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ON MINIMAL MODELS IN INTEGRAL HOMOTOPY THEORY

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(*communicated by Larry Lambe*)

Abstract

This paper takes its starting point in an idea of Grothendieck on the representation of homotopy types. We show that any locally finite nilpotent homotopy type can be represented by a simplicial set which is a finitely generated free group in all degrees and whose maps are given by polynomials with rational coefficients. Such a simplicial set is in some sense a universal localisation/completion as all localisations and completions of the homotopy is easily constructed from it. In particular relations with the Quillen and Sullivan approaches are presented. When the theory is applied to the Eilenberg-MacLane space of a torsion free finitely generated nilpotent group a close relation to the the theory of Passi polynomial maps is obtained.

To Jan–Erik Roos on his sixty–fifth birthday

Inspired by some ideas of A. Grothendieck ([**5**]) I shall in this article give an algebraic description of nilpotent homotopy types with finitely generated homology (in each degree).

From a homotopy theoretic perspective the main result says that every such nilpotent homotopy type may be represented by a simplicial set which is of the form $\mathbf{Z}^n$ for some n in each degree and for which the face and degeneracy maps are *numerical* maps; maps that are given by polynomials with rational coefficients. Furthermore, the cohomology may be computed using numerical cochains, any map between such models is homotopic to a numerical one and homotopic numerical maps are numerically homotopic. The construction of a model is rather straightforward; one first shows that the cohomology of an Eilenberg Mac-Lane space can be computed using numerical cochains and then uses induction over a principal Postnikov tower.

Localisation and completion fits very nicely into this framework. If R is either a subring of $\mathbf{Q}$ or is the ring of p-adic numbers for some prime p and $K := R\bigotimes\mathbf{Q}$ then a numerical function $\mathbf{Z}^m \to \mathbf{Z}^n$ clearly induces a map $K^m \to K^n$ but also takes R^m to R^n. Hence a numerical simplicial set gives rise to a numerical set obtained by replacing each $\mathbf{Z}^n$ that appears in some degree by R^n. For a model this new space is the R-localisation when R is a subring of $\mathbf{Q}$ and the p-completion when $R = \mathbf{Z}_p$. In this way a numerical model might be thought of as a universal localisation.

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If G is a finitely generated nilpotent torsion-free group, the theory may be applied to $K(G,1)$ but stronger results are available. In fact G has a canonical structure of numerical group and for that structure the numerical functions $G \to \mathbf{Z}$ are exactly the polynomial functions in Passi's sense (cf., [**10**]). Furthermore, the cohomology of the group (with $\mathbf{Z}$-coefficients) may be computed using numerical cochains (which are the same as the Passi polynomial maps).

The theory can be reformulated in terms of cosimplicial rings. The cosimplicial ring of numerical mappings of a model into $\mathbf{Z}$ has the property of being a free *numerical ring* in each degree, where a numerical ring is a ring together with certain extra operations. The cosimplicial ring of all cochains is a numerical ring and the numerical model can be obtained by the usual construction of a free cosimplicial object homology equivalent to a given one. As a further motivation for the relevance of numerical rings for homotopy we also note that the cohomology of cosimplicial numerical rings admit an action of all cohomology operations.

When passing to the rational localisation of a numerical model, the theory should be compared to the theories of Quillen ([**12**]) and Sullivan ([**14**]) of rational homotopy. In the case of Quillen's theory the first step in his construction of a differential Lie algebra model is to represent a nilpotent homotopy type by the simplicial classifying space of a simplicial group G that in each degree is the quotient of a finitely generated free group by some element of the descending central series. That means that G is a torsion free finitely generated nilpotent group and hence $K(G,1)$ is a numerical space. As for Sullivan's approach the closest connection I have found is by considering his spatial realisation of a differential graded algebra model. We will define a natural quotient of that spatial realisation that has a natural structure of $\mathbf{Q}$-numerical model.

To return to the starting point for the results of this article, Grothendieck's idea was to represent any (locally finite) homotopy type by a simplicial set that is $\mathbf{Z}^n$ and for which the face and degeneracy maps would be polynomial maps. While this seems reasonable for rational homotopy types as rational homotopy theory is fairly linear (and is a consequence of the results of this article) several people have pointed out problems with that idea (though I am not aware of any proof that it is impossible). It is the suggestion of the present article that polynomial maps should be replaced by numerical maps that are the same as polynomial maps over the rationals. Of course, numerical maps have appeared previously in homotopy theory in connection with polynomial functors but I do not know if there is any relations with that theory.

1. Preliminaries

It will be convenient to state a few preliminary results in some generality. If $\mathcal{A}$ is an additive category for which all idempotents have kernels, then the category of homological complexes in $\mathcal{A}$, concentrated in non-negative degrees, is equivalent, by the Dold-Puppe constructions, to the category of simplicial objects in $\mathcal{A}$. Following [**9**], which will be our general reference concerning simplicial results, we will use $N(-)$ for the functor going from simplicial objects to complexes and $\Gamma(-)$ for the functor going the other way. Similarly for cosimplicial objects and cohomological complexes concentrated in non-negative degrees.

If $\mathcal{C}$ now is a category having finite products (and in particular a final object $*$), then if X is a simplicial object in $\mathcal{C}$ we say that X is a Kan complex in $\mathcal{C}$ if the simplicial set X^K, where $(X^K)_n := \mathrm{Hom}_{\mathcal{C}}(K, X_n)$, is a Kan complex for all $K \in \mathcal{C}$. If $X, Y \in \mathbf{s}\mathcal{C}$, $\mathbf{s}\mathcal{C}$ being the simplicial objects of $\mathbf{s}\mathcal{C}$, then we can define the function complex Y^X as a simplicial set using the definition in terms of (p,q)-shuffles as in [**9**, 6.7]. Then [**9**, 6.9] goes through, so that if Y is Kan, then so is Y^X. Furthermore, we say that a sequence $F \to X \to Y$ of simplicial objects is a Kan fibration if, for all $K \in \mathcal{C}$, $F^K \to X^K \to Y^K$ is a Kan fibration in the usual sense. Finally, still following [**9**, 18.3], we define for $F, B \in \mathbf{s}\mathcal{C}$, G a group object in $\mathbf{s}\mathcal{C}$, and a group action $G \times F \to F$m a twisted cartesian product (TCP) to be an $E(\tau) \in \mathbf{s}\mathcal{C}$ s.t. $E(\tau)_n = F_n \times B_n$ and

$$\begin{aligned}
\partial_i(f,b) &= (\partial_i f, \partial_i b)\ i > 0, \ (i)\\
\partial_0(f,b) &= (\tau(b) \cdot \partial_0 f, \partial_0 b), \ (ii)\\
s_i(f,b) &= \quad (s_i f, s_i b), \quad\ (iii)
\end{aligned}$$

where $\tau\colon B_q \to G_{q-1}$ is a morphism fulfilling the identities of ([**9**]). If $F = G$ with the action being translation we will, still following ([**9**]), speak of a principal twisted tensor product ($PTCP$). It is then clear that if $T\colon \mathcal{C} \to \mathcal{C}'$ is a product preserving functor, then it takes TCP's to TCP's and PTCP's to PTCP's. In particular, if $K \in \mathcal{C}$ then $(F^K, B^K, G^K, E(\tau)^K)$ is a TCP and so, [**9**, 18.4], $F \to E(\tau) \to B$ is a Kan fibration if F is a Kan complex and, in particular, $E(\tau)$ is Kan if B is.

Suppose now that A is a ring object in $\mathcal{C}$ s. t. multiplication induces

$$\mathrm{Hom}_{\mathcal{C}}(X \times Y, A) = \mathrm{Hom}_{\mathcal{C}}(X, A) \bigotimes_{\mathbf{Z}} \mathrm{Hom}_{\mathcal{C}}(Y, A)$$

for all $X, Y \in \mathcal{C}$ and that $\mathrm{Hom}_{\mathcal{C}}(X, A)$ is torsion free for all $X \in \mathcal{C}$. If $F \in s\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(F, A)$ is a cosimplicial abelian group and we put $H^*_A(X) := H^*(N(\mathrm{Hom}_{\mathcal{C}}(F, A)))$ and more generally $H^*_A(X, M) := H^*(N(\mathrm{Hom}_{\mathcal{C}}(F, A)) \bigotimes_{\mathbf{Z}} M)$, for M an abelian group. Now, Szczarba's proof of the simplicial Brown's theorem [**15**] uses only universal expressions and hence goes through in this context. If τ is 1-trivial (i.e., $\tau_{|Bi} = *$, $i = 0, 1$) we therefore get a spectral sequence

$$E_2^{p,q} = H_A^{*,p}(B, H_A^{*,q}(F)) \Rightarrow H_A^{p+q}(E(\tau)). \tag{1}$$

This is functorial for product preserving functors $T\colon \mathcal{C} \to \mathcal{C}'$ and this is so also if $\mathcal{C}' = \mathcal{S}ets$ and we make no particular assumption on $T(A)$.

2. Numerical spaces

Having dealt with some preliminaries we can start getting down to business.

Definition 2.1. A *numerical function* from $\mathbf{Z}^m$ to $\mathbf{Z}^n$, $m, n \geqslant 0$, is a function $\mathbf{Z}^m \to \mathbf{Z}^n$ which can be expressed by polynomials with rational coefficients. If M and N are free abelian groups of rank m resp. n, then a numerical function from M to N is a function $M \to N$ such that for one (and hence any) choice of group isomorphisms $\mathbf{Z}^m \xrightarrow{\sim} M$ and $N \xrightarrow{\sim} \mathbf{Z}^n$, the composite $\mathbf{Z}^m \xrightarrow{\sim} M \to N \xrightarrow{\sim} \mathbf{Z}^n$ is a numerical function. The category $\mathcal{N}um$ has as objects finitely generated free abelian groups and as morphisms the numerical functions.

Remark 1. In the literature numerical functions sometimes appear under the name of polynomial maps. I dislike this terminology as it could easily be interpreted to mean maps given by polynomials with *integer* coefficients, indeed to me it seems the natural interpretation. Furthermore, it does not seem unlikely that polynomial maps will have a role to play in homotopy theory and hence deserve a name of their own.

It is clear that $M, N \mapsto M \times N$ is a product in $\mathcal{N}um$ and that the zero group is a final object so that $\mathcal{N}um$ has finite products. Furthermore, addition makes all objects in $\mathcal{N}um$ abelian group objects and addition and product makes $\mathbf{Z}$ a ring object. If $X \in s(\mathcal{N}um)$ then we will put $H^*_{\mathcal{N}um}(X, \mathbf{Z}) := H^*_{\mathbf{Z}}(X)$. Define $\mathcal{N}um_i := \mathrm{Hom}_{\mathcal{N}um}(\mathbf{Z}^i, \mathbf{Z})$ which thus is a ring, the ring of *numerical functions in i variables.*

Proposition 2.2. *i)* $\mathcal{N}um_1 = \bigoplus_{k \geqslant 0} \mathbf{Z}\binom{x}{k}$ *where* $\binom{x}{k}$ *is the numerical function* $n \mapsto \binom{n}{k}$.

ii) $\mathcal{N}um_i = \mathcal{N}um_1 \bigotimes \mathcal{N}um_1 \bigotimes \ldots \bigotimes \mathcal{N}um_1$ *(i times).*

iii) For $M, N \in \mathcal{N}um$,

$$\mathrm{Hom}_{\mathcal{N}um}(M \times N, \mathbf{Z}) = \mathrm{Hom}_{\mathcal{N}um}(M, \mathbf{Z}) \bigotimes_{\mathbf{Z}} \mathrm{Hom}_{\mathcal{N}um}(N, \mathbf{Z}).$$

iv) $\mathcal{N}um$ *is anti-equivalent to the full subcategory of the category of rings whose objects are the rings isomorphic to* $\mathcal{N}um_i$ *for some* i.

Proof. i) and ii) are well known. (Use $\Delta f(x) := f(x+1) - f(x)$ and induction on the degree for i) and $\Delta_{last\ variable}$ for ii).) Then iii) follows from ii) as any object in $\mathcal{N}um$ is isomorphic to some $\mathbf{Z}_i$. As for iv), $M \mapsto \mathrm{Hom}_{\mathcal{N}um}(M, \mathbf{Z})$ gives a functor in one direction and properly interpreted $R \mapsto \mathrm{Hom}_{Rings}(R, \mathbf{Z})$ will be a quasi-inverse. Now, evaluation gives a natural morphism

$$\phi_M \colon M \to \mathrm{Hom}_{Rings}(\mathrm{Hom}_{\mathcal{N}um}(M, \mathbf{Z}), \mathbf{Z})$$

in the category of sets. I claim that this map is a bijection. Indeed, we may assume that $M = \mathbf{Z}^i$ for some i and using ii) that $i = 1$. As already $x = \binom{x}{1}$ separates points $\phi := \phi_{\mathbf{Z}}$ is injective. Since $\mathcal{N}um_1 \bigotimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}[x]$, any ring homomorphism $\mathcal{N}um_1 \to \mathbf{Z}$ is determined by what it does to x and so ϕ is surjective. We now want to show that any ring homomorphism $\mathrm{Hom}_{\mathcal{N}um}(N, \mathbf{Z}) \to \mathrm{Hom}_{\mathcal{N}um}(M, \mathbf{Z})$ induces, through ϕ_M and ϕ_N, a numerical map $M \to N$. We reduce to $M = \mathbf{Z}_i$ and $N = \mathbf{Z}$ so we want to show that if $\rho \colon \mathcal{N}um_1 \to \mathcal{N}um_i$ is a ring homomorphism then $\mathbf{Z}^i \ni n \mapsto \rho(x)(n) \in \mathbf{Z}$ is a numerical function which is obvious by the definition of $\mathcal{N}um_i$. Hence $R \mapsto \mathrm{Hom}_{Rings}(R, \mathbf{Z})$ maps the subcategory of rings isomorphic to some $\mathcal{N}um_i$ into $\mathcal{N}um$ and by what we have just proved it is a quasi-inverse to $M \mapsto \mathrm{Hom}_{\mathcal{N}um}(M, \mathbf{Z})$. □

The proposition immediately gives the following corollary:

Corollary 2.3. *The category* $\mathbf{s}(\mathcal{N}um)$ *is anti-equivalent to the full subcategory of the category of cosimplicial rings consisting of those cosimplicial rings* R. *for which* R_n *is isomorphic to some* $\mathcal{N}um_i$ *for all* n.

Proof. □

Remark 2. It is this corollary that will allow us to interpret the results of this section in terms of cosimplicial rings and eventually an algebraic description of nilpotent homotopy types.

Let us now note that if $M, N \in \mathcal{N}um$, then any group homomorphism $M \to N$ is a numerical function so that the category $\mathcal{F}ree$ of free abelian groups of finite rank and homomorphisms embeds naturally in $\mathcal{N}um$. This is an additive category where all idempotents have kernels so the equivalences Γ and N are defined. From the explicit description of Γ, [**9**, p. 95], it follows that there is a unique function $T\colon \mathbf{N}^{\mathbf{N}} \to \mathbf{N}^{\mathbf{N}}$ such that if $\ldots \to C_n \to C_{n-1} \to \ldots$ is a complex in $\mathcal{F}ree$ and $r_{C_*} \in \mathbf{N}^{\mathbf{N}}$ is defined by $r_{C_*}(i) = \operatorname{rk} C_i$, then $T(r_{C_*})(i) = \operatorname{rk}\Gamma(C_*)_i$. If M is a finitely generated abelian group we let $g(M)$ be the minimal number of generators of M. If X is a nilpotent space with $H_i(X, \mathbf{Z})$ finitely generated for all $i \geqslant 0$, we put

$$g_n(X) := \sum_{i=0}^{\infty} g(\pi_n^i(X)/\pi_n^{i+1}(X)), \tag{2}$$

$$h_n(X) := \sum_{i=0}^{\infty} g(\mathrm{torsion}(\pi_n^i(X)/\pi_n^{i+1}(X))), \tag{3}$$

where $\pi_n^0(X) := \pi_n(X)$ and $\pi_n^{i+1}(X) := \langle \gamma(x)x^{-1} | x \in \pi_n^i(X);\ \gamma \in \pi_1(X)\rangle$. We note also that the rank of a f. g. free abelian group is invariant under numerical isomorphisms so we may unambigously speak of its rank as an object in $\mathcal{N}um$.

Definition 2.4. i) A *numerical space* is a connected numerical simplicial object X s. t. if F is the forgetful functor $\mathcal{N}um \to \mathcal{S}ets$, then the natural map $H^*_{\mathcal{N}um}(X, \mathbf{Z}) \to H^*(F(X), \mathbf{Z})$ is an isomorphism and the homology groups $H_*(F(X), \mathbf{Z})$ are finitely generated.

ii) A *fibred numerical space* is a connected simplicial object X which is an inverse limit

$$X = \varprojlim(\ldots \to X^{n+1} \to X^n \to \ldots \to X^0.)$$

where

1. X_0^n is a point for all n.
2. For all j there is an N s.t. $X_j^{n+1} \to X_j^n$ is an isomorphism for all $n \geqslant N$.
3. For all n, $X^{n+1} \to X^n$ is a PTCP (in $\mathcal{N}um$) with fiber some $\Gamma(M^n)$, M^n an $\mathcal{F}ree$-complex.
4. $X^0 = *$.

iii) A *special numerical space* is a fibred numerical simplicial object X s. t., using the notations of the previous definition, there is for each n integers (h_n, i_n) so that

$H^i(M^n) = 0$ for $i \neq h_n$ and $H^{h_n}(M^n) = \pi^{i_n}_{h_n}(X)/\pi^{i_n+1}_{h_n}(X)$. We will call the sequence $\ldots \to X^{n+1} \to X^n \to \ldots$ a *special tower.*

iv) A special numerical space X is *minimal* if

$$\forall n : \operatorname{rk} M^n = \sum_i T(\tau_i)(n),$$

where

$$\tau_i(j) = \begin{cases} g_i(X), & \text{if } j = i, \\ h_i(X), & \text{if } j = i+1, \\ 0, & \text{if } j \neq i, i+1. \end{cases}$$

We will call the sequence $\ldots \to X^{n+1} \to X^n \to \ldots$ a *minimal tower.*

Remark 3. For X a special numerical space

$$\forall n : \operatorname{rk} M^n \geqslant \sum_i T(\tau_i)(n),$$

which justifies the terminology minimal.

Lemma 2.5. *i) If $F \to E \to B$ is a* 1*-trivial TCP in* $\mathbf{s}(\mathcal{N}um)$ *and if F and B are numerical spaces then so is E.*

ii) A special numerical space X is a numerical space and a Kan object. In particular, $F(X) = \operatorname{Hom}_{\mathcal{N}um}(*, X)$ is Kan.

iii) If X is a special numerical space then

$$\forall n : \operatorname{rk} X_n \geqslant \sum_i T(\tau_i)(n).$$

Proof. Indeed, i) follows immediately from the map between the spectral sequences of (1) for the TCP and the induced TCP in $\mathbf{s}(\mathcal{S}ets)$. As for ii), the trivial nature of the limit in the definition of a special numerical space allows us to assume that X equals some X_n and then by i) that $X = \Gamma(M)$, for some $\mathcal{F}ree$-complex M. As $H^*(M)$ is concentrated in one degree, M is homotopic to a bounded complex M', indeed to one concentrated in 2 degrees, and then using the naive truncations of M' and i) again we may assume that $M = N[n]$ for some object N in $\mathcal{F}ree$. Now there is a numerical (indeed a $\mathcal{F}ree$-) PTCP $\Gamma(N[n-1]) \to X \to \Gamma(N[n])$ with X $\mathcal{F}ree$-contractible so by induction on n and Zeeman's comparison theorem (cf., [**16**]) we may assume that $n = 1$. We have a $\mathcal{F}ree$-PTCP

$$\Gamma(N_1[1]) \to \Gamma((N_1 \times N_2)[1]) \to \Gamma(N_2[1]),$$

so we may assume that $N = \mathbf{Z}$. Hence we are reduced to showing that $H^*_{\mathcal{N}um}(K(\mathbf{Z},1)) \to H^*(K(\mathbf{Z},1))$ is an isomorphism. A numerical 1-cocycle (for K($\mathbf{Z}$,1)) is just a numerical function $f\colon \mathbf{Z} \to \mathbf{Z}$ s.t. $f(x+y) = f(x) + f(y)$ so $f(x) = ax + b$ for some $a, b \in \mathbf{Z}$ and a 1-coboundary is of the form $f(x) = c$. As we have exactly the same description for set theoretical 1-cochains and 1-coboundaries we get an isomorphism for $* = 1$. As $* = 0$ is trivial it only remains to show that $H^i_{\mathcal{N}um}(K(\mathbf{Z},1)) = 0$ for $i \geqslant 2$ as this is true in the set case. Now $K(\mathbf{Z},1)_n = \mathbf{Z}^{n+1}$

and all the face operators are projections or sums of two adjacent coordinates. We can grade $\mathcal{N}um_i$ by

$$\deg \binom{x_1}{n_1}\binom{x_2}{n_2}\dots\binom{x_k}{n_k} = \sum_i n_i$$

and then $\mathrm{Hom}_{\mathcal{N}um}(K(\mathbf{Z},1),\mathbf{Z})$ becomes a graded complex as

$$\binom{x+y}{n} = \sum_{i+j=n}\binom{x}{i}\binom{y}{j}.$$

Hence to show the required vanishing we can replace

$$\mathcal{N}um_i = \sum \mathbf{Z}\binom{x_1}{n_1}\binom{x_2}{n_2}\dots\binom{x_k}{n_k}$$

by

$$\overline{\mathcal{N}um_i} = \prod \binom{x_1}{n_1}\binom{x_2}{n_2}\dots\binom{x_k}{n_k}$$

as the cohomology can be computed degree by degree. This gives us a complex T, say. Now, $\overline{\mathcal{N}um_i} = \mathrm{Hom}(\mathbf{N}^i,\mathbf{Z})$ where

$$\binom{x_1}{n_1}\binom{x_2}{n_2}\dots\binom{x_k}{n_k} \mapsto \text{char fct of}(n_1,n_2,\dots,n_k)$$

and this equality respects maps induced by projections and sums of coordinates (but it is not a ring isomorphism). Therefore, T is additively isomorphic to the standard cochain complex of $K(\mathbf{N},1)$ and so $H^*(T) = H^*(K(\mathbf{N},1),\mathbf{Z}) = Ext^*_{\mathbf{Z}[t]}(\mathbf{Z},\mathbf{Z})$ and the latter group is clearly concentrated in degree 0 and 1. That a special numerical space is Kan follows by induction on the Postnikov tower and a trivial passage to the limit. Finally, to prove iii) it suffices to prove that if M is a $\mathcal{F}ree$-complex with a single non-zero homology group $H_i(M)$, then $\mathrm{rk}\, M_i \geqslant g(H_i(M))$ and $\mathrm{rk}\, M_{i+1} \geqslant g(tor(H_i(M)))$. This, however, is clear by the principal divisor theorem. □

Remark 4. i) The next to last part of the proof of ii) looks somewhat mysterious and may be clarified by noting that $\mathcal{N}um_i$ is the ring of invariant differential operators on the formal i-dimensional torus. Its coproduct is therefore dual by Cartier duality [**3**, II,§4]) to the product on the coordinate ring of the formal i-dimensional torus. Similarly, $\mathrm{Hom}(\mathbf{N}^i,\mathbf{Z})$ is the Cartier dual of the i-dimensional formal additive group. As the formal i-dimensional torus and the i-dimensional formal additive group are isomorphic as formal schemes but not as formal group schemes, $\mathcal{N}um_i$ is isomorphic to $\mathrm{Hom}(\mathbf{N}^i,\mathbf{Z})$ as coalgebras but not as rings. However, in defining the differentials of chains on $K(-,1)$ only the coproduct is used.

ii) Had we worked with polynomial instead of numerical functions everything would have worked up to the statement $H^i_{pol}(K(\mathbf{Z},1)) = 0$ for $i > 1$. This statement is false however. As a matter of well known fact, the polynomial 2-cocycle $((x+y)^p - x^p - y^p)/p$, p prime, is not the coboundary of a polynomial 1-cochain. It is the boundary of the numerical 1-cochain $(x^p - x)/p$ as the lemma predicts.

Lemma 2.6. *Let Y be a special numerical space and X a numerical space. Then F, the forgetful functor, induces a bijection*

$$\mathrm{Hom}(X,Y)/(\text{numerical homotopy}) \to \mathrm{Hom}(F(X),F(Y))/(\text{homotopy})$$

where $\mathrm{Hom}(-,-)$ *means based maps.*

Proof. Indeed, it will be easier to prove a stronger statement. Let $Z, V \in \mathcal{N}um$ be pointed objects and V Kan and put $[Z,V]_n := \pi_n((V^Z)_b, *)$, where $(-)_b$ denotes based maps. As above, if $V_1 \to V_2 \to V_3$ is a TCP of Kan objects then $(V^{Z_1})_b \to (V^{Z_2})_b \to (V^{Z_3})_b$ is a fibration and we get a long exact sequence of homotopy. If $V_1 \to V_2 \to V_3$ is a PTCP then we get as usual the extra precision that the fibers of $[Z,V_2]_0 \to [Z,V_3]_0$ are the orbits under an action of $[Z,V_1]$ and the sequence extends to $[Z,V_2]_0 \to [Z,V_3]_0 \to [Z,\overline{W}V_1]_0$ (cf. [**9**, p. 87]). We now consider a sequence of PTCP's $Y^n \to Y^{n-1}$ as in the definition of a special numerical space. We want to prove by induction on n that $[X,Y^n]_m \to [F(X),F(Y^n)]_m$ is a bijection for all m. The case $n = 0$ certainly causes no problem and in general we have the PTCP $Y^n \to Y^{n-1}$ with fiber some $\Gamma(M^n)$. The extra precision given to the long exact sequence is exactly what is needed to make the 5-lemma work and we reduce hence to showing that $[X,\Gamma(M^n)]_m \to [F(X),F(\Gamma(M^n))]_m$ is a bijection. Now, M^n is homotopic to a bounded complex, so we may assume that M^n is bounded. By the same dvissage as before we reduce to $M^n = \mathbf{Z}[0]$ and then this bijection is true by the definition of numerical space. Putting $m = 0$ we then get the lemma for Y replaced by Y^n. To pass from Y^n to Y we use the Milnor exact sequence

$$* \to \varprojlim{}^1[X,Y^n]_1 \to [X,Y] \to \varprojlim[X,Y^n] \to *,$$

the similar sequence for $F(X)$ etc and the 5-lemma. □

We have now come to the main result of this section.

Theorem 2.7. *Let X be a simplicial set which is nilpotent (which to us will include being connected) of finite type (i. e., finitely generated homology in each degree).*

i) There is a minimal special numerical space Y and a homotopy equivalence $X \to F(Y)$.

ii) If Y' is a numerical space and $X \to F(Y')$ a homotopy equivalence, there is a unique, up to numerical homotopy, numerical morphism $Y' \to Y$, where Y is as in i), making the following diagram commute up to homotopy

$$\begin{array}{ccc} & & F(Y) \\ & \nearrow & \uparrow \\ X & \to & F(Y'). \end{array}$$

iii) If Y' is also special then $Y' \to Y$ is a numerical homotopy equivalence.

Remark 5. It is not true that a homology equivalence between minimal special numerical spaces is necessarily an isomorphism as is shown by the following example:

$$\Gamma(\mathbf{Z} \xrightarrow{2} \mathbf{Z}) \xrightarrow{3} \Gamma(\mathbf{Z} \xrightarrow{2} \mathbf{Z}).$$

Indeed, multiplication by 3 induces an isomorphism on the homology, $\mathbf{Z}/2$, but is clearly not an isomorphism.

Proof. Let $\ldots \to X^n \to X^{n-1} \to \ldots$ be a minimal principal Postnikov system, i.e., the $X^n \to X^{n-1}$ are principal fibrations with fiber some $K(\pi_m^i(X)/\pi_m^{i+1}(X), m)$ using the notations of (2) in order of increasing i and m. We will step by step replace, up to homotopy, X^n by some $F(Y^n)$ and $X^n \to X^{n-1}$ by $F(-)$ of a PTCP $Y^n \to Y^{n-1}$. Assume that we have done this up to $n-1$. We then have a cartesian diagram, where $K(M, m)$ is the fiber of $X^n \to X^{n-1}$,

$$\begin{array}{ccccc} & & & & K(M,m) \\ & & & & \downarrow \\ & & X^n & \to & T \\ & & \downarrow & & \downarrow \\ F(Y^{n-1}) & \stackrel{\sim}{\to} & X^{n-1} & \to & K(M, m-1). \end{array}$$

Here the right hand fibration is the standard one with $T \sim *$. Choose a resolution $F_* \to M$ by free f.g. abelian groups s.t. $\operatorname{rk} F_0 = g(M)$, $\operatorname{rk} F_1 = g(\operatorname{tor} M)$ and $F_i = 0$ for $i > 1$. There is then a numerical PTCP $\Gamma(F.[m]) \to I \to \Gamma(F.[m+1])$ s.t. $F(-)$ applied to it is homotopic to $K(M, m) \to T \to K(M, m+1)$. Hence by lemma 2.5 there is a morphism $\rho: Y^{n-1} \to \Gamma(F.[m+1])$ such that $F(\rho)$ is homotopic to $Y^{n-1} \to X^{n-1} \to K(M, m+1)$. Let $Y^n \to Y^{n-1}$ be the PTCP induced by ρ from $\Gamma(F.[m]) \to I \to \Gamma(F.[m+1])$. Then $F(Y^n) \to F(Y^{n-1})$ is homotopic to $X^n \to X^{n-1}$ and we put $Y := \varprojlim(\ldots \to Y^n \to Y^{n-1} \to \ldots)$. Then there is a morphism $X \to F(Y)$ which by construction is a homology equivalence and so a homotopy equivalence as X and $F(Y)$ are nilpotent. If $X \to F(Y')$ is a homology equivalence, where Y' is a numerical space, then by obstruction theory applied to the $F(Y^n) \to F(Y^{n-1})$ there is a map $F(Y') \to F(Y)$ s. t.

$$\begin{array}{ccc} & & F(Y') \\ & \nearrow & \downarrow \\ X & \to & F(Y) \end{array}$$

commutes up to homotopy. By lemma 2.6 there is a morphism $Y' \to Y$ s.t. $F(Y') \to F(Y)$ is homotopic to the given $F(Y') \to F(Y)$. In case Y' also is special another application of lemma 2.6 shows that $Y' \to Y$ is a numerical homotopy equivalence. $\square$

As we will see in an example in the next section, minimal models are not unique up to isomorphism. In the simple case we can however say that a minimal tower must be preserved.

Proposition 2.8. *Let X and Y be simple minimal numerical spaces and $X^\cdot$ and $Y^\cdot$ their minimal towers. Any map $f: X \to Y$ induces a map $f^\cdot: X^\cdot \to Y^\cdot$.*

Proof. The proof is easily reduced to the following statement. If $X' \to X$ is a PTCP for a simplicial group $\Gamma(M_.)$ with $M_i = 0$ if $i \leqslant n$ and $Y' \to Y$ is a PTCP for a simplicial group $\Gamma(N_.)$ with $N^i = 0$ for $i > n+1$ and for which $N_{n+1} \to N_n$

is injective then for any commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

there is a factorisation $X \to Y'$ of the diagram. This again amounts to saying that f' is constant on a fibre of g. In proving this we immediately reduce to the case when $X = Y = \Delta^m$, and then need only show that the map is constant on the fibre over e_m, the unique non-degenerate n-simplex of Δ^n. As PTCP's over a simplex are trivial we may assume that $X' \to X$ and $Y' \to Y$ are trivial PTCP's. Hence we reduce to showing that any map $\Gamma(M_.)\times\Delta^m \to \Gamma(N_.)$ is constant on $\Gamma(M_.)_m \times\{e_m\}$. In general a map $\Gamma(M_.) \times \Delta^m \to \Gamma(N_.)$ is the same thing as an additive map $\mathbf{Z}[\Gamma(M_.) \times \Delta^m] \to \Gamma(N_.)$ and what we want to show is that $[(k, e_m)] - [(0, e_m)]$ is mapped to zero for $k \in \Gamma(M_.)_m$. If $m \leqslant n$ then this is obvious. When $m \geqslant n$ want to show that the kernel K of the map $\mathbf{Z}[\Gamma(M_.) \times \Delta^m] \to \mathbf{Z}[\Delta^m]$ induced by projection maps to zero in $\Gamma(N_.)$. To prove this is equivalent to showing that $N(K) \to N(\Gamma(N_.)) = N_.$ is zero. Now, $N(K)$ is a complex of free abelian groups, $H_i(M_.)$ is zero when $i > n$ and K_i is zero when $i \leqslant n$ so that any map $N(K) \to N_.$ is null-homotopic. However, M_i is zero when $i > n+1$ so any homotopy is zero. □

3. Localisation and completion

We will now extend the theory presented so far to some other base rings than the (sometimes only implicitly mentioned) ring of rational integers. Our choice of rings is dictated on the one hand by which rings that are being used for defining localisation and completion in homotopy theory, on the other hand by which rings for which a straightforward generalisation of the notion of numerical function admits a description similar to the one given for the integers. Somewhat surprisingly these two requirements seem to give the same answer.

Definition-Lemma 3.1. Let R the rational numbers or the ring $\mathbf{Z}_p$ of p-adic integers. An (R-)numerical function $F \to G$ between finitely generated free R-modules is a function that can be defined by polynomials with coefficients in $R\bigotimes\mathbf{Q}$.

i) The R-algebra, $\mathcal{N}um_n(R)$, of numerical functions $R^n \to R$ is free as R-module on the functions $(r_1, r_2, \ldots, r_n) \mapsto \binom{r_1}{m_1}\ldots\binom{r_n}{m_n}$.

ii) The evaluation map $R^n \to \mathrm{Hom}_{R-algebras}(\mathcal{N}um_n(R), R)$ is a bijection.

Proof. Let us first prove that a polynomial with rational coefficients mapping $\mathbf{Z}^n$ to $\mathbf{Z}$ will map R^n to R. If R is a subring of $\mathbf{Q}$ then it is an intersection of the localisations $\mathbf{Z}_{(p)}$ that contains it so in that case one is reduced to $R = \mathbf{Z}_{(p)}$ and then to $R = \mathbf{Z}_p$ as $\mathbf{Z}_{(p)} = \mathbf{Q} \cap \mathbf{Z}_p$. Our polynomial defines a continuous function $\mathbf{Z}_p^n \to \mathbf{Q}_p$, $\mathbf{Z}_p \subset \mathbf{Q}_p$ is a closed subset and $\mathbf{Z}^n \subset \mathbf{Z}_p^n$ is a dense subset. As $\mathbf{Z}^n$ is mapped into $\mathbf{Z} \subset \mathbf{Z}_p$ it follows that $\mathbf{Z}_p^n$ is mapped into $\mathbf{Z}_p$. Conversely, if we have a polynomial with $R\bigotimes\mathbf{Q}$-coefficients mapping R^n into R it maps in particular $\mathbf{Z}^n$

into R and a slight modification of the argument in the proof of proposition 2.2 show that it is an R-linear combination of products of binomial polynomials.

The proof of the second part is entirely analogous to the same statement for $R = \mathbf{Z}$ given in the proof of proposition 2.2. $\square$

Remark 6. One may wonder whether torsion free rings R other than the ones mentioned in the lemma have the property that elements of $\mathcal{N}um_n$ maps R^n into R. Of course, $\mathcal{N}um_n$ itself or any ring containing $\mathbf{Q}$ is such an example but one can show that if R is a finitely generated ring and if $P(r) \in R$ for all $r \in R$ and all numerical polynomials $P \in \mathcal{N}um_1$ then $R \subset \mathbf{Q}$. Indeed, we have that $(x^p - x)/p \in \mathcal{N}um_1$ for all primes p and hence $r^p - r \in pR$. From this one concludes that r is algebraic for all $r \in R$ and hence that R is a subring of a number field K. The condition that $r^p - r \in pR$ for all primes p and $r \in R$ then implies that almost all primes in K are of degree 1 which implies that $K = \mathbf{Q}$ (details of this type of argument can be found in [**6**]).

We will say that a ring R is a *coefficient ring* if it is a subring of $\mathbf{Q}$ or equals the ring $\mathbf{Z}_p$ of p-adic integers for some prime p. By some abuse of language we will say that a nilpotent simplicial set is *R-local* if it is R-local in the usual sense if $R \subseteq \mathbf{Q}$ and is p-complete if $R = \mathbf{Z}_p$. Similarly we will speak of the R-localisation of a nilpotent simplicial set.

From the lemma one can continue almost verbatim and introduce, when R is a coefficient ring, (special) R-numerical spaces (the condition being that cohomology with R-coefficients can be computed using R-numerical cochains). One also, though we shall not use it, gets that one may represent any R-local nilpotent finite type space by a special R-numerical space. We will however note that if $F \to G$ is a numerical map between finitely generated free abelian groups and R is a subring of $\mathbf{Q}$ or $\mathbf{Z}_p$ then we get an induced R-numerical map $F \bigotimes R \to G \bigotimes R$. This gives a functor, also denoted by $-\bigotimes R$ from simplicial numerical objects to simplicial R-numerical objects. We also get a map of simplicial sets $F(X) \to F(X \bigotimes R)$. For special numerical spaces this is a localisation map:

Theorem 3.2. *Let R be a coefficient ring.*

i) A special R-numerical space is R-local.

ii) If X is a special numerical space then $F(X) \to F(X \bigotimes R)$ is an R-localisation map.

Proof. The first part uses the fact that locality is stable under fibrations, that $K(M, n)$ is local if M is a finitely generated free R-complex which is clear as its homotopy groups are and that by speciality one may reduce to such $K(M, n)$'s. For the second part we again reduce to $K(M, n)$'s for M a finitely generated free $\mathbf{Z}$-complex. $\square$

Remark 7. I do not know if $X \bigotimes R$ is a R-numerical space if X is a numerical space nor if it always is local.

Proposition 3.3. *Let X and Y be minimal R-numerical spaces where R is $\mathbf{Q}$, $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$. Then any homotopy equivalence $f\colon X \to Y$ is homotopic to a map that* *is the inverse limit of a map of inverse systems $X^{\cdot} \to Y^{\cdot}$ (where $X^{\cdot}$ and $Y^{\cdot}$ are sequences as required in the definition of minimality) such that each $X^n \to Y^n$ is an isomorphism at each point. In particular, a homotopy equivalence between minimal numerical spaces is homotopic to an isomorphism and even more particularly minimal models of the same space are isomorphic.*

Proof. Let p be the characteristic of R modulo its maximal proper ideal. For evident reasons we will have to carefully distinguish between equality versus homotopy of maps and we will start off with some observations. They will apply equally well to simplicial sets as to numerical spaces but for simplicity we will speak only of simplicial sets;N in any case the numerical case may be deduced from the set-theoretic one using (2.7). To begin with, if G is a simplicial abelian group with a single homotopy group M in degree $n \geqslant 0$ then isomorphism classes of PTCP's with structure group G over a simplicial set $X_{.}$ correspond to elements of $H^{n+1}(X, M)$ (cf., [**9**, class of PTCP's]). It follows from that proof together with the use of the mapping cone construction that if $X \to Y$ is a map of simplicial sets then the relative cohomology $H^{n+1}(Y, X, M)$ correspond to equivalence classes of PTCP's over Y together with a trivialisation of its pullback to X where two of them are equivalent if they are isomorphic over Y by an isomorphism whose pullback to X is homotopic to one that preserves the given trivialisations. Let us also note that as we are dealing with principal fibrations, giving a trivialisation is the same thing as giving a section.

The way the Postnikov tower $\{X^{\cdot}\}$ fits into this description is that $X^{n+1} \to X^n$ is universal for PTCP's in degree h_n over X^n that are provided with a trivialisation over X. From this we can construct the maps by induction over n. We therefore may assume we have the following diagram that is assumed to commute up to homotopy:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X^{n+1} & & Y^{n+1} \\ \downarrow & & \downarrow \\ X^n & \xrightarrow{f^n} & Y^n \end{array}$$

and f^n is an isomorphism. Let M_n resp. N_n be the complexes for which $X^{n+1} \to X^n$ resp. $Y^{n+1} \to Y^n$ are $\Gamma(M_n)$- resp. $\Gamma(N_n)$-PTCP's. As $H^n(M_{.})$ resp. $H^n(N_{.})$ are $\pi^{i_n}_{h_n}(X)$ resp. $\pi^{i_n}_{h_n}(Y)$, f induces an isomorphism between them. We lift this isomorphism to a map of complexes $M_{.} \to N_{.}$. Now, as $N_{.}$ is minimal, the image of N_{n+1} in N_n is contained in pN_n and hence by Nakayama's lemma the map $N_n \to N_n$ is a surjection. As $M_{.}$ also is minimal, the rank of M_n is the same as that of N_n and so the map $M_n \to N_n$ is an isomorphism. This implies that $M_{.} \to N_{.}$ is an isomorphism.

Now, the pullback of $Y^{n+1} \to Y^n$ along the composite $X \to Y \to Y^n$ has a section and hence a trivialisation. As the diagram is homotopy commutative we get a trivialisation of the pullback of $Y^{n+1} \to Y^n$ along the composite $X \to X^n \to Y^n$. This in turn, by the universality of $X^{n+1} \to X^n$, gives a mapping from $X^{n+1} \to X^n$ to the pullback of $Y^{n+1} \to Y^n$ along $X^n \to Y^n$ covering the isomorphism $\Gamma(M_.) \to \Gamma(N_.)$. This again is nothing but a map $f^{n+1}: X^{n+1} \to Y^{n+1}$ of PTCP's covering $\Gamma(M_.) \to \Gamma(N_.)$. As the latter as well as the base map, f^n, are isomorphisms so is f^{n+1}. By construction it gives rise to a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X^{n+1} & \xrightarrow{f^{n+1}} & Y^{n+1} \end{array}$$

and thus finishes the induction step. □

Remark 8. Uniqueness of minimal models is not true over the integers: Fix an integer $n > 1$ and consider a class α of order n in $H^4(K(\mathbf{Z}/n,1),\mathbf{Z}) = \mathbf{Z}/n$. This can be used as k-invariant for a fibration over $\Gamma(\mathbf{Z}[2] \xrightarrow{n} \mathbf{Z}[1])$ with fibre $K(\mathbf{Z},3)$. Now, multiplication by any invertible residue β modulo n on $H^2(\mathbf{Z}/n,\mathbf{Z}) = \mathbf{Z}/n$ can be induced by a homotopy equivalence of the base. Taking into account also the action of multiplication by -1 on $K(\mathbf{Z},4)$ we see that two k-invariants α and α' give homotopic total minimal models if (and only if) $\alpha' = \pm\beta^2\alpha$.

On the other hand it follows from (2.8) that any isomorphism between two such models induces an isomorphism over $\Gamma(\mathbf{Z}[2] \xrightarrow{n} \mathbf{Z}[1])$. Let us first consider the induced isomorphism on the base $B := \Gamma(\mathbf{Z}[2] \xrightarrow{n} \mathbf{Z}[1])$. As B consists of a point in degree 0 any map from B to itself preserves the base point 0. Then in degree we have a numerical isomorphism from $\mathbf{Z}$ to $\mathbf{Z}$ taking 0 to 0. This in turn is a polynomial isomorphism from $\mathbf{Q}$ to $\mathbf{Q}$ and as such is well known to have the form $x \mapsto ax + b$ and as 0 is preserved $b = 0$, as $\mathbf{Z}$ is preserved $a \in \mathbf{Z}$ and as its inverse has the same properties $a = \pm 1$. Now, it is easy to see that a map $B \to B$ is determined by what it does in degree 1 so any automorphism of B is given by multiplication by ± 1. Multiplication by -1 acts trivially on the k-invariants in question so we may assume that the induced map on B is the identity. For the rest of the argument we will ignore the numerical structure. Any $K(\mathbf{Z},3)$-PTCP over B is classified as fibration over B by a torsor over the simplicial set of automorphisms of the simplicial set $K(\mathbf{Z},3)$. On the one hand we have the translations which is isomorphic as simplicial set $K(\mathbf{Z},3)$. Using them we may concentrate on based isomorphisms. If we more generally consider the simplicial set of based endomorphisms of $K(\mathbf{Z},3)$ then the argument of (2.8) shows that they are determined by the action on the third homotopy group so that the simplicial group of based automorphisms is equal to the constant simplicial group $\{\pm 1\}$. Hence the simplicial group of automorphisms of $K(\mathbf{Z},3)$ is the split extension of $K(\mathbf{Z},3)$ by the constant group $\{\pm 1\}$ acting by multiplication. From this it follows that two $K(\mathbf{Z},3)$-PTCP's that are isomorphic as fibrations either are isomorphic as PTCP's or one is isomorphic to the transformation by multiplication of -1 on the other. In the notation above that means the relation $\alpha' = \pm 1\alpha$. Hence, there are in general minimal models that are homotopic but not isomorphic.

Remark 9. Note that the fibrations we get are exactly the first non-trivial step in the Postnikov tower of the three-dimensional lens spaces. I have no idea whether the fact that the isomorphism classes of minimal models coincides with the homeomorphism classes of these lens spaces has any significance.

4. Cosimplicial ring interpretation

We now want to interpret what we have proved in terms of cosimplicial rings. The rings that we have already encountered have the property of being closed under binomial, and not just polynomial, functions. We will need to formalise this property. Note first that by (2.2) there are unique polynomials $h^m_n(x)$, $f_n(x,y)$ and $g^m_n(x)$, which are linear, bilinear and linear respectively s.t.

$$\begin{aligned}\binom{x}{m}\binom{x}{n} &= h^m_n\left(\binom{x}{1},\binom{x}{2},\dots,\binom{x}{mn}\right)\\ \binom{xy}{n} &= f_n\left(\binom{x}{1},\dots,\binom{x}{n},\binom{y}{1},\dots,\binom{y}{n}\right)\\ \binom{\binom{x}{m}}{n} &= g^m_n\left(\binom{x}{1},\binom{x}{1},\dots,\binom{x}{mn}\right).\end{aligned}$$

Definition 4.1. A *numerical ring* is a commutative ring R together with functions $\binom{-}{n}\colon R \to R$, $n \geqslant 0$, s.t.

i) $\binom{0}{n} = 1$,

ii) $\binom{1}{n} = 0$, $n \geqslant 2$.

iii) $\binom{r}{1} = r$,

iv) $\binom{r+s}{n} = \sum_{i+j=n}\binom{r}{i}\binom{s}{i}$,

v) $\binom{r}{m}\binom{r}{n} = h^m_n\left(\binom{r}{1},\binom{r}{2},\dots,\binom{r}{mn}\right)$,

vi) $\binom{rs}{n} = f_n\left(\binom{r}{1},\dots,\binom{r}{n},\binom{s}{1},\dots,\binom{s}{n}\right)$,

vii) $\binom{\binom{r}{m}}{n} = g^m_n\left(\binom{r}{1},\binom{r}{1},\dots,\binom{r}{mn}\right)$.

Remark 10. i) In the presence of v), i-iv) and vi-vii) are equivalent to R being a λ-ring and one can in fact replace the polynomials f and g by those used in the theory of λ-rings (they are equal modulo vi)). Indeed, in the presence of vi) the polynomials appearing in the theory of λ-rings reduce to linear resp. bilinear polynomials. As f and g are characterised by iv) resp. v) being true for $r, s \in \mathbf{Z}$ and $\mathbf{Z}$ is a special λ-ring we see that they necessarily reduce to f and g.

ii) In terms of the ring homomorphism $\phi\colon R \to 1 + R\big[[t]\big]$ from the theory of λ-rings the extra axiom v) can be described as follows. The map ϕ can be thought of as giving an exponentiation of $1+t$ by elements of R through $(1+t)^r := \phi(r)$. For the exponentiation of an arbitrary element $1 + c(t)$ of $1 + R\big[[t]\big]$ there are two candidates. Either we can use the R-module structure on $1 + R\big[[t]\big]$ given by ϕ or we can substitute $c(t)$ for t in $(1+t)^r$. In the presence of the λ-ring axioms, v) is equivalent to these two constructions coinciding. The details are left to the reader.

As usual we can construct for any set S the free numerical ring $\mathcal{N}um(S)$ on S, i.e., $\mathrm{Hom}_{\mathcal{S}ets}(S,R) = \mathrm{Hom}_{\mathcal{N}um-rings}(\mathcal{N}um(S),R)$ for any numerical ring R and also the free numerical ring $\mathcal{N}um^g(M)$ on an abelian group M. Then $\mathcal{N}um(S) = \mathcal{N}um^g(\mathbf{Z}[S])$.

Lemma 4.2. *For any set S, $\mathbf{Z}^S$ is a numerical ring with pointwise operations. Let $x_j \in \mathcal{N}um_i$, $1 \leqslant j \leqslant i$, be the projection on the j'th factor. Then $\mathcal{N}um_i = \mathcal{N}um(\{x_j : 1 \leqslant j \leqslant i\})$.*

Proof. It is clear that $\mathbf{Z}$ with the binomial functions is a numerical ring, in fact the axioms were set up precisely to ensure this. Hence $\mathbf{Z}^S$ certainly is a numerical ring with pointwise operations. Furthermore, $\mathcal{N}um_i \subset \mathbf{Z}^{\mathbf{Z}^i}$ is clearly stable under all operations and is hence a sub-numerical ring. Therefore, if $S := \{x_j\}$ there is a map of numerical rings $\mathcal{N}um(S) \to \mathcal{N}um_i$ taking x_j to x_j. By (2.2) this map is surjective and if we prove that

$$\mathcal{N}um(S) = \sum \mathbf{Z}\binom{x_1}{n_1}\cdots\binom{x_i}{n_i} =: A,$$

where the sum is not necessarily direct, then we are finished. As $x_j \in A$ it is sufficient to show that A is a numerical subring of $\mathcal{N}um(S)$. That A is a subring follows from v) and stability under $\binom{-}{n}$ follows from the rest of the axioms. $\square$

For any complex $0 \to C^0 \to C^1 \to C^2 \to \dots$ of abelian groups we may construct a cosimplicial numerical ring as $\mathcal{N}um(C^{\cdot}) := \mathcal{N}um^g(\Gamma(C^{\cdot}))$, where $\mathcal{N}um(-)$ is extended pointwise to simplicial objects. In case C_i is f.g. free for all i, then $\mathcal{N}um(C^{\cdot}) = \mathrm{Hom}_{\mathcal{N}um}(\Gamma(\mathrm{Hom}_{\mathbf{Z}}(C^{\cdot})),\mathbf{Z})$ giving the relation with the preceding results.

Lemma 4.3. *Let $R \to S$ and $R \to T$ be morphisms of numerical rings. Then there is a structure of numerical ring on $S \bigotimes_R T$ making it the pushout, in the category of numerical rings, of $R \to S$ and $R \to T$.*

Proof. If $S \to S \bigotimes_R T$ and $T \to S \bigotimes_R T$ are to be morphisms of numerical rings then the definition of $\binom{-}{n}$ are forced by the axioms so we begin by showing that they are well-defined. As was remarked above any numerical ring is also a special λ-ring. This means that if U is a numerical ring and if we put

$$\begin{array}{rcl} \phi\colon U & \to & 1+tU\big[[t]\big] \\ r & \mapsto & \sum_{i=0}^{\infty}\binom{r}{i}t^i \end{array},$$

then ϕ is a ring homomorphism where $1+tU\big[[t]\big]$ is given the ring structure of [**2**, Exp. V,2.3] with multiplication denoted by $*$. This gives us ring homomorphisms $S \to 1+tS\bigotimes_R T\big[[t]\big]$ and $T \to 1+tS\bigotimes_R T\big[[t]\big]$ coinciding on R and hence we get a ring homomorphism $S\bigotimes_R T \to 1+tS\bigotimes RT\big[[t]\big]$ showing that the operations $\binom{-}{n}$ are well-defined on $S\bigotimes_R T$. To show that we get a numerical ring we reduce to $R = \mathbf{Z}$ and S and T free numerical rings on finite sets and conclude by (2.2) and (4.2) as these show that $S\bigotimes T$ then is again a (free) numerical ring. Clearly $S\bigotimes_R T$ has the required universal property. $\square$

We can now define twisted cartesian coproducts (TCcP's) of cosimplicial numerical rings (CNR's) etc by dualising section 1 using the coproduct of lemma 4.3.

Definition 4.4. A *fibred CNR* is a cosimplicial numerical ring Y s. t. $Y = \varinjlim(\ldots \to Y_n \to Y_{n+1} \to \ldots)$ and

i) $\forall i \exists N : Y_n^i \to Y_{n+1}^i$ is an isomorphism for $n \geqslant N$,

ii) $Y_n \to Y_{n+1}$ is a PTCcP with cofibre of the form $\mathcal{N}um(\Gamma(C_n^\cdot))$, where $C_n^\cdot$ is a bounded complex of free (not necessarily f. g.) abelian groups.

We will say that a cosimplicial numerical ring R is *connected* if $H^0(R) = \mathbf{Z}$ and $H^1(R)$ is torsion free and 1-connected if $H^0(R) = \mathbf{Z}$, $H^1(R) = 0$ and $H^2(R)$ is torsion free, where $H^*(R)$ denotes the cohomology of the corresponding complex.

Remark 11. i) In condition ii) we do not need the condition on the cohomology of $C_n^\cdot$; by refining and modifying the sequence $\ldots \to Y_n \to Y_{n+1} \to \ldots$ this condition can always be fulfilled.

ii) The conditions defining connectivity and 1-connectivity should be considered in the light of the universal coefficient sequence; connectivity and 1-connectivity refers to vanishing of homology.

Theorem 4.5. *i) Let X be a fibred CNR and $Y_1 \to Y_2$ a CNR-morphism which is a cohomology equivalence. Then every morphism $X \to Y_2$ can be lifted to Y_1.*

ii) Any cohomology equivalence between fibred CNR's is a homotopy equivalence.

iii) Let Y be a 1-connected CNR. Then there is a unique (up to homotopy) fibred CNR X and a numerical ring homomorphism $X \to Y$ which is a cohomology equivalence.

iv) Let Y be a connected CNR. Then there is a unique (up to homotopy) CNR X fulfilling the liftability with respect to cohomology equivalences as in i) and a numerical ring homomorphism $X \to Y$ which is a cohomology equivalence.

Proof. i) is proved by successive liftings (and is essentially a numerical cosimplicial version of the proof of the similar property for cdga's). The lifting at one stage is accomplished as follows. By the definition of 4.4 $C_n^\cdot$ will consist of normalised cochains in X^n and the subcomplex K of $N(X^n)$ generated by $N(X^{n-1})$ and $C_n^\cdot$ is the mapping cone of a map of complexes $N(X^{n-1})[1] \to C_n^\cdot$. We then extend the lifting of $N(X^{n-1}) \to Y_2$ to K. This in turn gives a lift of $\Gamma(C_n^\cdot) \to Y_2$ which then gives a lift of $\mathcal{N}um(\Gamma(C_n^\cdot)) \to Y_2$. Then ii) follows similarly. Let us turn to iii). We will build a Postnikov tower. Note to begin with that when building this tower we must kill homology and not cohomology. Hence we assume that we have $f\colon X^{n-1} \to Y$ s.t. if we consider X^{n-1} and Y as complexes then $H^i(C(f)) = 0$ if $i < n$ and $H^n(C(f))$ torsion free. Let $C^0 \to C^1$ be a free complex s.t. $H^0(C^\cdot) = H^n(C(f))$ and $H^1(C^\cdot) = \mathrm{tor} H^{n+1}(C(f))$. Then there is a morphism of complexes $C^\cdot[-n] \to C(f)$ inducing the identity on H^n and the natural inclusion on H^{n-1}. Hence there is a morphism of cosimplicial groups $\Gamma(C^\cdot[-n]) \to \Gamma(C(f))$ and by composition a morphism $\Gamma(C^\cdot[-n-1]) \to X^n$. (Note that $C(f)$ fits in to a distinguished triangle $N(X^n) \to N(Y) \to C(f)$.) The composite $\Gamma(C^\cdot[-n-1]) \to Y$ is, by construction, nullhomotopic. By adjunction we get $\mathcal{N}um(C^\cdot[-n-1]) \to X^n$ whose composite with $X^n \to Y$ is again nullhomotopic as adjunctions preserve homotopies. Therefore there is a PTCcP $X^n \to X^{n+1} \to \mathcal{N}um(C \cdot [-n])$ and a lifting $g\colon X^{n+1} \to Y$. I claim that $H^i(C(g)) = 0$ for $i \leqslant n$ and that $H^{n+1}(C(g))$ is torsion free. To do this one has to say something about the cohomology of $\mathcal{N}um(C^{\cdot}[-n])$. When $C^{\cdot}$ is finitely generated this has already been done and the general case is done by approximating $C^{\cdot}$ by finitely generated subcomplexes. After that one has to look at the Serre spectral sequence for the PTCcP constructed above. A small conceptual problem arises as we want to kill homology but are working with cohomology. This can certainly be overcome by brute force, but we will instead choose a hopefully more conceptual approach. This entails, however, the introduction of pro-(finitely generated abelian groups) and the reader who is unfamiliar with the concept of pro-objects will have no problem in translating the proof to follow into one using only cohomology. If $D^{\cdot}$ is a complex of torsion free abelian groups then we define its homology by $H^i(D^{\cdot}) := \text{"}\varprojlim\text{"}\{H_i(\mathrm{Hom}(D^{\cdot}_\alpha, \mathbf{Z}))\}$ (cf. [**1**, Exp. I,8]), where $D^{\cdot}_\alpha$ runs over all finitely generated subcomplexes of $D^{\cdot}$. We have the usual universal coefficient sequences expressing cohomology and homology in terms of each other if we put, for an abelian group M, $\mathrm{Hom}(M, \mathbf{Z})$ resp. $Ext1(M, \mathbf{Z})$ equal to $\text{"}\varprojlim\text{"}\{\mathrm{Hom}(M_\alpha, \mathbf{Z})\}$ resp. $\text{"}\varprojlim\text{"}\{Ext1(M_\alpha, \mathbf{Z})\}$, where M_α runs over all f.g. subgroups of M and, for a pro-object $\{M_\alpha\}$, $\mathrm{Hom}(\{M_\alpha\}, \mathbf{Z})$ resp. $Ext^1(\{M_\alpha\}, \mathbf{Z})$ equal to $\varinjlim\{\mathrm{Hom}(M_\alpha, \mathbf{Z})\}$ resp. $\varinjlim\{Ext^1(M_\alpha, \mathbf{Z})\}$. Hence our assumptions imply that $H_i(C(f)) = 0$ if $i < n$ and we want to prove that $H_i(C(g)) = 0$ if $i \leqslant n$. Furthermore, we may present $C^{\cdot}$ as a direct limit of complexes $C^{\cdot}_\alpha$ which are finitely generated free, concentrated in degrees 0 and 1 with H^0 free and H^1 torsion. Then

$$H_*(\mathcal{N}um(C^{\cdot}[-n])) = \text{"}\varprojlim\text{"}\{H_*(\mathcal{N}um(C^{\cdot}_\alpha[-n]))\}$$

and by (2.5 ii)

$$H_*(\mathcal{N}um(C^{\cdot}_\alpha[-n])) = H_*(K(H0(C^{\cdot}_\alpha), n)).$$

By the well-known computation of the cohomology of Eilenberg-MacLane spaces we get that $\tilde{H}_i(\mathcal{N}um(C^{\cdot}[-n])) = 0$ if $i < n$ or $= n+1$ (as $n \geqslant 2$) and $H_n(\mathcal{N}um(C^{\cdot}[-n])) = H_0(C^{\cdot}) = H_n(C(f))$. Finally, as above we get a Serre s.s.

$$H_i(X^n, H_j(\mathcal{N}um(C^{\cdot}[-n]))) \Rightarrow H_{i+j}(X^{n+1}).$$

This and the information we have on $H_j(\mathcal{N}um(C^{\cdot}[-n]))$ gives an exact sequence

$$\begin{array}{ccccccccc}
0 & \to & H_{n+1}(X^{n+1}) & & \to & & H_{n+1}(X^n) & \to & H_n(C(f)) \quad \to \\
& & & \nwarrow & & \nearrow & & & \\
& & & & H_{n+1}(Y) & & & & \\
& & \to & H_n(X^{n+1}) & \to & H_n(X^n) & \to & 0 & \\
& & & & \nwarrow & & \nearrow & & \\
& & & & & H_n(Y) & & &
\end{array}$$

and isomorphisms $H_i(X^{n+1}) \to H_i(X^n)$ for $i < n$. By construction we have an exact sequence

$$H_{n+1}(Y) \to H_{n+1}(X^n) \to H_n(C(f)) \to H_n(Y) \to H_n(X^n) \to 0.$$

Combining these two sequences we get that $H_{n+1}(Y) \to H_{n+1}(X^{n+1})$ is an epimorphism and that $H_i(Y) \to H_i(X^{n+1})$ is an isomorphism for $i \leqslant n$ i.e. that $H_i(C(g)) = 0$ for $i \leqslant n$. In case Y is only connected we only get that

$$\operatorname{coker}(H_{n+1}(Y) \to H_{n+1}(X^{n+1})) \to \operatorname{coker}(H_{n+1}(Y) \to H_{n+1}(X^n))$$

is zero so we may have to continue an infinite number of times just to kill homology in one degree and the end result will not necessarily be a special CNR. It will still have the lifting property of i) though. □

As should be no surprise there is a very tight relation between CNR's and simplicial numerical objects.

Proposition 4.6. *i) The functor that takes each CNR X and associates to it the simplicial scheme $\mathbf{Spec}\,X$ obtained by taking the spectrum in each degree is an equivalence of categories between CNR's that are free finitely generated numerical ring in each degree and a full subcategory $\mathcal{N}$ of the category of simplicial schemes. The inverse functor is taking global sections $\Gamma(X, \mathcal{O})$ of the structure sheaf.*

ii) The functor that to a simplicial scheme Y associates its simplicial set of $\mathbf{Z}$-points is an equivalence of categories from $\mathcal{N}$ to the category of simplicial numerical objects.

iii) The functor that to a simplicial numerical scheme X associates the CNR $\mathrm{Hom}_{Rings}(X, \mathbf{Z})$ induces an equivalence between the category of simplicial numerical rings and the full subcategory of CNR's that are finitely generated free in each degree.

Proof. The results clearly follow from (2.2). □

We will use these equivalences to think of a numerical space as the $\mathbf{Z}$-points $X(\mathbf{Z})$ of a simplicial scheme X. Localisation and completion has a particularly pleasant formulation in these terms; we have that $F(X(\mathbf{Z}) \bigotimes R) = X(R)$. In the case of completion we have the following rather striking fact which in particular shows that p-complete homotopy types can be described in terms of cosimplicial $\mathbf{Z}/p$-algebras whose cohomology is the cohomology of the type. We also add a rather curious fact saying that in the p-complete case we may use continuous chains to compute cohomology.

Proposition 4.7. *i) Let $X \in \mathcal{N}$. Then the reduction mod p map $X(\mathbf{Z}_p) \to X(\mathbf{Z}/p)$ is a bijection.*

ii) Assume X is a $\mathbf{Z}_p$-numerical space and give the components of $X(\mathbf{Z}_p)$ its p-adic topology. Then the cohomology of the complex of $\mathbf{Z}_p$-continuous $\mathbf{Z}_p$-valued cochains is isomorphic with ordinary cohomology of X with $\mathbf{Z}_p$-coefficients.

Proof. The first part clearly amounts to showing that the reduction mod p map $\mathrm{Hom}_{Rings}(\mathcal{N}um_i, \mathbf{Z}_p) \to \mathrm{Hom}_{Rings}(\mathcal{N}um_i, \mathbf{Z}/p)$ is a bijection. This can no doubt be done directly but the "real" reason why it is true is the following. As we have seen, $\mathbf{Z}^i = Spec\mathcal{N}um_i$ is the Cartier dual of $\hat{\mathbf{G}}_m^i$, the product of i copies of the formal multiplicative group, and so [**3**, II,§4] for any ring R, $\mathbf{Z}^i(R) = \mathrm{Hom}_{\text{R-formal groups}}(\hat{\mathbf{G}}_m^i, \hat{\mathbf{G}}_m)$ and it is well known [loc. cit.] that $\mathrm{Hom}_{\text{R-formal grps}}(\hat{\mathbf{G}}_m^i, \hat{\mathbf{G}}_m) = \mathbf{Z}_p^i$ for $R = \mathbf{Z}_p$ as well as $\mathbf{Z}/p$.

As for the second part we note that the ring of continuous functions from $\mathbf{Z}_p^n \to \mathbf{Z}_p$ equals the p-adic completion of $\mathcal{N}um_n$, this is Mahler's theorem ([**8**]). Hence the complex of continuous cochains is the completion of the complex of numerical cochains. This gives rise to short exact sequences

$$0 \to \varprojlim{}^1 H^{i-1}_{\mathcal{N}um}(X, \mathbf{Z}/p^n) \longrightarrow H^i_{cont}(X(\mathbf{Z}_p), \mathbf{Z}_p) \longrightarrow \varprojlim H^i_{\mathcal{N}um}(X, \mathbf{Z}/p^n) \to 0$$

but as the $H^i_{\mathcal{N}um}(X, \mathbf{Z}_p)$ are finitely generated $\mathbf{Z}_p$ the left hand side is zero and the right hand side is $H^i_{\mathcal{N}um}(X, \mathbf{Z}_p)$. $\square$

Remark 12. i) The first part of the proposition shows that the category of simplicial $\mathbf{Z}_p$-numerical objects is equivalent to a category cosimplicial $\mathbf{Z}/p$-algebras which in the case of $\mathbf{Z}_p$-numerical spaces computes the $\mathbf{Z}/p$-cohomology of the space. This accords more with the usual view of p-complete spaces where $\mathbf{Z}/p$-cohomology reflects isomorphisms. I do not however know of an intrinsic characterisation of the algebras of the form $\mathcal{N}um_i\,/p$ in the style of characterising $\mathcal{N}um_i$ as free numerical algebras. It should be noted that there is a Stone type duality between $\mathbf{Z}/p$-algebras R fulfilling $r^p = r$ for each $r \in R$ and totally disconnected compact spaces; the space is the set of ring homomorphisms into $\mathbf{Z}/p$ and the ring is the set of continuous maps into $\mathbf{Z}/p$. For $p = 2$ this is the usual Stone duality.

ii) By [**7**, V,Thm 2.3.10] we get that the cohomology of a p-complete finitely generated torsion free nilpotent group can also be computed using analytical cochains.

Let us end this section with an observation that shows that disregarding the rest of the section cosimplicial numerical rings are related to homotopy theory. Thus let R be a cosimplicial numerical ring and $\alpha \in H^i(R)$. If z is a representing cocycle in $N(R)$ for α it is represented by a map $\mathbf{Z}[-i] \to N(R)$ and hence a map $\Gamma(\mathbf{Z}[-i]) \to R$ and again by a map $\mathcal{N}um(\Gamma(\mathbf{Z}[-1])) \to R$. This induces a map on cohomology $H^*(\mathcal{N}um(\Gamma(\mathbf{Z}[-1]))) \to H^*(R)$. By (2.5) this means that all cohomology operations will operate on the cohomology of cosimplicial numerical rings with all relations being preserved.

5. Nilpotent groups

We will spend some time considering the case of $K(G,1)$'s or equivalently nilpotent groups. More precisely we will only consider those that are torsion free. It can be concluded from the results of the previous section that each such group G may be identified as a set with $\mathbf{Z}^n$ for some n in such a way that the multiplication and inverse are given by numerical maps and that the cohomology may be computed using numerical cochains. We will now see that there is a *canonical* way to define the structure of object in $\mathcal{N}um$ on the set underlying a nilpotent group such that the group structure is given by a group object in $\mathcal{N}um$. Indeed, let G be a nilpotent f. g. torsion free group and let $\mathbf{Z}[G]$ be its group algebra. Any function $\phi\colon G \to \mathbf{Z}$ gives rise to an additive function, also denoted ϕ $\phi\colon \mathbf{Z}[G] \to \mathbf{Z}$ using the fact that $\mathbf{Z}[G]$ is free on G. We say that ϕ is $\cdot$*-numerical*, where $\cdot$ is the product on G, if ϕ vanishes on some power of the augmentation ideal of $\mathbf{Z}[G]$. We denote by $\mathcal{N}um.(-,\mathbf{Z})$ the set of $\cdot$-numerical functions.

Remark 13. These functions were first introduced by Passi ([**10**]) and are also known as "Passi polynomial" maps. We have chosen a different terminology because $+$-numerical functions on $\mathbf{Z}^n$ are exactly numerical functions and because of subsequent results.

We need a preliminary result giving a characterisation of $\cdot$-numerical functions that may be of independent interest. For that recall that a module for a group G is said to be *unipotent* if it is a successive extension of modules with trivial action.

Lemma 5.1. *Let $(G,\cdot)$ be a torsion-free f. g. nilpotent group. Then a map $G \to \mathbf{Z}$ is $\cdot$-numerical if and only if it generates a $\mathbf{Z}$-finitely generated unipotent submodule of $\mathbf{Z}^G$.*

Proof. It is clear that a G-module M is unipotent if and only if it is annihilated by some power of the augmentation ideal of the group ring $\mathbf{Z}[G]$. By definition a map $G \to \mathbf{Z}$ is $\cdot$-numerical if and only if it is annihilated by a power of the augmentation ideal and as the augmentation ideal is two-sided this is true if and only if it generates a submodule that is. Finally, as $\mathbf{Z}[G]$ modulo any power of the augmentation ideal is a finitely generated $\mathbf{Z}$-module, any unipotent submodule of $\mathbf{Z}^G$ generated by one element is finitely generated. □

We will need the following result seemingly unrelated result. Recall that a (smooth) connected algebraic group is *unipotent* if it is an algebraic subgroup of the group of unipotent upper triangular $n \times n$-matrices for some n and that in that case every linear representation of it is unipotent (cf. [**13**, Cor. 3.4]). In particular the points of the group over a base field is a nilpotent group. Furthermore, if that base field is the rational numbers, if g is a point defined over the it and f is a polynomial vanishing on the group then we may introduce the polynomial in x $g^x := \exp(x\log(g))$, where the logarithm is a finite series mapping unipotent upper triangular matrices to nilpotent ones and the exponential is a finite series mapping nilpotent upper triangular matrices to unipotent ones. This polynomial vanishes on all integers and hence is identically zero. In particular it vanishes on g^r for $r \in \mathbf{Q}$. As the group is closed in the Zariski topology it is equal to the common zero set of all such f and g^r is in the group. This means that the group of rational points is a uniquely divisible nilpotent group.

Lemma 5.2. *An algebraic group whose underlying algebraic variety is isomorphic to affine space is unipotent.*

Proof. By a standard specialisation argument we may assume that the base field $\mathbf{F}$ is a finite field. By [**13**, Cor. 3.8] if the group, G say, is not unipotent it will, after possible extending the base field, contain a non-unipotent element. It has a non-trivial order prime to the characteristic p but the cardinality of $G(\mathbf{F})$ is a power of p as G is an affine space and is hence a p-group that cannot contain a non-trivial element of order prime to p. □

Remark 14. The author distinctly remembers having seen this result mentioned together with a notice that the proof used reduction to a finite base field but has been unable to find either that attribution or the actual reference.

One of the most natural questions on the relation between nilpotent torsion-free groups and numerical groups is answered by the following result.

Lemma 5.3. *Let G be a group object in $\mathcal{N}um$. Then the underlying group is finitely generated torsion-free nilpotent.*

Proof. If we extend the scalars of $\mathrm{Hom}_{\mathcal{N}um}(G,\mathbf{Z})$ to $\mathbf{Q}$ we get a polynomial ring over $\mathbf{Q}$ which is the affine algebra of an algebraic group $\mathcal{G}$ over $\mathbf{Q}$. Hence (5.2) applies and we conclude that $\mathcal{G}$ is unipotent and hence that $\mathcal{G}(\mathbf{Q})$ is a uniquely divisible nilpotent group. Clearly, G is a subgroup of $\mathcal{G}(\mathbf{Q})$ and hence is nilpotent and torsion-free. To prove finite generation we note that there as a finite dimensional faithful subrepresentation V of the representation of $\mathcal{G}$ on its affine algebra. We now want show that there is finitely generated subgroup M of V stable under the action of G. As V is finite dimensional it contains a finite number of vectors spanning it as $\mathbf{Q}$-vector space. After possibly multiplying them by a non-zero integer we may assume that they are contained in $\mathrm{Hom}_{\mathcal{N}um}(G,\mathbf{Z})$. It is therefore sufficient to show that each $f\in\mathrm{Hom}_{\mathcal{N}um}(G,\mathbf{Z})$ lies in a finitely generated subgroup of $\mathrm{Hom}_{\mathcal{N}um}(G,\mathbf{Z})$ invariant under G. For this we consider the product map $\varphi\colon G\times G\to G$ and write the pullback $\varphi^*f\in\mathrm{Hom}_{\mathcal{N}um}(G,\mathbf{Z})\bigotimes\mathrm{Hom}_{\mathcal{N}um}(G,\mathbf{Z})$ as $\sum_i f_i\otimes g_i$. This means that for $g,h\in G$ $f(g\cdot h)=\sum_i f_i(g)g_i(h)$ and by definition $h\in G$ acts on f by $(hf)(g)=f(gh)$. Hence we get that $hf=\sum_i g_i(h)f_i$ so that the translates of f by the elements of G lies in the finitely generated group spanned by the f_i and hence is finitely generated.

Thus, G is a subgroup of the subgroup $\mathcal{G}_M$ of elements of $\mathcal{G}(\mathbf{Q})$ stabilising M. As a subgroup of a finitely generated nilpotent group is finitely generated it is enough to show that $\mathcal{G}_M$ is finitely generated. This will be done by induction over the dimension of V (with $\mathcal{G}$ changing during the induction). As $\mathcal{G}$ is unipotent V contains a 1-dimensional subspace U on which $\mathcal{G}$ acts trivially. We may use the induction hypothesis on the image of $\mathcal{G}$ in $\mathrm{Aut}(V/U)$ and the image M' of M in V/U to conclude that the image of $\mathcal{G}_M$ in $\mathrm{Aut}(V/U)$ and it is then enough to show that the kernel of this map is finitely generated. However, that kernel is a subgroup of the abelian group of additive maps $\mathrm{Hom}(M',U\cap M)$ which is finitely generated. $\square$

Proposition 5.4. *Let $(G,\cdot)$ be a finitely generated torsion-free nilpotent group.*

i) G may be identified with $\mathbf{Z}^n$ in such a way that multiplication and inverse on G are numerical functions and $K(G,1)$ is a numerical space.

ii) $\mathcal{N}um\cdot(G,\mathbf{Z})$ is a numerical subring of the numerical ring of all functions $G\to\mathbf{Z}$.

iii) If G has been given the structure of group object in $\mathcal{N}um$ for which $K(G,1)$ is a numerical space then a function $G\to\mathbf{Z}$ is $\cdot$-numerical precisely when it is numerical with respect to the given numerical structure.

iv) The numerical ring $\mathcal{N}um\cdot(G,\mathbf{Z})$ is isomorphic to the free numerical ring on a finite number of generators and the natural map $G\to\mathrm{Hom}(\mathcal{N}um\cdot(G,\mathbf{Z}),\mathbf{Z})$ is a bijection.

v) The product map $G \times G \to G$ induces a map $\mathcal{N}um\cdot(G, \mathbf{Z}) \to \mathcal{N}um\cdot(G, \mathbf{Z}) \bigotimes \mathcal{N}um\cdot(G, \mathbf{Z})$, where $\mathcal{N}um\cdot(G, \mathbf{Z}) \bigotimes \mathcal{N}um\cdot(G, \mathbf{Z})$ is thought of as a subring of the set of functions $G \times G \to \mathbf{Z}$. The inverse $G \to G$ induces a map $\mathcal{N}um\cdot(G, \mathbf{Z}) \to \mathcal{N}um\cdot(G, \mathbf{Z})$.

Proof. i) follows from lemma 2.5 and induction over the length of the ascending central series,. ii) is obvious. As for iii) consider first a function $f: G \to \mathbf{Z}$ that is numerical with respect to the given numerical structure. Let $\mathcal{G}$ be the algebraic group over $\mathbf{Q}$ whose ring of regular function is $\mathrm{Hom}_{\mathcal{N}um}(G, \mathbf{Z}) \bigotimes \mathbf{Q}$. Then f generates a finite dimensional subrepresentation of $\mathrm{Hom}_{\mathcal{N}um}(G, \mathbf{Z}) \bigotimes \mathbf{Q}$ which is unipotent as $\mathcal{G}$ is by (5.2). Hence f $\cdot$-numerical by (5.1). Assume conversely that $f: G \to \mathbf{Z}$ is $\cdot$-numerical. Again by lemma 5.1 it generates a unipotent module M. We may choose a G-invariant filtration of M whose successive quotients are free of rank 1 with trivial G-action. Having done this, the G-action on M corresponds to a group homomorphism from G to U, the group of unipotent upper triangular integer $n \times n$-matrices, where n is the rank of M. Furthermore, U is a numerical group (in fact an algebraic one) and f is the composite of a numerical map $U \to \mathbf{Z}$ and the group homomorphism $G \to U$. It will therefore suffice to show that the group homomorphism $G \to U$ is numerical. The ascending central series of U is given by $\{U_m\}$, where U_m is defined by $\{(a_{ij}) \mid a_{ij} = 0 \text{ if } j < i <= j + m\}$ and U/U_m is clearly also a numerical group. We now prove by descending induction on m that the composite $G \to U \to U/U_m$ is numerical. The homomorphism $U/U_m \to U/U_{m-1}$ is a central extension. The obstruction for lifting the homomorphism $G \to U/U_{m-1}$ to U/U_m is an element of $H^2(G, U_{m-1}/U_m)$ that is zero as the morphism is known to lift and the obstruction for lifting it to a numeric homomorphism is an element of $H^2_{\mathcal{N}um}(G, U_{m-1}/U_m)$. The latter group maps bijectively to the former by (2.5) and hence the numerical homomorphism $G \to U/U_{m-1}$ lifts to a numerical homomorphism $G \to U/U_m$. The set of group homomorphism liftings are classified by $H^1(G, U_{m-1}/U_m)$ and the set of numerical group homomorphism liftings are classified by $H^1_{\mathcal{N}um}(G, U_{m-1}/U_m)$. Again by (2.5) the map between these groups is a bijection and hence every lifting is numerical. In particular the given one is which finishes the induction step.

Finally, iv) and v) follow from i) and iii). □

We gather together the main results of this section in the following theorem.

Theorem 5.5. *i) The abstract group underlying a group object in $\mathcal{N}um$ is a finitely generated torsion free nilpotent group.*

ii) The forgetful functor from the category of group objects of $\mathcal{N}um$ to the category of finitely generated torsion free nilpotent groups is an equivalence of categories.

iii) Let G be a finitely generated torsion-free nilpotent group and let S be the ring of $\cdot$-numerical functions on G. Then S is a free numerical ring on the rank of G generators. The product, inverse and unit element of G induces a Hopf algebra structure on S, the evaluation map $G \to \mathrm{Hom}_{Rings}(S, \mathbf{Z})$ is a bijection and through this bijection, the Hopf algebra structure on S induces the given group structure on G.

iv) With notations as in the previous part, the group cohomology $H^*(G,\mathbf{Z})$ of G may be computed using $\cdot$-numerical cochains.

v) With notations as in iii) let R be a coefficient ring. Then the group structure induced on $G_R := \mathrm{Hom}_{Rings}(S,R)$ by the Hopf algebra structure of S together with the group homomorphism $G \to G_R$ given by the composite of the isomorphism $G \to \mathrm{Hom}_{Rings}(S,\mathbf{Z})$ given by iii) and the map induced by the inclusion $\mathbf{Z} \to R$ is an R-localisation.

Proof. This follows from the previous results of this section together with (3.2). □

Apart from possible group theoretic applications we can apply our results to more general homotopy types.

Proposition 5.6. *Let $G.$ be a simplicial group all of whose components G_n are f.g. torsion free nilpotent groups. Then $G.$ has a natural structure of simplicial group in $\mathcal{N}um$ and using this structure $K(G.)$, the simplicial classifying space of $G.$, has a natural structure of simplicial object in $\mathcal{N}um$. As such it is a numerical space.*

Proof. This follows directly from the spectral sequence

$$E_1^n = H^*_{\mathbf{Z}}(K(G_n,1)) \Rightarrow H^*_{\mathbf{Z}}(K(G.,1))$$

and the corollary. □

This result gives a relation between the present approach and one given by Quillen to rational homotopy theory (cf. [**12**]). Indeed, there Quillen represents any finitely generated complex up to homotopy by exactly a $K(G.)$ as in the proposition. What the proposition shows is that it also gives a representation of the complex as a numerical space. The next step in Quillen's construction is to pass to the Malcev completion of the components of $G.$ which fits very well in our context as taking the $\mathbf{Q}$-points of the G_n's considered as group schemes.

6. Sullivan models

We now want to see how Sullivan's theory (cf., [**14**]) of minimal models fits in with the present theory. His theory is a rational so throughout this section the coefficient ring will be the ring of rational numbers. As numerical functions then are the same as polynomial ones we will call them just that. A numerical space then can be seen as a simplicial scheme which in each degree is an affine space. Let us introduce some notation appropriate to the situation. We let Δ_a be the cosimplicial scheme that in each degree n is the algebraic n-simplex $\mathbf{Spec}\,\mathbf{Q}[x_0,\ldots,x_n]/(x_0+\ldots+x_n-1)$ with the obvious face and degeneracy operators and let Ω_a be the simplicial graded commutative differential graded algebra (*cdga*) of algebraic forms on Δ_a. Let us recall that Sullivan associates to each simplicial set X the differential graded algebra $\mathcal{E}(X)$ consisting of a choice of forms on $X_n \times \Delta_a^n$ with appropriate compatibility conditions with respect to face and degeneracy operations, where X_n is thought of as a zero-dimensional scheme being the disjoint union of copies of $\mathbf{Spec}\,\mathbf{Q}$, one for each point of X_n. To generalise this to the case of a numerical space X we consider relative forms on $X_n \times \Delta^n_a$, relative to the projection on the first factor, thus obtaining a cdga $\mathcal{E}_a(X)$. Another way to think of this is to consider the set of simplicial algebraic maps from X to Ω_a, where an algebraic map from an affine space Y over $\mathbf{Q}$ to a $\mathbf{Q}$-vector space V is a map $Y \to V$ whose image lies in a finite dimensional subspace U of V and is algebraic as a map $Y \to U$. Now, one proves as in [**14**, Thm. 7.1] that the cohomology of $\mathcal{E}_a(X)$ computes the numerical cohomology of X. In particular, if X is a numerical space the inclusion map $\mathcal{E}_a(X) \to \mathcal{E}(X)$ is a quasi-isomorphism. Note that even though $\mathcal{E}_a(X)$ is considerably smaller than $\mathcal{E}(X)$, even when X is a minimal numerical space $\mathcal{E}_a(X)$ is far from being a minimal model (or even a model), in fact in general $\mathcal{E}_a(X)$ will not be connected (i. e., $\mathbf{Q}$ in degree 0). It would interesting to have a modification of this construction that would give a minimal model from a minimal numerical space...

Remark 15. Instead of looking at relative forms on $X_n \times \Delta^n_a$ one could look at all forms. This would give a complex analogous to the de Rham complex of a simplicial manifold and its cohomology does in fact compute the algebraic de Rham cohomology of the simplicial scheme X. However, as each component X_n is contractible this de Rham complex is acyclic.

The relation with Sullivan's geometric realisation functor (cf., [**14**, §8]) seems to be more interesting. By investigating it a little bit more closely than is done ([**14**]) we will find a way of giving a direct construction of a minimal numerical space from a minimal cdga. For that we need some preliminary results. We begin by recalling that the *canonical truncation*, $\tau_{\leqslant n}$, of a complex $C^{\cdot}$ is the subquotient of $C^{\cdot}$ obtained by first taking the quotient by the subcomplex $C^{>n}$ and then the subcomplex which is unchanged in degrees $< n$ and the kernel of the differential $d^n\colon C^n \to C^{n+1}$ in degree n. Normally $\tau_{\leqslant n}C^{\cdot}$ is seen as a subcomplex of $C^{\cdot}$ but done in this fashion it is clear that if $C^{\cdot}$ is a cdga then we get an induced structure of cdga on $\tau_{\leqslant n}$. Note that the cohomology groups of $\tau_{\leqslant n}C^{\cdot}$ are the same as those for $C^{\cdot}$ in degrees $\leqslant n$ and 0 otherwise.

Definition-Lemma 6.1. i) For each $n \geqslant 0$ the simplicial group Ω^n is acyclic.

ii) Putting $Z^n := \ker d : \Omega^n \to \Omega^{n+1}$, the simplicial abelian group Z^n is 0 in degrees $< n$, $\pi_n(Z^n) = \mathbf{Q}$ and $\pi_i(Z^n) = 0$ if $i > n$. As $Z^n_i = 0$ for $i > n$ there is a mapping $\int\colon Z^n \to K(\mathbf{Q}, n)$ inducing the identity on π_n. We let T^n be the simplicial cdga that is the quotient of $\tau_{\leqslant n}\Omega$ by the kernel of $\int$.

Proof. The first part is implicit in [**14**] but can be found explicitly in [**4**, Lemma 10.7 & §17, Ex. 3]. The second part is clear when $n = 0$ as Z^0 is the constant simplicial object with constant value $\mathbf{Q}$. For larger n it follows from the first part, induction and the exact sequences

$$0 \to Z^n \longrightarrow \Omega^n \longrightarrow Z^{n+1} \to 0$$

coming from the acyclicity of (Ω, d). $\square$

Remark 16. It is easily seen that the identification $\pi_n(Z^n)$ with $\mathbf{Q}$ can also be given by the map $Z^n_n \to \mathbf{Q}$ given by integration over the standard simplex hence justifying the terminology.

Recall now, ([**14**, §8]), that the spatial realisation of a commutative differential graded $\mathbf{Q}$-algebra A, that we will assume is locally finite, i. e., finite-dimensional in each degree, is the simplicial set $\langle A\rangle$ that in degree n is the set of cdga-maps from A to Ω_n. The set of such maps is in a natural fashion the $\mathbf{Q}$-points of a formal $\mathbf{Q}$-scheme. More precisely, it is a closed subset of the formal affine space of graded linear maps from $A^{\leqslant n}$ to Ω_n (formal as the target space is infinite dimensional when $n > 0$). Hence, the spatial realisation is a simplicial formal scheme. As will be seen in a moment it is very big when A is a (minimal) model but we will want to cut it down to reasonable size. For two m-simplices $f, g\colon A \to \Omega$ we define equivalence relations for each $n > 0$

$f \sim_n g \iff$ The two induced maps $f', g'\colon \tau_{\leqslant n}\gamma_n A \to \tau_{\leqslant n+1}\Omega$, where $\gamma_{\leqslant n}A$ denotes the sub-cdga of A generated by the elements of degree $\leqslant n$, coincide when composed by the surjection $\tau_{\leqslant n+1}\Omega \to T^{n+1}$.

We then define a quotient $\langle A\rangle_t$ of $\langle A\rangle$ defined by the intersection of all these equivalence relations. Clearly $\langle A\rangle_t$ is contravariantly functorial in A and the quotient map $\langle -\rangle \to \langle -\rangle_t$ is a natural transformation.

Proposition 6.2. *Let A be a locally finite cdga, V a graded finite dimensional vector space concentrated in degree $n > 0$ and $A\bigotimes\Lambda V$ a cdga which as a graded algebra is the graded tensor product of A and the free graded commutative algebra on V and such that d maps V into $A\bigotimes\mathbf{Q}$. By functoriality the inclusion mapping $A \to A\bigotimes\Lambda V$ then induces maps $\langle A\bigotimes\Lambda V\rangle \to \langle A\rangle$ and $\langle A\bigotimes\Lambda V\rangle_t \to \langle A\rangle_t$.*

i) $\langle A\bigotimes\Lambda V\rangle \to \langle A\rangle$ is a PTCP of simplicial formal schemes with structure group $\mathrm{Hom}(V, Z^n)$.

ii) $\langle A\bigotimes\Lambda V\rangle_t \to \langle A\rangle_t$ is a PTCP of simplicial formal schemes with structure group
$K(\mathrm{Hom}(V,\mathbf{Q}), n)$. *The quotient mapping $\langle A\bigotimes\Lambda V\rangle \to \langle A\bigotimes\Lambda V\rangle_t$ is a mapping of PTCP's over $\langle A\rangle \to \langle A\rangle_t$ with respect to the structure group map induced by the natural surjection $\mathbf{Z}^n \to K(\mathbf{Q}, n)$.*

Proof. If we begin with the first part and we consider a fibre of $\langle A\bigotimes\Lambda V\rangle \to \langle A\rangle$ over an m-simplex $\phi\colon A \to \Omega_n$, an extension to $A\bigotimes\Lambda V$ is completely determined by the restriction of d to V. Furthermore, d applied to any element v of V is determined as it has to be the already prescribed image of $dv \in A$. That means that if f and g are two extensions of ϕ then $f - g$ maps V into Z^n_m. Conversely given an extension f and a linear map $V \to Z^n_m$ there is an extension g of ϕ such that the restriction of $f - g$ to V is the given map. Hence, the map $\langle A\bigotimes\Lambda V\rangle \to \langle A\rangle$ is a principal homogeneous space over the simplicial group Z^n. To show that it is a PTCP we need to find a section of the restriction of $\langle A\bigotimes\Lambda V\rangle \to \langle A\rangle$ to the subcategory Δ_* of Δ consisting of those increasing maps $\{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ that take 0 to 0 (cf., [**9**, 18.7]). This is obtained by the following observations

- Given a map $\phi\colon A \to \Omega_m$ to extend it to $A\bigotimes\Lambda V$ one needs to find a map $f\colon V \to \Omega_m$ such that $df(v) = \phi(dv)$ for all $v \in V$. This is possible as $d\phi(dv) = 0$ and Ω_m is acyclic. It can be done explicitly given a contraction of Ω.

- Given a **Q**-point $b \in \Delta_a^m$ we may use it as origin and use the algebraic contraction $x \to t(x-b)+b$ and the usual integration formulas to construct a contraction of Ω_m. This contraction is natural for affine maps preserving the chosen basepoints.
- The restriction of the cosimplical scheme Δ_a to Δ_* has a base point and hence the restriction of Ω to Δ_* has a contraction.

To turn to the second part it is clear that the only of the equivalence relations $\sim_m$ on $\langle A \bigotimes \Lambda V\rangle$ that does not factor through $\langle A \bigotimes \Lambda V\rangle \to \langle A\rangle$ is $\sim_n$. As for $\sim_n$ it is clear that two m-simplices f and g are equivalent if their restrictions to A are and if the restrictions to V of $\int \circ d \circ f$ and $\int \circ d \circ g$ are equal. This combined with the first part now gives the second. □

From the proposition the main result of this section immediately follows.

Theorem 6.3. *Let A be a nilpotent cdga model. Then the natural map $\langle A\rangle \to \langle A\rangle_t$ is a homotopy equivalence and $\langle A\rangle_t$ has a natural structure of special* **Q***-numerical space. It is minimal if A is.*

Proof. We leave to the reader to prove, in a fashion analogous to [**14**], that if $A' \to A$ is a minimal model then $\langle A\rangle_t \to \langle A'\rangle_t$ is a homotopy equivalence and that the formal scheme structure on $\langle A\rangle$ induces a special numerical space structure on $\langle A\rangle_t$ and will assume that A is minimal. That means that there is a filtration A^n of A by sub-cdga's such that $A^n = A^{n-1} \bigotimes \Lambda V_n$, where V_n is concentrated in a single degree and d maps V_n into A^{n-1}. Furthermore, the degree of V_n tends monotonically to infinity with n. One now proves by induction that $\langle A^n\rangle \to \langle A^n\rangle_t$ is a homotopy equivalence and that $\langle A^n\rangle_t$ is a minimal **Q**-numerical space using proposition 6.2. One then concludes by noting that $\langle A\rangle_t$ is the inverse limit of $\ldots \to \langle A^n\rangle_t \to \langle A^{n-1}\rangle_t \to \ldots$ and that this system is eventually constant in each degree. □

7. Fibrations

As a simple, and not very original, application of the ideas of this paper we will study fibrations. We now note that we can relativise our constructions; a morphism $R \to S$ of cosimplicial numerical rings may be factored $R \to S' \to S$ where $R \to S'$ is a direct limit of a succession cTCP's with fibers as before and $S' \to S$ is a cohomology equivalence. We call such a factorisation a *special resolution* of the map. We get as before that any two special resolutions are homotopic and for any map $R \to T$ of cosimplicial numerical rings we will call $T \to T \bigotimes_R S'$ the homotopy pushout of $R \to S$.

Definition 7.1. Let $\phi\colon R \to S$ be a morphism of cosimplicial numerical rings. We say that ϕ is a cofibration if for one (and hence any) special resolution $R \to S' \xrightarrow{\rho} S$ of ϕ and every morphism $R \to T$ of cosimplicial numerical rings, $\rho \otimes_R T$ is a cohomology equivalence.

We then have the following result.

Proposition 7.2. *i) Any morphism $R \to S$ of cosimplicial numerical rings which is flat in each degree (i.e. $R_n \to S_n$ is a flat ring homomorphism for each n) is a cofibration.*

ii) Let $f\colon X \to Y$ be a morphism of numerical spaces. The homotopy pullback of the map of simplicial sets underlying f by any map of numerical spaces has cohomology equal to the homotopy pushout of the corresponding cosimplicial numerical rings of numerical functions.

iii) If $R \to S$ is a fibration and $R \to T$ a morphism, then there is a spectral sequence

$$Tor_*^{H^*(R)}(H^*(S), H^*(T)) \Rightarrow H^*(S \bigotimes_R T).$$

Proof. i) is clear as a special morphism is flat so $S \bigotimes_R (-)$ and $S' \bigotimes_R (-)$ are both exact. As for ii) one verifies it by induction over a Postnikov tower of $X \to Y$. Finally, iii) follows as in the simplicial case [**11**, II, Thm 5]. □

Remark 17. In the case of a diagram of spaces this gives the Eilenberg-Moore spectral sequence.

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