Homology, Homotopy and Applications, vol. 10(3), 2008, pp.151–179

NONIMMERSIONS OF RP^n IMPLIED BY tmf, REVISITED

DONALD M. DAVIS AND MARK MAHOWALD

(*communicated by J. Michael Boardman*)

Abstract

In a 2002 paper, the authors and Bruner used the new spectrum tmf to obtain some new nonimmersions of real projective spaces. In this note, we complete/correct two oversights in that paper.

The first is to note that in that paper a general nonimmersion result was stated which yielded new nonimmersions for RP^n with n as small as 48, and yet it was stated there that the first new result occurred when $n = 1536$. Here we give a simple proof of those overlooked results.

Secondly, we fill in a gap in the proof of the 2002 paper. There it was claimed that an axial map f must satisfy $f^*(X) = X_1 + X_2$. We realized recently that this is not clear. However, here we show that it is true up multiplication by a unit in the appropriate ring, and so we retrieve all the nonimmersion results claimed in the 2002 paper.

Finally, we completely determine $\mathrm{tmf}^{8*}(RP^\infty \times RP^\infty)$ and $\mathrm{tmf}^*(CP^\infty \times CP^\infty)$ in positive dimensions.

1. Introduction

In [**6**], the authors and Bruner described a proof of the following theorem, along with some additional nonimmersion results.

Theorem 1.1 ([**6**, 1.1])**.** *Assume that M is divisible by the smallest 2-power greater than or equal to h.*

- *If $\alpha(M) = 4h - 1$, then $P^{8M+8h+2}$ cannot be immersed in ($\not\subseteq$) $\mathbb{R}^{16M-8h+10}$.*
- *If $\alpha(M) = 4h - 2$, then $P^{8M+8h} \not\subseteq \mathbb{R}^{16M-8h+12}$.*

Here and throughout, $\alpha(M)$ denotes the number of 1's in the binary expansion of M, and P^n denotes real projective space.

In [**6**], the theorem is followed by a comment that this is new provided $\alpha(M) \geqslant 6$, i.e., $h \geqslant 2$, and the first new result occurs for P^{1536}. In this note, we point out

We thank Steve Wilson for causing us to take a look at these matters.
Received April 5, 2007, revised October 1, 2008; published on December 5, 2008.
2000 Mathematics Subject Classification: 57R42, 55N20.
Key words and phrases: immersion, projective space, topological modular forms.
This article is available at `http://intlpress.com/HHA/v10/n3/a8`

that 1.1 is valid when $h = 1$, and these results are new when M is even, including new nonimmersions of P^n for n as small as 56. A remark in [**6**, p. 66] that the nonimmersions when $h = 1$ were implied by earlier work of the authors was incorrect. Letting $h = 1$ in 1.1, we have the following result.

Corollary 1.2.

(a) If $\alpha(M) = 3$, then $P^{8M+10} \not\subseteq \mathbb{R}^{16M+2}$.

(b) If $\alpha(M) = 2$, then $P^{8M+8} \not\subseteq \mathbb{R}^{16M+4}$.

Part (a) is new when M is even. It is two better than the previous best result, proved in [**4**], and the nonembedding result that it implies is also new, one better than the previous best, proved in [**3**]. In [**7**], a table of known nonimmersions, immersions, nonembeddings, and embeddings of P^n is presented, arranged according to $n = 2^i + d$ with $0 \leqslant d < 2^i$ and $d < 64$. Part (a) enters the table with a new result for $d = 58$, applying first to P^{122}.

If M is even, then 1.2(b) is new, one better than the previous best result, of [**13**], and the nonembedding result implied is also new. It enters [**7**] at $d = 24$ and 40, with a new result for P^n with n as small as 56. The result of 1.2(b) with $M = 2^i + 1$ was also proved very recently by Kitchloo and Wilson in [**16**]. This result for P^{2^k+16}, two better than the previous result of [**4**] and also new as a nonembedding, enters [**7**] at $d = 16$, and applies for n as small as 48.

In Section 2, we present a self-contained proof of Corollary 1.2. The primary reason for doing this, which amounts to a reproof of part of [**6**, 1.1], is that the proof of the general case in [**6**] requires some extremely elaborate arguments and calculations. Our proof here, which is just for the case $h = 1$, is much more comprehensible.

The proof in [**6**] contained an oversight which we shall correct here. The argument there was that an immersion of P^n in $\mathbb{R}^{n+k}$ implies existence of an axial map $P^n \times P^m \xrightarrow{f} P^{m+k}$ for an appropriate value of m, and obtains a contradiction for certain n, m, and k by consideration of $\mathrm{tmf}^*(f)$. Here tmf is the spectrum of topological modular forms, which was discussed in [**6**]. A class $X \in \mathrm{tmf}^8(P^n)$ was described, along with $X_1 = X \times 1$ and $X_2 = 1 \times X$ in $\mathrm{tmf}^8(P^n \times P^m)$. It was asserted that $f^*(X) = X_1 + X_2$, and a contradiction obtained by showing that, for certain values of the parameters, we might have $X^\ell = 0$ but $(X_1 + X_2)^\ell \neq 0$. We recently realized that it is conceivable that $f^*(X)$ might contain other terms coming from $\mathrm{tmf}^8(P^n \wedge P^m)$.

In Section 3 (see Thm. 3.5), we perform a complete calculation of $\mathrm{tmf}^*(P^\infty \times P^\infty)$ in positive gradings divisible by 8, and in Section 4 we use it to show that effectively $f^*(X) = \mathbf{u}(X_1 + X_2)$, where $\mathbf{u}$ is a unit in $\mathrm{tmf}^*(P^\infty \times P^\infty)$, which enables us to retrieve all the nonimmersions of [**6**].

In Section 5, we compute $\mathrm{tmf}^*(CP^\infty \times CP^\infty)$ in positive gradings. The original purpose of doing this was, prior to our obtaining the argument of Section 4, to see whether we might mimic the argument of [**2**] and [**8**] to conclude that if f is an axial map, then $f^*(X)$ might necessarily equal $u(X_1 - X_2)$, where u is a unit in $\mathrm{tmf}^*(CP^\infty \times CP^\infty)$. This approach to retrieving the nonimmersions of [**6**] did not yield the desired result, but the later approach given in Section 4 did. Nevertheless the nice result for $\mathrm{tmf}^*(CP^\infty \times CP^\infty)$ obtained in Theorem 5.16 should be of independent interest.

2. Proof of Corollary 1.2

We begin by proving 1.2(a). The following standard reduction goes back at least to [**15**]. If $P^{8M+10} \subseteq \mathbb{R}^{16M+2}$, then $\mathrm{gd}((2^{L+3} - 8M - 11)\xi_{8M+10}) \leqslant 8M - 8$; hence this bundle has $(2^{L+3} - 16M - 3)$ linearly independent sections, and thus there is an axial map

$$P^{8M+10} \times P^{2^{L+3}-16M-4} \xrightarrow{f} P^{2^{L+3}-8M-12}.$$

The bundle here is the stable normal bundle, L is a sufficiently large integer, and gd refers to geometric dimension. Let X, X_1, and X_2 be elements of $\mathrm{tmf}^8(-)$ described in [**6**] and also in Section 1. In Section 4, we will show that we may assume that $f^*(X) = X_1 + X_2$, as was done in [**6**], since this is true up to multiplication by a unit. Since $\mathrm{tmf}^{2^{L+3}-8M-8}(P^{2^{L+3}-8M-12}) = 0$, we have

$$\begin{aligned} 0 = f^*(0) &= f^*(X^{2^L-M-1}) \\ &= (X_1 + X_2)^{2^L-M-1} \in \mathrm{tmf}^{2^{L+3}-8M-8}(P^{8M+10} \times P^{2^{L+3}-16M-4}). \end{aligned}$$

Expanding, we obtain $\binom{2^L-M-1}{M+1} X_1^{M+1} X_2^{2^L-2M-2} + \binom{2^L-M-1}{M} X_1^M X_2^{2^L-2M-1}$ as the only terms which are possibly nonzero. Next we note that, with all u's representing odd integers,

$$\binom{2^L-M-1}{M+1} = u_1 \binom{2M+1}{M+1} = 2^{\alpha(M)-\nu(M+1)} u_2 = 2^{3-\nu(M+1)} u_2,$$

where we have used $\alpha(M) = 3$ at the last step. Here and throughout, $\nu(2^e u) = e$. Similarly, $\binom{2^L-M-1}{M} = u_3 \binom{2M}{M} = 2^{\alpha(M)} u_4 = 2^3 u_4$. Thus an immersion implies that in $\mathrm{tmf}^{2^{L+3}-8M-8}(P^{8M+10} \times P^{2^{L+3}-16M-4})$, we have

$$2^{3-\nu(M+1)} u_2 X_1^{M+1} X_2^{2^L-2M-2} + 2^3 u_4 X_1^M X_2^{2^L-2M-1} = 0. \tag{1}$$

We recall [**6**, 2.6], which states that there is an equivalence of spectra

$$P_{b+8}^{k+8} \wedge \mathrm{tmf} \simeq \Sigma^8 P_b^k \wedge \mathrm{tmf}\,.$$

Combining this with duality, we obtain

$$\mathrm{tmf}^{8M+8}(P^{8M+10}) \approx \mathrm{tmf}_{-8M-9}(P_{-8M-11}^{-2}) \approx \mathrm{tmf}_{-1}(P_{-3}^{8M+6}) \approx \mathrm{tmf}_{-1}(P_{-3}) \approx \mathbb{Z}/8,$$

using [**12**, p.367] for the final isomorphism. Hence $8X_1^{M+1} X_2^{2^L-2M-2} = 0$. Here and throughout, $P_n = P_n^\infty = RP^\infty / RP^{n-1}$. Similarly,

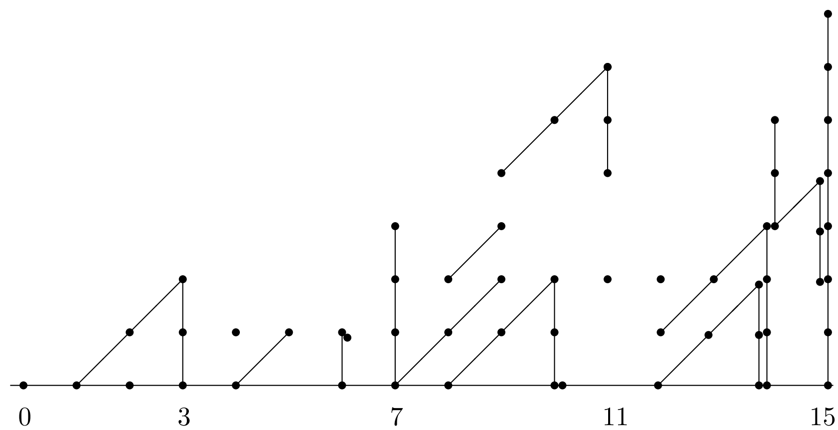
$$\mathrm{tmf}^{2^{L+3}-16M-8}(P^{2^{L+3}-16M-4}) \approx \mathrm{tmf}_7(P_3) \approx \mathbb{Z}/16,$$

and hence $16X_1^M X_2^{2^L-2M-1} = 0$. Duality also implies

$$\mathrm{tmf}^{2^{L+3}-8M-8}(P^{8M+10} \times P^{2^{L+3}-16M-4}) \approx \mathrm{tmf}_{14}(P_{-3} \wedge P_3).$$

Calculations such as $E_2(\mathrm{tmf}_*(P_{-3} \wedge P_3))$, the E_2-term of the Adams spectral sequence (ASS), were made by Bruner's minimal-resolution computer programs in our work on [**6**]. This one is in a small enough range to actually do by hand. The result is given in Diagram 2.1.

Diagram 2.1. $E_2(\mathrm{tmf}_*(P_{-3} \wedge P_3))$, $* \leqslant 15$:



The $\mathbb{Z}/8 \oplus \mathbb{Z}/16$ arising from filtration 0 in grading 14 in 2.1 is not hit by a differential from the class in $(15, 0)$ because, as explained in the last paragraph of page 54 of [**6**], the class in $(15, 0)$ corresponds to an easily-constructed nontrivial map. The monomials $X_1^{M+1} X_2^{2^L - 2M - 2}$ and $X_1^M X_2^{2^L - 2M - 1}$ are detected in mod-2 cohomology, and so their duals emanate from filtration 0. We saw in the previous paragraph that 8 and 16, respectively, annihilate these monomials, and hence also their duals. Since the chart shows that the subgroup of $\mathrm{tmf}_{14}(P_{-3} \wedge P_3)$ generated by classes of filtration 0 is $\mathbb{Z}/8 \oplus \mathbb{Z}/16$, we conclude that 8 and 16, respectively, are the precise orders of the monomials. In particular, the order of $X_1^M X_2^{2^L - 2M - 1}$ is 16, and hence the class in (1) is nonzero since it has a term $8uX_1^M X_2^{2^L - 2M - 1}$, and so (1) contradicts the hypothesized immersion.

Part (b) of Corollary 1.2 is proved similarly. If P^{8M+8} immerses in $\mathbb{R}^{16M+4}$, then there is an axial map

$$P^{8M+8} \times P^{2^{L+3} - 16M - 6} \xrightarrow{f} P^{2^{L+3} - 8M - 10},$$

and hence, up to odd multiples,

$$\begin{aligned} 2^{2-\nu(M+1)} X_1^{M+1} X_2^{2^L - 2M - 2} &+ 2^2 X_1^M X_2^{2^L - 2M - 1} \\ &= 0 \in \mathrm{tmf}^{2^{L+3} - 8M - 8}(P^{8M+8} \wedge P^{2^{L+3} - 16M - 6}), \quad (2) \end{aligned}$$

since $\alpha(M) = 2$. We have $\mathrm{tmf}^{8M+8}(P^{8M+8}) \approx \mathrm{tmf}_{-1}(P_{-1}) \approx \mathbb{Z}/2$, and

$$\mathrm{tmf}^{2^{L+3} - 16M - 8}(P^{2^{L+3} - 16M - 6}) \approx \mathrm{tmf}_{-1}(P_{-3}) \approx \mathbb{Z}/8.$$

Thus the two monomials in (2) have order at most 2 and 8, respectively. On the other hand, the group in (2) is isomorphic to $\mathrm{tmf}_6(P_{-1} \wedge P_{-3})$. A minimal resolution calculation easier than the one in Diagram 2.1 shows that $\mathrm{tmf}_6(P_{-1} \wedge P_{-3})$ has $\mathbb{Z}/2 \oplus \mathbb{Z}/8$ emanating from filtration 0 (and another $\mathbb{Z}/2 \oplus \mathbb{Z}/8$ in higher filtration). The monomials of (2) are generated in filtration 0, and since the above upper bound for their orders equals the order of the subgroup generated by filtration-0 classes, we conclude that the orders of the monomials in (2) are precisely 2 and 8, respectively, and so the term $4X_1^M X_2^{2^L-2M-1}$ in (2) is nonzero, contradicting the immersion.

3. tmf-cohomology of $P^\infty \times P^\infty$

In this section, we compute $\mathrm{tmf}^*(P^\infty)$ and $\mathrm{tmf}^{8*}(P^\infty \times P^\infty)$ in positive gradings. These will be used in the next section in studying the axial class in tmf-cohomology.

There is an element $c_4 \in \pi_8(\mathrm{tmf})$ which reduces to $v_1^4 \in \pi_8(bo)$; it has Adams filtration 4. It acts on $\mathrm{tmf}^*(X)$ with degree -8. Recall also that $\pi_*(bo) = bo_*$ is as depicted in Diagram 5.1. We denote $bo^* = bo_{-*}$. We use P_1 and P^∞ interchangeably.

Theorem 3.1. *There is an element $X \in \mathrm{tmf}^8(P_1)$ of Adams filtration 0, described in [**6**], such that, in positive dimensions divisible by 8, $\mathrm{tmf}^*(P_1)$ is isomorphic as an algebra over $\mathbb{Z}_{(2)}[c_4]$ to $\mathbb{Z}_{(2)}[c_4][X]$. In particular, each $\mathrm{tmf}^{8i}(P_1)$ with $i > 0$ is a free abelian group with basis $\{c_4^j X^{i+j} : j \geqslant 0\}$. There is a class $L \in \mathrm{tmf}^0(P_1)$ such that*

- *$\mathrm{tmf}^0(P_1)$ is a free abelian group with basis $\{L, c_4^j X^j : j \geqslant 1\}$, and*
- *$L^2 = 2L$ and $LX = 2X$.*

Moreover, in positive dimensions $\mathrm{tmf}^*(P_1)$ is isomorphic as a graded abelian group to $bo^*[X]$, and is depicted in Diagram 3.4.

Remark 3.2. No claim is made about the action of elements of tmf_* other than c_4 on $\mathrm{tmf}^*(P_1)$. A complete description of $\mathrm{tmf}^*(P_1)$ as a graded abelian group could probably be obtained using the analysis in the proof which follows, together with the computation of the E_2-term of the ASS converging to $\mathrm{tmf}_*(P_{-1})$, which was given in [**10**]. However, this is quite complicated and unnecessary for this paper, and so will be omitted.

Proof. We begin with the structure as a graded abelian group. There are isomorphisms

$$\mathrm{tmf}^*(P_1) \approx \lim_{\leftarrow} \mathrm{tmf}^*(P_1^n) \approx \lim_{\leftarrow} \mathrm{tmf}_{-*-1}(P_{-n-1}^{-2}) = \mathrm{tmf}_{-*-1}(P_{-\infty}^{-2}). \tag{3}$$

Since $H^*(\mathrm{tmf}; \mathbf{Z}_2) \approx A//A_2$, there is a spectral sequence converging to $\mathrm{tmf}_*(X)$ with $E_2(X) = \mathrm{Ext}_{A_2}(H^*X, \mathbf{Z}_2)$. Here A_2 is the subalgebra of the mod 2 Steenrod algebra A generated by Sq^1, Sq^2, and Sq^4. Also $\mathbf{Z}_2 = \mathbb{Z}/2$.

We compute $E_2(P_{-\infty}^{-2})$ from the exact sequence

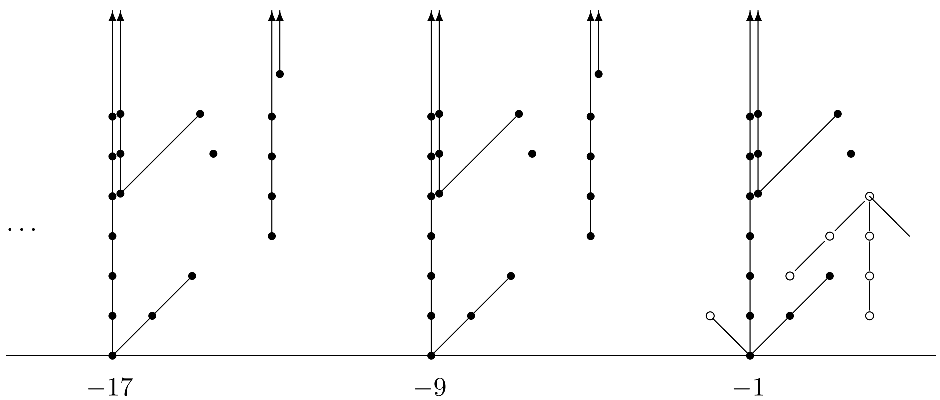
$$\to E_2^{s-1,t}(P_{-1}^\infty) \to E_2^{s,t}(P_{-\infty}^{-2}) \to E_2^{s,t}(P_{-\infty}^\infty) \xrightarrow{q_*} E_2^{s,t}(P_{-1}^\infty) \to . \tag{4}$$

It was proved in [**18**] that

$$\mathrm{Ext}_{A_2}(\mathbf{P}_{-\infty}^\infty, \mathbf{Z}_2) \approx \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}_{A_1}(\Sigma^{8i-1}\mathbf{Z}_2, \mathbf{Z}_2).$$

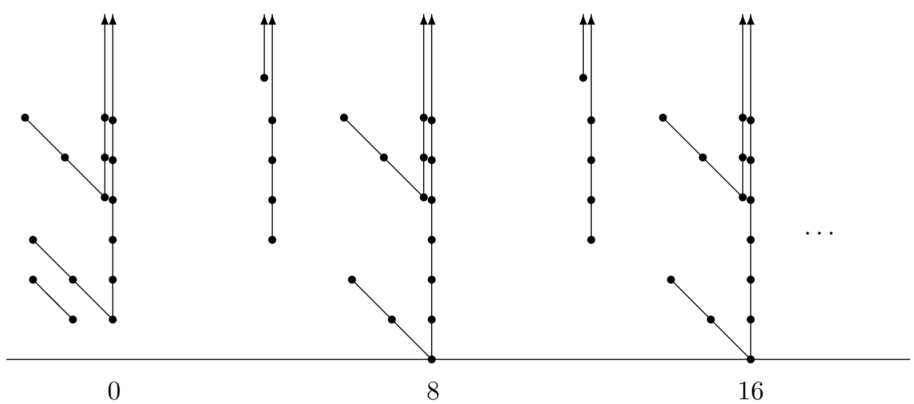
Here we have initiated a notation that $\mathbf{P}_n^m := H^*(P_n^m)$. A complete calculation of $\mathrm{Ext}_{A_2}(\mathbf{P}_{-1}^\infty, \mathbf{Z}_2)$ was performed in [**10**], but all we need here are the first few groups. We can now form a chart for $E_2(P_{-\infty}^{-2})$ from (4), as in Diagram 3.3, where $\circ$ indicate elements of $\mathrm{Ext}_{A_2}(\mathbf{P}_{-1}^{\infty}, \mathbf{Z}_2)$ suitably positioned, and lines of negative slope correspond to cases of $q_* \neq 0$ in (4).

Diagram 3.3. $\mathrm{tmf}_*(P_{-\infty}^{-2})$, $-17 \leqslant * \leqslant 2$:



Dualizing, we obtain Diagram 3.4 for the desired $\mathrm{tmf}^*(P_1^{\infty})$.

Diagram 3.4. $\mathrm{tmf}^*(P_1^{\infty})$, $* \geqslant -2$:



Naming of the generators X^i is clear since X has filtration 0. The free action of c_4 is also clear. The class L is (up to sign) the composite $P_1 \xrightarrow{\lambda} S^0 \to \mathrm{tmf}$, where λ is the well-known Kahn-Priddy map. Thus L is the image of a class $\hat{L} \in \pi^0(P_1)$. Lin's theorem ([**17**]) says that $\pi^0(P_1) \approx \mathbf{Z}_2^{\wedge}$, generated by $\hat{L}$. Since $\pi^0(P_1) \to ko^0(P_1)$ is an isomorphism, and, since $(1-\xi)^2 = 2(1-\xi)$ for a generator $(1-\xi)$ of $ko^0(P_1)$, we obtain $\hat{L}^2 = 2\hat{L}$, and hence also for L. We chose the generator to be $(1-\xi)$ rather than $(\xi - 1)$ to avoid minus signs later in the paper.

To prove the claim about LX, first note that, by the structure of $\mathrm{tmf}^8(P_1)$, we must have $LX = p(c_4X)X$ for some polynomial p. Multiply both sides by L and apply the result about L^2 to get $2LX = p(c_4X)LX$; hence $2p = p^2$, from which we conclude $p = 2$. $\square$

In $\mathrm{tmf}^*(P_1 \times P_1)$, for $i = 1, 2$, let L_i and X_i denote the classes L and X in the ith factor. Note that there is an isomorphism as tmf_*-modules, but not as rings,

$$\mathrm{tmf}^*(P_1 \times P_1) \approx \mathrm{tmf}^*(P_1 \wedge P_1) \oplus \mathrm{tmf}^*(P_1 \times *) \oplus \mathrm{tmf}^*(* \times P_1).$$

Theorem 3.5. *In positive dimensions divisible by* 8, $\mathrm{tmf}^*(P_1 \wedge P_1)$ *is isomorphic as a graded abelian group to a free abelian group on monomials* $X_1^i X_2^j$ *with* $i, j > 0$ *direct sum with a free* $\mathbb{Z}[c_4]$*-module with basis* $\{L_1 X_2^i,\ X_1^i L_2 \colon i \geqslant 1\}$*. The product and* $\mathbb{Z}[c_4]$*-module structure is determined from Theorem 3.1 and*

$$c_4(X_1X_2) = (c_4X_1)X_2 = X_1(c_4X_2) = \sum_{i \geqslant 0} \gamma_i c_4^i (L_1 X_2^{i+1} + X_1^{i+1} L_2),$$

for certain integers γ_i *with* γ_0 *divisible by* 8.

The proof of this theorem involves a number of subsidiary results. They and it occupy the remainder of this section. We will use duality and exact sequences similar to (4), but to get started, we need $\mathrm{Ext}_{A_2}(\mathbf{P} \otimes \mathbf{P}, \mathbf{Z}_2)$. Here we have begun to abbreviate $\mathbf{P} := \mathbf{P}_{-\infty}^{\infty}$. We begin with a simple lemma. Throughout this section, x_1 and x_2 denote nonzero elements coming from the factors in $H^1(P_1 \times P_1; \mathbf{Z}_2)$.

Lemma 3.6 ([**9**])**.** *There is a split short exact sequence of* A*-modules*

$$0 \to \mathbf{Z}_2 \otimes \mathbf{P} \to \mathbf{P} \otimes \mathbf{P} \to (\mathbf{P}/\mathbf{Z}_2) \otimes \mathbf{P} \to 0.$$

Proof. The $\mathbf{Z}_2$ is, of course, the subgroup generated by x^0, which is an A-submodule. A splitting morphism $\mathbf{P} \otimes \mathbf{P} \xrightarrow{g} \mathbf{Z}_2 \otimes \mathbf{P}$ is defined by $g(x_1^i \otimes x_2^j) = x_1^0 \otimes x_2^{i+j}$. This is A-linear since

$$\begin{aligned} g(\mathrm{Sq}^k(x_1^i \otimes x_2^j)) &= \sum_{\ell} \tbinom{i}{\ell}\tbinom{j}{k-\ell} x_1^0 \otimes x_2^{i+j+k} \\ &= \tbinom{i+j}{k} x_1^0 \otimes x_2^{i+j+k} = \mathrm{Sq}^k\, g(x_1^i \otimes x_2^j). \end{aligned}$$ □

The following result is more substantial. We will prove it at the end of this section.

Proposition 3.7. *There is a short exact sequence of* A_2*-modules*

$$0 \to C \to (\mathbf{P}/\mathbf{Z}_2) \otimes \mathbf{P} \to B \to 0,$$

where C *has a filtration with*

$$F_p(C)/F_{p-1}(C) \approx \Sigma^{8p} A_2/\mathrm{Sq}^2,\ p \in \mathbb{Z},$$

and B *has a filtration with*

$$F_p(B)/F_{p-1}(B) \approx \bigoplus_{\mathbb{Z}\ copies} \Sigma^{4p-2} A_2/\mathrm{Sq}^1,\ p \in \mathbb{Z}.$$

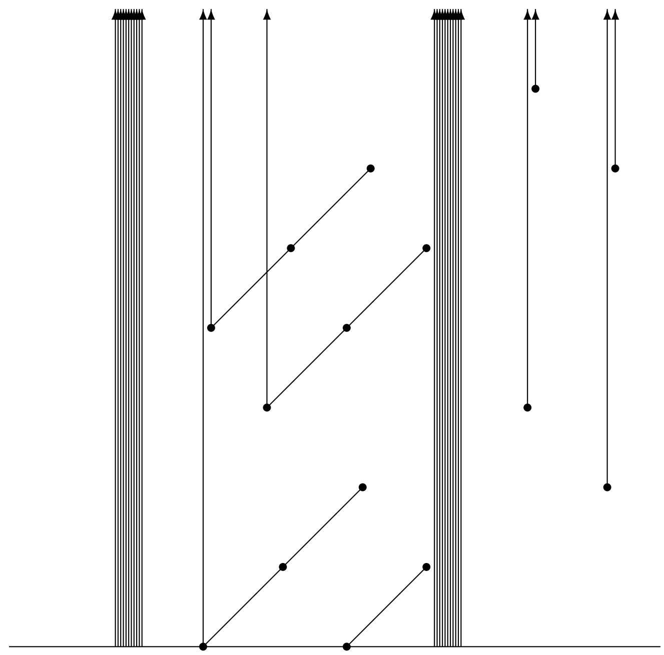
The generator of $F_p(C)/F_{p-1}(C)$ *is* $x_1^1 x_2^{8p-1}$*; a basis over* $\mathbf{Z}_2$ *for* C *is*

$$\{x_1^2 x_2^{i+2} + x_1^4 x_2^i, x_1^4 x_2^i + x_1^8 x_2^{i-4}, i \in \mathbb{Z}\} \\ \cup \{x_1^1 x_2^{i-1} + x_1^2 x_2^{i-2}, i \not\equiv 0(8)\} \cup \{x_1^1 x_2^{8p-1} p \in \mathbb{Z}\}.$$

A minimal set of generators as an A_2*-module for the filtration quotients of* B *is* $\{x_1^{8i-1} x_2^{4j-1} \colon i, j \in \mathbb{Z}\}$.

Corollary 3.8. *A chart for* $\mathrm{Ext}^{s,t}_{A_2}(\mathbf{P}\otimes\mathbf{P},\mathbf{Z}_2)$ *in* $8p-3\leqslant t-s\leqslant 8p+4$ *is as suggested in Diagram 3.9, for all integers p. The big batch of towers in each grading* $\equiv 2(4)$ *represents an infinite family of towers. The pattern of the other classes is repeated with vertical period* 4*. Thus, for example, in* $8p-1$ *there is an infinite tower emanating from filtration* 4 *for each* $i\geqslant 0$*.*

Diagram 3.9. $\mathrm{Ext}^{s,t}_{A_2}(\mathbf{P}\otimes\mathbf{P},\mathbf{Z}_2)$ *in* $8p-3\leqslant t-s\leqslant 8p+4$:



Proof of Corollary 3.8. We first note that $\mathrm{Ext}_{A_2}(\mathbf{P},\mathbf{Z}_2)$ is identical to the left portion of Diagram 3.3 extended periodically in both directions. Also,

$$\mathrm{Ext}_{A_2}(A_2/\mathrm{Sq}^1,\mathbf{Z}_2)\approx\mathrm{Ext}_{A_0}(\mathbf{Z}_2,\mathbf{Z}_2)$$

is just an infinite tower, and

$$\mathrm{Ext}_{A_2}(A_2/\mathrm{Sq}^2,\mathbf{Z}_2)\approx\mathrm{Ext}_{A_1}(A_1/\mathrm{Sq}^2,\mathbf{Z}_2)$$

is given as in Diagram 3.10. We will show at the end of this proof that

$$\mathrm{Ext}_{A_2}(C,\mathbf{Z}_2)\approx\bigoplus_{p\in\mathbb{Z}}\mathrm{Ext}_{A_2}(\Sigma^{8p}A_2/\mathrm{Sq}^2,\mathbf{Z}_2)\tag{5}$$

and similarly

$$\mathrm{Ext}_{A_2}(B,\mathbf{Z}_2)\approx\bigoplus_{p}\bigoplus_{\mathbb{Z}}\mathrm{Ext}_{A_2}(\Sigma^{4p-2}A_2/\mathrm{Sq}^1,\mathbf{Z}_2).$$

These would follow by induction on p once you get started, but since p ranges over all integers, that is not automatic.

Thus $\mathrm{Ext}_{A_2}(\mathbf{P}\otimes\mathbf{P},\mathbf{Z}_2)$ is formed from

$$\mathrm{Ext}_{A_2}(\mathbf{P},\mathbf{Z}_2)\oplus\bigoplus\mathrm{Ext}_{A_2}(\Sigma^{8p}A_2/\,\mathrm{Sq}^2,\mathbf{Z}_2)\oplus\bigoplus\mathrm{Ext}_{A_2}(\Sigma^{4p-2}A_2/\,\mathrm{Sq}^1,\mathbf{Z}_2),$$

using the sequences in Lemma 3.6 and Proposition 3.7. The Ext sequence of 3.6 must split, and there are no possible boundary morphisms in the Ext sequence of 3.7, yielding the claim of the corollary.

To prove (5), let (s,t) be given, and choose p_0 so that $8p_0<t-23s+2$. Since the highest degree element in A_2 is in degree 23, $\mathrm{Ext}_{A_2}^{s,t}(F_{p_0}(C),\mathbf{Z}_2)=0$. Actually a much sharper lower vanishing line can be established, but this is good enough for our purposes. Thus, for this (s,t),

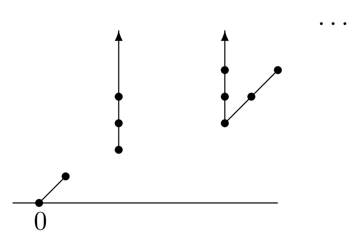
$$\mathrm{Ext}_{A_2}^{s,t}(F_{p_1}(C),\mathbf{Z}_2)\approx\bigoplus_{p\leqslant p_1}\mathrm{Ext}_{A_2}^{s,t}(\Sigma^{8p-2}A_2/\,\mathrm{Sq}^2) \tag{6}$$

for $p_1\leqslant p_0$, as both are 0. Let p_1 be minimal such that (6) does not hold. Then comparison of exact sequences implies that

$$\mathrm{Ext}_{A_2}^{s-1,t}(F_{p_1-1}(C),\mathbf{Z}_2)\to\mathrm{Ext}_{A_2}^{s,t}(F_{p_1}(C)/F_{p_1-1}(C),\mathbf{Z}_2)$$

must be nonzero. But one or the other of these groups is always 0,[1] as both charts $\mathrm{Ext}_{A_2}^{*,*}(F_{p_1-1}(C),\mathbf{Z}_2)$ and $\mathrm{Ext}_{A_2}^{*,*}(F_{p_1}(C)/F_{p_1-1}(C),\mathbf{Z}_2)$ are copies of Diagram 3.10 displaced by four vertical units from one another. Thus (6) is true for all p_1, and hence (5) holds. A similar proof works when C is replaced by B. □

Diagram 3.10. $\mathrm{Ext}_{A_2}(A_2/\,\mathrm{Sq}^2,\mathbf{Z}_2)$:



Now we can prove a result which will, after dualizing, yield Theorem 3.5. The groups $\mathrm{Ext}_{A_1}(\mathbf{Z}_2,\mathbf{Z}_2)$ to which it alludes are depicted in Diagram 5.1. The content of this result is pictured in Diagram 3.14.

Proposition 3.11. *In dimensions $t-s\equiv 2\mod 4$ with $t-s\leqslant -10$, the chart of $\mathrm{Ext}_{A_2}(\mathbf{P}_{-\infty}^{-2}\otimes\mathbf{P}_{-\infty}^{-2},\mathbf{Z}_2)$ consists of i infinite towers emanating from filtration 0 in dimensions $-8i-6$ and $-8i-10$, together with the relevant portion of two copies of $\mathrm{Ext}_{A_1}(\mathbf{Z}_2,\mathbf{Z}_2)$ beginning in filtration 1 in each dimension $-8i-2$. The generators of*

---

[1]Actually this is not quite true; for one family of elements we need to use h_0-naturality.

the towers in $-8i-10$ correspond to cohomology classes $x_1^{-9}x_2^{-8i-1},\dots,x_1^{-8i-1}x_2^{-9}$. The generators of the two copies of $\mathrm{Ext}_{A_1}(\mathbf{Z}_2,\mathbf{Z}_2)$ in $-8i-2$ arise from h_0 times classes corresponding to $x_1^{-1}x_2^{8i-1}$ and $x_1^{-8i-1}x_2^{-1}$.

Proof. Using exact sequences like (4) on each factor, we build

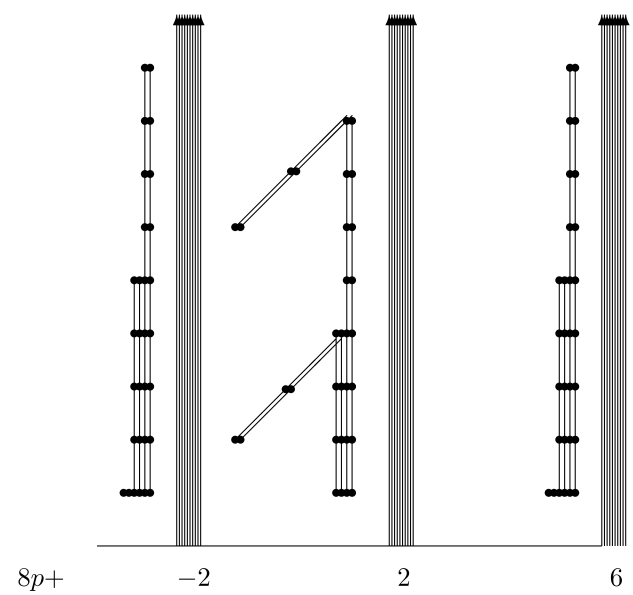
$$\mathrm{Ext}_{A_2}^{*,*}(\mathbf{P}_{-\infty}^{-2}\otimes\mathbf{P}_{-\infty}^{-2},\mathbf{Z}_2)$$

from

$$\mathbf{A}:=\mathrm{Ext}_{A_2}^{*,*}(\mathbf{P}\otimes\mathbf{P},\mathbf{Z}_2),\quad \mathbf{B}:=\mathrm{Ext}_{A_2}^{*-1,*}(\mathbf{P}_{-1}^{\infty}\otimes\mathbf{P},\mathbf{Z}_2),$$
$$\mathbf{C}:=\mathrm{Ext}_{A_2}^{*-1,*}(\mathbf{P}\otimes\mathbf{P}_{-1}^{\infty},\mathbf{Z}_2),\quad \mathbf{D}:=\mathrm{Ext}_{A_2}^{*-2,*}(P_{-1}^{\infty}\otimes P_{-1}^{\infty},\mathbf{Z}_2),$$

with possible d_1-differential from **A** and into **D**. In the range of concern, $t-s\leqslant -9$, the **D**-part will not be present, and the part of Diagram 3.9 in dimension $\not\equiv 2$ mod 4 will not be involved in d_1. Using [**18**] for **B** and **C**, the relevant part, namely the portion of **A** in dimension $\equiv 2$ mod 4, together with **B** and **C**, is pictured in Diagram 3.12.

Diagram 3.12. *Portion of* $\mathbf{A}+\mathbf{B}+\mathbf{C}$:



In dimension $8p-2$, the towers in **A** arise from all classes $x_1^{-8i-1}x_2^{-8j-1}$ with $i+j=-p$, while in dimension $8p+2$, they arise from

$$x_1^{8i-1}x_2^{8j+3}\sim x_1^{8i+3}x_2^{8j-1}.$$

The finite towers in **B** arise from $x_1^{4i-1}x_2^{8j-1}$ with $i\geqslant 0$, and those from **C** from

$x_1^{8i-1}x_2^{4j-1}$ with $j \geqslant 0$. The homomorphism

$$\mathrm{Ext}^0_{A_2}(\mathbf{P}\otimes\mathbf{P},\mathbf{Z}_2) \to \mathrm{Ext}^0_{A_2}(\mathbf{P}^{\infty}_{-1}\otimes\mathbf{P},\mathbf{Z}_2)\oplus\mathrm{Ext}^0_{A_2}(\mathbf{P}\otimes\mathbf{P}^{\infty}_{-1},\mathbf{Z}_2),$$

which is equivalent to the d_1-differential mentioned above, sends classes to those with the same name. In dimension $\leqslant -10$, this is surjective, with kernel spanned by classes with both components < -1. In dimension $-8i-6$ and $-8i-10$, there will be i such classes. We illustrate by listing the classes in the first few gradings:

$$\begin{aligned}
&-14\colon x_1^{-9}x_2^{-5} \sim x_1^{-5}x_2^{-9}\\
&-18\colon x_1^{-9}x_2^{-9}\\
&-22\colon x_1^{-17}x_2^{-5} \sim x_1^{-13}x_2^{-9},\ x_1^{-9}x_2^{-13} \sim x_1^{-5}x_2^{-17}\\
&-26\colon x_1^{-17}x_2^{-9},\ x_1^{-9}x_2^{-17}.
\end{aligned}$$

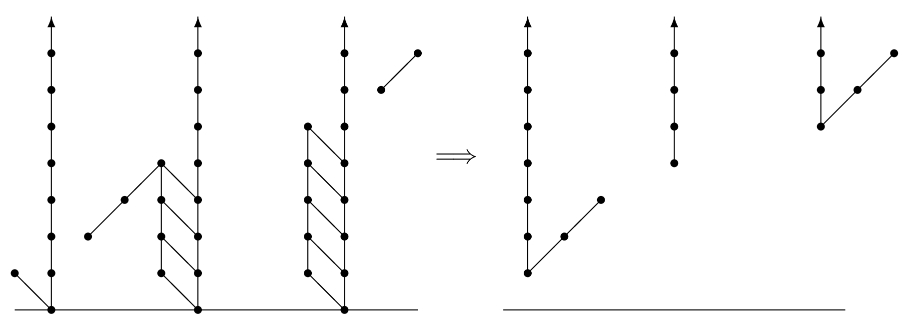
These kernel classes yield infinite towers emanating from filtration 0.

For each $p < 0$, the towers arising from $x_1^{4j-1}x_2^{8p-1}$, $j \geqslant 0$, in $\mathbf{A}$ combine with those in the p-summand of

$$\mathbf{B} \approx \bigoplus_{p\in\mathbb{Z}} \mathrm{Ext}_{A_1}(\Sigma^{8p-1}P^{\infty}_{-1},\mathbf{Z}_2),$$

as in Diagram 3.13 to yield one of the copies of $\mathrm{Ext}_{A_1}(\mathbf{Z}_2,\mathbf{Z}_2)$ arising from filtration 1. An identical picture results when the factors are reversed. □

Diagram 3.13. *Part of* $\mathrm{Ext}_{A_2}(\mathbf{P}^{-2}_{-\infty}\otimes\mathbf{P}^{-2}_{-\infty},\mathbf{Z}_2)$:

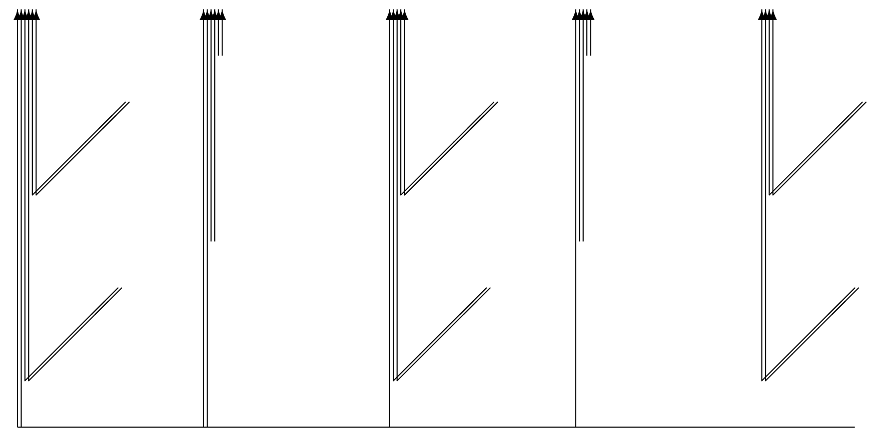


Putting things together, we obtain that in dimensions less than -8,

$$\mathrm{Ext}_{A_2}(\mathbf{P}^{-2}_{-\infty}\otimes\mathbf{P}^{-2}_{-\infty},\mathbf{Z}_2)$$

consists of a chart described in Proposition 3.11 and partially illustrated in Diagram 3.14 together with the classes in Diagram 3.9, which are not part of the infinite sums of towers in dimension $\equiv 2 \bmod 4$.

Diagram 3.14. *Illustration of Proposition 3.11*:



The only possible differentials in the Adams spectral sequence of $P_{-\infty}^{-2} \wedge P_{-\infty}^{-2} \wedge \mathrm{tmf}$ involving the classes in dimensions $8p-2$ with $p<0$ are from the towers in $8p-1$ in Diagram 3.9, but these differentials are shown to be 0 as in [**6**, p. 54]. Similarly to (3), we have

$$\mathrm{tmf}^*(P_1 \wedge P_1) \approx \mathrm{tmf}_{-*-2}(P_{-\infty}^{-2} \wedge P_{-\infty}^{-2}),$$

and so we obtain a turned-around version of Diagram 3.14, of the same general sort as Diagram 3.4, as a depiction of a relevant portion of $\mathrm{tmf}^*(P_1 \wedge P_1)$, with the labeled columns in Diagram 3.14 corresponding to cohomology gradings 24, 16, and 8.

The classes $X_1^i X_2^j$ described in Theorem 3.5 are detected by the S-duals of the classes from which the filtration-0 towers in dimensions $8p-2$ in Diagram 3.14 arise, and so they can be chosen to be the corresponding elements of $\mathrm{tmf}^{8*}(P_1 \wedge P_1)$. Similarly, the classes $L_1X_2^i$ and $X_1^iL_2$ have Adams filtration 1, and so one would anticipate that they represent the duals of the generators of the two towers in dimension $8p-2$ with $p<0$ in Diagram 3.14. This seems a bit harder to prove using the Adams spectral sequence; however, the Atiyah-Hirzebruch spectral sequence shows this quite clearly. The class X_1^i is detected by $H^{8i}(P_1; \pi_0(\mathrm{tmf}))$, while L is detected by $H^1(P_1; \pi_1(\mathrm{tmf}))$. Under the pairing, their product is detected in $H^{8i+1}(P_1; \pi_1(\mathrm{tmf}))$, clearly of Adams filtration 1.

The last part of Theorem 3.5 deals with the action of c_4 on the monomials $X_1^iX_2^j$. Since tmf is a commutative ring spectrum, $\mathrm{tmf}^*(P_1 \wedge P_1)$ is a graded commutative algebra over tmf_*. The action $c_4(X_1X_2)$ must be of the form

$$\sum_{i \geqslant 0} \gamma_i c_4^i (L_1X_2^i + X_1^iL_2)$$

as these are the only elements in $\mathrm{tmf}^8(P_1 \wedge P_1)$, and the class must be invariant under reversing factors. The divisibility of γ_0 by 8 follows since c_4 has Adams filtration 4.

Having just completed the proof of Theorem 3.5, we conclude this section with the postponed proof of Proposition 3.7.

Proof of Proposition 3.7. Let C denote the A_2-submodule of $(\mathbf{P}/\mathbf{Z}_2)\otimes\mathbf{P}$ generated by all $x_1^1x_2^{8p-1}$, $p\in\mathbb{Z}$. Note that $\mathrm{Sq}^2(x_1^1x_2^{8p-1})=\mathrm{Sq}^4\,\mathrm{Sq}^6(x_1^1x_2^{8p-9})$. Thus a basis of $A_2/\,\mathrm{Sq}^2$ acting on all $x_1^1x_2^{8p-1}$ spans C. The 24 elements in a basis of $A/\,\mathrm{Sq}^2$ acting on $x_1^1x_2^7$ yield

$$\begin{array}{llll}
x_1^1x_2^7, & x_1^1x_2^8+x_1^2x_2^7, & x_1^2x_2^9+x_1^4x_2^7, & x_1^1x_2^{11}+x_1^2x_2^{10},\\
x_1^1x_2^{12}+x_1^2x_2^{11}, & x_1^1x_2^{13}+x_1^2x_2^{12}, & x_1^1x_2^{14}+x_1^2x_2^{13}, & x_1^2x_2^{13}+x_1^4x_2^{11},\\
x_1^4x_2^{11}+x_1^8x_2^7, & x_1^2x_2^{14}+x_1^4x_2^{12}, & x_1^1x_2^{16}+x_1^4x_2^{13}, & x_1^1x_2^{17}+x_1^2x_2^{16},\\
x_1^2x_2^{16}+x_1^4x_2^{14}, & x_1^1x_2^{18}+x_1^2x_2^{17}, & x_1^2x_2^{17}+x_1^8x_2^{11}, & x_1^2x_2^{18}+x_1^8x_2^{12},\\
x_1^1x_2^{20}+x_1^4x_2^{17}, & x_1^4x_2^{17}+x_1^8x_2^{13}, & x_1^2x_2^{20}+x_1^4x_2^{18}, & x_1^4x_2^{18}+x_1^8x_2^{14},\\
x_1^4x_2^{20}+x_1^8x_2^{16}, & x_1^1x_2^{24}+x_1^8x_2^{17}, & x_1^2x_2^{24}+x_1^8x_2^{18}, \text{ and} & x_1^4x_2^{24}+x_1^8x_2^{20}.
\end{array}$$

These classes with second components shifted by all multiples of 8 exactly comprise the basis for C described in the proposition.

The procedure to establish the structure of $B=((\mathbf{P}/\mathbf{Z}_2)\otimes\mathbf{P})/C$ is similar but more elaborate. For the 32 elements θ in a basis of $A_2/\,\mathrm{Sq}^1$, we list $\theta(x_1^{-1}x_2^{-1})$ and $\theta(x_1^{-1}x_2^3)$. Then we show that these, with each component allowed to vary by multiples of 8, together with C, fill out all of $(\mathbf{P}/\mathbf{Z}_2)\otimes\mathbf{P}$.

It is convenient to let $\mathbf{Q}$ denote the quotient of $(\mathbf{P}/\mathbf{Z}_2)\otimes\mathbf{P}$ by C and all elements $\theta(x_1^{8i-1}x_2^{8j-1})$ and $\theta(x_1^{8i-1}x_2^{8j+3})$. We will show $\mathbf{Q}=0$. This will complete the proof of Proposition 3.7, implying in particular that $\mathrm{Sq}^1(x_1^{8i-1}x_2^{8j-1})$ and $\mathrm{Sq}^1(x_1^{8i-1}x_2^{8j+3})$ are decomposable over A_2.

A separate calculation is performed for each mod 8 value of the degree. Here we use repeatedly that the A_2-action on x^i depends only on i mod 8. We illustrate with the case in which degree $\equiv 0$ mod 8. The other seven congruences are handled similarly, although some are a bit more complicated.

A basis of $A_2/\,\mathrm{Sq}^1$ in degree $\equiv 2$ mod 8 acting on $x_1^{-1}x_2^{-1}$ yields the following elements: $x_1^{-1}x_2^1+x_1^0x_2^0+x_1^1x_2^{-1}$, $x_1^2x_2^6+x_1^6x_2^2$, $x_1^{-1}x_2^9+x_1^3x_2^5+x_1^4x_2^4+x_1^5x_2^3+x_1^9x_2^{-1}$, and $x_1^4x_2^{12}+x_1^{12}x_2^4$. A basis of $A_2/\,\mathrm{Sq}^1$ in degree $\equiv 6$ mod 8 acting on $x_1^{-1}x_2^3$ yields the following elements: $x_1^2x_2^6+x_1^3x_2^5+x_1^4x_2^4+x_1^5x_2^3$, $x_1^{-1}x_2^9+x_1^2x_2^6+x_1^5x_2^3$, $x_1^4x_2^{12}+x_1^6x_2^{10}+x_1^{10}x_2^6+x_1^{12}x_2^4$, and $x_1^8x_2^{16}+x_1^{16}x_2^8$. Because we allow both components to vary by multiples of 8, we will list just the first component of the ordered pairs. These are considered as relations in $\mathbf{Q}$. Thus the relation R_1 below really means that all $x_1^{8i-1}x_2^{8j+1}+x_1^{8i}x_2^{8j}+x_1^{8i+1}x_2^{8j-1}$ become 0 in $\mathbf{Q}$.

$$\begin{aligned}
&R_1: X_{-1}+X_0+X_1,\\
&R_2: X_2+X_6,\\
&R_3: X_{-1}+X_3+X_4+X_5+X_9,\\
&R_4: X_4+X_{12},\\
&R_5: X_2+X_3+X_4+X_5,\\
&R_6: X_{-1}+X_2+X_5,\\
&R_7: X_4+X_6+X_{10}+X_{12},\\
&R_8: X_8+X_{16}.
\end{aligned}$$

We will use these relations to show that all classes (in degree $\equiv 0$ mod 8) are 0 in $\mathbf{Q}$. First, R_8 implies that all classes X_{8i} are congruent to one another. Since X_0 is 0 in the quotient due to $\mathbf{P}/\mathbf{Z}_2$, we conclude that all classes X_{8i} are 0 in $\mathbf{Q}$. Next, R_4 implies that all X_{8i+4} are congruent to one another. Since $X_4 + X_8 \in C$, and we have just shown that $X_8 \equiv 0$ in $\mathbf{Q}$, we deduce that all X_{8i+4} are 0 in $\mathbf{Q}$. Now we use $R_2 + R_7$ to see that all $X_{8i+2} + X_{8i+4}$ are congruent to one another, then that $X_2 + X_4 \in C$ to deduce all $X_{8i+2} + X_{8i+4} \equiv 0$, and finally the result of the previous sentence to conclude all $X_{8i+2} \equiv 0$. Then R_2 implies all $X_{8i+6} \equiv 0$. Now $R_1 + R_3 + R_5$, together with relations previously obtained, implies all X_{8i+1} are congruent to one another, and since $X_1 \in C$, we conclude all $X_{8i+1} \equiv 0$. Finally, R_1 implies $X_{8i-1} \equiv 0$, R_6 implies $X_{8i+5} \equiv 0$, and then R_3 implies $X_{8i+3} \equiv 0$. $\square$

4. Careful treatment of the axial class

In this section, we fill the gap in the proof in [**6**] of its Theorem 1.1 by careful consideration of the possible "other terms" in the axial class discussed in the introduction. We show that, at least as far as the monomials $cX_1^iX_2^j$ in its powers are concerned, the axial class equals $\mathbf{u}(X_1 + X_2)$, where $\mathbf{u}$ is a unit in $\mathrm{tmf}^0(P^\infty \times P^\infty)$. Thus the ℓth power of the axial class is nonzero in $\mathrm{tmf}^{8\ell}(P^n \times P^m)$ if and only if $(X_1 + X_2)^\ell$ is nonzero there, and the latter is the condition which yielded the nonimmersions of [**6**, 1.1]. Thus we have a complete proof of [**6**, 1.1].

If $P^n \times P^m \xrightarrow{f} P^{m+k}$ is an axial map, then there is a commutative diagram

$$\begin{array}{ccc} P^n \times P^m & \xrightarrow{f} & P^{m+k} \\ \downarrow & & \downarrow \\ P^\infty \times P^\infty & \xrightarrow{g} & P^\infty, \end{array}$$

where g is the standard multiplication of P^∞, since $P^\infty = K(\mathbf{Z}_2, 1)$. Since $X \in \mathrm{tmf}^8(P^{m+k})$ has been chosen to extend over P^∞, we obtain that $f^*(X)$ is the restriction of $g^*(X)$. By Theorem 3.5 and the symmetry of g, we must have

$$g^*(X) = X_1 + X_2 + \sum_{i \geqslant 0} \kappa_i c_4^i (L_1 X_2^{i+1} + X_1^{i+1} L_2), \tag{7}$$

for some integers κ_i. This is what we call the "axial class." Then $g^*(X^\ell)$ equals the ℓth power of (7). Using the formulas for L_i^2, L_iX_i, and $c_4(X_1X_2)$ in Theorems 3.1 and 3.5 and the binomial theorem, this ℓth power can be written in terms of the basis described in 3.5. If some κ_i's are nonzero, then the coefficients of $X_1^iX_2^{\ell-i}$ in $g^*(X^\ell)$ will not equal $\binom{\ell}{i}$, as was claimed in [**6**]. We will study this possible deviation carefully.

One simplification is to treat L_1 and L_2 as being just 2. Note that L_i acts like 2 when multiplying by X_i, and if, for example, L_1 is present without X_1, then the terms $c_4^iL_1X_2^j$ cannot cancel our $X_1^kX_2^\ell$-classes because both are separate parts of the basis. You have to carry the terms along, because they might get multiplied by an X_1, and then it is as if $L_1 = 2$. We will incorporate this important simplification throughout the remainder of this section.

For example, one easily checks that, using $L_1^2 = 2L_1$ and $L_1X_1 = 2X_1$, we obtain

$$(X_1 + X_2 + L_1X_2)^4 = (X_1 + 3X_2)^4 - 80X_2^4 + 40L_1X_2^4.$$

The exponent of 2 in each monomial of $(X_1 + 3X_2)^4 - 80X_2^4$ is the same as that in $(X_1 + X_2)^4$, and $L_1X_2^4$ is a separate basis element.

With this simplification, the axial class in (7) becomes

$$X_1 + X_2 + 2\sum_{i>0} \kappa_i c_4^i (X_1^{i+1} + X_2^{i+1}) \tag{8}$$

for some integers κ_i. There was another term $2\kappa_0(X_1 + X_2)$, but it can be incorporated into the leading $(X_1 + X_2)$. The odd multiple that it can create is not important.

From Theorem 3.5, we have

$$c_4(X_1X_2) = 16(X_1 + X_2) + 2\sum_{k>0} \gamma_k c_4^k (X_1^{k+1} + X_2^{k+1}), \tag{9}$$

for some integers γ_k. The 16 comes from $\gamma_0 = 8$ and $L_i = 2$. Actually we do not really know that $\gamma_0 = 8$, even just up to multiplication by a unit, but it is divisible by 8 and the possibility of equality must be allowed for. This gives

$$c_4(X_1^{i+1}X_2^{j+1}) = 16(X_1^{i+1}X_2^j + X_1^iX_2^{j+1}) + 2\sum_{k>0} \gamma_k c_4^k (X_1^{i+k+1}X_2^j + X_1^iX_2^{j+k+1}). \tag{10}$$

Here we use that in a graded tmf_*-algebra $\mathrm{tmf}^*(X)$ with even-degree elements, $c(xy) = cx \cdot y$, for $c \in \mathrm{tmf}_*$ and $x, y \in \mathrm{tmf}^*(X)$.

There is an iterative nature to the action of c_4 in (10), but the leading coefficient 16 enables us to keep track of 2-exponents of leading terms in the iteration. (As observed above, the leading coefficient might be an even multiple of 16, which would make the terms even more highly 2-divisible. We assume the worst, that it equals 16.) We obtain the following key result about the action of c_4 on monomials in X_1 and X_2.

Theorem 4.1. *There are 2-adic integers A_i such that*

$$c_4 = \sum_{i\geqslant 0} 2^{4+i} A_i \left(\frac{1}{X_1}\Big(\frac{X_2}{X_1}\Big)^i + \frac{1}{X_2}\Big(\frac{X_1}{X_2}\Big)^i \right).$$

Remark 4.2. This formula will be evaluated on (i.e. multiplied by) monomials $X_1^kX_2^\ell$. One might worry that the negative powers of X_1 or X_2 in Theorem 4.1 will cause nonsensical negative powers in $c_4X_1^kX_2^\ell$. This will, in fact, not occur because the monomials on which we act always have total degree greater than the dimension of either factor. Thus if, after multiplication by c_4, a term with negative exponent of X_i appears, then the accompanying X_{3-i}^j-term will be 0 for dimensional reasons.

Proof of Theorem 4.1. The defining equation (9) may be written, with $\theta = c_4\sqrt{X_1X_2}$ and $z = \sqrt{X_1/X_2}$, as

$$\theta = 16(z + z^{-1}) + \sum_{i>0} 2\gamma_i\theta^i(z^{i+1} + z^{-(i+1)}). \tag{11}$$

Let $p_i = z^i + z^{-i}$. We will show that

$$\theta = \sum_{i\geqslant 0} 2^{4+i}A_i p_{2i+1} \tag{12}$$

for certain 2-adic integers A_i, which interprets back to the claim of 4.1.

Note that $p_ip_j = p_{i+j} + p_{|i-j|}$, and hence

$$p_1^{e_1}\cdots p_k^{e_k} = p_{\Sigma ie_i} + \mathcal{L},$$

where $\mathcal{L}$ is a sum of integer multiples of p_j with $j < \sum ie_i$ and $j \equiv \sum ie_i \bmod 2$. We will ignore for awhile the coefficients γ_i which occur in (11). This is allowable if we agree that when collecting terms, we only make crude estimates about their 2-divisibility. We have

$$\begin{aligned}\theta &= 16p_1 + 2\theta p_2 + 2\theta^2 p_3 + 2\theta^3 p_4 + \cdots \\ &= 16p_1 + 2p_2(16p_1 + 2p_2(16p_1 + \cdots) + 2p_3(16p_1 + \cdots)^2 + \cdots) \\ &\quad +2p_3(16p_1 + 2p_2(16p_1 + \cdots) + \cdots)^2 + \cdots .\end{aligned}$$

Note that the only terms that actually get evaluated must end with a $16p_1$ factor.

Now let $T_1 = 16p_1$ and, for $i \geqslant 2$, let $T_i = 2\theta^{i-1}p_i$. Each term in the expansion of θ involves a sequence of choices. First choose T_i for some $i \geqslant 1$, and then if $i > 1$ choose $(i-1)$ factors T_j, one from each factor of θ^{i-1}. For each of these T_j with $j > 1$, choose $j-1$ additional factors, and continue this procedure. This builds a tree, and we do not get an explicit product term until every branch ends with T_1. Each selected factor T_j with $j > 1$ contributes a factor $2p_j$. There will also be binomial coefficients and the omitted γ_i's occurring as additional factors.

For example, Diagram 4.3 illustrates the choices leading to one term in the expansion of θ. This yields the term $2p_2 \cdot 2p_4 \cdot 16p_1 \cdot 2p_2 \cdot 16p_1 \cdot 2p_3 \cdot 16p_1 \cdot 2p_2 \cdot 16p_1$, which equals $2^{21}(p_{17} + \mathcal{L})$, where $\mathcal{L}$ is a sum of p_i with $i < 17$ and i odd. By induction, one sees in general that the sum of the subscripts emanating from any node, including the subscript of the node itself, is odd.

Diagram 4.3. *A possible choice of terms*:

$$\begin{tikzcd}[row sep=small]
 & & T_1 & & \\
T_2 \arrow[r, dash] & T_4 \arrow[ur, dash] \arrow[r, dash] \arrow[ddr, dash] & T_2 \arrow[r, dash] & T_1 & \\
 & & & T_1 & \\
 & & T_3 \arrow[ur, dash] \arrow[dr, dash] & & \\
 & & & T_2 \arrow[r, dash] & T_1 .
\end{tikzcd}$$

The important terms are those in which T_2 is chosen k times ($k \geqslant 0$) and then T_1 is chosen. These give $(2p_2)^k p_1$ with no binomial coefficient. This term is

$2^{k+4}(p_{2k+1}+\mathcal{L})$. Note that a term $2^{k+4}p_{2i+1}$ with $i<k$ obtained from $\mathcal{L}$ will be more 2-divisible than the $2^{i+4}p_{2i+1}$ term that was previously obtained. Thus it may be incorporated into the coefficient of that term.

All other terms will be more highly 2-divisible than these. For example, the first would arise from choosing T_3 then two copies of T_1. This would give $2p_3 \cdot 2^4 p_1 \cdot 2^4 p_1 = 2^9 p_5 + \mathcal{L}$, and the $2^9 p_5$ can be combined with the $2^6 p_5$ obtained from choosing T_2 then T_2 then T_1. Incorporating γ_i's may make terms even more divisible, but the claim of (12) is only that p_{2i+1} occurs with coefficient divisible by 2^{4+i}. $\square$

Now we incorporate Theorem 4.1 into (8) to obtain the following key result, which we prove at the end of the section.

Theorem 4.4. *The monomials $c_i X_1^i X_2^{n-i}$ in the nth power of the axial class in $\mathrm{tmf}^{8n}(P^\infty \times P^\infty)$ are equal to those in the nth power of*

$$(X_1+X_2)\Big(u+\sum_{i\geqslant 1} 2^{4+i}\alpha_i\big((\frac{X_1}{X_2})^i+(\frac{X_2}{X_1})^i\big)\Big), \tag{13}$$

where u is an odd 2-adic integer and α_i are 2-adic integers.

The factor which accompanies (X_1+X_2) in (13) is a unit in $\mathrm{tmf}^*(P^\infty \times P^\infty)$; we referred to it earlier as $\mathbf{u}$. Indeed, its inverse is a series of the same form, obtained by solving a sequence of equations. This justifies the claim in the first paragraph of this section regarding retrieval of the nonimmersions of [**6**, 1.1].

We must also observe that restriction to $\mathrm{tmf}^{8\ell}(P^n \times P^m)$ of the non-$X_1^i X_2^{\ell-i}$ parts of the basis of $\mathrm{tmf}^{8\ell}(P^\infty \times P^\infty)$ cannot cancel the $X_1^i X_2^{\ell-i}$ terms essential for the nonimmersion. This is proved by noting that these elements such as $L_1 X_2^\ell$ and $c_4^i L_1 X_2^{\ell+i}$ will restrict to a class of the same name in $\mathrm{tmf}^{8\ell}(P^n \times P^m)$, and will be 0 there for dimensional reasons, since $8\ell > n$.

Proof of Theorem 4.4. Let $g^*(X)$ denote the axial class as in (7). From (8) and Theorem 4.1, the difference $g^*(X)-(X_1+X_2)$ equals

$$2\sum_{i\geqslant 1}\kappa_i(X_1^{i+1}+X_2^{i+1})2^{4i}\Big(\sum_{j\geqslant 0}2^j A_j\Big(\frac{1}{X_1}(\frac{X_2}{X_1})^j+\frac{1}{X_2}(\frac{X_1}{X_2})^j\Big)\Big)^i.$$

We let $z=\sqrt{X_1/X_2}$ and $p_j = z^j + z^{-j}$ as in the proof of 4.1.

The summand with $i=2t$ becomes

$$\begin{aligned}2\kappa_i(X_1+X_2)\frac{\sum_s X_1^{2t-s}X_2^s}{X_1^t X_2^t}2^{4i}\Big(\sum_{j\geqslant 0}2^j A_j p_{2j+1}\Big)^i \\ = 2\kappa_i(X_1+X_2)(p_{2t}+\mathcal{L})2^{4i}\sum_k c_k 2^k(p_{2k+i}+\mathcal{L}).\end{aligned}$$

Here k is a sum of j-values taken from the various factors in the ith power. Also, in $p_j+\mathcal{L}$, $\mathcal{L}$ denotes a combination of p_t's with $t<j$. Noting $(p_{2t}+\mathcal{L})(p_{2k+i}+\mathcal{L}) =$

$p_{2k+2i} + \mathcal{L}$, this becomes

$$2(X_1 + X_2)2^{4i} \sum c'_k 2^k (p_{2k+2i} + \mathcal{L}). \tag{14}$$

The argument when $i = 2t + 1$ is similar but slightly more complicated because $(X_1^{i+1} + X_2^{i+1})$ is not divisible by $(X_1 + X_2)$. We obtain

$$2\kappa_i \frac{X_1^{i+1} + X_2^{i+1}}{(\sqrt{X_1 X_2})^{2t+1}} 2^{4i} \Big(\sum_{j \geqslant 0} 2^j A_j p_{2j+1}\Big)^i.$$

For one of the factors of the ith power, say the first, we treat p_{2j+1} as $\frac{X_1+X_2}{\sqrt{X_1X_2}}(p_{2j} + \mathcal{L})$. The expression then becomes

$$2(X_1 + X_2)p_{i+1}2^{4i} \sum c_k 2^k (p_{2k+i-1} + \mathcal{L}),$$

where k is obtained as in the previous case. We again obtain (14).

Thus when $g^*(X) - (X_1 + X_2)$ is written as $(X_1 + X_2)\sum \beta_j p_{2j}$, the coefficient β_j satisfies $\nu(\beta_j) \geqslant (j - 1) + 4 + 1$. Here the $(j - 1) + 4$ comes from the case $i = 1$, $k = j - 1$ in (14), and the extra $+1$ is the factor 2 which has been present all along. This yields the claim of (13). $\square$

5. tmf-cohomology of $CP^\infty \times CP^\infty$

In [**2, 4**], and [**8**], it was noted, first by Astey, that the axial class using BP (or $BP\langle 2\rangle$) was $u(X_2 - X_1)$, where u is a unit in $BP^*(RP^\infty \wedge RP^\infty)$. In this section, we review that argument and consider the possibility that it might be true when BP is replaced by tmf, which would render the considerations of the previous section unnecessary. To do this, we calculate $\mathrm{tmf}^*(CP^\infty)$ and $\mathrm{tmf}^*(CP^\infty \times CP^\infty)$ in positive dimensions. (See Theorems 5.13 and 5.16.) Although our conclusion will be that Astey's BP-argument cannot be adapted to tmf, nevertheless these calculations may be of independent interest.

We begin by reviewing Astey's argument. Whereas in previous sections we have used P to denote real projective spaces, in this section we use RP, to distinguish them from complex projective spaces, which are denoted by CP. There is a commutative diagram

$$\begin{tikzcd}
RP^\infty \arrow[r, "d_R"] \arrow[d, "h"'] & RP^\infty \times RP^\infty \arrow[r, "m_R"] \arrow[d, "h\times h"'] & RP^\infty \arrow[d, "h"'] \\
CP^\infty \arrow[r, "d_C"] & CP^\infty \times CP^\infty \arrow[d, "1\times(-1)"'] & CP^\infty \arrow[d, "1"'] \\
& CP^\infty \times CP^\infty \arrow[r, "m_C"] & CP^\infty.
\end{tikzcd}$$

The generator $X_R \in BP^2(RP^\infty)$ satisfies $X_R = h^*(X)$. We also have that

$$m_C \circ (1 \times (-1)) \circ d_C$$

is null-homotopic. The key fact, which will fail for tmf, is

$$BP^*(CP^\infty \times CP^\infty) \approx BP^*[X_1, X_2].$$

The axial class is $m_R^*(X_R)$. It equals $(h \times h)^*(1 \times (-1))^* m_C^*(X)$, but

$$(1 \times (-1))^* m_C^*(X) \in \ker(d_C^*).$$

By the above "key fact," d_C^* is the projection $BP^*[X_1, X_2] \to BP^*[X]$ in which each $X_i \mapsto X$. The kernel of this projection is the ideal $(X_2 - X_1)$. To see this, just note that in grading $2n$ a kernel element must be $\sum c_i X_1^i X_2^{n-i}$ with $\sum c_i = 0$, and hence is

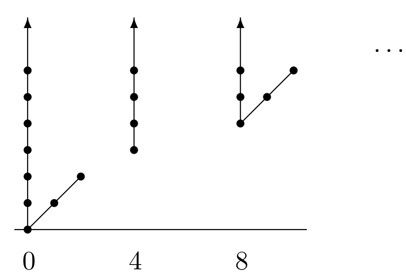
$$\sum_{i<n} c_i(X_1^i X_2^{n-i} - X_1^n) = \sum_{i<n} c_i X_1^i (X_2 - X_1) \sum X_1^j X_2^{n-i-1-j}.$$

Thus $(1 \times (-1))^* m_C^*(X) = (X_2 - X_1)u$ for some $u \in BP^*(CP^\infty \times CP^\infty)$. This u is a unit by consideration of its reduction to $H^*(-;\mathbb{Z})$, as in [**2**]. Since $h^*(u)$ will then be a unit in $BP^*(RP^\infty \times RP^\infty)$ and $h^*(X_i) = X_{Ri}$, we obtain the claim about the axial class being a unit times $X_{R2} - X_{R1}$.

In order to see if there is any chance of adapting this to tmf, we compute $\mathrm{tmf}^*(CP^\infty)$ and $\mathrm{tmf}^*(CP^\infty \times CP^\infty)$ in positive gradings. We begin with the relevant Ext calculations.

Let $\mathbf{bo} = \mathrm{Ext}_{A_1}^{*,*}(\mathbf{Z}_2, \mathbf{Z}_2)$. Recall that a chart for this is given as in Diagram 5.1, extended with period $(t-s, s) = (8, 4)$.

Diagram 5.1. $\mathrm{Ext}_{A_1}^{*,*}(\mathbf{Z}_2, \mathbf{Z}_2)$:



Let M_{10} denote the A_2-module $\langle 1, \mathrm{Sq}^4, \mathrm{Sq}^2\,\mathrm{Sq}^4, \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4 \rangle$.

Lemma 5.2. *There is an additive isomorphism*

$$\mathrm{Ext}_{A_2}^{*,*}(M_{10}, \mathbf{Z}_2) \approx \mathbf{bo}[v_2],$$

where $v_2 \in \mathrm{Ext}^{1,7}(-)$.

Thus the chart for $\mathrm{Ext}_{A_2}^{*,*}(M_{10}, \mathbf{Z}_2)$ consists of a copy of $\mathbf{bo}$ shifted by $(t-s, s) = (6i, i)$ units for each $i \geqslant 0$.

Proof. There is a short exact sequence of A_2-modules

$$0 \to \Sigma^7 M_{10} \to A_2//A_1 \to M_{10} \to 0.$$

This yields a spectral sequence which builds $\mathrm{Ext}_{A_2}^{*,*}(M_{10}, \mathbf{Z}_2)$ from

$$\bigoplus_{i \geqslant 0} \mathrm{Ext}_{A_2}^{*-i,*-7i}(A_2//A_1, \mathbf{Z}_2).$$

Since $\mathrm{Ext}_{A_2}^{*,*}(A_2//A_1, \mathbf{Z}_2) \approx \mathbf{bo}$, one easily checks that there are no possible differentials in this spectral sequence. □

Let $\mathbf{C}_n^m = H^*(CP_n^m; \mathbf{Z}_2)$.

Theorem 5.3. *There is an additive isomorphism*

$$\mathrm{Ext}_{A_2}^{*,*}(\mathbf{C}_{-\infty}^{\infty}, \mathbf{Z}_2) \approx \bigoplus_{p \in \mathbb{Z}} \Sigma^{8p-2} \mathbf{bo}[v_2].$$

Of course Σ applied to a module or an Ext group just means to increase the t-grading by 1.

Proof. There is a filtration of $\mathbf{C}_{-\infty}^{\infty}$ with $F_p/F_{p-1} \approx \Sigma^{8p-2} M_{10}$ for $p \in \mathbb{Z}$. We have $\mathrm{Sq}^2\, \iota_{8p-2} = \mathrm{Sq}^4\, \mathrm{Sq}^2\, \mathrm{Sq}^4\, \iota_{8p-10}$. The same argument used in the last paragraph of the proof of Corollary 3.8 works to initiate an inductive proof of the Ext-isomorphism claimed in the theorem. □

Corollary 5.4. *In gradings* $(t-s)$ *less than* -1*,*

$$\mathrm{Ext}_{A_2}^{*,*}(\mathbf{C}_{-\infty}^{-2}, \mathbf{Z}_2) \approx \bigoplus_{p<0} \Sigma^{8p-2} \mathbf{bo}[v_2].$$

Proof. There is an exact sequence

$$\to \mathrm{Ext}_{A_2}^{s-1,t}(\mathbf{C}_{-1}^{\infty}, \mathbf{Z}_2) \to \mathrm{Ext}_{A_2}^{s,t}(\mathbf{C}_{-\infty}^{-2}, \mathbf{Z}_2) \to \mathrm{Ext}_{A_2}^{s,t}(\mathbf{C}_{-\infty}^{\infty}, \mathbf{Z}_2) \xrightarrow{q_*} \mathrm{Ext}_{A_2}^{s,t}(\mathbf{C}_{-1}^{\infty}, \mathbf{Z}_2).$$

The result is immediate from this and Theorem 5.3, since q_* sends the initial tower in F_0/F_{-1} isomorphically to the initial tower in $\mathrm{Ext}_{A_2}(\mathbf{C}_{-1}^{\infty}, \mathbf{Z}_2)$. □

The A-modules $\mathbf{C}_1^{\infty}$ and $\Sigma^2 \mathbf{C}_{-\infty}^{-2}$ are dual. Thus, by [**9**, Prop. 4],

$$\mathrm{Ext}_{A_2}^{s,t}(\mathbf{Z}_2, \mathbf{C}_1^{\infty}) \approx \mathrm{Ext}_{A_2}^{s,t}(\Sigma^2 \mathbf{C}_{-\infty}^{-2}, \mathbf{Z}_2).$$

There is a ring structure on $\mathrm{Ext}_{A_2}^{*,*}(\mathbf{Z}_2, \mathbf{C}_1^{\infty})$. We deduce the following result, which is pictured in Diagram 5.10.

Corollary 5.5. *In* $(t-s)$ *gradings* $\leqslant 0$*, there is a ring isomorphism*

$$\mathrm{Ext}_{A_2}^{*,*}(\mathbf{Z}_2, \mathbf{C}_1^{\infty}) \approx \mathbf{bo}[v_2][X],$$

where $X \in \mathrm{Ext}^{0,-8}$.

Proof. We apply the duality isomorphism to 5.4. The multiplicative structure is obtained from the observation that the powers of the class in $\mathrm{Ext}^{0,-8}$ equal the class in $\mathrm{Ext}^{0,-8i}$ for each $i > 0$. □

The Ext groups computed here are E_2 of the ASS converging to $\mathrm{tmf}^{-*}(CP^\infty)$. We will consider the differentials in this spectral sequence after performing the Ext calculation relevant for $\mathrm{tmf}^*(CP^\infty \times CP^\infty)$.

Now we consider $\mathbf{C}_{-\infty}^{-2} \otimes \mathbf{C}_{-\infty}^{-2}$, and let x_1 and x_2 denote elements of $H^2(CP^\infty; \mathbf{Z}_2)$. Let E_2 denote the exterior subalgebra generated by the Milnor primitives of grading 1, 3, and 7. Note that $A_2//E_2$ has a basis with elements of grading 0, 2, 4, 6, 6, 8, 10, and 12. Finally we note that for any $j \equiv -2 \bmod 8$ with $j \leqslant -10$, there is a nontrivial A_2-morphism $\mathbf{C}_{-\infty}^{-2} \xrightarrow{\rho} \Sigma^j \mathbf{Z}_2$.

Lemma 5.6. *Let*

$$K = \ker(\mathbf{C}_{-\infty}^{-2} \otimes \mathbf{C}_{-\infty}^{-2} \xrightarrow{\rho} \mathbf{C}_{-\infty}^{-2} \otimes \Sigma^{-10}\mathbf{Z}_2).$$

Let S denote the set of all classes $x_1^{8i-2}x_2^{8j-2}$ with $i \leqslant -1$ and $j \leqslant -2$, together with the classes $x_1^{8i-2}x_2^{8j+2}$ with $i \leqslant -1$ and $j \leqslant -1$. Then K is the direct sum of a free $A_2//E_2$-module on S with a single relation $\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4(x_1^{-10}x_2^{-6}) = 0$.

Proof. Since the generators of E_2 have odd grading, $A_2//E_2$ acts on any element of these evenly-graded modules. The action of $A_2//E_2$ on $x_1^{-2}x_2^{-2}$ yields the additional elements $x_1^{-2}x_2^0 + x_1^0x_2^{-2}$, $x_1^{-2}x_2^2 + x_1^0x_2^0 + x_1^2x_2^{-2}$, $x_1^{-2}x_2^4 + x_1^4x_2^{-2}$, $x_1^0x_2^2 + x_1^2x_2^0$, $x_1^0x_2^4 + x_1^4x_2^0$, $x_1^{-2}x_2^8 + x_1^2x_2^4 + x_1^4x_2^2 + x_1^8x_2^{-2}$, and $x_1^0x_2^8 + x_1^8x_2^0$. The action of $A_2//E_2$ on $x_1^{-2}x_2^2$ yields the additional elements $x_1^0x_2^2 + x_1^{-2}x_2^4$, $x_1^0x_2^4 + x_1^2x_2^2$, $x_1^2x_2^4 + x_1^4x_2^2$, $x_1^2x_2^4 + x_1^{-2}x_2^8$, $x_1^0x_2^8 + x_1^4x_2^4$, $x_1^2x_2^8 + x_1^8x_2^2$, and $x_1^4x_2^8 + x_1^8x_2^4$. Each exponent can be decreased by any multiple of 8.

One can easily check that in each grading all classes in $\mathbf{C}_{-\infty}^{-2} \otimes \mathbf{C}_{-\infty}^{-2}$ are obtained exactly once from the described elements in K together with $\mathbf{C}_{-\infty}^{-2} \otimes \Sigma^{-10}\mathbf{Z}_2$. There are four cases, for the four even mod 8 values. We illustrate with the case of grading 4 mod 8. We will just consider the specific value -28, but it will be clear that it generalizes to all gradings $\equiv 4$ mod 8. Letting X_i denote $x_1^i x_2^{-28-i}$, we have:

1. From generators in -28, we obtain just X_{-10} in K. The class X_{-18} is in $\mathbf{C}_{-\infty}^{-2} \otimes \Sigma^{-10}\mathbf{Z}_2$.
2. From generators in -32, we obtain $X_{-8} + X_{-6}$, $X_{-16} + X_{-14}$, and $X_{-24} + X_{-22}$.
3. From generators in -36, we obtain $X_{-8} + X_{-4}$ and $X_{-16} + X_{-12}$.
4. From generators in -40, we obtain X_{-4}, $X_{-12} + X_{-8}$, $X_{-20} + X_{-16}$, and X_{-24}.

Note in (4) that X_0 and X_{-28} do not appear because each component must be $\leqslant -4$ and the components sum to -28.

One easily checks that the 11 classes listed above, including X_{-18}, form a basis for the space spanned by $X_{-4}, \ldots, X_{-24}$, in an orderly fashion that clearly generalizes to any grading $\equiv 4$ mod 8. A similar argument works in the other three congruences. There are some minor variations in the top few dimensions. $\square$

Now we dualize. There is a pairing

$$\mathrm{Ext}_{A_2}(\mathbf{Z}_2, \mathbf{C}_1^\infty) \otimes \mathrm{Ext}_{A_2}(\mathbf{Z}_2, \mathbf{C}_1^\infty) \to \mathrm{Ext}_{A_2}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty).$$

Let X_i denote the class in grading -8 coming from the ith factor. Then we obtain

Theorem 5.7. *The algebra* $\mathrm{Ext}_{A_2}^{0,*}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty)$ *in gradings* $\leqslant -8$ *is isomorphic to* $\mathbf{Z}_2[X_1, X_2]\langle X_1X_2, y_{-12}\rangle$ *with* $y_{-12}^2 = X_1^2X_2 + X_1X_2^2$*. The monomials of the form* $X_1^iX_2^jy_{-12}$ *are acted on freely by* $\mathbf{Z}_2[v_0, v_1, v_2]$*. Let* S_n *denote the* $\mathbf{Z}_2$*-vector space with basis the monomials* $X_1^iX_2^{n-i}$*, and define a homomorphism* $\epsilon : S_n \to \mathbf{Z}_2$ *by sending each monomial to* 1*. Then* $\mathbf{Z}_2[v_0, v_1, v_2]$ *acts freely on* $\ker(\epsilon)$*, while* $\mathbf{bo}[v_2]$ *acts freely on* $S_n/\ker(\epsilon)$*. Thus in dimensions* $t - s \leqslant -8$*,* $\mathrm{Ext}_{A_2}^{*,*}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty)$ *has, for each* $i > 0$*,* i *copies of* $\Sigma^{-8i-4}\mathbf{Z}_2[v_0, v_1, v_2]$ *and* i *copies of* $\Sigma^{-8i-16}\mathbf{Z}_2[v_0, v_1, v_2]$*, and also one copy of* $\Sigma^{-8i-8}\mathbf{bo}[v_2]$*.*

Here by $\mathbf{Z}_2[X_1, X_2]\langle X_1X_2, y_{-12}\rangle$ we mean a free $\mathbf{Z}_2[X_1, X_2]$-module with basis $\{X_1X_2, y_{-12}\}$.

Proof. The structure as graded abelian group is straightforward from Lemma 5.6, Corollary 5.5, and the duality isomorphism

$$\mathrm{Ext}_{A_2}^{*,*}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty) \approx \mathrm{Ext}_{A_2}^{*,*-4}(\mathbf{C}_{-\infty}^{-2} \otimes \mathbf{C}_{-\infty}^{-2}, \mathbf{Z}_2).$$

We use that $\mathrm{Ext}_{A_2}(A_2/\!/E_2, \mathbf{Z}_2) \approx \mathbf{Z}_2[v_0, v_1, v_2]$. The reason that we only assert the structure in dimension $\leqslant -8$ is due to the Σ^{-10} in the cokernel part of Lemma 5.6, and that Theorem 5.5 was only valid in dimension $\leqslant 0$. In the range under consideration, the relation on the top class in Lemma 5.6 does not affect Ext.

The ring structure in filtration 0 comes from $\mathrm{Hom}_{A_2}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty)$ being isomorphic to elements of $\mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty$ annihilated by Sq^2 and Sq^4, which has as basis all elements $x_1^{4i} \otimes x_2^{4j}$ and $(x_1^{4i} \otimes x_2^{4j})(x_1^4 \otimes x_2^2 + x_1^2 + x_2^4)$.

Now we show that $\mathrm{Ext}_{A_2}^{1,-8n+2}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty) = \mathbf{Z}_2$, and h_1 times each monomial in $\mathrm{Ext}_{A_2}^{0,-8n}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty)$ equals the nonzero element here. An element in $\mathrm{Ext}_{A_2}^{1,-8n+2}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty) = \mathbf{Z}_2$ is an equivalence class of morphisms

$$\Sigma^2 A_2 \oplus \Sigma^4 A_2 \xrightarrow{h} \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty,$$

which increase grading by $8n - 2$, and yield a trivial composite when preceded by

$$\Sigma^4 A_2 \oplus \Sigma^8 A_2 \xrightarrow{\begin{pmatrix} \mathrm{Sq}^2 & \mathrm{Sq}^6 \\ 0 & \mathrm{Sq}^4 \end{pmatrix}} \Sigma^2 A_2 \oplus \Sigma^4 A_2.$$

Morphisms h which can be factored as

$$\Sigma^2 A_2 \oplus \Sigma^4 A_2 \xrightarrow{\mathrm{Sq}^2, \mathrm{Sq}^4} A_2 \xrightarrow{k} \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty \tag{15}$$

are equivalent to 0 in Ext.

We illustrate with the case $n = 3$. There are A_2-morphisms increasing grading by 22 sending either $\Sigma^2 A_2$ or $\Sigma^4 A_2$ to any one of the following classes:

$$x_1^1x_2^{12},\ x_1^2x_2^{10},\ x_1^4x_2^9,\ x_1^4x_2^8,\ x_1^5x_2^8,\ x_1^6x_2^6,\ x_1^8x_2^5,\ x_1^8x_2^4,\ x_1^9x_2^4,\ x_1^{10}x_2^2,\ x_1^{12}x_2^1. \tag{16}$$

The classes are listed in this order because any two adjacent monomials are equivalent using as k in (15) the morphism sending the generator to the indicated classes in succession:

$$x_1^1x_2^{10},\ x_1^2x_2^9,\ x_1^4x_2^7,\ x_1^3x_2^8,\ x_1^5x_2^6,\ x_1^6x_2^5,\ x_1^8x_2^3,\ x_1^7x_2^4,\ x_1^9x_2^2,\ x_1^{10}x_2^1.$$

For example, $(\mathrm{Sq}^2, \mathrm{Sq}^4)(x_1^1x_2^{10}) = (x_1^2x_2^{10}, x_1^1x_2^{12})$. Thus all classes in (16) are equivalent to one another.

Usual Yoneda product considerations show that h_1 times any monomial $X_1^iX_2^{n-i}$ equals this nonzero element of $\mathrm{Ext}_{A_2}^{1,8n+2}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty)$. Indeed, if

$$0 \leftarrow \mathbf{Z}_2 \leftarrow C_0 \leftarrow C_1 \leftarrow$$

is the beginning of a minimal A_2-resolution, with $C_1 = \Sigma^1A_2 \oplus \Sigma^2A_2 \oplus \Sigma^4A_2$, then $h_1X_1^iX_2^{n-i}$ is represented by the composite $C_1 \to C_0 \to \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty$ sending $\iota_2 \mapsto \iota \mapsto X_1^iX_2^{n-i}$, and this is equivalent to the element described in the previous paragraph. $\square$

Here is a schematic way of picturing Theorem 5.7. We first list the generators in grading greater than -32. Then for each of the two types of generators, we list the structure arising from them in the first ten dimensions. The $\mathbf{bo}[v_2]$-structure in the left half of Diagram 5.9 arises from one tower in dimensions -24 and -16, while the $\mathbf{Z}_2[v_0, v_1, v_2]$-structure in the right half of Diagram 5.9 arises from the other towers in Diagram 5.8.

Diagram 5.8. *Generators of* $\mathrm{Ext}_{A_2}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty)$:

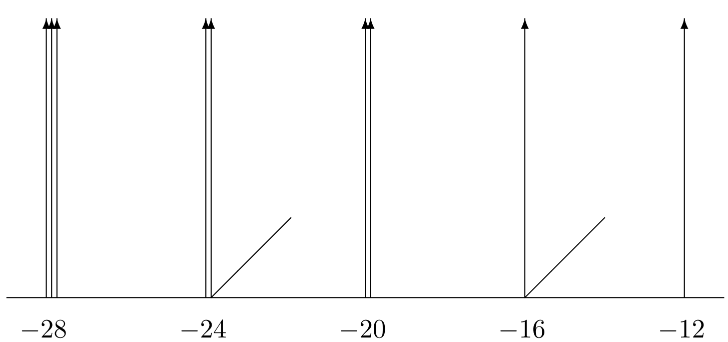
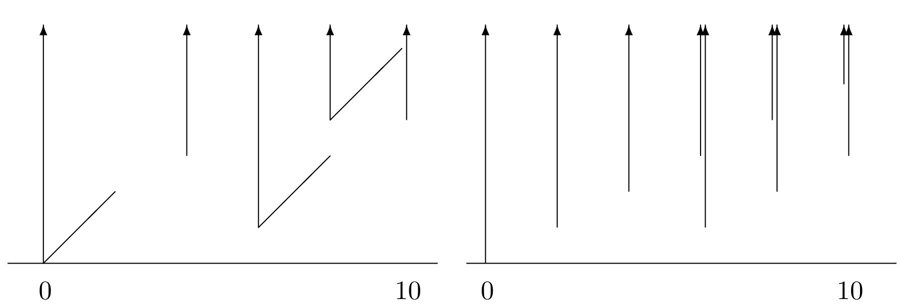
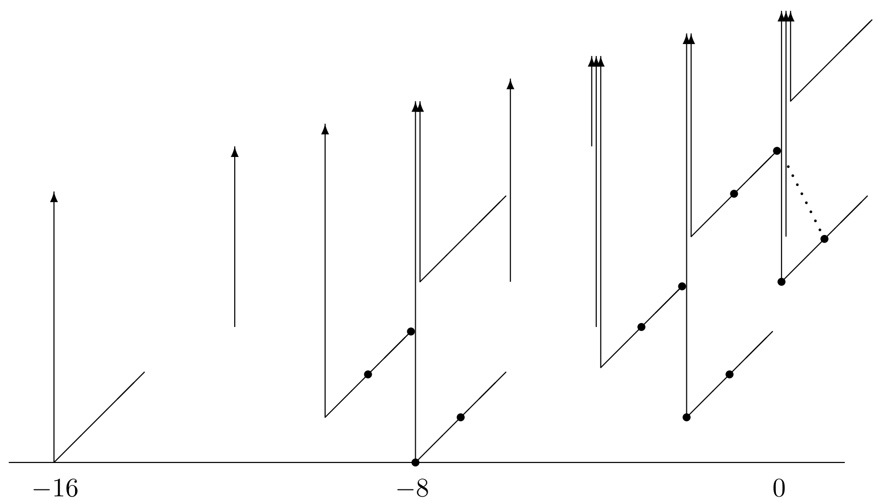


Diagram 5.9. *Structure on two types of generators*:



Now we consider the differentials in the ASS converging to $\mathrm{tmf}^*(CP^\infty)$ and then for $\mathrm{tmf}^*(CP^\infty \wedge CP^\infty)$. The gradings are negated when considered as tmf-cohomology groups. Corollary 5.5 gives the E_2-term converging to $[\Sigma^* CP_1^\infty, \mathrm{tmf}] \approx \mathrm{tmf}^{-*}(CP_1^\infty)$. We will maintain the homotopy gradings until just before the end. In Diagram 5.10, we depict a portion of the E_2-term of this ASS in gradings -16 to 1. There are also classes in higher filtration arising from powers of v_1^4 and v_2 acting on generators in lower grading. The elements indicated by $\bullet$'s are involved in differentials, as explained later.

Diagram 5.10. *A portion of E_2 for $[\Sigma^* CP^\infty, \mathrm{tmf}]$:*



We will prove the following key result about differentials in this ASS.

Theorem 5.11. *The nonzero differentials in the ASS converging to $[\Sigma^* CP^\infty, \mathrm{tmf}]$, $* < 1$, are given by*

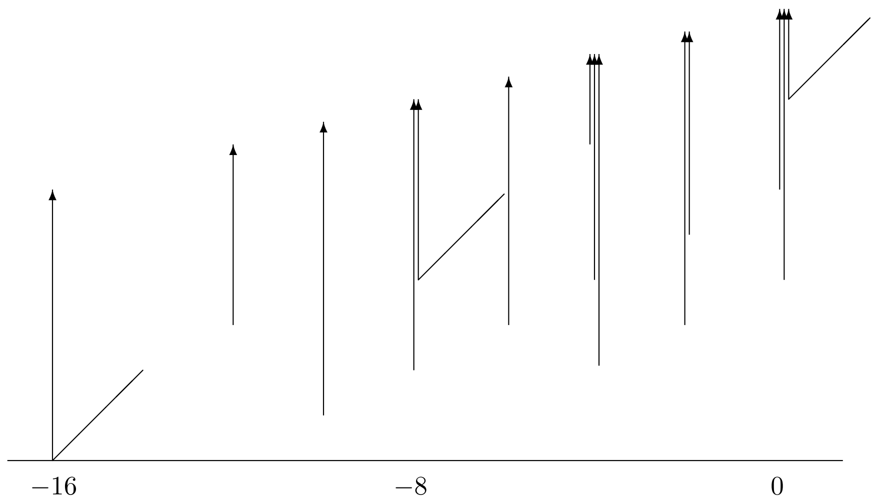
$$d_2(h_1^\epsilon v_1^{4i} v_2^j X^{-2k+1}) = h_1^{\epsilon+1} v_1^{4i} v_2^{j+1} X^{-2k}$$

for $\epsilon = 0, 1$, $i, j \geqslant 0$, $k \geqslant 1$.

Here h_1, v_1^4, and v_2 have the usual $\mathrm{Ext}^{s,t}$ gradings $(s,t) = (1,2)$, $(4,12)$, and $(1,7)$, respectively.

Diagram 5.10 pictures the situation for $k = 1$ and small values of i and j. The elements indicated by $\bullet$'s are involved in the differentials. The resulting picture is nicer if the filtrations of all classes built on X^{-2k+1} are increased by 1. There is a nontrivial extension (multiplication by 2) in dimension -6 due to the preceding differential. This is equivalent to the way that bu_* is formed from bo_* and $\Sigma^2 bo_*$. We obtain Diagram 5.12 from Diagram 5.10 after the differentials, extensions, and filtration shift are taken into account.

Diagram 5.12. *Diagram 5.10 after differentials and filtration shift*:



The regular sequence of towers in the chart beginning in filtration 1 in dimension -10 is interpreted as $v_1^i v_2$, $i \geqslant 0$.

After negating dimensions to switch to cohomology indexing, we obtain the following result, which is immediate from Theorem 5.11 after the extensions such as just seen are taken into account.

Theorem 5.13. *In positive gradings, there is an isomorphism of graded abelian groups*

$$\mathrm{tmf}^*(CP_1^\infty) \approx \mathbb{Z}_{(2)}[Z_{16}](bo^* \oplus v_2\mathbb{Z}_{(2)}[v_1, v_2]).$$

Here $Z_{16} \in \mathrm{tmf}^{16}(CP_1^\infty)$*, and* $|v_1| = -2$ *and* $|v_2| = -6$.

Recall that $bo^* = bo_{-*}$ with bo_* as suggested in Daigram 5.1. Much of the ring structure of $\mathrm{tmf}^*(CP_1^\infty)$ is described in 5.13, since bo_* and $v_2\mathbb{Z}_{(2)}[v_1, v_2]$ are rings, and it is quite clear how to multiply an element in bo_* by one in $v_2\mathbb{Z}_{(2)}[v_1, v_2]$. Because of the filtration shift that led to the identification of some of the classes in $v_2\mathbb{Z}_{(2)}[v_1, v_2]$, we hesitate to make any complete claims about the ring structure.

A complete computation of $\mathrm{tmf}^*(CP^\infty)$ was made in [**5**]; see especially Theorem 7.1 and Diagram 7.1. At first glance, the two descriptions appear quite different, but they seem to be compatible.

Proof of Theorem 5.13. We first prove that there is a nontrivial class in $[\Sigma^{-16}CP^\infty, \mathrm{tmf}]$ detected in filtration 0. This is obtained by using the virtual bundle $8(H-1) - (H^3 - H)$, where H denotes the complex Hopf bundle. Considered as a real bundle θ, this bundle satisfies $w_2(\theta)$ and $p_1(\theta) = 0$. Here we use from [**19**] that p_1 generates the infinite cyclic summand in $H^4(BSO;\mathbb{Z})$ and satisfies $r^*(p_1) = c_1^2 - 2c_2$ under $BU \xrightarrow{r} BSO$, and $\rho^*(p_1) = 2e_1$ under $BSpin \xrightarrow{\rho} BSO$, where $H^4(BSpin;\mathbb{Z})$ is an infinite cyclic group generated by e_1. The total Chern class of $9H - H^3$ is

$$(1+x)^9(1+3x)^{-1} = 1 + 6x + 18x^2 + \cdots ,$$

and hence

$$r^*(p_1(\theta)) = (c_1(9H - H^3))^2 - 2c_2(9H - H^3) = (6x)^2 - 2 \cdot 18x^2 = 0.$$

Thus $e_1(\theta) = 0$, hence $CP^\infty \xrightarrow{\theta} BSpin \to K(\mathbb{Z}, 4)$ is trivial, and so θ lifts to a map $CP^\infty \to BO[8]$. Hence its Thom spectrum induces a degree-1 map $T(\theta) \to MO[8]$. Since $\psi^3(H) = H^3 - H$, by [**20**] θ is $J_{(2)}$-equivalent to $8(H-1)$, and hence its Thom spectrum is $T(8(H-1)) = \Sigma^{-16}CP_8^\infty$. Using the Ando-Hopkins-Rezk orientation ([**1**]) $MO[8] \to \text{tmf}$, we obtain our desired class as the composite

$$\Sigma^{-16}CP_1^\infty \xrightarrow{\text{col}} \Sigma^{-16}CP_8^\infty \xrightarrow{T(\theta)} MO[8] \to \text{tmf}\,. \tag{17}$$

We will deduce our differentials from the d_3-differential $E_3^{4,21} \to E_3^{7,23}$ in the ASS converging to $\pi_*(\text{tmf})$. This can be seen in [**14**, p. 37] or [**11**, Theorem 2.2]; see Remark 5.14 for additional explanation. It is not difficult to show that, with M_{10} as in Lemma 5.2, the morphism

$$\text{Ext}_{A_2}^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2) \to \text{Ext}_{A_2}^{s,t}(M_{10}, \mathbf{Z}_2),$$

induced by the nontrivial A_2-map $M_{10} \to \mathbf{Z}_2$, sends the $\mathbf{Z}_2$ in $\text{Ext}_{A_2}^{7,23}(\mathbf{Z}_2, \mathbf{Z}_2)$ which is not part of the infinite tower to $h_1^2 v_1^4 v_2$.

We prefer to think about the ASS for $\text{tmf}_*(\Sigma^2 CP_{-\infty}^{-2})$, which, as we have noted, is isomorphic to that of $[\Sigma^* CP_1^\infty, \text{tmf}]$. The E_2-term was described in 5.4. Let $S^{-16} \to \Sigma^2 CP_{-\infty}^{-2} \wedge \text{tmf}$ correspond to the map in (17). Since $E_2(CP_{-\infty}^{-2} \wedge \text{tmf})$ in negative dimensions is built from copies of $\text{Ext}_{A_2}(M_{10}, \mathbf{Z}_2)$, we deduce from the previous paragraph that $h_1^2 v_1^4 v_2 g_{-16}$ in the ASS for $\text{tmf}_*(\Sigma^2 CP_{-\infty}^{-2})$ must be hit by a d_2- or d_3-differential, since it is the image of a class hit by a d_3. The only possibility is that it be d_2 from $h_1 v_1^4 g_{-8}$, as indicated by the dotted line in Diagram 5.10. Naturality of differentials with respect to h_1 and v_1^4 implies the differentials of 5.11 for $\epsilon = 0, 1$, all i, $j = 0$, and $k = 1$. Using the diagonal map of CP_1^∞ and the multiplication of tmf, powers of (17) give similar nontrivial elements in $[\Sigma^{-16k} CP_1^\infty, \text{tmf}]$ for all $k \geqslant 1$, and by the argument just presented, we establish the differentials of Theorem 5.11 for all k (with $j = 0$ still).

The only possible differentials on $v_2 g_{-16}$ would be some d_r with $r > 2$ hitting an element which is acted on nontrivially by h_1. However $h_1 v_2 g_{-16}$ has become 0 in E_3 since it was hit by a d_2-differential. Thus a nonzero differential on $v_2 g_{-16}$ would contradict naturality of differentials with respect to h_1-action. Hence there is a map $S^{-10} \to \Sigma^2 CP_{-\infty}^{-2} \wedge \text{tmf}$ hitting $v_2 g_{-16}$, and the argument of the previous paragraph implies that $d_2(h_1 v_1^4 v_2 g_{-8}) = h_1^2 v_1^4 v_2^2 g_{-16}$ and then other related differentials. This now establishes the differentials of 5.11 when $j = 1$, and sets in motion an inductive argument to establish these differentials for all $j \geqslant 1$.

No further differentials in the spectral sequence are possible, by dimensional and h_1-naturality considerations. □

Remark 5.14. The proof of the key d_3-differential in the ASS of tmf from the 17-stem to the 16-stem, which was cited above, has not had a thorough proof in the literature.

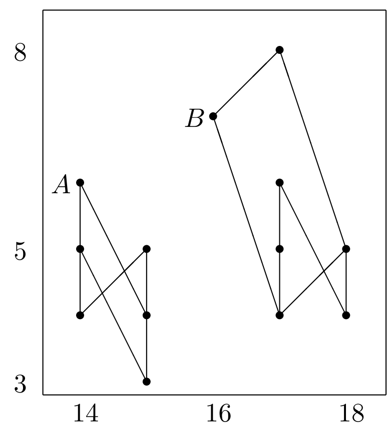
Giambalvo's original argument was incorrect and his correction merely refers to "a homotopy argument." The current authors cited Giambalvo's result in [**11**] without additional argument. We provide some more detail here regarding this differential.

The relevant portion of the ASS of tmf appears in Diagram 5.15. In [**11**] and [**14**], this was pictured as the ASS of $MO[8]$, but through dimension 18,

$$\mathrm{Ext}_A^{*,*}(H^*(MO[8]),\mathbf{Z}_2)\approx \mathrm{Ext}_{A_2}^{*,*}(\mathbf{Z}_2\oplus\Sigma^{16}\mathbf{Z}_2,\mathbf{Z}_2).$$

One way of obtaining the differentials from 15 to 14, as in [**14**], is to note that the [8]-cobordism group of 14-dimensional manifolds is $\mathbf{Z}_2$, and so the top two elements must be killed by differentials. It is not difficult to compute in Ext the Massey product formula $B=\langle A,h_0,h_1\rangle$, where A and B are as in Diagram 5.15. This can be seen as v_1^4 times a similar formula between classes in dimensions 6 and 8. Since A is 0 in homotopy, the associated Toda bracket formula says that B must be divisible by η. But only 0 can be divisible by η in dimension 16 here. Thus B must be killed by a differential, and the depicted way is the only way this can happen.

Diagram 5.15. *Portion of ASS of* tmf:



The differentials in the ASS converging to $\mathrm{tmf}_*(CP_{-\infty}^{-2}\wedge CP_{-\infty}^{-2})$ are implied by the same considerations that worked for $CP_{-\infty}^{-2}$. The $\mathbf{Z}_2[v_0,v_1,v_2]$-parts in Theorem 5.7 cannot support differentials by dimensionality and h_1-naturality. For the *bo*-like part, we prefer thinking about it as $[\Sigma^{*+4}CP_1^\infty\wedge CP_1^\infty,\mathrm{tmf}]\approx \mathrm{tmf}^{-*-4}(CP_1^\infty\wedge CP_1^\infty)$, where the product structure is more apparent.

Let Z_n denote the nonzero element of $\mathrm{Ext}_{A_2}^{0,-8n}(\mathbf{Z}_2,\mathbf{C}_1^\infty\otimes\mathbf{C}_1^\infty)/\ker(h_1)$. By Theorem 5.7, Z_n can be represented by $X_1^iX_2^{n-i}$ for any $1\leqslant i<n$. If n is even and $n\geqslant 4$, choosing i even, Z_n is an infinite cycle because it is an external product of infinite cycles. Hence by the proof of Theorem 5.11,

$$d_2(h_1^\epsilon v_1^{4i}v_2^jZ_{2k-1})=h_1^{\epsilon+1}v_1^{4i}v_2^{j+1}Z_{2k}$$

for $\epsilon=0,1$, $i,j\geqslant 0$, and $k\geqslant 2$.

Finally, X_1X_2 is an infinite cycle since there is nothing that it can hit. Also, $h_1v_2X_1X_2$ and $h_1^2v_2X_1X_2$ are not hit by differentials since

$$\mathrm{Ext}_{A_2}^{0,-8}(\mathbf{Z}_2, \mathbf{C}_1^\infty \otimes \mathbf{C}_1^\infty) = 0$$

by Theorem 5.7. We obtain the following.

Theorem 5.16. *In grading $\geqslant 10$, there is an isomorphism of graded abelian groups*

$$\mathrm{tmf}^*(CP_1^\infty \wedge CP_1^\infty) \approx y\mathbb{Z}_{(2)}[v_1, v_2, X_1, X_2] \oplus \bigoplus_{n\geqslant 3} I_n \cdot \mathbb{Z}_{(2)}[v_1, v_2] \oplus \mathbb{Z}_{(2)}[Z](bo^* \oplus v_2\mathbb{Z}_{(2)}[v_1, v_2]),$$

where $|y| = 12$, $|X_i| = 8$, $|Z| = 16$, $|v_1| = -2$, and $|v_2| = -6$. Here $I_n = \ker(F_n \xrightarrow{\epsilon} \mathbb{Z})$, where F_n is a free abelian group with basis $\{X_1^iX_2^{n-i} \colon 1 \leqslant i < n\}$, and $\epsilon(X_1^iX_2^{n-i}) = 1$.

Thus I_n consists of all polynomials of grading n with sum of coefficients equal to 0. We could have extended the description in 5.16 down to grading 8, but the description would have been slightly more complicated, since it would include h_1v_2Z and $h_1^2v_2Z$.

The motivation for this section was to see if perhaps

$$\ker(\mathrm{tmf}^*(CP^\infty \times CP^\infty) \xrightarrow{d^*} \mathrm{tmf}^*(CP^\infty))$$

might be something nice like the $I(X_1 - X_2)$, which was the case for $BP^*(-)$. In Theorem 5.16, we described $\mathrm{tmf}^*(CP^\infty \wedge CP^\infty)$. To obtain $\mathrm{tmf}^*(CP^\infty \times CP^\infty)$, we add on two copies of $\mathrm{tmf}^*(CP^\infty)$, which was described in Theorem 5.13. Denote by Z_1 and Z_2 the generators in $\mathrm{tmf}^{16}(CP^\infty \times CP^\infty)$. Monomials $Z_1^iZ_2^{n-i}$ should equal Z^n of 5.16 plus perhaps elements of I_{2n} of 5.16. The class y of 5.16 plus perhaps a sum of elements of higher filtration is in $\ker(d^*)$ and not in the ideal generated by $(Z_1 - Z_2)$. Thus, as expected, $\ker(d^*)$ does not have the nice form that it did for $BP^*(-)$, and so we cannot use this argument to show that the axial class in $\mathrm{tmf}^*(RP^\infty \times RP^\infty)$ is $u(X_1 - X_2)$. However, we showed something like this by a completely different method in Theorem 4.4. We feel that the results obtained in Theorems 5.13 and 5.16 should be of independent interest.

References

[1] M. Ando, M.J. Hopkins, and C. Rezk, Multiplicative orientations of KO-theory and of the spectrum of topological modular forms, preprint, `http://www.math.uiuc.edu/~mando/papers/koandtmf.pdf`.

[2] L. Astey, Geometric dimension of bundles over real projective spaces, *Quar. J. Math. Oxford* **31** (1980), no. 122, 139–155.

[3] L. Astey, A cobordism obstruction to embedding manifolds, *Illinois J. Math.* **31** (1987), no. 2, 344–350.

[4] L. Astey and D.M. Davis, Nonimmersions of real projective spaces implied by BP, *Bol. Soc. Mat. Mexicana.* **25** (1980), no. 1, 15–22.

[5] T. Bauer, Elliptic cohomology and projective spaces–a computation, preprint, `http://www.few.vu.nl/~tilman/cpinfty.pdf`.

[6] R.R. Bruner, D.M. Davis, and M. Mahowald, Nonimmersions of real projective spaces implied by tmf, in *Recent progress in homotopy theory* (Baltimore, MD, 2000), *Contemp. Math.* **293**, 45–68, A. M. S., Providence, RI (2002).

[7] D.M. Davis, Table of immersions and embeddings of real projective spaces, `http://www.lehigh.edu/∼dmd1/immtable`.

[8] D.M. Davis, A strong nonimmersion theorem for real projective spaces, *Annals of Math.* **120** (1984), no. 3, 517–528.

[9] D.M. Davis, On the Segal conjecture for $\mathbf{Z}_2 \times \mathbf{Z}_2$, *Proc. Amer. Math. Soc.* **83** (1981), no. 3, 619–622.

[10] D.M. Davis and M. Mahowald, Ext over the subalgebra A_2 of the Steenrod algebra for stunted projective spaces, in *Current trends in algebraic topology, Part 1* (London, Ont., 1981), *Canadian Math. Soc. Conf. Proc.* **2**, 297–342, Amer. Math. Soc., Providence, RI, 1982.

[11] D.M. Davis, A new spectrum related to 7-connected cobordism, in *Algebraic topology* (Arcata, CA, 1986), *Lecture Notes in Math.* **1370**, 126–134, Springer-Verlag, New York, 1989.

[12] D.M. Davis, The immersion conjecture for $RP^{8\ell+7}$ is false, *Trans. Amer. Math. Soc.* **236** (1978,) 361–383.

[13] D.M. Davis and V. Zelov, Some new embeddings and nonimmersions of real projective spaces, *Proc. Amer. Math. Soc.* **128** (2000), no. 12, 3731–3740.

[14] V. Giambalvo, On $\langle 8 \rangle$-cobordism, *Illinois J. Math.* **15** (1971), 533–541; Correction in *Illinois J. Math.* **16** (1972), 704.

[15] I.M. James, On the immersion problem for real projective spaces, *Bull. Amer. Math. Soc.* **69** (1963), 231–238.

[16] N. Kitchloo and W.S. Wilson, The second real Johnson-Wilson theory and nonimmersions of RP^n, *Homology, Homotopy, and Applications,* this volume.

[17] W.H. Lin, On conjectures of Mahowald, Segal and Sullivan, *Math. Proc. Cambridge Philos. Soc.* **87** (1980), no. 3, 449–458.

[18] W.H. Lin, D.M. Davis, M. Mahowald, and J.F. Adams, Calculation of Lin's Ext groups, *Math. Proc. Cambridge Philos. Soc.* **87** (1980), no. 3, 459–469.

[19] J. Milnor and J.D. Stasheff, *Characteristic classes*, *Annals of Math. Studies*, **70**, Princeton Univ. Press, Princeton, NJ (1974).

[20] D. Sullivan, Genetics of homotopy theory and the Adams conjecture, *Annals of Math.* **100** (1974), no. 1, 1–79.

Donald M. Davis `dmd1@lehigh.edu`

Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA

Mark Mahowald `markmah@mac.com`

Department of Mathematics, Northwestern University, Evanston, IL 60208, USA