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A UNIVERSAL PROPERTY FOR $Sp(2)$ AT THE PRIME 3

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Abstract

We study a universal property of $Sp(2)$ in the category of 3-local homotopy associative, homotopy commutative H-spaces. We show that while $Sp(2)$ fails to be universal in the full category, there is a subcategory in which it is universal for its 7-skeleton.

1. Introduction

If V is a graded vector space then the tensor algebra $T(V)$ is universal for V in the category of associative algebras. That is, if M is an associative algebra then there is a one-to-one correspondence between linear maps $V \longrightarrow M$ and algebra maps $T(V) \longrightarrow M$. On the level of spaces, if X is a path-connected space then the Bott-Samelson theorem implies that $H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$. This suggests that $\Omega\Sigma X$ ought to be universal for X in the category of homotopy associative H-spaces; that is, if Y is a homotopy associative H-space, then there ought to be a one-to-one correspondence between continuous maps $X \longrightarrow Y$ and H-maps $\Omega\Sigma X \longrightarrow Y$. James [**J**] proved that this is the case.

The next step is to consider universality in the category of associative, commutative spaces. The algebra is easy. If V is a graded vector space then the symmetric algebra $S(V)$ is universal for V in the category of associative, commutative algebras. However, realizing this on the level of spaces is problematic. If X is a path-connected space, then a universal space $S(X)$ is characterized by the property that there is a one-to-one correspondence between continuous maps $X \longrightarrow Z$ and H-maps $S(X) \longrightarrow Z$ whenever Z is a homotopy associative, homotopy commutative H-space. In particular, $S(X)$ ought to have the property that $H_*(S(X)) \cong S(\widetilde{H}_*(X))$. However, the most basic example is a sphere S^{2n+1}, whose homology suggests that it ought to be universal for itself, implying that the sphere ought to be a homotopy associative, homotopy commutative H-space, which is false. After localizing at a prime $p \geqslant 5$, though, S^{2n+1} is a homotopy associative, homotopy commutative H-space and so it is more appropriate to consider this universal property in a p-local setting. Recently, there has been considerable interest in p-local universal spaces. There is no known functorial construction, but several useful examples have been established [**G1, G2, GT, Gr, T1, T2**]. In all of these, a key ingredient in proving

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the universal property has been the existence of a retraction of $S(X)$ from $\Omega\Sigma X$, which for technical reasons requires $p \geqslant 5$.

The repeated restrictions to $p \geqslant 5$ begs the question of what happens at the prime 3. The purpose of this paper is to investigate a 3-primary example in detail to try to better understand to what extent a universal property can succeed or fail. From here on, we assume that all spaces and maps have been localized at 3 and take homology with mod-3 coefficients. An appropriate example $S(X)$ must be an H-space which is homotopy associative, homotopy commutative, and whose homology is a symmetric algebra generated by $H_*(X)$. Most 3-local H-spaces fail at least one of these three requirements. An odd dimensional sphere, for example, satisfies the last two but not the first, while a double loop space satisfies the first two but not the last. One choice stands out: the Lie group $Sp(2)$. It is a loop space and so is homotopy associative, and McGibbon [**Mc**] showed that the standard loop multiplication is homotopy commutative at 3. The 7-skeleton A of $Sp(2)$ is a two-cell complex with cells in dimensions 3 and 7. Homologically, $H_*(Sp(2)) \cong \Lambda(\widetilde{H}_*(A))$ $(\cong S(\widetilde{H}_*(A)))$. In Theorem 1.1 we show that, modulo conditions on the homotopy groups π_{13} and π_{17}, $Sp(2)$ is universal for A. In Section 5, we go on to show that $Sp(2)$ cannot be universal for A without qualification, and give examples to show that the partial universal property does require some condition on both π_{13} and π_{17}.

If X and Z are spaces, let $[X, Z]$ be the set of homotopy classes of maps from X to Z, and if X and Z are H-spaces, let $H[X, Z]$ be the set of homotopy classes of H-maps from X to Z. Notice that $[X, Z]$ is a group if Z is homotopy associative and has a homotopy inverse, but $H[X, Z]$ need not be since the multiplication on Z need not be an H-map. If Z is also homotopy commutative, then the multiplication on Z is an H-map and so $H[X, Z]$ is an abelian group.

Theorem 1.1. *Let Z be a homotopy associative, homotopy commutative H-space with the property that $\pi_{13}(Z) = \pi_{17}(Z) = 0$. Then any map $A \longrightarrow Z$ extends to an H-map $Sp(2) \longrightarrow Z$, which is unique up to homotopy. Moreover, the one-to-one correspondence*

$$[A, Z] \cong H[Sp(2), Z]$$

is an isomorphism of abelian groups.

A useful example is when $Z = Sp(2)$. By [**MT**], $\pi_{13}(Sp(2)) = \pi_{17}(Sp(2)) = 0$, so Theorem 1.1 can be applied to obtain the following:

Corollary 1.2. *There is a group isomorphism $[A, Sp(2)] \cong H[Sp(2), Sp(2)]$.*

Corollary 1.2 lets us determine all the homotopy classes of multiplicative self-maps of $Sp(2)$ by calculating $[A, Sp(2)]$. This is fairly simple to do. Observe that the inclusion $Sp(2) \longrightarrow Sp(\infty)$ is 9-connected, so as A is 7-dimensional there is an isomorphism $[A, Sp(2)] \cong [A, Sp(\infty)]$. Adjointing, we have

$$[A, Sp(\infty)] \cong [\Sigma A, BSp(\infty)] \cong \widetilde{KSp}(\Sigma A),$$

where $\widetilde{KSp}(\ )$ is (3-local) reduced quaternionic K-theory. It is straightforward to show that $\widetilde{KSp}(\Sigma A)$ is a free group on two generators, and under this isomorphism an explicit generating set of $[A, Sp(2)]$ is given by the maps $i\colon A \longrightarrow Sp(2)$ and

$g\colon A \xrightarrow{q} S^7 \xrightarrow{c} Sp(2)$, where i is the inclusion, q is the pinch map to the top cell, and c is the characteristic map, which has the property that its composition with the quotient map $Sp(2) \longrightarrow Sp(2)/Sp(1) \simeq S^7$ is of degree 3. Theorem 1.1 implies that i and g extend to H-maps $\iota\colon Sp(2) \longrightarrow Sp(2)$ and $\gamma\colon Sp(2) \longrightarrow Sp(2)$ respectively. Since the identity map on $Sp(2)$ is also an H-map which extends i, the uniqueness condition in Theorem 1.1 implies that ι is homotopic to the identity map. Thus we obtain a group isomorphism $H[Sp(2), Sp(2)] \cong \mathbb{Z}_{(3)}\langle \iota, \gamma\rangle$, where the right side is the free group on the indicated generators.

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2. A construction of finite H-spaces

Cohen and Neisendorfer [**CN**] gave a construction of finite H-spaces, for which the statement is as follows: Fix an odd prime p. Let X be a CW-complex consisting of l odd dimensional cells, where $l < p-1$. Now localize at p and take homology with mod-p coefficients. Then there is an H-space Y with the property that $H_*(Y) \cong \Lambda(\widetilde{H}_*(X))$.

At first glance this does not seem relevant to our case. We are considering $Sp(2)$ at 3 which is already known to be an H-space. Moreover, while A has only odd dimensional cells and $H_*(Sp(2)) \cong \Lambda(\widetilde{H}_*(A))$, we have fallen outside the allowable parameters as A has 2 cells and we are localizing at the prime 3.

However, there are aspects of the construction which are useful in our case. To discuss these, we review the work in [**CN**], beginning with some algebra. For a graded vector space V, let $L = L\langle V\rangle$ be the free Lie algebra generated by V. Let UL be the universal enveloping algebra. Let $L_{ab} = L_{ab}\langle V\rangle$ be the free abelian Lie algebra generated by V, that is, the bracket in L_{ab} is identically zero. Let $[L, L]$ be the kernel of the quotient map $L \longrightarrow L_{ab}$. The short exact sequence of Lie algebras

$$0 \longrightarrow [L, L] \longrightarrow L \longrightarrow L_{ab} \longrightarrow 0$$

induces a split short exact sequence of Hopf algebras

$$0 \longrightarrow U[L, L] \longrightarrow UL \longrightarrow UL_{ab} \longrightarrow 0$$

for which there is an isomorphism $UL \cong U[L, L] \otimes UL_{ab}$ of left $U[L, L]$-modules and right UL_{ab}-comodules.

When the elements in V are all of odd degree, an explicit Lie basis for $[L, L]$ is given by the following:

Lemma 2.1. *Suppose $V = \{u_1, \ldots, u_l\}$ where each u_i is of odd degree and l is a positive integer. Let $L = L\langle V\rangle$. Then a Lie basis for $[L, L]$ is given by the elements*

$$[u_i, u_j], [u_{k_1}, [u_i, u_j]], [u_{k_2}, [u_{k_1}, [u_i, u_j]]], \ldots,$$

where $1 \leqslant j \leqslant i \leqslant l$ and $1 \leqslant k_t < k_{t-1} < \cdots < k_2 < k_1 < i$.

Let V_k consist of those Lie basis elements of bracket length k. Lemma 2.1 implies that $V_m = 0$ for all $m \geqslant l+2$, so a basis for $[L, L]$ is given by $\oplus_{k=2}^{l+1} V_k$.

We now bring in the topology. Let X be a CW-complex consisting of l odd dimensional cells and localize at p. Note that we do not impose a restriction on l at this point. Let $V = \widetilde{H}_*(X)$. We wish to geometrically realize the Lie basis elements of $[L, L]$ in Lemma 2.1 as certain Whitehead products on ΣX. Let $X^{(k)}$ be the k-fold smash of X with itself and let

$$w_k\colon \Sigma X^{(k)} \longrightarrow \Sigma X$$

be the k-fold Whitehead product of the identity map on ΣX with itself. For $k \geqslant 2$, define the map

$$\beta_k\colon \Sigma X^{(k)} \longrightarrow \Sigma X^{(k)}$$

inductively by letting $\beta_2 = 1 - (1, 2)$ and $\beta_k = (1 - (k, k-1, \dots, 2, 1)) \circ (1 \wedge \beta_{k-1})$, where $(1, 2)$ and $(k, k-1, \dots, 2, 1)$ are permutations of the smash factors. In homology, $(\beta_k)_*(\sigma(x_1 \otimes \cdots \otimes x_k)) = \sigma[x_1, [x_2, \dots, [x_{k-1}, x_k] \cdots]]$, and we have

$$(\beta_k)_* \circ (\beta_k)_* = k \cdot (\beta_k)_*.$$

If k is not a multiple of p, then $\overline{\beta}_k = \frac{1}{k} \cdot \beta_k$ is an idempotent in homology. One consequence of this is that, after looping and restricting to $k < p$, the composite $\Omega\Sigma X^{(k)} \xrightarrow{\Omega\overline{\beta}_k} \Omega\Sigma X^{(k)} \xrightarrow{\Omega w_k} \Omega\Sigma X$ has the property that its image in homology is $UL\langle V_k\rangle$. In such cases, let R_k and S_k be the mapping telescopes of $\overline{\beta}_k$ and $(1 - \overline{\beta}_k)$ respectively. Define λ_k as the composite

$$\lambda_k\colon R_k \longrightarrow \Sigma X^{(k)} \xrightarrow{w_k} \Sigma X.$$

The idempotent property of $(\overline{\beta}_k)_*$ is used to prove the following lemma:

Lemma 2.2. *For $k < p$, there is a homotopy decomposition $\Sigma X^{(k)} \simeq R_k \vee S_k$, where the cells of R_k are in one-to-one correspondence with the elements in the module ΣV_k. Further, $(\Omega\lambda_k)_*$ maps $H_*(\Omega R_k)$ isomorphically onto the submodule $UL\langle V_k\rangle$ of UL.*

Now a restriction is imposed on the number of odd dimensional cells l. If $l < p - 1$, then Lemma 2.1 implies that a Lie basis for L consists of brackets of length k for $2 \leqslant k < p$. Collecting the $2 \leqslant k < p$ cases, let $R = \bigvee_{k=2}^{p-1} R_k$ and define $\lambda\colon R \longrightarrow \Sigma X$ as the wedge sum of the maps λ_k. Then the image of $(\Omega\lambda)_*$ is $U[L, L]$. Define Y as the homotopy fiber of λ, so we get an induced homotopy fibration $\Omega R \xrightarrow{\Omega\lambda} \Omega\Sigma X \xrightarrow{r} Y$ which defines the map r. A homological model for this fibration is the short exact sequence of Hopf algebras $0 \longrightarrow U[L, L] \longrightarrow UL \longrightarrow UL_{ab} \longrightarrow 0$. In particular,

$$H_*(Y) \cong UL_{ab} \cong \Lambda(\widetilde{H}_*(X)).$$

Lemma 2.3 will imply that r has a right homotopy inverse. Thus $\Omega\Sigma X \simeq Y \times \Omega R$, and so Y is an H-space. It is important to note that Lemma 2.3 holds in slightly more generality, when the number l of odd dimensional cells satisfies $l < p$ rather than $l < p - 1$ as before.

Lemma 2.3. *Fix a prime p. Suppose X is a CW-complex consisting of l odd dimensional cells, $l < p$. Localize at p. Suppose there is a map $\Omega\Sigma X \xrightarrow{r} Y$ which in homology is the abelianization $T(\widetilde{H}_*(X)) \longrightarrow \Lambda(\widetilde{H}_*(X))$. Then r has a right homotopy inverse.*

The example of interest is the following:

Example 2.4. Let $p = 3$ and consider the 7-skeleton A of $Sp(2)$. Let $V = \widetilde{H}_*(A) = \{u, v\}$, where $|u| = 3$ and $|v| = 7$. By Lemma 2.1, a Lie basis for $L = L\langle u, v\rangle$ is given by the length 2 brackets $V_2 = \{[u, u], [u, v], [v, v]\}$ and the length 3 brackets $V_3 = \{[u, [u, v]], [v, [u, v]]\}$. These can be geometrically realized by Lemma 2.2 only when $k = 2$. In this case there is a homotopy decomposition $\Sigma A^{(2)} \simeq R_2 \vee S_2$, where $H_*(R_2) \cong \Sigma V_2$. Since $\widetilde{H}_*(\Sigma A^{(2)})$ is a 4-dimensional vector space while $\widetilde{H}_*(R_2)$ is a 3-dimensional vector space, comparing the degrees of the generators shows that $\widetilde{H}_*(S_2) \cong \widetilde{H}_*(S^{11})$ and so $S_2 \simeq S^{11}$. The failure of Lemma 2.2 when $k = 3$ implies that there is no analogous decomposition of $\Sigma A^{(3)}$ which gives a space R_3 that can be used to geometrically realize the brackets in V_3. However, in Section 3 we will show that a space R_3 realizing V_3 can be defined in a different manner.

3. A fibration for $Sp(2)$

We begin by recalling some basic properties of $Sp(2)$ and its 7-skeleton A. We have $H_*(Sp(2)) \cong \Lambda(\widetilde{H}_*(A)) \cong \Lambda(u, v)$ where $|u| = 3$, $|v| = 7$, and there is a dual Steenrod operation given by $\mathcal{P}^1_*(v) = u$. Since $\mathcal{P}^1_*$ detects the homotopy class α_1 that generates the stable 3-stem, there is a homotopy cofibration

$$S^6 \xrightarrow{\alpha_1} S^3 \longrightarrow A.$$

As $Sp(2) \simeq \Omega BSp(2)$, there is an evaluation map $ev\colon \Sigma Sp(2) \longrightarrow BSp(2)$. Let j be the composite $j\colon \Sigma A \xrightarrow{\Sigma i} \Sigma Sp(2) \xrightarrow{ev} BSp(2)$. Define the space $\overline{R}$ and the map $\overline{\lambda}$ by the homotopy fibration

$$\overline{R} \xrightarrow{\overline{\lambda}} \Sigma A \xrightarrow{j} BSp(2).$$

Looping we obtain a homotopy fibration

$$\Omega\overline{R} \xrightarrow{\Omega\overline{\lambda}} \Omega\Sigma A \xrightarrow{\Omega j} Sp(2).$$

The usual method for proving a universal property for an H-space X, as in Theorem 1.1, involves trying to show that there is a split fibration

$$\Omega R(X) \xrightarrow{\Omega\lambda(X)} \Omega\Sigma A(X) \xrightarrow{j(X)} X,$$

where $\lambda(X)$ factors through Whitehead products. Then an H-map from $\Omega\Sigma A(X)$ to a homotopy associative, homotopy commutative space Z has the property that it composes trivially with $\Omega\lambda(X)$, and so it factors through $j(X)$. In our case, we do not have enough control over the space $\overline{R}$ to identify $\overline{\lambda}$ as factoring through Whitehead products. So we aim to replace $\overline{R}$ by a homotopy equivalent space R and construct a map $R \xrightarrow{\lambda} \Sigma A$ which, although it does not quite factor through Whitehead products, deviates from this in a manner over which we have control.

While the statements of the following lemmas are phrased in terms of Whitehead products, it will often be helpful in the proofs to adjoint and use Samelson products instead. In general, for a space X, let $s_k\colon X^{(k)} \longrightarrow \Omega\Sigma X$ be the adjoint of the Whitehead product w_k. If $E\colon X \longrightarrow \Omega\Sigma X$ is the suspension map, then s_2 is homotopic to the Samelson product $\langle E, E\rangle$, and s_k for $k > 2$ is homotopic to the Samelson product $\langle E, s_{k-1}\rangle$.

We begin with the following observation:

Lemma 3.1. *For each $k \geqslant 2$, there is a lift*

$$\begin{array}{ccc} & & \Sigma A^{(k)} \\ & \overset{\widetilde{w}_k}{\swarrow} & \big\downarrow{\scriptstyle w_k} \\ \overline{R} & \xrightarrow{\ \overline{\lambda}\ } & \Sigma A \end{array}$$

for some map $\widetilde{w}_k$.

Proof. By adjointing, it is equivalent to show that the Samelson product

$$A^{(k)} \xrightarrow{s_k} \Omega\Sigma A$$

lifts through the map

$$\Omega\overline{R} \xrightarrow{\Omega\overline{\lambda}} \Omega\Sigma A.$$

Such a lift exists if and only if the composite $A^{(k)} \xrightarrow{s_k} \Omega\Sigma A \xrightarrow{\Omega j} Sp(2)$ is null homotopic. Samelson products are natural for H-maps between homotopy associative H-spaces, so $\Omega j \circ s_k$ is homotopic to $\bar{s}_k \circ i^{(k)}$, where $\bar{s}_k$ is the k-fold Samelson product of the identity map on $Sp(2)$ with itself. But as $Sp(2)$ is homotopy commutative, $\bar{s}_k$ is null homotopic. □

We need to focus on the $k = 2$ and $k = 3$ cases of Lemma 3.1. When $k = 2$, Example 2.4 says that there is a homotopy decomposition $\Sigma A^{(2)} \simeq R_2 \vee S^{11}$. Define λ_2 as the composite

$$R_2 \longrightarrow \Sigma A^{(2)} \xrightarrow{w_2} \Sigma A.$$

The following two lemmas give the relevant properties of λ_2; the first is a special case of Lemma 2.2:

Lemma 3.2. *In homology, $\Omega R_2 \xrightarrow{\Omega\lambda_2} \Omega\Sigma A$ has image $UL\langle V_2\rangle$, where*

$$V_2 = \{[u, u], [u, v], [v, v]\}$$

is the Lie basis of length 2 brackets in $L\langle u, v\rangle$.

Lemma 3.3. *There is a lift*

$$\begin{array}{ccc} & & R_2 \\ & \overset{\widetilde{\lambda}_2}{\swarrow} & \big\downarrow{\scriptstyle \lambda_2} \\ \overline{R} & \xrightarrow{\ \overline{\lambda}\ } & \Sigma A \end{array}$$

for some map $\widetilde{\lambda}_2$.

Proof. This is immediate from the $k = 2$ case of Lemma 3.1. Simply define $\widetilde{\lambda}_2$ as the composite $R_2 \longrightarrow \Sigma A^{(2)} \xrightarrow{\widetilde{w}_2} \overline{R}$. □

The $k = 3$ case is more delicate. As we are localized at 3, we have $(k, p) = 3$ and so Lemma 2.2 no longer applies. The way we get around this will essentially boil down to Lemma 3.4. Let $t\colon S^3 \longrightarrow A$ be the inclusion of the bottom cell. Notice that $S^{10} \xrightarrow{\Sigma t^{(3)}} \Sigma A^{(3)}$ is the inclusion of the bottom cell.

Lemma 3.4. *The composite* $S^{10} \xrightarrow{\Sigma t^{(3)}} \Sigma A^{(3)} \xrightarrow{w_3} \Sigma A$ *is null homotopic.*

Proof. By the naturality of the Whitehead product, it is equivalent to show that the composite $S^{10} \xrightarrow{w_3} S^4 \xrightarrow{\Sigma t} \Sigma A$ is null homotopic. In fact, more is true. By [**To**], the three-primary component of $\pi_{10}(S^4)$ is the direct sum of two cyclic groups of order 3, generated by the suspension of $\alpha_1 \circ \alpha_1$ on S^3, and by the Whitehead product $[\alpha_1, \iota]$, where ι is the identity map on S^4. Both maps compose trivially into ΣA since it is the homotopy cofiber of the map $S^7 \xrightarrow{\alpha_1} S^4$. Thus $\pi_{10}(S^4)$ composes trivially into ΣA. $\square$

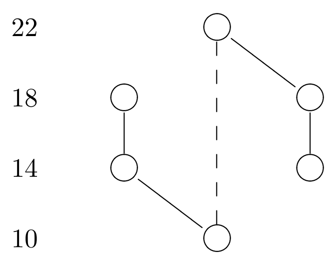
Now observe that as $\Sigma A^{(2)} \simeq R_2 \vee S^{11}$, we have

$$\Sigma A^{(3)} \simeq A \wedge (R_2 \vee S^{11}) \simeq (A \wedge R_2) \vee (\Sigma^{11} A).$$

Let $\widetilde{w}_3$ be the composite

$$\widetilde{w}_3\colon A \wedge R_2 \longrightarrow \Sigma A^{(3)} \xrightarrow{w_3} \Sigma A.$$

The space $A \wedge R_2$ consists of six cells and has a cell diagram



(∗)

Here, the left column records the dimensions of the cells, and the lines between cells record the presence of nontrivial Steenrod operations, as determined by the Cartan formula. Solid lines represent the operation $\mathcal{P}^1_*$ and the dashed line represents the operation $\mathcal{P}^3_*$. By a change of basis in homology if necessary, we can regard the left strand as the image of the inclusion $S^3 \wedge R_2 \longrightarrow A \wedge R_2$. We wish to consider the adjoint of the composite $S^3 \wedge R_2 \longrightarrow A \wedge R_2 \xrightarrow{\widetilde{w}_3} \Sigma A$ and determine its image in homology.

To simplify matters, observe that the 11-skeleton of R_2 is homotopy equivalent to $\Sigma^4 A$, and consider the further restriction of $S^3 \wedge R_2$ to $S^3 \wedge \Sigma^4 A \simeq \Sigma^7 A$. By Lemma 3.2, the adjoint of the composite $\Sigma^4 A \longrightarrow R_2 \xrightarrow{\lambda_2} \Sigma A$ sends the degree 6 and 10 generators of $H_*(\Sigma^3 A)$ to the elements $[u, u]$ and $[u, v]$ in $H_*(\Omega\Sigma A)$. Since the Whitehead product w_3 is defined as the iterated Whitehead product $[1, w_2]$, the adjoint of the composite $S^3 \wedge \Sigma^4 A \longrightarrow S^3 \wedge R_2 \longrightarrow A \wedge R_2 \xrightarrow{\widetilde{w}_3} \Sigma A$ sends the generators in degrees 9 and 13 of $H_*(\Sigma^6 A)$ to the elements $[u, [u, u]]$ and $[u, [u, v]]$ in $H_*(\Omega\Sigma A)$. The element $[u, [u, u]]$ is zero for degree reasons, but $[u, [u, v]]$ is nonzero. Moreover, taking into account the Steenrod operation $\mathcal{P}^1_*$ we have the following:

Lemma 3.5. *The adjoint of the composite* $S^3 \wedge R_2 \longrightarrow A \wedge R_2 \xrightarrow{\widetilde{w}_3} \Sigma A$ *sends the generators in degrees* 13 *and* 17 *of* $\Sigma^{-1}H_*(S^3 \wedge R_2)$ *to the elements* $[u,[u,v]]$ *and* $2[v,[u,v]]$ *in* $H_*(\Omega\Sigma A)$.

Proof. It remains to show that the degree 17 generator in $\Sigma^{-1}H_*(S^3 \wedge R_2)$ has the asserted image. Observe that for degree reasons, $\mathcal{P}^1_*([v,[u,v]]) = 2[u,[u,v]]$. So the Steenrod operation implies that the degree 17 generator of $\Sigma^{-1}H_*(S^3 \wedge R_2)$ is sent to $2[v,[u,v]] + t$, where t is primitive and $\mathcal{P}^1_*(t) = 0$. But the submodule of primitives in $H_{17}(\Omega\Sigma A)$ is generated only by the element $[v,[u,v]]$, so $t = 0$. □

Define X as the quotient space obtained by collapsing out the bottom cell in $A \wedge R_2$. This gives a homotopy cofibration

$$S^{10} \longrightarrow A \wedge R_2 \longrightarrow X.$$

Lemma 3.6. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} A \wedge R_2 & \xrightarrow{\widetilde{w}_3} & \Sigma A \\ \big\downarrow & & \big\| \\ X & \xrightarrow{\epsilon} & \Sigma A \end{array}$$

for some map ϵ.

Proof. By Lemma 3.4, the composite $S^{10} \hookrightarrow \Sigma A^{(3)} \xrightarrow{w_3} \Sigma A$ is null homotopic. The inclusion of the bottom cell into $\Sigma A^{(3)}$ factors through the map $A \wedge R_2 \longrightarrow \Sigma A^{(3)}$ by connectivity. Thus the composite $S^{10} \hookrightarrow A \wedge R_2 \xrightarrow{\widetilde{w}_3} \Sigma A$ is null homotopic, and so the lemma follows. □

The cell diagram for $A \wedge R_2$ in $(*)$ implies that X has five cells, two each in dimensions 14 and 18 and one in dimension 22. Moreover, when the inclusion of the left strand $S^3 \wedge R_2 \longrightarrow A \wedge R_2$ has the bottom cell pinched out we obtain a composite $\Sigma^{11}A \longrightarrow X$. Let $R_3 = \Sigma^{11}A$, and define λ_3 as the composite

$$\lambda_3 \colon R_3 \longrightarrow X \xrightarrow{\epsilon} \Sigma A.$$

Lemma 3.5 lets us determine the image of $(\Omega\lambda_3)_*$.

Lemma 3.7. *In homology,* $\Omega R_3 \xrightarrow{\Omega\lambda_3} \Omega\Sigma A$ *has image* $UL\langle V_3\rangle$, *where*

$$V_3 = \{[u,[v,u]],[v,[v,u]]\}$$

is the Lie basis of length 3 *brackets in* $L\langle u,v\rangle$.

Proof. We have $R_3 = \Sigma^{11}A$; thus, by the Bott-Samelson theorem, there is an algebra isomorphism $H_*(\Omega R_3) \cong T(\widetilde{H}_*(\Sigma^{10}A)) \cong T(x,y)$, where x and y are in degrees 13 and 17 respectively. By Lemma 3.5, $(\Omega\lambda_3)_*$ sends x and y to $[u,[u,v]]$ and $2[v,[u,v]]$ respectively. That is, $(\Omega\lambda_3)_*$ sends $\{x,y\}$ isomorphically onto V_3. Extending multiplicatively gives an isomorphism from $T(x,y)$ to the submodule $T(V_3) \cong UL\langle V_3\rangle$ of $UL\langle u,v\rangle$. □

We also want the analogue of Lemma 3.3 for λ_3. To get this, we need a preliminary lemma which will also be used later in Section 4.

Lemma 3.8. *Let $k \geqslant 0$. Suppose X is a space with the property that $\pi_{3+k}(X) = \pi_{7+k}(X) = 0$. Then any map $\Sigma^k A \longrightarrow X$ is null homotopic.*

Proof. There is a homotopy cofibration $S^{3+k} \xrightarrow{r} \Sigma^k A \xrightarrow{q} S^{7+k}$, where r is the inclusion of the bottom cell and q is the pinch onto the top cell. Suppose there is a map $f\colon \Sigma^k A \longrightarrow X$. The composite $f \circ r$ represents an element of $\pi_{3+k}(X)$. Since this homotopy group is zero, $f \circ r$ is null homotopic. Thus f extends across q to a map $g\colon S^{7+k} \longrightarrow X$. Since $\pi_{7+k}(X) = 0$, g is null homotopic. Hence f is null homotopic. □

Lemma 3.9. *There is a lift*

$$\begin{array}{ccc} & & R_3 \\ & \swarrow^{\widetilde{\lambda}_3} & \downarrow{\scriptstyle \lambda_3} \\ \overline{R} & \xrightarrow{\overline{\lambda}} & \Sigma A \end{array}$$

for some map $\widetilde{\lambda}_3$.

Proof. The space $\overline{R}$ was defined by the homotopy fibration $\overline{R} \longrightarrow \Sigma A \xrightarrow{j} BSp(2)$. So the asserted lift $\widetilde{\lambda}_3$ exists if the composite $j \circ \lambda_3$ is null homotopic. But by definition $R_3 = \Sigma^{11} A$ and by [**MT**] we have $\pi_{14}(BSp(2)) = \pi_{18}(BSp(2)) = 0$, so Lemma 3.8 implies that $j \circ \lambda_3$ is null homotopic. □

Now we combine the results for R_2 and R_3. Let $R = R_2 \vee R_3$ and define

$$\lambda\colon R \longrightarrow \Sigma A$$

as the wedge sum of λ_2 and λ_3. Define

$$\tilde{\lambda}\colon R \longrightarrow \overline{R}$$

as the wedge sum of $\widetilde{\lambda}_2$ and $\widetilde{\lambda}_3$. Note that Lemmas 3.3 and 3.9 imply that $\widetilde{\lambda}$ is a lift of λ through the map $\overline{R} \xrightarrow{\overline{\lambda}} \Sigma A$.

Lemma 3.10. *The map $R \xrightarrow{\widetilde{\lambda}} \overline{R}$ is a homotopy equivalence.*

Proof. Since R and $\overline{R}$ are simply connected, $\widetilde{\lambda}$ is a homotopy equivalence if and only if $\Omega\widetilde{\lambda}$ is a homotopy equivalence. For the latter, it suffices to show that $(\Omega\widetilde{\lambda})_*$ is an ismorphism in homology. We will do this by showing that both $\Omega R \xrightarrow{\Omega\lambda} \Omega\Sigma A$ and $\Omega\overline{R} \xrightarrow{\Omega\overline{\lambda}} \Omega\Sigma A$ are injections in homology and have isomorphic images.

As in Section 2, with $L = L\langle u, v\rangle$, there is a short exact sequence of Lie algebras $0 \longrightarrow [L,L] \xrightarrow{f} L \xrightarrow{g} L_{ab} \longrightarrow 0$ which determines a split short exact sequence of Hopf algebras $0 \longrightarrow U[L,L] \xrightarrow{Uf} UL \xrightarrow{Ug} UL_{ab} \longrightarrow 0$ such that there is an isomorphism $UL \cong U[L,L] \otimes UL_{ab}$ as left $U[L,L]$-modules. Note in our case that as u, v are in odd dimensions, $UL_{ab} \cong \Lambda(u,v)$.

First, consider the homotopy fibration $\Omega\overline{R} \xrightarrow{\Omega\overline{\lambda}} \Omega\Sigma A \xrightarrow{\Omega j} Sp(2)$ and the Eilenberg-Moore spectral sequence which converges to $H_*(\Omega\overline{R})$. Since $(\Omega j)_*$ sends u and v to the generators of $H_*(Sp(2))$, multiplicativity implies that $(\Omega j)_*$ is the abelianization of the tensor algebra. Thus $\Omega\Sigma A \xrightarrow{\Omega j} Sp(2)$ is modelled homologically by $UL \xrightarrow{Ug} UL_{ab}$. Since UL is a free left $U[L,L]$-module, the Eilenberg-Moore spectral sequence under consideration collapses, and therefore there is an isomorphism $H_*(\Omega\overline{R}) \cong U[L,L]$ and an identification of $(\Omega\overline{\lambda})_*$ with Uf.

Next, consider the map $\Omega R \xrightarrow{\Omega\lambda} \Omega\Sigma A$. Since $R = R_2 \vee R_3$ and λ is the wedge sum of λ_2 and λ_3, Lemmas 3.2 and 3.7 imply that $(\Omega\lambda)_*$ is an isomorphism onto the subalgebra $UL\langle V_2 \oplus V_3\rangle$ of $UL = UL\langle u, v\rangle$. By Lemma 2.1, $V_2 \oplus V_3$ is a Lie basis for $U[L,L]$, and so $(\Omega\lambda)_*$ is an isomorphism onto $U[L,L]$. Hence $(\Omega\overline{\lambda})_*$ and $(\Omega\lambda)_*$ have isomorphic images, as required. □

The homotopy equivalence in Lemma 3.10 lets us replace the space $\overline{R}$ and the map $\overline{\lambda}$ in the homotopy fibration $\overline{R} \xrightarrow{\overline{\lambda}} \Sigma A \xrightarrow{j} BSp(2)$ with R and λ to get the following:

Proposition 3.11. *There is a homotopy fibration $R \xrightarrow{\lambda} \Sigma A \xrightarrow{j} BSp(2)$.*

4. The universal property

We begin by stating a theorem due to James [**J**].

Theorem 4.1. *Let X be a path-connected space and let Y be a homotopy associative H-space. Let $f\colon X \longrightarrow Y$ be any map. Then there is a unique H-map $\overline{f}\colon \Omega\Sigma X \longrightarrow Y$ such that $\overline{f} \circ E \simeq f$.*

In our case, suppose we are given a map $f\colon A \longrightarrow Z$ where Z is a homotopy associative, homotopy commutative H-space. The homotopy associativity property of Z lets us apply Theorem 4.1 to extend f in a unique way to an H-map $\overline{f}\colon \Omega\Sigma A \longrightarrow Z$. In Lemmas 4.2 and 4.3 we consider how $\overline{f}$ behaves with respect to the homotopy fibration $\Omega R \xrightarrow{\Omega\lambda} \Omega\Sigma A \xrightarrow{\Omega j} Sp(2)$.

Lemma 4.2. *Let Z be a homotopy associative, homotopy commutative H-space with the property that $\pi_{13}(Z) = \pi_{17}(Z) = 0$. Then the composite $\Omega R \xrightarrow{\Omega\lambda} \Omega\Sigma A \xrightarrow{\overline{f}} Z$ is null homotopic.*

Proof. In general, let X and Y be spaces. Let ev_X be the composite

$$\Sigma\Omega X \xrightarrow{ev} X \xrightarrow{i_X} X \vee Y,$$

where ev is the evaluation map and i_X is the inclusion. Define ev_Y similarly with respect to Y. It is well known that the inclusion of the wedge into the product gives a homotopy fibration $\Sigma\Omega X \wedge \Omega Y \xrightarrow{W} X \vee Y \longrightarrow X \times Y$, where W is the Whitehead product of the maps ev_X and ev_Y. After looping, the maps Ωi_X, Ωi_Y, and ΩW multiply to give a homotopy equivalence

$$e\colon \Omega X \times \Omega Y \times \Omega(\Sigma\Omega X \wedge \Omega Y) \longrightarrow \Omega(X \vee Y).$$

In our case we can apply the previous paragraph to $R = R_2 \vee R_3$. Since $\Omega\lambda$ and $\overline{f}$ are H-maps, the composite $\overline{f} \circ \Omega\lambda \circ e$ is determined by its restrictions to the factors

ΩR_2, ΩR_3, and $\Omega(\Sigma\Omega R_2 \wedge \Omega R_3)$. So to prove the lemma it is equivalent to show that each of these three restrictions is null homotopic.

Next, in general, suppose $\Sigma X \xrightarrow{w} Y$ is a Whitehead product. After looping, Theorem 4.1 implies that $\Omega\Sigma X \xrightarrow{\Omega w} \Omega Y$ is determined by the restriction $\Omega w \circ E$. Note that $s = \Omega w \circ E$ is a Samelson product. Now suppose that Ωw is composed with an H-map $\Omega Y \xrightarrow{g} Z$, where Z is homotopy associative. Theorem 4.1 implies that $g \circ \Omega w$ is determined by $g \circ \Omega w \circ E$, that is, by $g \circ s$. Moreover, since g is an H-map and Z is homotopy associative, $g \circ s$ is a Samelson product. So if Z is also homotopy commutative then $g \circ s$ is null homotopic, implying that $g \circ \Omega w$ is null homotopic.

We apply this twice to our case. First, consider $\overline{f} \circ \Omega\lambda \circ e$ restricted to ΩR_2. By the definitions of λ and e, the restriction of $\Omega\lambda \circ e$ to ΩR_2 is $\Omega\lambda_2$. By the definition of λ_2, it factors through the Whitehead product $\Sigma A^{(2)} \xrightarrow{w_2} \Sigma A$. The argument in the previous paragraph therefore implies that $\overline{f} \circ \Omega\lambda_2$ is null homotopic. Second, consider $\overline{f} \circ \Omega\lambda \circ e$ restricted to $\Omega(\Sigma\Omega R_2 \wedge \Omega R_3)$. By the definition of e, this restriction is homotopic to $\overline{f} \circ \Omega\lambda \circ \Omega W$. Since W is a Whitehead product, the argument in the previous paragraph implies that $\overline{f} \circ \Omega\lambda \circ \Omega W$ is null homotopic.

Third, consider $\overline{f} \circ \Omega\lambda \circ e$ restricted to ΩR_3. By the definitions of λ and e, the restriction of $\Omega\lambda \circ e$ to ΩR_3 is $\Omega\lambda_3$. Recall that $R_3 = \Sigma^{11} A$. Theorem 4.1 therefore implies that $\Omega\Sigma^{11} A \xrightarrow{\Omega\lambda_3} \Omega\Sigma A \xrightarrow{\overline{f}} Z$ is determined by $\overline{f} \circ \Omega\lambda_3 \circ E$. This restriction is a map $\Sigma^{10} A \longrightarrow Z$. By hypothesis, $\pi_{13}(Z) = \pi_{17}(Z) = 0$, and so Lemma 3.8 implies that any map $\Sigma^{10} A \longrightarrow Z$ is null homotopic. Hence $\overline{f} \circ \Omega\lambda_3$ is null homotopic. □

The null homotopy in Lemma 4.2 will now be used to factor $\overline{f}$ through a map $Sp(2) \longrightarrow Z$ which can be chosen to be an H-map. First observe that in homology the map $\Omega\Sigma A \xrightarrow{\Omega j} Sp(2)$ induces the abelianization of the tensor algebra. So by Lemma 2.3, Ωj has a right homotopy inverse $s\colon Sp(2) \longrightarrow \Omega\Sigma A$. Thus there is a homotopy equivalence

$$Sp(2) \times \Omega R \xrightarrow{s \times \Omega\lambda} \Omega\Sigma A \times \Omega\Sigma A \xrightarrow{\mu} \Omega\Sigma A,$$

where μ is the loop multiplication. Define g as the composite

$$g\colon Sp(2) \xrightarrow{s} \Omega\Sigma A \xrightarrow{\overline{f}} Z.$$

The decomposition of $\Omega\Sigma A$ and the null homotopy for $\overline{f} \circ \Omega\lambda$ in Lemma 4.2 lets us argue exactly as in [**G2**, 4.2] or [**T1**, 5.2] to prove the following:

Lemma 4.3. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega\Sigma A & \xrightarrow{\Omega j} & Sp(2) \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle g} \\ Z & = & Z \end{array}$$

and g is an H-map.

Proof of Theorem 1.1. Lemma 4.3 proves the first assertion that any map

$$f\colon A \longrightarrow Z$$

can be extended to an H-map

$$g\colon Sp(2) \longrightarrow Z.$$

To show uniqueness, suppose that $g, h\colon Sp(2) \longrightarrow Z$ are H-maps extending f. Consider the composites $\Omega\Sigma A \xrightarrow{\Omega j} Sp(2) \xrightarrow{g,h} Z$. Note that by adjunction the inclusion $A \xrightarrow{i} Sp(2)$ of the bottom two cells is homotopic to the composite

$$A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\Omega j} Sp(2).$$

Thus there is a string of homotopies

$$g \circ \Omega j \circ E \simeq g \circ i \simeq f \simeq h \circ i \simeq h \circ \Omega j \circ E.$$

Therefore, both $g \circ \Omega j$ and $h \circ \Omega j$ are H-maps extending f. As Z is homotopy associative, Theorem 4.1 implies that there is a unique H-map extending f, and so $g \circ \Omega j \simeq h \circ \Omega j$. Since s is a right homotopy inverse of Ωj, we have

$$g \simeq g \circ \Omega j \circ s \simeq h \circ \Omega j \circ s \simeq h.$$

At this point, we have proved that the map $\theta\colon [A, Z] \longrightarrow H[Sp(2), Z]$, defined by sending f to g, is a one-to-one correspondence. Note that the homotopy associativity and homotopy commutativity of Z implies that both $[A, Z]$ and $H[Sp(2), Z]$ are abelian groups. So to show θ is a group isomorphism it suffices to show it is a group homomorphism. This is done exactly as in [**GT**, Lemma 2.3]. □

5. The necessity of the π_{13} and π_{17} homotopy group conditions

Theorem 1.1 states that $Sp(2)$ is universal for A provided Z satisfies a condition on the homotopy groups π_{13} and π_{17}. In this section we give examples to show that these conditions are necessary and that $Sp(2)$ fails to be universal for A in full generality. These require two lemmas as preparation. In Lemma 5.1 we show that certain maps are not H-maps. In Lemma 5.3 we give an extension property of $Sp(2)$ for A which is weaker than the universal one. Namely, given a map $f\colon A \longrightarrow Y$, where Y is a homotopy associative H-space, there is an extension to a map $f'\colon Sp(2) \longrightarrow Y$. Compared to the universal framework, Y need not be homotopy commutative and f' need not be an H-map.

As reported in [**H**], Harper and Zabrodsky showed that any map $q\colon Sp(2) \longrightarrow S^7$ which is onto in homology cannot be an H-map. A succinct way of seeing this is to compare the projective planes of $Sp(2)$ and S^7 and see that the action of the Steenrod operation $\mathcal{P}^1$ is incompatible with q being an H-map. We make use of this by composing q with the double suspension $E^2\colon S^7 \longrightarrow \Omega^2 S^9$. *A priori*, it may be possible that $E^2 \circ q$ is an H-map even if q is not. However, Lemma 5.1, based on [**H**, Proposition 3], shows that this is not the case.

In general, if $f\colon A \longrightarrow B$ is a map between H-spaces, let $\overline{D}(f)\colon A \times A \longrightarrow B$ be the difference $f \circ \mu_A - \mu_B \circ (f \times f)$, where μ_A and μ_B are the multiplications on A

and B respectively. Observe that $\overline{D}(f)$ is null homotopic when restricted to $A \vee A$, and so it factors through a map $D(f)\colon A \wedge A \longrightarrow B$ whose homotopy class is uniquely determined by that of $\overline{D}(f)$. In particular, f is an H-map if and only if $\overline{D}(f)$ is null homotopic, which is the case if and only if $D(f)$ is null homotopic. Also, if $g\colon B \longrightarrow C$ is an H-map then $D(g \circ f) \simeq g \circ D(f)$.

Lemma 5.1. *Let $Sp(2) \xrightarrow{q} S^7$ be any map which is onto in homology. Then the composite $Sp(2) \xrightarrow{q} S^7 \xrightarrow{E^2} \Omega^2 S^9$ is not an H-map.*

Proof. Suppose $E^2 \circ q$ is an H-map. Then $D(E^2 \circ q)$ is null homotopic. It is well known that E^2 is an H-map when localized at an odd prime, so

$$D(E^2 \circ q) \simeq E^2 \circ D(q).$$

Thus $E^2 \circ D(q)$ is null homotopic and so there is a lift

$$\begin{array}{ccccc} & & Sp(2) \wedge Sp(2) & & \\ & \swarrow^{\gamma} & \downarrow{\scriptstyle D(q)} & & \\ W & \longrightarrow & S^7 & \xrightarrow{E^2} & \Omega^2 S^9 \end{array}$$

for some map γ, where W is the homotopy fiber of E^2. As W is 20-connected and the dimension of $Sp(2) \wedge Sp(2)$ is 20, the map γ is null homotopic. Hence $D(q)$ is null homotopic and so q is an H-map, a contradiction. □

Next, recall from Section 4 that there is a homotopy equivalence

$$Sp(2) \times \Omega R \xrightarrow{s \times \Omega\lambda} \Omega\Sigma A \times \Omega\Sigma A \xrightarrow{\mu} \Omega\Sigma A,$$

where μ is the loop multiplication.

Lemma 5.2. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{E} & \Omega\Sigma A \\ \downarrow{\scriptstyle i} & & \| \\ Sp(2) & \xrightarrow{s} & \Omega\Sigma A. \end{array}$$

Proof. It is equivalent to show that the composite $h\colon A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\pi_2} \Omega R$ is null homotopic, where π_2 is the projection. As A is of dimension 7, it suffices to consider the 7-skeleton of ΩR. By definition, $R = R_2 \vee R_3$, where R_2 has cells in dimensions 7, 11 and 15, and R_3 is 13-connected. Thus the 7-skeleton of ΩR_2 is S^6. Since $\pi_3(S^6) = \pi_7(S^6) = 0$, Lemma 3.8 implies that h is null homotopic. □

Lemma 5.3. *Suppose there is a map $f\colon A \longrightarrow Y$ where Y is a homotopy associative H-space. Then there is a map $f'\colon Sp(2) \longrightarrow Y$ such that $f' \circ i \simeq f$.*

Proof. By Theorem 4.1, f extends to an H-map $\overline{f}\colon \Omega\Sigma A \longrightarrow Y$ such that $\overline{f} \circ E \simeq f$. By Lemma 5.2, $E \simeq s \circ i$, so if we define $f' = \overline{f} \circ s$ then the lemma follows. □

In Example 5.4 we will show that $Sp(2)$ cannot be universal for A without qualification. Example 5.5 will build on Example 5.4 to show that some condition on π_{13} is necessary in Theorem 1.1 and Example 5.6 will show that some condition on π_{17} is also necessary.

Example 5.4. Let f be the composite $A \longrightarrow S^7 \xrightarrow{E^2} \Omega^2 S^9$, where the left map is the pinch onto the top cell. Since $\Omega^2 S^9$ is a homotopy associative, homotopy commutative H-space, if $Sp(2)$ were universal for A there would be a homotopy commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & \Omega^2 S^9, \\ \big\downarrow{\scriptstyle i} & \nearrow{\scriptstyle \overline{f}} & \\ Sp(2) & & \end{array}$$

where $\overline{f}$ is an H-map. Since $Sp(2)$ is 10-dimensional and $S^7 \xrightarrow{E^2} \Omega^2 S^9$ is a homotopy equivalence through dimension 20, $\overline{f}$ lifts through E^2 to a map $q\colon Sp(2) \longrightarrow S^7$. As f is onto in $H_7(\ )$, the commutativity of the previous diagram implies that $\overline{f}$ is too, and therefore so is q. Now q is onto in homology and $q \circ E^2$ is an H-map as it is homotopic to $\overline{f}$, contradicting Lemma 5.1. Thus f cannot extend to an H-map, and so $Sp(2)$ is not universal for A.

Example 5.5. Continuing Example 5.4, by Lemma 5.3, the map f does extend to a map $f'\colon Sp(2) \longrightarrow \Omega^2 S^9$. Example 5.4 states that f' cannot be chosen to be an H-map. We wish to pinpoint the obstruction that prevents this. Toda's [**To**] calculations of the homotopy groups of spheres in low dimensions show that $\pi_{13}(\Omega^2 S^9) \cong \pi_{15}(S^9) = 0$ and $\pi_{17}(\Omega^2 S^9) \cong \pi_{19}(S^9) = \mathbb{Z}/3\mathbb{Z}$. Had $\pi_{17}(\Omega^2 S^9)$ been 0 as well, Theorem 1.1 would have implied that the map f did extend to an H-map. As this cannot be the case, the obstruction to extending f to an H-map lies in $\pi_{17}(\Omega^2 S^9)$. Thus the partial universal property in Theorem 1.1 does require some condition on π_{17}.

Example 5.6. By [**H**], there is a homotopy fibration $S^9 \longrightarrow B(9,13) \longrightarrow S^{13}$, where $H^*(B(9,13)) \cong \Lambda(x_9, x_{13})$ and $\mathcal{P}^1(x_9) = x_{13}$. Looping six times, there is a map $f\colon A \longrightarrow \Omega^6 B(9,13)$ which includes the bottom two cells. By Lemma 5.3, f extends to a map $f'\colon Sp(2) \longrightarrow \Omega^6 B(9,13)$. Suppose that f' can be chosen to be an H-map. Let g' be the composite $g'\colon Sp(2) \xrightarrow{f'} \Omega^6 B(9,13) \longrightarrow \Omega^6 S^{13}$, where the right side is the 6-fold loop map. Now g' is an H-map, and arguing as in Example 5.4 shows that g' factors as a composite $Sp(2) \xrightarrow{q} S^7 \xrightarrow{E^6} \Omega^6 S^{13}$, where q is onto in homology and E^6 is the 6-fold suspension. Arguing as in the proof of Lemma 5.1 shows that q cannot be an H-map. Thus the extension f' cannot be chosen to be an H-map. By [**MNT**], $\pi_{13}(\Omega^6 B(9,13)) = \pi_{19}(B(9,13)) \cong \mathbb{Z}/3\mathbb{Z}$ and $\pi_{17}(\Omega^6 B(9,13)) = \pi_{23}(B(9,13)) \cong 0$. Had $\pi_{13}(\Omega^6 B(9,13))$ been 0 as well, then Theorem 1.1 would have implied that the map f did extend to an H-map. As this is not the case, the obstruction to extending f to an H-map lies in $\pi_{13}(\Omega^6 B(9,13))$. Thus the partial universal property in Theorem 1.1 also requires some condition on π_{13}.

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