldots, n\}$. For $n \geqslant 0$, the *cycle graph* $\mathbf{C}_n$ is the graph produced by identifying the nodes 0 and n of $\mathbf{P}_n$. We have $\mathbf{C}_0 = \mathbf{P}_0 = \mathbf{D}$, the graph with one node and no arcs; and $\mathbf{C}_1$ is the graph with one node, and one arc with source equal to target. Let $C_n(X)$ denote the set of graph morphisms from $\mathbf{C}_n$ to X; we may call this the set of n-cycles in X.

Definition 4.2. A graph morphism $f \colon X \to Y$ is *Acyclic* when

$$C_n(f) \colon C_n(X) \to C_n(Y)$$

is bijective for all $n > 0$.

Here we exclude $n = 0$, since we do not want to require that $f_0 \colon X_0 \to Y_0$ is a bijection.

The Acyclics contain the Whiskerings, and many other useful graph morphisms.

Now we are ready to define the morphism classes $\mathcal{W}$ and $\mathcal{C}$ and $\mathcal{F}$ for our Quillen model structure on Gph.

Definition 4.3. Let $\mathcal{W}$ be the Acyclics. Let $\mathcal{F}$ be the Surjectings, so that $\underline{\mathcal{F}} = \mathcal{W} \cap \mathcal{F}$ must be the Acyclic Surjectings. Let $\mathcal{C}$ be ${}^{\dagger}\underline{\mathcal{F}}$.

In the next few propositions we show directly that this does indeed define a model structure on Gph. We use the following facts, which are easy to verify directly from the definition $\mathcal{C} = {}^{\dagger}\underline{\mathcal{F}}$:

(1) Every composition of graph morphisms in $\mathcal{C}$ is in $\mathcal{C}$.
(2) Every Whiskering is in $\mathcal{C}$.
(3) For any set I and $n > 0$, if $A_i \to B_i$ is in $\mathcal{C}$ for all $i \in I$, then $\sum_{i \in I} A_i \to \sum_{i \in I} B_i$ is in $\mathcal{C}$.
(4) For any set I and $n > 0$, the graph morphism $\mathbf{i} \colon 0 \to I \times \mathbf{C}_n$ is in $\mathcal{C}$.
(5) For any set I, the graph morphism $\mathbf{j} \colon I \times \mathbf{C}_n \to \mathbf{C}_n$ is in $\mathcal{C}$.

In (4) and (5) we view the set I as a discrete graph and we view the graph $I \times \mathbf{C}_n$ as a sum of copies of $\mathbf{C}_n$. Let $\mathbf{i}_n$ denote $\mathbf{i} : 0 \to \mathbf{C}_n$ and let $\mathbf{j}_n$ denote $\mathbf{j} : \mathbf{C}_n + \mathbf{C}_n \to \mathbf{C}_n$.

Proposition 4.4. *The Acyclics satisfy the "three for two" property.*

Proof. This is easy, since Acyclics are defined functorially. Consider $h = f \circ g$. Then $C_n(h) = C_n(f) \circ C_n(g)$ for all n. But the class of bijective functions in the category of sets satisfies the "three for two" property. □

Proposition 4.5. $(\mathcal{C}, \underline{\mathcal{F}})$ *is a weak factorization system in* Gph.

Proof. We have already shown in Section 3 that (Whiskering, Surjecting) is a weak factorization system in Gph. Let us show that the class of Whiskerings is equal to $\underline{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}$. Every Whiskering is in $\underline{\mathcal{C}}$. Suppose $f \in \underline{\mathcal{C}}$. By Proposition 3.5 we have $f = p \circ w$ with $p \in \mathcal{F}$, so the diagram

$$\begin{array}{ccc} X & \xrightarrow{w} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

has a filler $h \colon Y \to Z$ with $h \circ f = w$ and $p \circ h = \text{id}_Y$. It follows that

$$\begin{array}{ccccc} X & \xrightarrow{\text{id}} & X & \xrightarrow{\text{id}} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{h} & Z_F & \xrightarrow{p} & Y \end{array}$$

commutes, making f a retract of the Whiskering w. Thus f is a Whiskering, by Lemma 3.8. □

Proposition 4.6. *Every graph morphism g factors as $g = f \circ c$ with*

$$c \in \mathcal{C} \quad \textit{and} \quad f \in \underline{\mathcal{F}}.$$

Proof. Given any graph morphism $g\colon X \to Y$, we factor g in three steps.

First, we let C be the disjoint union of a copy of $\mathbf{C}_n$ for each element of $C_n(Y)$ which is not in the image of $C_n(g)\colon C_n(X) \to C_n(Y)$. Let $h\colon C \to Y$ be the graph morphism which sends each summand cycle of C to its image in Y. Let $X' = X + C$, let $g'\colon X \to X'$ denote the inclusion $X \to X + C$, and let $f'\colon X' \to Y$ denote the graph morphism $X + C \to Y$ determined by $g\colon X \to Y$ and $h\colon C \to Y$. Then $g = f' \circ g'$, and $g' \in \mathcal{C}$, and $C_n(f')\colon C_n(X') \to C_n(Y)$ is surjective for all $n > 0$.

Next, we let $J = \{(c, n)\colon c \in C_n(Y)\}$, with $j\colon \sum_J I_c \times \mathbf{C}_n \to \sum_J \mathbf{C}_n$, where I_c is the preimage of c for the function $C_n(f')\colon C_n(X') \to C_n(Y)$. Also let $k\colon \sum_J I_c \times \mathbf{C}_n \to X'$ be the graph morphism which sends each summand cycle to the corresponding cycle in X', and let $\ell\colon \sum_J \mathbf{C}_n \to Y$ be the graph morphism which sends each summand cycle to the corresponding cycle in Y. Let $g''\colon X' \to X''$ denote the pushout of j along k. Let $f''\colon X'' \to Y$ be the pushout graph morphism induced by ℓ and f'. Then $g'' \in \mathcal{C}$ and $f' = f'' \circ g''$, and $C_n(f'')\colon C_n(X'') \to C_n(Y)$ is bijective for all $n > 0$, so that $f'' \in \mathcal{W}$.

Finally, we factor $f'' = f''' \circ g'''$ with Whiskering $g'''\colon X'' \to X'''$ and Surjecting $f'''\colon X''' \to Y$, as in Section 3. Then $g''' \in \mathcal{C}$ and $f''' \in \mathcal{W} \cap \mathcal{F}$.

Thus, $g = c \circ f$ with $c = g''' \circ g'' \circ g'$ in $\mathcal{C}$, and $f = f'''$ in $\underline{\mathcal{F}}$. ☐

Proposition 4.7. $(\mathcal{C}, \underline{\mathcal{F}})$ *is a weak factorization system in* Gph*.*

Proof. We have $\mathcal{C} = {}^{\dagger}\underline{\mathcal{F}}$, by definition. This shows also that $\underline{\mathcal{F}} \subseteq \mathcal{C}^{\dagger}$. It remains only to show that $\mathcal{C}^{\dagger} \subseteq \underline{\mathcal{F}}$. But this is easy. Consider the Whiskering $\mathbf{s}\colon \mathbf{D} \to \mathbf{A}$ from Section 3, and the graph morphisms $\mathbf{i}_n\colon 0 \to \mathbf{C}_n$, $\mathbf{j}_n\colon \mathbf{C}_n + \mathbf{C}_n \to \mathbf{C}_n$ as in (4) and (5) above. These are all in $\mathcal{C}$, since each can be lifted against any graph morphism in $\underline{\mathcal{F}}$. But if $g \notin \underline{\mathcal{F}}$, then we can show failure of lifting for either $\mathbf{s}$ or some $\mathbf{i}_n$ or $\mathbf{j}_n$. ☐

Corollary 4.8. *Our morphism classes $\mathcal{W}$ and $\mathcal{C}$ and $\mathcal{F}$ provide a Quillen model structure for the category* Gph*.*

The above proofs show how the graph morphisms $\mathbf{s}\colon \mathbf{D} \to \mathbf{A}$, together with $\mathbf{i}_n\colon 0 \to \mathbf{C}_n$ and $\mathbf{j}_n\colon \mathbf{C}_n + \mathbf{C}_n \to \mathbf{C}_n$, for $n > 0$, generate our class $\mathcal{C}$ of cofibrations. This situation is a special case of a general principle in presheaf categories; see Proposition 7.5 in Joyal and Tierney [**8**], for instance.

5. Zeta series and almost isospectral graphs

Ihara zeta functions of graphs are usually discussed in a setting of "unoriented" or "symmetric" graphs; see Kotani and Sunada [**9**], for instance. We need a version suitable for *directed* graphs (in this section we may refer to objects of our category Gph as directed graphs, for emphasis). There is a nice treatment of zeta series of finite directed graphs in Section 2 of Kotani and Sunada [**9**]; we will follow them here, but with our own terminology.

Definition 5.1. A *finite graph* is one with finitely many nodes and arcs. The *zeta series* of a finite directed graph X is the formal power series

$$Z(u) = \exp\Big(\sum_{m=1}^{\infty} c_m \frac{u^m}{m}\Big),$$

where $c_m = |C_m(X)|$ for $m > 0$.

See the appendix for some motivation for this definition, including how it relates to an Euler product expansion in terms of "primes".

Example 5.2. If X is the graph with one node and n arcs, then $c_m = n^m$ and

$$\sum_{m=1}^{\infty} c_m \frac{u^m}{m} = \sum_{m=1}^{\infty} \frac{n^m u^m}{m} = -\log(1-nu) \quad \text{so that} \quad Z(u) = \frac{1}{1-nu}.$$

Definition 5.3. Let X be a finite graph. Let $\mathrm{R}X_0$ denote the real vector space with basis the nodes of X. The *adjacency operator A for X* is the linear transformation $A\colon \mathrm{R}X_0 \to \mathrm{R}X_0$ determined by

$$A(x) = \sum_{a\in X(x,*)} t(a)$$

for $x \in X_0$. The *characteristic polynomial* of X is defined as $a(x) = \det(xI - A)$, the characteristic polynomial of the adjacency operator A for X. If X has n nodes, then $a(x)$ is a monic polynomial of degree n, and the *reversed characteristic polynomial* of X is defined to be $u^n a(u^{-1}) = \det(I - uA)$.

Note that the reversed characteristic polynomial of a finite graph X has constant term 1, and is thus a unit in the ring of formal power series with integer coefficients.

If we totally order the nodes of X, then the adjacency operator is represented by the square matrix A with entry $A_{j,i}$ equal to the number of arcs in X from the i^{th} node to the j^{th} node. It follows that $c_m = |C_m(X)|$ is the trace of the matrix A^m.

Proposition 5.4. *If X is a finite graph with n nodes and $Z(u)$ is the zeta series of X, then*

$$Z(u) = \det(I - uA)^{-1} = \frac{1}{u^n a(u^{-1})}.$$

Proof. Let A be any endomorphism of an n-dimensional real vector space V. We have $\det(I - uA) = u^n \det(u^{-1}I - A)$, which proves the second equality. The first equality follows from

$$\exp\Big(\sum_{m=1}^{\infty} \mathrm{Trace}(A^m)\frac{u^m}{m}\Big) = \det(1 - uA)^{-1}.$$

One can check this linear algebra identity by induction on the dimension of V, since both sides are multiplicative for short exact sequences of vector spaces endowed with endomorphisms. Or, when V has a basis of eigenvectors for A with eigenvalues $\lambda_1, \ldots, \lambda_n$, the identity follows from $-\log(1-x) = \sum_k \frac{x^k}{k}$ and

$$\exp\Big(\sum_{m=1}^{\infty}\sum_{i=1}^{n} \lambda_i^m \frac{u^m}{m}\Big) = \prod_{i=1}^{n} \exp(-\log(1-\lambda_i u) = \prod_{i=1}^{n} \frac{1}{1-\lambda_i u} = \det(I - uA)^{-1}. \quad \square$$

Example 5.5 (continued). If X is the graph with one node and n arcs, then $a(x) = x - n$ and $u^1 a(u^{-1}) = 1 - nu$, which agrees with $Z(u) = \frac{1}{1-nu}$.

Proposition 5.6. *If X and Y are finite graphs and $f\colon X \to Y$ is an acyclic morphism, then $Z_X = Z_Y$.*

Proof. This is clear from the definition, since $|C_m(X)| = |C_m(Y)|$ for all $m > 0$. □

Definition 5.7. The eigenvalues of the adjacency operator for X may be called the *spectrum* of X (even though the adjacency operator is not necessarily a diagonalizable operator). We say that two finite graphs X and Y are *isospectral* if they have the same characteristic polynomial. We say that X and Y are *almost isospectral* if they have the same reversed characteristic polynomial.

Loosely speaking, X and Y are almost isospectral if and only if they have the same non-zero eigenvalues.

Corollary 5.8. *If X and Y are finite graphs with $Z_X = Z_Y$, then X and Y are almost isospectral.*

Proof. This follows immediately from the preceding two propositions. □

Appendix A. An appendix on the zeta series and its Euler product expansion

Here is a little history, based on Thomas [**16**], of how our zeta series for finite directed graphs relate to the famous zeta functions from number theory.

The zeta function of Euler and Riemann.

Let p range over the prime numbers. Then

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}.$$

Dedekind's zeta function for algebraic number fields.

Let A be the ring of integers in an algebraic number field K (so K is a finite extension over the field Q of rational numbers). Let $N(I) = |A/I|$ for any non-zero ideal in A. Then

$$\zeta(s) = \sum_I \frac{1}{N(I)^s} = \prod_P (1 - N(P)^{-s})^{-1},$$

where I ranges over the non-zero principal ideals in A and P ranges over the prime ideals in A.

A zeta function for algebraic function fields.

Let A be the ring of integers in an algebraic function field (so K is a finite extension over the field $\mathrm{F}_q(x)$ of rational functions with coefficients in the field F_q with q elements). But here $N(I) = |A/I| = q^{\nu(I)}$ for any non-zero ideal in A, where $\nu(I)$ is the dimension of A/I as a finite-dimensional vector space over F_q, so that

$$\zeta(s) = Z(u)|_{u=q^{-s}} \quad \text{for} \quad Z(u) = \prod_P (1 - u^{\nu(P)})^{-1}.$$

The zeta function for a projective variety over a finite field from Weil [**18**] is a completed version of this.

The form of zeta series that we use in Section 5 seems ultimately based on the following observation. If A is an endomorphism of a finite-dimensional vector space over a field of characteristic zero, then the knowledge of the trace of A^n for all n is equivalent to the knowledge of the reversed characteristic polynomial of A, as we see from the identity

$$\det(1 - uA)^{-1} = \exp\Big(\sum_{n=1}^{\infty} \operatorname{Trace}(A^n)\frac{u^n}{n}\Big),$$

used in the proof of Proposition 5.4.

If A is a matrix expressing a function $\tau\colon S \to S$ with S finite, then the trace of A^n counts the number of fixed points of τ^n. This is one of the ideas behind the Lefschetz fixed point theorem, the Weil zeta function used in the Weil conjectures, and other dynamical zeta functions such as the Selberg zeta function in Riemannian geometry and the Ihara zeta function (see Ruelle [**15**] for instance).

In our setting of graphs, we merely use that if A is the adjacency matrix of a finite graph X, then the trace of A^n counts the number of cycles of length n in X. This leads to the following Euler product expansion, analogous to the one for algebraic function fields.

The cycle graph $\mathbf{C}_n$ has nodes i for $0 \leqslant i < n$. If m divides n then we have a graph morphism $\pi\colon \mathbf{C}_n \to \mathbf{C}_m$ given by sending node i to node $i \bmod m$.

Definition A.1. A cycle $c\colon \mathbf{C}_{km} \to X$ is a *k-multiple* if $c = c' \circ \pi$ for some cycle $c'\colon \mathbf{C}_m \to X$. A *prime cycle of length n in X* is a cycle $c\colon \mathbf{C}_n \to X$ which is not a k-multiple for any $k > 1$. Let us say that two cycles $c, c'\colon \mathbf{C}_n \to X$ are *shift equivalent* if $c' = c \circ \tau^i$ for some i, where $\tau^i\colon \mathbf{C}_n \to \mathbf{C}_n$ is the *shift morphism* sending node j to node $j + i \bmod n$. Let us say that a *prime P in X* is an equivalence class of prime cycles in X, and that $\nu(P)$ is the length of the prime P.

This makes sense, since shift equivalence is an equivalence relation on $C_n(X)$, and on prime cycles of length n in X.

Proposition A.2. *The Euler product expansion for the zeta function of a finite graph is given by*

$$Z(u) = \prod_P (1 - u^{\nu(P)})^{-1},$$

where P ranges over all primes in X and $\nu(P)$ is the length of P.

Proof. Let $\bar{c}_k$ be the number of primes P of length k. Then $c_m = \sum_{\{k:k|m\}} k\bar{c}_k$ and we have

$$\log \prod_P (1-u^{\nu(P)})^{-1} = \sum_P \sum_{k=1}^{\infty} \frac{u^{k|p|}}{k} = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{\nu(P)=\ell} \frac{u^{k\ell}}{k}$$

$$= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \bar{c}_\ell \frac{u^{k\ell}}{k} = \sum_{m=1}^{\infty} \frac{1}{m}\sum_{\ell|m} \ell\bar{c}_\ell u^m = \sum_{m=1}^{\infty} c_m \frac{u^m}{m},$$

which is $\log Z(u)$. Here equality one and equality four follow from

$$-\log(1-u^v) = \sum_k \frac{u^{kv}}{k} \qquad \text{and} \qquad m = k\ell \iff \frac{\ell}{m} = \frac{1}{k}. \qquad \square$$

Example A.3 (*Example 5.2* (continued)). For X the graph with one node and two arcs, we must have

$$\frac{1}{1-2u} = (1-u)^{-2}(1-u^2)^{-1}(1-u^3)^{-2}\cdots = \prod (1-u^k)^{-\bar{c}_k}.$$

This is related to the cyclotomic (necklace) identity; see Dress and Siebeneicher [2], for instance.

References

[1] D-C. Cisinski, Théories homotopiques dans les topos, *J. Pure Appl. Algebra* **174** (2002), no. 1, 43–82.

[2] A.W.M. Dress and C. Siebeneicher, The Burnside ring of the infinite cyclic group and its relations to the necklace algebra, λ-rings, and the universal ring of witt vectors, *Adv. Math.* **78** (1989), 1–41.

[3] W.G. Dwyer and J. Spalinski, Homotopy theories and model categories, in *Handbook of algebraic topology* (I.M. James, ed.), 73–126, Elsevier B V, Amsterdam, 1995.

[4] E.E. Enochs and I. Herzog, A homotopy of quiver morphisms with applications to representations, *Canadian J. Math.* **51** (1999), no. 2, 294–308.

[5] L. Fajstrup and J. Rosický, A convenient category for directed homotopy, in *Theory and applications of categories* **21** (2008), no. 1, 7–20.

[6] M. Hovey, *Model categories*, Math. Surveys and Monographs **63**, Amer. Math. Soc., Providence, RI, 1999.

[7] A. Joyal and M. Tierney, Strong stacks and classifying spaces, in *Category theory* (Como, 1990), Lecture Notes in Math. **1488**, 213–236, Springer-Verlag, New York, 1991.

[8] A. Joyal and M. Tierney, Quasi-categories vs Segal spaces, in *Categories in algebra, geometry and mathematical physics*, Contemp. Math. **431**, 277–326, Amer. Math. Soc., Providence, RI, 2007.

[9] M. Kotani and T. Sunada, Zeta functions of finite graphs, *J. Math. Sci. Univ. Tokyo* **7** (2000), no. 1, 7–25.

[10] F.W. Lawvere, Qualitative distinctions between some toposes of generalized graphs, in *Categories in computer science and logic* (Boulder 1987), Contemp. Math. **92**, 261–299, Amer. Math. Soc., Providence, RI, 1989.

[11] F.W. Lawvere and S.H. Schanuel, *Conceptual mathematics: a first introduction to categories*, Cambridge University Press, Cambridge, 1997.

[12] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic: a first introduction to topos theory*, Universitext, Springer-Verlag, New York (1994).

[13] F. Morel and V. Voevodsky, A^1-homotopy of schemes, *Inst. Hautes Études Sci. Publ. Math.* **90** (1999), 45–143.

[14] D.G. Quillen, *Homotopical algebra*, Lecture Notes in Math. **43**, Springer-Verlag, New York, 1967.

[15] D. Ruelle, Dynamical zeta functions and transfer operators, *Notices Amer. Math. Soc.* **49** (2002), no. 8, 887–895.

[16] A.D. Thomas, *Zeta-functions: an introduction to algebraic geometry*, Pitman, London, 1977.

[17] R.W. Thomason, Cat as a closed model category, *Cahiers Topologie Géom. Différentielle* **21** (1980), no. 3, 305–324.

[18] A. Weil, Number of solutions of equations in finite fields, *Bull. Amer. Math. Soc.* **55** (1949), 497–508.

Terrence Bisson `bisson@canisius.edu`

Department of Mathematics and Statistics, Canisius College, 2001 Main St., Buffalo, NY 14208, USA

Aristide Tsemo `tsemo58@yahoo.ca`

Mathematics, College Boreal, 351 Carlaw Ave., Toronto, Ontario M4K 3M2, Canada
Or: 102 Goodwood Park, Apt. 614, Toronto, Ontario M4C 2G8, Canada