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AN ALGEBRAIC GENERALIZATION OF IMAGE J

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Abstract

As is well known, the image of the J-homomorphism in the stable homotopy groups of spheres is described in terms of the first line of the Adams-Novikov E_2-term. In this paper we consider an algebraic analogue of the image of J using the spectrum $T(m)_{(j)}$ defined by Ravenel and determine the Adams-Novikov first line for small values of j.

1. Introduction

As usual, let BP be the Brown-Peterson spectrum and $BP_*(-)$ be the BP-homology functor from the category of spectra to that of abelian groups. The stable homotopy groups and the BP-homology groups of BP are known to be polynomial algebras

$$\begin{aligned} \pi_*(BP) &= \mathbf{Z}_{(p)}[v_1, \ldots, v_n \ldots] \\ \text{and} \quad BP_*(BP) &= BP_*[t_1, \ldots, t_n \ldots] \end{aligned}$$

with $|v_i| = |t_i| = 2(p^i - 1)$.

In [**2**] Ravenel has shown the existence of p-local spectra $T(m)$ and $T(m)_{(j)}$ for non-negative integers m and j, each of which satisfies

$$\begin{aligned} BP_*(T(m)) &= BP_*[t_1, \ldots, t_m] \\ \text{and} \quad BP_*(T(m)_{(j)}) &= BP_*(T(m))\{t_{m+1}^{\ell} \colon 0 \leqslant \ell < p^j\}. \end{aligned}$$

In [**2**] and [**5**] these spectra are extensively used for analyzing the stable homotopy groups of spheres by *the method of infinite descent in homotopy theory*, which is first introduced in [**1**]: Once we have information on $T(m)_{(j)}$, then we can obtain information on $T(m)_{(j-1)}$ using *the small descent spectral sequence* induced in [**2**] Theorem 1.17 and 1.21. Iterated use of this spectral sequence gives information on $T(m)_{(0)} = T(m)$ and finally $T(0) = S^0$. In this sense the Adams-Novikov E_2-term

$$\mathrm{Ext}^{i,*}_{BP_*(BP)}(BP_*, BP_*(T(m)_{(j)})) \tag{1}$$

can be regarded as the starting point for doing calculations on the stable homotopy groups of spheres.

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In this paper we present computations of the Ext groups (1) in lower dimensions. For example, the $j=0$ case (4), the $i=0$ case (Theorem 4.1) and the $(i,j)=(1,1)$ case (Theorem 5.2).

Even for $i=1$, the computations are complicated for higher j in general. For example, we exhibit calculations of the $j=2$ case in Theorem 5.4 (for $p=2$) and Proposition 5.5 (for $p>2$). Note that the result for $(j,m)=(0,0)$ is not new and is well known: the group (1) is equal to $\mathrm{Ext}^1_{BP_*(BP)}(BP_*, BP_*)$ which is isomorphic to the p-primary component of the image of J-homomorphism $\pi_*(SO)\to\pi_*(S)$ for odd prime p (cf. [**4**] Theorem 2.2).

The main task in this paper is to analyze the complicated structure of the comodule W_{m+1} (Definition 2.3) by defining elements $\widehat{\alpha}_{i,p^j-1}$ (Lemma 3.5). The computations for the second or higher lines of (1) will be given in [**5**] and those require quite independent calculations. *That is the reason why we separate this paper from [**5**].*

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2. Notations and method

We will use the following notations (cf. [**5**]):

$$\begin{aligned} A(k) &= \mathbf{Z}_{(p)}[v_1,\dots,v_k], & \Gamma(k) &= BP_*(BP)/(t_i\colon 0<i<k),\\ G(k) &= A(k)[t_k] \quad\text{and} & T_m^{(j)} &= BP_*\left\{t_{m+1}^{\ell}\colon 0\leqslant \ell<p^j\right\}. \end{aligned}$$

The pairs $(BP_*,\Gamma(k))$ and $(A(k),G(k))$ form Hopf algebroids. We will omit the first variable of the associated Ext groups like $\mathrm{Ext}_{\Gamma(m+i)}(-)$ or $\mathrm{Ext}_{G(m+1)}(-)$. It is shown that $T_m^{(j)}$ is in fact a $\Gamma(m+1)$-comodule and we have

Theorem 2.1 ([**2**] Lemma 1.15)**.** *The Adams-Novikov E_2-term for $T(m)_{(j)}$ (1) is isomorphic to* $\mathrm{Ext}^{i,*}_{\Gamma(m+1)}(T_m^{(j)})$. □

We will abbreviate v_{m+i} by $\widehat{v}_i$ and t_{m+i} by $\widehat{t}_i$ for short. In [**2**] (3.10) Ravenel defined elements $\widehat{\lambda}_n\in p^{-1}BP_*$ for $n>0$ satisfying

$$\widehat{v}_1=p\widehat{\lambda}_1,\quad \widehat{v}_2=p\widehat{\lambda}_2+(1-p^{p-1})v_1\widehat{\lambda}_1^p-\begin{cases} v_1^{p^{m+1}}\widehat{\lambda}_1 & (m>0)\\ 0 & (m=0),\end{cases} \tag{2}$$

and so on. He also defined the subcomodule D^0_{m+1} of $p^{-1}BP_*$ by

$$D^0_{m+1}=A(m)[\widehat{\lambda}_1,\dots,\widehat{\lambda}_n,\dots]$$

and E^1_{m+1} by the short exact sequence of comodules

$$0\longrightarrow BP_*\rightarrowtail D^0_{m+1}\overset{\varphi}{\twoheadrightarrow} E^1_{m+1}\longrightarrow 0.$$

Then he showed:

Lemma 2.2 ([**2**] Theorem 3.11)**.** *The comodule D^0_{m+1} is weak injective over $\Gamma(m+1)$ and the inclusion map $BP_* \to D^0_{m+1}$ induces an isomorphism of $\mathrm{Ext}^0_{\Gamma(m+1)}$, i.e., we have*

$$\mathrm{Ext}^i_{\Gamma(m+1)}(D^0_{m+1}) = \begin{cases} A(m) & (i=0), \\ 0 & (i>0). \end{cases} \qquad \square$$

For a given $\Gamma(m+2)$-comodule M we will denote $\mathrm{Ext}^0_{\Gamma(m+2)}(M)$ by $\overline{M}$ for short. In particular we have

$$\begin{aligned} \overline{D}^0_{m+1} &= A(m)[\widehat{\lambda}_1] \\ \text{and} \quad \overline{T}^{(j)}_m &= A(m+1)\left\{\widehat{t}^{\ell}_1 \colon 0 \leqslant \ell < p^j\right\}. \end{aligned}$$

Define elements $\widehat{\beta}'_{i/j}$ and $\widehat{\beta}_{i/j}$ by

$$\widehat{\beta}'_{i/j} = \widehat{v}^i_2/ipv^j_1 \quad \text{and} \quad \widehat{\beta}_{i/j} = \widehat{v}^i_2/pv^j_1,$$

and the $A(m+1)$-submodule B_{m+1} of $\overline{E}^1_{m+1}/(v_1^\infty)$ by

$$B_{m+1} = A(m+1)\left\{\widehat{\beta}'_{i/i} \colon i > 0\right\},$$

which turns out to be a $G(m+1)$-comodule and is invariant over $\Gamma(m+2)$ (cf. [**6**]). The following is proved in [**5**]:

Proposition 2.3. *Define the subcomodule W_{m+1} of $v_1^{-1}\overline{E}^1_{m+1}$ as an extension of B_{m+1} by $\overline{E}^1_{m+1}$:*

$$\begin{tikzcd} 0 \arrow[r] & \overline{E}^1_{m+1} \arrow[r, tail, "f"] \arrow[d, equal] & W_{m+1} \arrow[r, two heads, "\rho"] \arrow[d, tail] & B_{m+1} \arrow[r] \arrow[d, tail] & 0 \\ 0 \arrow[r] & \overline{E}^1_{m+1} \arrow[r, tail] & v_1^{-1}\overline{E}^1_{m+1} \arrow[r, two heads] & \overline{E}^1_{m+1}/(v_1^\infty) \arrow[r] & 0. \end{tikzcd} \tag{3}$$

Then W_{m+1} is weak injective over $G(m+1)$ with

$$\mathrm{Ext}^0_{G(m+1)}(W_{m+1}) = \mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$$

i.e., the map f induces an isomorphism in Ext^0. $\square$

Remark 2.4. The case for $m=0$ has been essentially treated in [**3**], where it is shown that there exists a weak injective comodule D^1_1 and that we may set $W_{m+1} = \mathrm{Ext}^0_{\Gamma(2)}(D^1_1)$ for $p>2$ (cf. [**3**] Lemma 7.2.1, (7.2.17) and Lemma 7.2.19).

Let $C^{*,s}(T^{(j)}_m)$ denote the cochain complex obtained by applying the functor $\mathrm{Ext}^s_{G(m+1)}(\overline{T}^{(j)}_m \otimes -)$ to the sequence

$$\overline{D}^0_{m+1} \xrightarrow{f\circ\varphi} W_{m+1} \xrightarrow{\rho} B_{m+1}$$

and $H^{*,s}(T^{(j)}_m)$ the associated cohomology group. The following is proved in [**5**].

Proposition 2.5. *For $i = 0$ and 1, the cohomology group $H^{i,0}(T_m^{(j)})$ is isomorphic to* $\mathrm{Ext}^i_{\Gamma(m+1)}(T_m^{(j)})$, *and* $H^{2,0}(T_m^{(j)}) = \mathrm{Ext}^1_{G(m+1)}(\overline{T}_m^{(j)} \otimes \overline{E}^1_{m+1})$. □

This allows us to compute the 0-th and the 1-st line of (1) in terms of the cochain complex $C^{*,s}(T_m^{(j)})$. Note that the case for $j = 0$ is easy: By definition we have $T_m^{(0)} = A(m+1)$ and

$$
\begin{aligned}
C^{0,0}(T_m^{(0)}) &= \mathrm{Ext}^0_{G(m+1)}(\overline{D}^0_{m+1}) \;=\; A(m),\\
C^{1,0}(T_m^{(0)}) &= \mathrm{Ext}^0_{G(m+1)}(W_{m+1}) \;=\; A(m)\left\{\frac{\widehat{v}_1^{\ell}}{\ell p} \colon \ell > 0\right\} \quad \text{for } (p,m) \neq (2,0),\\
C^{0,s}(T_m^{(0)}) &= C^{1,s}(T_m^{(0)}) \;=\; 0 \quad \text{for } s > 0\\
\text{and} \quad C^{2,s}(T_m^{(0)}) &= \mathrm{Ext}^s_{G(m+1)}(B_{m+1}) \quad \text{for } s \geqslant 0.
\end{aligned}
$$

All differentials in the cochain complex are trivial and so $H^{i,s}(T_m^{(0)}) = C^{i,s}(T_m^{(0)})$ for all i and s, which gives

$$
H^{0,0}(T_m^{(0)}) = A(m) \quad \text{and} \quad H^{1,0}(T_m^{(0)}) = A(m)\left\{\frac{\widehat{v}_1^{\ell}}{\ell p} \colon \ell > 0\right\}. \tag{4}
$$

In particular, when $m = 0$, it is isomorphic to the Adams-Novikov E_2-terms for the sphere spectrum and thus the old result for the image J is recovered.

3. Structure of the cochain complex $C^{*,0}(T_m^{(j)})$

Hereafter we assume that $m > 0$. Given a $G(m+1)$-comodule M and the structure map $\psi_M \colon M \to G(m+1) \otimes M$, define Quillen operations $\widehat{r}_n \colon M \to M$ $(n \geqslant 0)$ on each element $x \in M$ by $\psi_M(x) = \sum_n \widehat{t}_1^n \otimes \widehat{r}_n(x)$. It is easy to see that

Lemma 3.1. *For $x, y \in M$, we have the following relations:*

1. $\widehat{r}_i \widehat{r}_j(x) = \binom{i+j}{i} \widehat{r}_{i+j}(x)$,
2. *(Cartan formula)* $\widehat{r}_n(xy) = \displaystyle\sum_{i+j=n} \widehat{r}_i(x)\widehat{r}_j(y)$.

Proof. The first relation follows from the coassociativity of the comodule M: Observe that

$$
(\Delta \otimes 1)\left(\sum_{n \geqslant 0} \widehat{t}_1^n \otimes \widehat{r}_n(x)\right) = (1 \otimes \psi_M)\left(\sum_{j \geqslant 0} \widehat{t}_1^j \otimes \widehat{r}_j(x)\right)
$$

$$
\text{and so} \quad \sum_{0 \leqslant i \leqslant n} \binom{n}{i} \widehat{t}_1^{n-i} \otimes \widehat{t}_1^i \otimes \widehat{r}_n(x) = \sum_{i,j \geqslant 0} \widehat{t}_1^j \otimes \widehat{t}_1^i \otimes \widehat{r}_i(\widehat{r}_j(x))
$$

where Δ is the coproduct map of $G(m+1)$. The second one follows from the equality

$\psi_M(xy) = \psi_M(x)\psi_M(y)$: Observe that

$$\begin{aligned}\sum_{n\geqslant 0} \widehat{t}_1^n \otimes \widehat{r}_n(xy) &= \left(\sum_{i\geqslant 0} \widehat{t}_1^i \otimes \widehat{r}_i(x)\right)\left(\sum_{j\geqslant 0} \widehat{t}_1^j \otimes \widehat{r}_j(y)\right)\\ &= \sum_{0\leqslant i\leqslant n} \widehat{t}_1^n \otimes \widehat{r}_i(x)\widehat{r}_{n-i}(y). \qquad \square\end{aligned}$$

We will compute Quillen operations on some BP_*-based comodules, whose structure maps are given by the right unit on Hazewinkel generators.

Lemma 3.2. *The right unit $\eta_R \colon BP_* \to \Gamma(m+1)$ on Hazewinkel generators v_n ($n \leqslant m+2$) are given by*

$$\begin{aligned}&\eta_R(v_k) = v_k \quad \textit{for } 0 \leqslant k \leqslant m,\\ &\eta_R(\widehat{v}_1) = \widehat{v}_1 + p\widehat{t}_1,\\ &\eta_R(\widehat{v}_2) \equiv \widehat{v}_2 + v_1\widehat{t}_1^p - v_1^{p^{m+1}}\widehat{t}_1 \qquad \bmod (p).\end{aligned}$$

This gives $\widehat{r}_\ell(\widehat{v}_1^n) = p^\ell\binom{n}{\ell}\widehat{v}_1^{n-\ell}$. By definition we also have $\eta_R(\widehat{\lambda}_1) = \widehat{\lambda}_1 + \widehat{t}_1$.

Proof. The equations of η_R follow from recursive calculations using relations [**4**] (1.1) and (1.3). Then other statements are obvious. $\square$

To obtain generators of $\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(j)} \otimes M)$, it is indeed enough to consider a certain subgroup of M.

Proposition 3.3. *Denote the subgroup $\bigcap_{n\geqslant p^j} \ker \widehat{r}_n$ of M by $L_j(M)$, and assume that M is weak injective over $G(m+1)$. If there is an element $z_{\lambda,p^j-1} \in M$ for each $A(m)$-module generator $z_{\lambda,0} \in \mathrm{Ext}^0_{G(m+1)}(M)$ satisfying $\widehat{r}_{p^j-1}(z_{\lambda,p^j-1}) = z_{\lambda,0}$, then $\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(j)} \otimes M)$ is isomorphic as an $A(m)$-module to $L_j(M)$ which is spanned by elements $z_{\lambda,k}$ ($0 \leqslant k < p^j$) defined by*

$$z_{\lambda,k} = \binom{p^j-1}{k}^{-1} \widehat{r}_{p^j-1-k}(z_{\lambda,p^j-1}). \tag{5}$$

Proof. By [**2**] Lemmas 1.12 and 1.14, we have isomorphisms

$$\widetilde{T}_m^{(j)} \otimes \mathrm{Ext}^0_{G(m+1)}(M) \xrightarrow[\cong]{\theta} \mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(j)} \otimes M) \xleftarrow[\cong]{(c\otimes 1)\psi_M} L_j(M)$$

where $\widetilde{T}_m^{(j)} = A(m) \otimes_{A(m+1)} \overline{T}_m^{(j)}$ and the map $c \colon G(m+1) \to G(m+1)$ is the conjugation. The left map θ is the convergence of spectral sequence based on the skeletal filtration of $T_m^{(j)}$ and $\theta(\widehat{t}_1^k \otimes z_{\lambda,0})$ corresponds to $(c\otimes 1)\psi_M(z_{\lambda,k})$ by solving the extension problem.

The elements $z_{\lambda,k}$ (5) are in $L_j(M)$ since

$$\begin{aligned}
\psi_M(z_{\lambda,k}) &= \sum_{\ell\geqslant 0} \widehat{t}_1^{\ell} \otimes \widehat{r}_\ell(z_{\lambda,k}) \\
&= \sum_{\ell\geqslant 0} \binom{p^j-1}{k}^{-1} \widehat{t}_1^{\ell} \otimes \widehat{r}_\ell\left(\widehat{r}_{p^j-1-k}(z_{\lambda,p^j-1})\right) \\
&= \sum_{\ell\geqslant 0} \binom{p^j-1}{k}^{-1} \widehat{t}_1^{\ell} \otimes \binom{p^j-1-k+\ell}{\ell} \widehat{r}_{p^j-1-k+\ell}(z_{\lambda,p^j-1}) \\
&= \sum_{\ell\geqslant 0} \binom{k}{\ell} \widehat{t}_1^{\ell} \otimes \binom{p^j-1}{k-\ell}^{-1} \widehat{r}_{p^j-1-k+\ell}(z_{\lambda,p^j-1}) \\
&= \sum_{\ell\geqslant 0} \binom{k}{\ell} \widehat{t}_1^{\ell} \otimes z_{\lambda,k-\ell},
\end{aligned} \tag{6}$$

and so we can choose $z_{\lambda,k}$ $(0 \leqslant k < p^j)$ as generators of $L_j(M)$ as desired. □

Proposition 3.3 allows us to compute the cohomology group $H^{i,0}(T_m^{(j)})$ for $i \leqslant 1$ using the following commutative diagram of $A(m)$-modules:

$$\begin{tikzcd}
C^{0,0}(T_m^{(j)}) \arrow[r, "d^0"] & C^{1,0}(T_m^{(j)}) \arrow[r, "d^1"] & C^{2,0}(T_m^{(j)}) \\
L_j\left(\overline{D}^0_{m+1}\right) \arrow[u, "(c\otimes 1)\psi_{\overline{D}^0_{m+1}}", "\cong"'] \arrow[r, "d^0"] & L_j(W_{m+1}) \arrow[u, "(c\otimes 1)\psi_{W_{m+1}}", "\cong"'] \arrow[r, "d^1"] & L_j(B_{m+1}). \arrow[u, hook, "(c\otimes 1)\psi_{B_{m+1}}"]
\end{tikzcd} \tag{7}$$

Note that the explicit structure of $L_j(B_{m+1})$ is not needed here since it is a subgroup of $BP_*/(p^\infty, v_1^\infty)$ and we can judge the triviality of the d^1-image there.

The first cochain group $C^{0,0}(T_m^{(j)})$ is described as follows:

Corollary 3.4. $C^{0,0}(T_m^{(j)}) = A(m)\left\{\widehat{\lambda}_1^{\ell} \colon 0 \leqslant \ell < p^j\right\}$.

Proof. If we put $M = \overline{D}^0_{m+1}$ in Proposition 3.3, then we have

$$\mathrm{Ext}^0_{G(m+1)}(M) = \mathrm{Ext}^0_{G(m+1)}(A(m)[\widehat{\lambda}_1]) = A(m).$$

So we have only one $A(m)$-module generator, namely $z_0 = 1$, and we may set $z_\ell = \widehat{\lambda}_1^{\ell}$ for $0 \leqslant \ell < p^j$ which clearly satisfies the formula (6) by Lemma 3.2. □

In order to describe $C^{1,0}(T_m^{(j)})$, we put $M = W_{m+1}$. Corresponding to each $A(m)$-module generator $\widehat{v}_1^i/ip$ of $\mathrm{Ext}^0_{G(m+1)}(W_{m+1})$, we need to construct elements $\widehat{\alpha}_{i,p^j-1}$ of $L_j(W_{m+1})$ satisfying the condition of Proposition 3.3.

Lemma 3.5. *For all primes p and integers $i > 0$ and $j \geqslant 0$, define*

$$\widehat{\alpha}_{i,p^j-1} = \frac{\widehat{v}_1^{i-1}}{p^j}\left(c_{i,j}\widehat{\lambda}_1^{p^j} + \mu^{p^{j-1}} - (v_1^{-1}\widehat{v}_2)^{p^{j-1}}\right)$$

$$\text{where } c_{i,j} = \frac{1}{\binom{i+p^j-1}{p^j}} - 1 \quad \text{and} \quad \mu = (1-p^{p-1})\widehat{\lambda}_1^p - v_1^{p^{m+1}-1}\widehat{\lambda}_1.$$

Then it is in $v_1^{-1}\overline{E}^1_{m+1}$ and it satisfies

$$\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1}) = 0$$

$$\text{and} \quad \widehat{r}_{p^j-1}(\widehat{\alpha}_{i,p^j-1}) \equiv \widehat{v}_1^i/ip \mod (v_1) \quad \text{up to the unit scalar.}$$

In particular, we have $\widehat{\alpha}_{i,0} \equiv \widehat{v}_1^i/ip$ mod (v_1) up to unit scalar.

Proof. It is $\Gamma(m+2)$-invariant since the possible $\widehat{t}_2$-multiples in the comodule expansion arise from the term $-\widehat{v}_1^{i-1}(v_1^{-1}\widehat{v}_2)^{p^{j-1}}/p^j$, which is in fact $\Gamma(m+2)$-invariant.

By (2) we have $v_1^{-1}\widehat{v}_2 = pv_1^{-1}\widehat{\lambda}_2 + \mu$ and so $\mu^{p^{j-1}} - (v_1^{-1}\widehat{v}_2)^{p^{j-1}}$ is divisible by p^j in $v_1^{-1}\overline{E}^1_{m+1}$. The first term is also in $v_1^{-1}\overline{E}^1_{m+1}$: For $i = 1$ it is trivial since $c_{1,j} = 0$. For $i > 1$ it is expressed as

$$\frac{(p\widehat{\lambda}_1)^{i-1}}{p^j}\left(\frac{1}{\binom{i+p^j-1}{p^j}} - 1\right)\widehat{\lambda}_1^{p^j} = \frac{p^{i-1}}{p^j}\cdot\frac{(i-1)! - c(i,j)}{c(i,j)}\widehat{\lambda}_1^{p^j+i-1}$$

where $c(i,j) = \prod_{1\leqslant k\leqslant i-1}(p^j+k)$ and the coefficient is in $\mathbf{Z}_{(p)}$, by the congruences

$$(i-1)! - c(i,j) \;\equiv\; 0 \mod (p^j) \quad \text{and} \quad c(i,j) \;\not\equiv\; 0 \mod (p^{i-1}).$$

For Quillen operations, note that we can ignore elements belonging to BP_* in $v_1^{-1}\overline{E}_{m+1}$ since $E^1_{m+1} = D^0_{m+1}/BP_*$ by definition. For example, we can ignore $\widehat{\lambda}_1$ when it is multiplied by p since $p\widehat{\lambda}_1 = \widehat{v}_1 \in BP_*$.

When $i \leqslant p$, it is enough to do calculations mod $(v_1^{p^{m+1}-1})$ by degree reason: We have

$$|\widehat{\alpha}_{i,p^j-1}| = (i-1+p^j)|\widehat{\lambda}_1| \;=\; 2(i-1+p^j)(p^{m+1}-1),$$

$$|v_1^{p^{m+1}-1}| = 2(p-1)(p^{m+1}-1),$$

$$\text{and} \quad |\widehat{t}_1^{p^j}| = 2p^j(p^{m+1}-1).$$

Since $|\widehat{\alpha}_{i,p^j-1}| - |\widehat{t}_1^{p^j}|$ is less than or equal to $|v_1^{p^{m+1}-1}|$ and there is not a $v_1^{p^{m+1}-1}\widehat{t}_1^{p^j}$ in $\eta_R(\widehat{\alpha}_{i,p^j-1})$, we lose no information even if we calculate $\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1})$ mod $(v_1^{p^{m+1}-1})$.

Observe that

$$\begin{aligned}\widehat{\alpha}_{i,p^j-1} &\equiv \frac{\widehat{v}_1^{i-1}}{p^j}\left(c_{i,j}\widehat{\lambda}_1^{p^j} + ((1-p^{p-1})\widehat{\lambda}_1^p)^{p^{j-1}} - (v_1^{-1}\widehat{v}_2)^{p^{j-1}}\right)\\ &= \frac{\widehat{v}_1^{i-1}}{p^j}\left(\bar{c}_{i,j}\widehat{\lambda}_1^{p^j} - (v_1^{-1}\widehat{v}_2)^{p^{j-1}}\right) \quad \text{where } \bar{c}_{i,j} = (c_{i,j} + (1-p^{p-1})^{p^{j-1}})\\ &= \bar{c}_{i,j}\frac{\widehat{v}_1^{i-1+p^j}}{p^{j+p^j}} - \frac{\widehat{v}_1^{i-1}(v_1^{-1}\widehat{v}_2)^{p^{j-1}}}{p^j},\end{aligned}$$

which gives

$$\begin{aligned}\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1}) &= \binom{i+p^j-1}{p^j}\bar{c}_{i,j}\frac{\widehat{v}_1^{i-1}}{p^j} - \frac{\widehat{v}_1^{i-1}}{p^j} \\ &= \binom{i+p^j-1}{p^j}(c_{i,j}+1)\frac{\widehat{v}_1^{i-1}}{p^j} - \frac{\widehat{v}_1^{i-1}}{p^j} \\ &= \frac{\widehat{v}_1^{i-1}}{p^j} - \frac{\widehat{v}_1^{i-1}}{p^j} \;=\; 0.\end{aligned}$$

When $i > p$, each $v_1^{p^{m+1}-1}$-multiple in $\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1})$ has higher p-exponent in the coefficient since its $\widehat{t}_1$-exponent almost comes from $\eta_R(\widehat{v}_1^{i-1}) = (\widehat{v}_1 + p\widehat{t}_1)^{i-1}$. Consequently, all elements in $\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1})$ in fact belong to BP_* and are forgettable.

Similarly we observe

$$\eta_R(\widehat{\alpha}_{i,p^j-1}) \equiv \bar{c}_{i,j}\frac{(\widehat{v}_1 + p\widehat{t}_1)^{i-1+p^j}}{p^{j+p^j}} - \frac{(\widehat{v}_1 + p\widehat{t}_1)^{i-1}(v_1^{-1}\widehat{v}_2 + \widehat{t}_1^{p})^{p^{j-1}}}{p^j}$$

mod (v_1) and so

$$\widehat{r}_{p^j-1}(\widehat{\alpha}_{i,p^j-1}) \;\equiv\; \bar{c}_{i,j}\frac{\binom{i-1+p^j}{i}\widehat{v}_1^{i}}{p^{j+1}} \;\equiv\; \bar{c}_{i,j}\frac{\widehat{v}_1^{i}}{ip}. \qquad \square$$

Lemma 3.6. *The image of* $\widehat{\alpha}_{i,p^j-1} \in v_1^{-1}\overline{E}_{m+1}^1$ *under the differential* d^1 (7) *is* $-\widehat{v}_1^{i-1}\widehat{\beta}'_{p^{j-1}/p^{j-1}} \in B_{m+1}$, *and the elements* $\widehat{\alpha}_{i,k}$ $(0 \leqslant k < p^j)$ *defined in the same way as* (5) *for each* i *are in* $L_j(W_{m+1})$.

Proof. The first statement follows easily from the definition of $\widehat{\alpha}_{i,p^j-1}$. We can conclude that $\widehat{\alpha}_{i,k} \in W_{m+1}$ for $0 \leqslant k < p^j - 1$ by (3), (6) and Lemma 3.5. $\square$

By Proposition 3.3 and Lemma 3.6, we have:

Proposition 3.7. $C^{1,0}(T_m^{(j)}) = A(m)\left\{\widehat{\alpha}_{i,k} \colon i > 0 \text{ } \textit{and} \text{ } 0 \leqslant k < p^j\right\}$. $\square$

4. Computing differentials in the cochain complex

Based on the results in the previous section, we have

Theorem 4.1. *For any prime* p, *the 0-th line of the Adams-Novikov* E_2*-term for* $T(m)_{(j)}$ *is isomorphic to the* $A(m)$*-module generated by* $\widehat{v}_1^k$ $(0 \leqslant k < p^j)$.

Proof. The d^0-image of $\widehat{\lambda}_1^k \in C^{0,0}(T_m^{(j)})$ $(0 \leqslant k < p^j)$ is $\widehat{v}_1^k/p^k$, which is annihilated by p^k but not by p^{k-1}. So we have

$$H^{0,0}(T_m^{(j)}) \;=\; A(m)\left\{p^k\widehat{\lambda}_1^k \colon 0 \leqslant k < p^j\right\} \;=\; A(m)\left\{\widehat{v}_1^k \colon 0 \leqslant k < p^j\right\}$$

as claimed. $\square$

To detect elements in $\ker d^1$, we do not need to determine the explicit form of $\widehat{\alpha}_{i,k}$: By the commutativity of the diagram (7) and Lemma 3.6 we have an equality

$$d^1(\widehat{\alpha}_{i,k}) = -\binom{p^j-1}{k}^{-1} \widehat{r}_{p^j-1-k}\left(\widehat{v}_1^{i-1}\widehat{\beta}'_{p^{j-1}/p^{j-1}}\right), \tag{8}$$

i.e., the only term relevant to this calculation is the last term of $\widehat{\alpha}_{i,p^j-1}$ (Lemma 3.5) and we can ignore any other terms. More precisely, we have:

Lemma 4.2. *Assume that $j \leqslant m+2$. Then we have*

$$d^1(\widehat{\alpha}_{i,k}) = -\binom{p^j-1}{k}^{-1} \sum_{a+bp=p^j-1-k} p^a \binom{i-1}{a}\binom{p^{j-1}-1}{b} \widehat{v}_1^{i-1-a}\widehat{\beta}'_{p^{j-1}-b/p^{j-1}-b}.$$

Proof. Since $j \leqslant m+2$, we have $\eta_R(\widehat{v}_2) \equiv \widehat{v}_2 + v_1\widehat{t}_1^p \bmod (v_1^{p^{j-1}})$ and

$$\psi\left(\frac{\widehat{v}_2^{p^{j-1}}}{p^j v_1^{p^{j-1}}}\right) = \frac{(\widehat{v}_2 + v_1\widehat{t}_1^p)^{p^{j-1}}}{p^j v_1^{p^{j-1}}} = \sum_{0<\ell\leqslant p^{j-1}} \binom{p^{j-1}-1}{\ell-1}\frac{\widehat{v}_2^{\ell}\widehat{t}_1^{p(p^{j-1}-\ell)}}{\ell p v_1^{\ell}}$$

which gives that $\widehat{r}_n(\widehat{\beta}'_{p^{j-1}/p^{j-1}}) = 0$ for $n \not\equiv 0 \bmod (p)$ and

$$\widehat{r}_{p(p^{j-1}-\ell)}(\widehat{\beta}'_{p^{j-1}/p^{j-1}}) = \binom{p^{j-1}-1}{\ell-1}\widehat{\beta}'_{\ell/\ell} \qquad \text{for } 0<\ell\leqslant p^{j-1}. \tag{9}$$

Then the right hand side of the equality (8) is modified using Lemma 3.1, 3.2 and the formula (9). □

Corollary 4.3. *$H^{2,0}(T_m^{(j)})$ is isomorphic to the quotient of*

$$C^{2,0}(T_m^{(j)}) = \operatorname{Ext}^0_{G(m+1)}(\overline{T}_m^{(j)} \otimes B_{m+1})$$

by $A(m+1)$-module generated by $(c\otimes 1)\psi_{B_{m+1}}(\widehat{\beta}'_{i/i})$ $(0<i\leqslant p^{j-1})$.

Proof. By Lemma 3.6 and the formula (9) the d^1-image (7) of $L_j(W_{m+1})$ is an $A(m+1)$-module spanned by $\widehat{\beta}'_{\ell/\ell}$ $(0<\ell\leqslant p^{j-1})$. □

Proposition 4.4. *When $j=1$, we have*

$$d^1(\widehat{\alpha}_{i,k}) = \begin{cases} -\widehat{v}_1^{i-1}\widehat{\beta}_{1/1} & \textit{for } k=p-1,\\ 0 & \textit{for } 0\leqslant k<p-1.\end{cases}$$

When $j=2$, we have

$$d^1(\widehat{\alpha}_{i,k}) = \begin{cases} (-1)^{p+1-\ell}\dfrac{\binom{p}{\ell}}{p}\widehat{v}_1^{i-1}\widehat{\beta}_{\ell/\ell} & \textit{for } k=p\ell-1 \textit{ with } 0<\ell\leqslant p,\\ -(i-1)\widehat{v}_1^{i-2}\widehat{\beta}_{p/p} & \textit{for } k=p^2-2,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Whenever $j \leqslant 2$, we can use Lemma 4.2 with no restriction on m. The $j = 1$ case is easy. For $j = 2$ we have

$$d^1(\widehat{\alpha}_{i,(p-b)p-1}) = -\binom{p^2-1}{bp}^{-1} \frac{\binom{p}{b}}{p} \widehat{v}_1^{i-1} \widehat{\beta}_{p-b/p-b} \quad \text{for } 0 \leqslant b < p.$$

Note that $\binom{p^2-1}{bp} \equiv (-1)^b \mod (p)$. □

By Corollary 3.4 we have

$$d^0\left(C^{0,0}(T_m^{(j)})\right) = A(m)\left\{\frac{\widehat{v}_1^{\ell}}{p^{\ell}} : 0 \leqslant \ell < p^j\right\} \tag{10}$$

and so a cocycle $\widehat{\alpha}_{i,k} \in \ker d^1$ satisfying $i + k < p^j$ is in $\operatorname{Im} d^0$ if and only if it has no $\widehat{v}_2$-multiple. We have:

Proposition 4.5. *When $j = 1$, $\widehat{\alpha}_{i,k}$ does not have any non-trivial $\widehat{v}_2$-multiple except for $k = p - 1$. When $j = 2$, $\widehat{\alpha}_{i,k}$ has a non-trivial $\widehat{v}_2$-multiple if and only if $n(p-1) \leqslant k < np$ with $0 < n \leqslant p$, whose order is p for $k \neq p^2 - 1$ and p^2 for $k = p^2 - 1$.*

Proof. The statement for $j = 1$ is obvious since $\widehat{r}_{p-1-k}(\widehat{v}_1^{i-1} v_1^{-1} \widehat{v}_2/p)$ does not have any $\widehat{v}_2$-multiple except for $k = p - 1$. For $j = 2$, the comodule expansion of $-\widehat{v}_1^{i-1}(v_1^{-1}\widehat{v}_2)^p/p^2$ is

$$-(i-1)\frac{v_1^{-p}\widehat{v}_1^{i-2}\widehat{v}_2^{p}\widehat{t}_1}{p} - \frac{v_1^{-p}\widehat{v}_1^{i-1}}{p^2} \sum_{0 \leqslant n \leqslant \ell < p} (-1)^n \binom{p}{\ell}\binom{\ell}{n} v_1^{\ell+(p^{m+1}-1)n} \widehat{v}_2^{p-\ell} \widehat{t}_1^{p(\ell-n)+n}.$$

Replacing $\ell - n$ with t $(0 \leqslant t < p)$ and assuming that the exponent of $\widehat{v}_2$ is positive (i.e., $0 \leqslant n < p - t$), we have an inequality

$$pt \leqslant (\text{the exponent of } \widehat{t}_1) < pt + p - t.$$

This implies that the coefficient of $\widehat{t}_1^{p^2-1-k}$ (i.e., $\widehat{\alpha}_{i,k}$ up to scalar by unit) has a $\widehat{v}_2$-multiple of order p if and only if $(p-t)(p-1) \leqslant k < (p-t)p$. □

5. Structure of the cohomology $H^{1,0}(T_m^{(j)})$

In this section we finally determine the structure of $H^{1,0}(T_m^{(j)})$ for $j = 1$ and 2 using some tools arranged in the previous section.

First we take care of the possibility of linear relations among cocycles obtained by Proposition 4.4 since the leading term of each $\widehat{\alpha}_{n-\ell,\ell}$ is always a scalar multiple of $\widehat{\lambda}_1^n$. In fact, as the simplest case, we have

Proposition 5.1. *In $C^{1,0}(T_m^{(1)})$, there are relations*

$$\frac{ip}{k}\widehat{\alpha}_{i,k} = \widehat{\alpha}_{i+1,k-1} \quad \textit{for } 0 < k \leqslant p - 1. \tag{11}$$

Proof. Definition of $\widehat{\alpha}_{i,p-1}$ (Lemma 3.5) and direct calculations show that

$$\widehat{\alpha}_{i,k} = \begin{cases} \dfrac{p^{i-1}}{i\binom{i+p-2}{i}}\widehat{\lambda}_1^{i+p-2} + (i-1)\widehat{v}_1^{i-2}(v_1^{p^{m+1}-1}\widehat{\lambda}_1 + p^{p-1}\widehat{\lambda}_1^p) & \text{for } k = p-2, \\ \dfrac{p^{i-1}}{i\binom{i+k}{i}}\widehat{\lambda}_1^{i+k} & \text{for } 0 < k \leqslant p-3 \end{cases}$$

where the second term for $k = p-2$ has the order p. $\square$

Consequently, we have the following result for $H^{1,0}(T_m^{(1)})$.

Theorem 5.2. *For any prime p, the first line of the Adams-Novikov E_2-term for $T(m)_{(1)}$ is isomorphic to the $A(m)$-submodule of W_{m+1} generated by $p\widehat{\alpha}_{i,p-1}$ and $v_1\widehat{\alpha}_{i,p-1}$ for $i \geqslant 1$.*

Proof. By Proposition 4.4 the kernel of d^1 (7) consists of

$$\widehat{\alpha}_{i,k}\ (0 \leqslant k < p-1) \qquad p\widehat{\alpha}_{i,p-1} \qquad \text{and} \qquad v_1\widehat{\alpha}_{i,p-1}.$$

Proposition 5.1 tells us that the elements $\widehat{\alpha}_{i,k}$ with $k < p-1$ are not needed. $\square$

The situation is more complicated for $j = 2$. First we consider the most accessible case, $p = 2$. By Lemma 4.2 we find that

$$d^1(\widehat{\alpha}_{i,3}) = -\frac{\widehat{v}_1^{i-1}\widehat{v}_2^2}{4v_1^2}, \quad d^1(\widehat{\alpha}_{i,2}) = -\frac{(i-1)\widehat{v}_1^{i-2}\widehat{v}_2^2}{6v_1^2}, \quad d^1(\widehat{\alpha}_{i,1}) = -\frac{\widehat{v}_1^{i-1}\widehat{v}_2}{6v_1}$$

and $d^1(\widehat{\alpha}_{i,0}) = 0$. Thus we have cocycles

$$\begin{array}{rl} \widehat{\alpha}_{i,2} & \text{for odd } i, \\ \widehat{\alpha}_{i,0}, \quad 4\widehat{\alpha}_{i,3}, \quad 2\widehat{\alpha}_{i,2}, \quad 2\widehat{\alpha}_{i,1}, \quad 2i\widehat{\alpha}_{i,3} - 3\widehat{\alpha}_{i+1,2} & \text{for all } i \geqslant 1, \\ v_1^2\widehat{\alpha}_{i,3}, \quad v_1^2\widehat{\alpha}_{i,2}, \quad v_1\widehat{\alpha}_{i,1} & \text{for all } i \geqslant 1. \end{array} \tag{12}$$

For linear relations among these cocycles we have

Proposition 5.3. *For $(p, j) = (2, 2)$ we have the following relations:*

1. *$\widehat{\alpha}_{i,0}$ is a scalar multiple of $\widehat{\alpha}_{i-1,1}$ if $2 \leqslant i \equiv 1, 2$ mod (4), $\widehat{\alpha}_{i-2,2}$ if $3 \leqslant i \equiv 1$ mod (2), and $\widehat{\alpha}_{i-3,3}$ if $4 \leqslant i \equiv 0, 1$ mod (4),*
2. *$2\widehat{\alpha}_{i,1}$ is a scalar multiple of $\widehat{\alpha}_{i-1,2}$ if $3 \leqslant i \equiv 3$ mod (4), and $\widehat{\alpha}_{i-2,3}$ if $4 \leqslant i \equiv 0$ mod (2).*
3. *$8\widehat{\alpha}_{i,2}$ is a scalar multiple of $\widehat{\alpha}_{i-1,3}$ for all $i \geqslant 2$.*

Proof. Tedious routine calculations give the following:

$$\begin{aligned}
\widehat{\alpha}_{i,3} &= \frac{\widehat{v}_1^{i+3}}{16i\binom{i+3}{3}} + \frac{v_1^{2^{m+1}-1}\widehat{v}_1^{i+2}}{16} + \frac{v_1^{2^{m+2}-2}\widehat{v}_1^{i+1}}{16} - \frac{v_1^{-2}\widehat{v}_1^{i-1}\widehat{v}_2^2}{4},\\
\widehat{\alpha}_{i,2} &= \frac{\widehat{v}_1^{i+2}}{24\binom{i+2}{3}} + \frac{(i+2)v_1^{2^{m+1}-1}\widehat{v}_1^{i+1}}{24} + \frac{(i+1)v_1^{2^{m+2}-2}\widehat{v}_1^{i}}{24}\\
&\quad - \frac{(i-1)v_1^{-2}\widehat{v}_1^{i-2}\widehat{v}_2^2 - v_1^{2^{m+1}-2}\widehat{v}_1^{i-1}\widehat{v}_2}{6},\\
\widehat{\alpha}_{i,1} &= \frac{\widehat{v}_1^{i+1}}{8\binom{i+1}{2}} + \frac{\binom{i+2}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i}}{12} + \frac{\binom{i+1}{2}-1}{12}v_1^{2^{m+2}-2}\widehat{v}_1^{i-1} + \frac{v_1^{-1}\widehat{v}_1^{i-1}\widehat{v}_2}{6},\\
\widehat{\alpha}_{i,0} &= \frac{\widehat{v}_1^{i}}{2i} + \frac{\binom{i+2}{3}-1}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1} + \frac{\binom{i+1}{3}-(i-1)}{2}v_1^{2^{m+2}-2}\widehat{v}_1^{i-2},
\end{aligned}$$

and we consequently obtain the following relations in W_{m+1}:

$$\begin{aligned}
\widehat{\alpha}_{i,0} - 2(i-1)\widehat{\alpha}_{i-1,1} &= \frac{(i-1)(i+2)}{4}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1},\\
\widehat{\alpha}_{i,0} - 2(i-1)(i-2)\widehat{\alpha}_{i-2,2} &= \frac{i^2-1}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1},\\
\widehat{\alpha}_{i,0} - \frac{4}{3}(i-1)(i-2)(i-3)\widehat{\alpha}_{i-3,3} &= \frac{3i(i-1)}{4}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1},\\
2(\widehat{\alpha}_{i,1} - (i-1)\widehat{\alpha}_{i-1,2}) &= \frac{i+1}{4}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1} + \frac{i-1}{6}v_1^{2^{m+2}-2}\widehat{v}_1^{i-2},\\
2\left(\widehat{\alpha}_{i,1} - \frac{2}{3}(i-1)(i-2)\widehat{\alpha}_{i-2,3}\right) &= \frac{i}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1},\\
4\left(\widehat{\alpha}_{i,2} - \frac{2}{3}(i-1)\widehat{\alpha}_{i-1,3}\right) &= \frac{1}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1}.
\end{aligned}$$

□

By these observations we have

Theorem 5.4. *For $p = 2$, $H^{1,0}(T_m^{(2)})$ is isomorphic to the $A(m)$-submodule of W_{m+1} generated by the elements listed in* (12) *with relations obtained in Proposition 5.3.* □

As showed in Proposition 5.3, calculation experience tells us that unfortunately it does not seem likely that there will be brief formulas for the relations among cocycles $\widehat{\alpha}_{n-\ell,\ell} \in H^{1,0}(T_m^{(2)})$ for general prime p.

We can still find all cocycles of $H^{1,0}(T_m^{(2)})$ for an arbitrary prime using Proposition 4.4 and 4.5.

Proposition 5.5. *For any prime p, $H^{1,0}(T_m^{(2)})$ consists of the following cocycles:*

1. $\widehat{\alpha}_{i,k}$ *with $k \not\equiv -1$ mod (p), $k \neq p^2-2$, $i+k \geqslant p^2$ and $k < p^2$,*
2. $\widehat{\alpha}_{i,k}$ *with $i+k < p^2$ and $n(p-1) \leqslant k < np-1$ for an integer $0 < n < p$,*
3. $\widehat{\alpha}_{i,k}$ *with $i+k < p^2$ and $p(p-1) \leqslant k < p^2-2$,*
4. $\widehat{\alpha}_{i,p^2-2}$ *with $i \equiv 1$ mod (p),*

5. $p^2\widehat{\alpha}_{i,p^2-1}$,
6. $p\widehat{\alpha}_{i,p\ell-1}$ *with* $0<\ell<p$ *and* $i \geqslant p(p-\ell)+1$,
7. $p\widehat{\alpha}_{i,p^2-2}$ *with* $i \not\equiv 1$ *mod* (p),
8. $ip\widehat{\alpha}_{i,p^2-1}+\widehat{\alpha}_{i+1,p^2-2}$ *with* $i \not\equiv 0$ *mod* (p),
9. $v_1^{\ell}\widehat{\alpha}_{i,p\ell-1}$ *with* $0<\ell\leqslant p$,
10. $v_1^{p}\widehat{\alpha}_{i,p^2-2}$.

Proof. By Proposition 4.4 we have the following elements of $\ker d^1$: (1) $\widehat{\alpha}_{i,k}$ with $k \not\equiv -1 \bmod (p)$ and $k \neq p^2-2$; (2) $\widehat{\alpha}_{i,p^2-2}$ with $i \equiv 1 \bmod (p)$; (3) $p^2\widehat{\alpha}_{i,p^2-1}$; (4) $p\widehat{\alpha}_{i,p\ell-1}$ with $0<\ell<p$; (5) $p\widehat{\alpha}_{i,p^2-2}$ with $i \not\equiv 1 \bmod (p)$; and (6) $ip\widehat{\alpha}_{i,p^2-1}+\widehat{\alpha}_{i+1,p^2-2}$ with $i \not\equiv 0 \bmod (p)$; (7) $v_1^{\ell}\widehat{\alpha}_{i,p\ell-1}$ with $0<\ell\leqslant p$ and (8) $v_1^{p}\widehat{\alpha}_{i,p^2-2}$.

We need to remove elements of $\mathrm{Im}\, d^0$ using (10), and the case (1) consequently splits into the cases (1), (2) and (3).

Notice that all possible $\widehat{v}_2$-multiples are order p, which are vanished in cases (3), (4) and (5), and that elements of these cases are not in $\mathrm{Im}\, d^0$ when the sum of two subscripts is larger than p^2-1. □

Concluding remark. We are hopeful that it is still likely that all cocycles of $H^{1,0}(T_m^{(j)})$ for $j>2$ can be obtained after arranging the results similar to Proposition 4.4 and 4.5. However, we do not know how we can describe all linear relations among cocycles systematically for general value of j.

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