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TRUNCATIONS OF THE RING OF NUMBER-THEORETIC FUNCTIONS

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Abstract

We study the ring Γ of all functions $\mathbb{N}^+ \to K$, endowed with the usual convolution product. Γ, which we call *the ring of number-theoretic functions*, is an inverse limit of the "truncations"

$$\Gamma_n = \{ f \in \Gamma | \forall m > n : f(m) = 0 \}.$$

Each Γ_n is a zero-dimensional, finitely generated K-algebra, which may be expressed as the quotient of a finitely generated polynomial ring with a *stable* (after reversing the order of the variables) monomial ideal. Using the description of the free minimal resolution of stable ideals given by Eliahou-Kervaire, and some additional arguments by Aramova-Herzog and Peeva, we give the Poincaré-Betti series for Γ_n.

1. Introduction

Cashwell and Everett [**2**] studied "the ring of number-theoretic functions"

$$\Gamma = \{ f | \mathbb{N}^+ \to K \} \tag{1}$$

where $\mathbb{N}^+$ is the set of positive natural numbers (we denote by $\mathbb{N}$ the set of all natural numbers) and K is a field containing the rational numbers. Γ is endowed with component-wise addition and multiplication with scalars, and with the convolution (or Cauchy) product

$$fg(n) = \sum_{\substack{(a,b)\in(\mathbb{N}^+)\times(\mathbb{N}^+) \\ ab=n}} f(a)g(b) \tag{2}$$

With these operations, Γ becomes a commutative K-algebra. It is immediate that it is a local domain; less obvious is the fact that it is a unique factorisation domain. Cashwell and Everett proved this in [**2**] using the isomorphism

$$\begin{aligned} \Phi : \Gamma &\to K[[X]] \\ f &\mapsto \sum f(n) x_1^{\alpha_1} x_2^{\alpha_2} \cdots \end{aligned} \tag{3}$$

where $X = \{x_1, x_2, x_3, \dots\}$, $K[[X]]$ is the "large" power series ring of all functions from the free abelian monoid $\mathcal{M} = [X]$ (the free abelian monoid generated by X) to K, and where the summation extends over all $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots \in \mathbb{N}^+$. Here, and henceforth, we denote by p_i the i'th prime number, with $p_1 = 2$, and by $\mathcal{P}$ the set of all prime numbers. That (3) is an isomorphism is immediate from the following isomorphism of commutative monoids, implied by the fundamental theorem of arithmetics:

$$(\mathbb{N}^+, \cdot) \simeq \coprod_{p\in\mathcal{P}} (\mathbb{N}, +) \tag{4}$$

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The following number-theoretic functions are of particular interest (whenever possible, we use the same notation as in [**2**]):

1. The multiplicative unit ϵ given by $\epsilon(1) = 1$, $\epsilon(n) = 0$ for $n > 1$,
2. $\lambda : \mathbb{N}^+ \to \mathbb{N}$ given by $\lambda(1) = 0$, $\lambda(q_1 \cdots q_l) = l$ if $q_1, \ldots, q_l$ are any (not necessarily distinct) prime numbers.
3. $\tilde{\lambda} : \mathbb{N}^+ \to \mathbb{N}$ given $\tilde{\lambda}(1) = 0$, $\tilde{\lambda}(p_1^{a_1} \ldots p_r^{a_r}) = \sum a_r p_r$.
4. The Möbius function $\mu(1) = 1$, $\mu(n) = (-1)^v$ if n is the product of v distinct prime factors, and 0 otherwise,
5. For any $i \in \mathbb{N}^+$, $\chi_i(p_i) = 1$, and $\chi_i(m) = 0$ for $m \neq p_i$. Note that under the isomorphism (3), $\Phi(\chi_i) = x_i$.

The topic of this article is the study of the "truncations" Γ_n, where for each $n \in \mathbb{N}^+$,

$$\Gamma_n = \{ f \in \Gamma | m > n \implies f(m) = 0 \} \tag{5}$$

With the modified multiplication given by

$$fg(n) = \sum_{\substack{(a,b) \in \{1,\ldots,n\} \times \{1,\ldots,n\} \\ ab=n}} f(a)g(b) \tag{6}$$

Γ_n becomes a K-algebra, isomorphic to Γ / J_n, where J_n is the ideal

$$J_n = \{ f \in \Gamma | \forall m \leqslant n : f(m) = 0 \}.$$

If we define

$$\pi_n : \Gamma \to \Gamma_n \tag{7}$$

$$\pi_n(f)(m) = \begin{cases} f(m) & m \leqslant n \\ 0 & m > n \end{cases} \tag{8}$$

then π_n is a K-algebra epimorphism, and J_n is the kernel of π_n. We note furthermore that J_n is generated by *monomials* in the elements χ_i.

To describe the main idea of this paper, we need a few additional definitions. First, for any $n \in \mathbb{N}^+$ we denote by $r(n) \in \mathbb{N}$ the largest integer such that $p_{r(n)} \leqslant n$. In other words, $r(n)$ is the number of prime numbers $\leqslant n$ (this number is often denoted $\pi(n)$). Secondly, for a monomial $m = x_1^{\alpha_1} \cdots x_w^{\alpha_w}$, we define the *support* $\operatorname{Supp}(m)$ as the set of positive integers i such that $\alpha_i > 0$. We define $\max(m)$ and $\min(m)$ as the maximal and minimal elements in the support of m.

Definition 1.1. A monomial ideal $I \subset K[x_1, \ldots, x_r]$ is said to be *strongly stable* if whenever m is a monomial such that $x_j m \in I$, then $x_i m \in I$ for all $i \leqslant j$. If this condition holds at least for all $i \leqslant j = \max(m)$ then I is said to be *stable*.

We can now state our main theorem:

Theorem 1.2. *Let $n \in \mathbb{N}^+$ and $r = r(n)$. Then the following holds:*

(I) $\Gamma_n \simeq \frac{K[x_1,\ldots,x_r]}{I_n}$, *where I_n is a strongly stable monomial ideal, with respect to the reverse order of the variables.*

(II) *Γ_n is artinian, with $\dim_K(\Gamma_n) = n$. Furthermore, if it is given the natural grading with $|\chi_i| = 1$, then its Hilbert series is $\sum_i d_i t^i$ where d_i is the number of $w \leqslant n$ with $\lambda(w) = i$.*

(III) *There is a 1-1 bijection between the minimal monomial generators of I_n of minimal support v, and the solutions in non-negative integers to the equation*

$$\log n - \log p_v < \sum_{i=v}^{r} b_i \log p_i \leqslant \log n \tag{9}$$

(IV) *If we denote by $C_{n,v}$ the number of such solutions, then the Poincaré-Betti series of the free minimal resolution of K as a cyclic module over Γ_n is the following rational function:*

$$P(\mathrm{Tor}_*^{\Gamma_n}(K,K),t) = \frac{(1+t)^r}{1-t^2\left(\sum_{i=1}^{r}(1+t)^{(i-1)}C_{n,r-i+1}\right)} \tag{10}$$

We will show this result, and also give the graded Poincaré-Betti series. For this, we define the number $C_{n,v,d}$ which counts the number of minimal generators of I_n of minimal support v and total degree d. We determine some elementary properties of the numbers $C_{n,v,d}$ and $C_{n,v}$.

2. The ring of number-theoretic functions and its truncations

2.1. Norms, degrees, and multiplicativity

For a monomial $\mathcal{M} \ni m = x_1^{a_1}\dots x_n^{a_n}$ we define the *weight* of m as $w(m) = p_1^{a_1}\dots p_n^{a_n}$ (we put $w(1) = 1$). Hence w gives a bijection between $\mathcal{M}$ and $\mathbb{N}^+$. Furthermore, we can define a term order on $\mathcal{M}$ by $m > m'$ iff $w(m) > w(m')$. If we define the *initial monomial* $\mathrm{in}(f)$ of $f \in K[[X]]$ as the monomial in $\mathrm{Supp}(f)$ minimal with respect to $>$, then $\mathrm{in}(f)$ is easily seen to correspond to the *norm* $N(\alpha)$ of a number-theoretic function α, defined as the smallest n such that $\alpha(n) \neq 0$. Here, we must use w and Φ to identify $\mathcal{M}$ and $\mathbb{N}^+$ and $K[[X]]$ and Γ. As observed in [**2**], the norm is multiplicative: $N(\alpha\beta) = N(\alpha)N(\beta)$.

Cashwell and Everett also define the *degree* $D(\alpha)$ to mean the smallest d such that there exists an n with $\lambda(n) = d$ and $\alpha(n) \neq 0$. This corresponds the smallest *total degree* of a monomial in $\mathrm{Supp}(f)$. Furthermore, the norm $M(\alpha)$, defined as the smallest integer n with $\lambda(n) = D(\alpha)$, $\alpha(n) \neq 0$, corresponds to the initial monomial of f under the term order obtained by refining the total degree partial order with the term order $>$.

A *multiplicative function* is an element $\alpha \in \Gamma$ such that $\alpha(1) = 1$ and $\alpha(ab) = \alpha(a)\alpha(b)$ whenever a and b are relatively prime. Cashwell and Everett observes that a multiplicative function is necessarily a unit in Γ. One can further observe that if α is multiplicative, then $f = \Phi(\alpha)$ can be written

$$f(x_1,x_2,x_3,\dots) = f_1(x_1)f_2(x_2)f_3(x_3)\cdots$$

where each $f_i(x_i) \in K[[x_i]]$ is invertible. In particular, the constant function $\Gamma \ni \nu_0$ with $\nu_0(n) = 1$ for all n, corresponds to

$$\sum_{m\in\mathcal{M}} m = \frac{1}{1-x_1}\frac{1}{1-x_2}\frac{1}{1-x_3}\cdots$$

Since the Möbius function is defined to be the inverse of this function, we get that it corresponds to

$$(1-x_1)(1-x_2)(1-x_3)\cdots = 1 - (\sum_{i=1}^{\infty} x_i) + (\sum_{i<j} x_ix_j) - (\sum_{i<j<k} x_ix_jx_k) + \cdots$$

2.2. Truncations of the ring of number-theoretic functions

Let $n, n' \in \mathbb{N}^+$, $n' > n$. Then there is a K-algebra epimorphism

$$\varphi_n^{n'} : \Gamma_{n'} \to \Gamma_n$$

$$\varphi_n^{n'}(f)(m) = \begin{cases} f(m) & m \leqslant n \\ 0 & m > n \end{cases}$$

Hence, the Γ_n's form an inverse system.

Lemma 2.1. $\varprojlim \Gamma_n \simeq \Gamma$.

Proof. Given any $f \in \Gamma$, the sequence $(\pi_1(f), \pi_2(f), \pi_3(f), \dots)$ is coherent. Conversely, given any coherent sequence $(g_1, g_2, g_3, \dots)$, we can define $g : \mathbb{N} \to K$ by $g(m) = g_i(m)$ where $i \geqslant m$. $\square$

As a side remark, we note that

Lemma 2.2. *The decreasing filtration*

$$J_1 \supsetneq J_2 \supsetneq J_3 \supsetneq \cdots \tag{11}$$

is separated, that is, $\cap_n J_n = (0)$.

Definition 2.3. We define

$$I_n = K[[X]]\,\{\, m \in \mathcal{M} | w(m) > n \,\}, \tag{12}$$

that is, as the monomial ideal in $K[[X]]$ generated by all monomials of weight strictly higher than n. We put $A_n = \frac{K[[X]]}{I_n}$.

Proposition 2.4. *A K-basis of A_n is given by all monomials of weight $\leqslant n$. Hence A_n is an artinian algebra, with* $\dim_K(A_n) = n$. *Putting $r = r(n)$, we have that*

$$A_n = \frac{K[[X]]}{I_n} \simeq \frac{K[x_1, \dots, x_r]}{I_n \cap K[x_1, \dots, x_r]} \tag{13}$$

Proof. As a vector space, $K[[X]] \simeq U \oplus I_n$, where U consists of all functions supported on monomials of weight $\leqslant n$. It follows that $A_n \simeq U$ as K vector spaces. Of course, there are exactly n monomials of weight $\leqslant n$. Finally, if $s > r$ then $w(x_s) = p_s > n$, hence $x_s \in I_n$. $\square$

We will abuse notations and identify I_n and its contraction $I_n \cap K[x_1, \dots, x_r]$.

Lemma 2.5. $\Gamma_n \simeq A_n$.

Proof. Since A_n has a K-basis is given by all monomials of weight $\leqslant n$, the two K-algebras are isomorphic as K-vector spaces. The multiplication in A_n is induced from the multiplication in $K[[X]]$, with the extra condition that monomials of weight $> n$ are truncated. This is the same multiplication as in Γ_n. $\square$

Proposition 2.6. *I_n is a strongly stable ideal, with respect to the reverse order of the variables.*

Proof. We must show that if $m \in I_n$, and $x_i | m$, then $m x_j / x_i \in I$ for $i \leqslant j \leqslant r$. We have that $w(m x_j / x_i) = w(m) p_j / p_i > w(m) > n$. $\square$

Part I of the main theorem is now proved.

We give $K[x_1, \dots, x_r]$ an $\mathbb{N}^2$-grading by giving the variable x_i bi-degree $(1, p_i)$. Since each I_n is bihomogeneous, this grading is inherited by A_n.

Theorem 2.7. *The bi-graded Hilbert series of A_n is given by*

$$A_n(t, u) = \sum_{i,j} c_{ij} t^i u^j,$$

where c_{ij} is the number of $p_1^{a_1} \dots p_r^{a_r} \leqslant n$ with $\sum a_r = i$ and $\sum a_r p_r = j$. Furthermore,

$$A_n(t, 1) = \sum_i d_i t^i$$

$$A_n(1, u) = \sum_j e_j u^j$$

where d_i is the number of $w \leqslant n$ with $\lambda(w) = i$, and e_i is the number of $w \leqslant n$ with $\tilde{\lambda}(w) = i$. In particular, the t^1-coefficient of $A_n(t, 1)$ is the number of prime numbers $\leqslant n$.

Proof. The monomial $x_1^{a_1} \cdots x_n^{a_n}$ has bi-degree $(\sum_{i=1}^n a_i, \sum a_i p_i)$. □

This establishes part II of the main theorem.

3. Minimal generators for I_n

Let $n \in \mathbb{N}^+$, and let $r = r(n)$. We have that

$$x_1^{a_1} \dots x_r^{a_r} = m \in I_n \iff w(m) > n \iff \prod_{i=1}^{r} p_i^{a_i} > n. \tag{14}$$

We denote by $G(I_n)$ the set of minimal monomial generators of I_n. For $m = x_1^{a_1} \dots x_r^{a_r}$ to be an element of $G(I_n)$ it is necessary and sufficient that $m \in I_n$ and that for $1 \leqslant v \leqslant r$, $x_v \,|m \implies m/x_v \notin I_n$. In other words,

$$1 \leqslant j \leqslant n,\, a_j > 0 \implies n < \prod_{i=1}^{r} p_i^{a_i} \leqslant p_j n. \tag{15}$$

Definition 3.1. For n, v, d positive integers, we define:

$$C_n = \#G(I_n) \tag{16}$$
$$C_{n,v} = \#\{\, m \in G(I_n) | \min(m) = v \,\} \tag{17}$$
$$C_{n,v,d} = \#\{\, m \in G(I_n) | \min(m) = v,\, |m| = d \,\} \tag{18}$$

Theorem 3.2. *$C_{n,v}$ is the number of solutions $(b_1, \dots, b_r) \in \mathbb{N}^r$ to the equation*

$$\log n - \log p_v < \sum_{i=v}^{r} b_i \log p_i \leqslant \log n. \tag{19}$$

Equivalently, $C_{n,v}$ is the number of integers x such that $n/p_v < x \leqslant n$ and such that no prime factors of x are smaller than p_v.

Similarly, $C_{n,v,d}$ is the number of solutions $(b_1, \dots, b_r) \in \mathbb{N}^r$ to the system of equations

$$\begin{aligned} \log n - \log p_v < \sum_{i=v}^{r} b_i \log p_i \leqslant \log n \\ \sum_{i=1}^{r} b_i = d - 1. \end{aligned} \tag{20}$$

or equivalently, $C_{n,v,d}$ is the number of integers x such that $n/p_v < x \leqslant n$ and such that no prime factors of x are smaller than p_v, and with the additional constraint that $\lambda(x) = d$.

Proof. We have that $a_v > 0$, $a_w = 0$ for $w < v$. Hence equation (15) implies that

$$n < \prod_{j=v}^{r} p_i^{a_i} \leqslant p_v n.$$

Putting $b_v = a_v - 1$, $b_j = a_j$ for $j > v$ we can write this as

$$n < p_v \prod_{j=v}^{r} p_i^{b_i} \leqslant p_v n \iff n/p_v < \prod_{j=v}^{r} p_i^{b_i} \leqslant n$$

from which (19) follows by taking logarithms. This implies (20) as well. □

We have now proved part III of the main theorem.

Figure 1: The numbers C_n and $C_{n,i}$.

n	Σ	$i = 1$	$i = 2$	3	4	5	6	7	8	9	10
2	1	1									
3	3	2	1								
4	3	2	1								
5	6	3	2	1							
6	6	3	2	1							
7	10	4	3	2	1						
8	10	4	3	2	1						
9	11	5	3	2	1						
10	11	5	3	2	1						
11	16	6	4	3	2	1					
12	16	6	4	3	2	1					
13	22	7	5	4	3	2	1				
14	22	7	5	4	3	2	1				
15	23	8	5	4	3	2	1				
16	23	8	5	4	3	2	1				
17	30	9	6	5	4	3	2	1			
18	30	9	6	5	4	3	2	1			
19	38	10	7	6	5	4	3	2	1		
20	38	10	7	6	5	4	3	2	1		
21	39	11	7	6	5	4	3	2	1		
22	39	11	7	6	5	4	3	2	1		
23	48	12	8	7	6	5	4	3	2	1	
24	48	12	8	7	6	5	4	3	2	1	
25	50	13	9	7	6	5	4	3	2	1	
26	50	13	9	7	6	5	4	3	2	1	
27	51	14	9	7	6	5	4	3	2	1	
28	51	14	9	7	6	5	4	3	2	1	
29	61	15	10	8	7	6	5	4	3	2	1
30	61	15	10	8	7	6	5	4	3	2	1

Figure 2: The numbers $C_{n,i,g}$.

n	$i = 1$	$i = 2$	3	4	5	6	7	8	9	
2	1									
3	2	1								
4	$u+1$	1								
5	$u+2$	2	1							
6	$2u+1$	2	1							
7	$2u+2$	3	2	1						
8	u^2+u+2	3	2	1						
9	u^2+2u+2	$u+2$	2	1						
10	u^2+3u+1	$u+2$	2	1						
11	u^2+3u+2	$u+3$	3	2	1					
12	$2u^2+2u+2$	$u+3$	3	2	1					
13	$2u^2+2u+3$	$u+4$	4	3	2	1				
14	$2u^2+3u+2$	$u+4$	4	3	2	1				
15	$2u^2+4u+2$	$2u+3$	4	3	2	1				
16	u^3+u^2+4u+2	$2u+3$	4	3	2	1				
17	u^3+u^2+4u+3	$2u+4$	5	4	3	2	1			
18	u^3+2u^2+3u+3	$2u+4$	5	4	3	2	1			
19	u^3+2u^2+3u+4	$2u+5$	6	5	4	3	2	1		
20	u^3+3u^2+2u+4	$2u+5$	6	5	4	3	2	1		
21	u^3+3u^2+3u+4	$3u+4$	6	5	4	3	2	1		
22	u^3+3u^2+4u+3	$3u+4$	6	5	4	3	2	1		
23	u^3+3u^2+4u+4	$3u+5$	7	6	5	4	3	2	1	
24	$2u^3+2u^2+4u+4$	$3u+5$	7	6	5	4	3	2	1	
25	$2u^3+2u^2+5u+4$	$4u+5$	$u+6$	6	5	4	3	2	1	
26	$2u^3+2u^2+6u+3$	$4u+5$	$u+6$	6	5	4	3	2	1	
27	$2u^3+3u^2+6u+3$	u^2+3u+5	$u+6$	6	5	4	3	2	1	
28	$2u^3+4u^2+5u+3$	u^2+3u+5	$u+6$	6	5	4	3	2	1	
29	$2u^3+4u^2+5u+4$	u^2+3u+6	$u+7$	7	6	5	4	3	2	1
30	$2u^3+5u^2+4u+4$	u^2+3u+6	$u+7$	7	6	5	4	3	2	1

Example 3.3. The first few I_n's are as follows: $I_2=(x_1^2)$, $I_3=(x_1^2,x_2^2,x_1x_2)$, $I_4=(x_1^3,x_2^2,x_1x_2)$, $I_5=(x_1^3,x_2^2,x_1x_2,x_3^2,x_1x_3,x_2x_3)$.

We tabulate $C_{n,i}$ and $C_{n,i,d}$, the latter in form of the polynomial $u^{-2}\sum_j C_{n,i,j}u^j$ in the tables 1 and 2.

Theorem 3.4. *(1)* $C_{n,v}=0$ *for* $v>r(n)$

(2) $\forall n\in\mathbb{N}:\ \forall v\leqslant r(n):\ C_{n,1+r(n)-v}\geqslant v$,

(3) $\forall n\in\mathbb{N}: C_n\geqslant\binom{r(n)+1}{2}$,

(4) $\forall v\in\mathbb{N}:\exists N:\forall n\geqslant N: C_{n,1+r(n)-v}=v$.

(5) If n *is even, then* $C_{n,v}=C_{n-1,v}$ *for all* v,

(6) $C_{n,1}=\lceil n/2\rceil$.

Proof. (1) Obvious.

(2) and (3) It suffices to show that for any subset $S\subset\{1,\dots,r\}$ of cardinality 1 or 2, there is an $m\in G(I_n)$ with $\mathrm{Supp}(m)=S$. If $S=\{i\}$ then there is an unique positive integer a such that $p_i^{b-1}\leqslant n<p_i^b$, and $m=x_i^b$ is the desired generator. If $S=\{i,j\}$ with $i<j$ then we claim that there is a positive integer a such that $x_i^a x_j\in G(I_n)$. Namely, choose b such that $p_i^{b-1}\leqslant n<p_i^b$, then since $p_i<p_j$ one has $n<p_i^{b-1}p_j$. Hence $x_i^{b-1}x_j\in I_n$, so it is a multiple of some minimal generator. By the definition of b, this minimal generator must be of the form $x_i^a x_j$ for some a, which establishes the claim.

(6) We must show that the number of solutions in $\mathbb{N}^r$ to

$$\frac{n}{2}<\prod_{i=1}^{r}{p_i}^{b_i}\leqslant n$$

is precisely $\lceil\frac{n}{2}\rceil$. Obviously, any integer $\in(\frac{n}{2},n]$ fits the bill; there are $\lceil\frac{n}{2}\rceil$ of those.

(5) The case $v=1$ follows from (6). Hence, it suffices to show that if $v>1$, $x\in(\frac{n}{p_v},n]\cap\mathbb{N}$, and if x has no prime factor $<p_v$, then $x\in(\frac{n-1}{p_v},n-1]\cap\mathbb{N}$. The only way this can fail to happen is if $x=n$, but then x is even, and has the prime factor $2=p_1<p_v$, a contradiction.

(4) For large enough n, the only integers $x\leqslant n$ with all prime factors $\geqslant 1+r(n)-v$ are $p_{1+r(n)-v},\dots,p_{r(n)}$. There is v of these, and they are all $>\frac{n}{p_v}$. □

Theorem 3.5. *1.* $C_{n,v,d}=0$ *for* $v>r(n)$*, and for* $d<2$,

2. $\forall v\in\mathbb{N}:\exists N:\forall n\geqslant N: C_{n,1+r(n)-v,2}=v$, $C_{n,1+r(n)-v,d}=0$ *for* $d\neq 2$,

3. $\binom{r(n)}{2}=\#\{m\in\mathbb{N}^+\,|m\leqslant n,\ \lambda(m)=2\}$.

Proof. The first and the last assertions are obvious. The second one follows from the proof of (4) in the previous lemma. □

4. Poincaré series

In [**3**], a minimal free multi-graded resolution of a I over S is given, where $S=K[x_1,\dots,x_r]$ is a polynomial ring, and $I\subset(x_1,\dots,x_r)^2$ is a stable ideal. As a consequence, the following formula for the Poincaré-Betti series is derived:

$$P(\mathrm{Tor}_*^S(I,K),t)=\sum_{a\in G(I)}(1+t)^{\max(a)-1} \tag{21}$$

where $G(I)$ is the minimal generating set of I. Since the resolution is multi-graded, (21) can be modified to yield a formula for the graded Poincaré-Betti series (we here consider S as $\mathbb{N}$-graded, with each variable given weight 1):

$$P(\mathrm{Tor}_{*,*}^S(I,K),t,u)=\sum_{a\in G(I)}u^{|a|}(1+t)^{\max(a)-1} \tag{22}$$

We will use the following variant of this result:

Theorem 4.1 (Eliahou-Kervaire). *Let $I \subset (x_1, \ldots, x_r)^2 \subset K[x_1, \ldots, x_r] = S$ be a stable monomial ideal. Put*

$$b_{i,d} = \#\{\, m \in G(I) | \max(m) = i,\ |m| = d \,\} \tag{23}$$
$$b_i = \#\{\, m \in G(I) | \max(m) = i \,\} \tag{24}$$

Then

$$P(\mathrm{Tor}_*^S(I, K), t) = \sum_{i=1}^{r} b_i (1+t)^{(i-1)} \tag{25}$$
$$P(\mathrm{Tor}_{*,*}^S(I, K), t, u) = \sum_{i=1}^{r} \left((1+tu)^{(i-1)} \sum_j b_{i,j} u^j \right). \tag{26}$$

For the Betti-numbers we have that

$$\beta_q = \dim_K \left(\mathrm{Tor}_q^S(I, K) \right) = \sum_{i=1}^{r} b_i \binom{i-1}{q}. \tag{27}$$

From Proposition 2.6 we have that the ideals I_n are stable after reversing the order of the variables. Hence, replacing max by min, and hence b_i with $C_{n,1+r-i}$, we get:

Corollary 4.2. *Let $n \in \mathbb{N}^+$, $r = r(n)$, $S = K[x_1, \ldots, x_r]$. Then*

$$P(\mathrm{Tor}_*^S(I_n, K), t) = \sum_{i=1}^{r} C_{n,1+r-i} (1+t)^{(i-1)} \tag{28}$$
$$P(\mathrm{Tor}_{*,*}^S(I_n, K), t, u) = \sum_{i=1}^{r} (1+tu)^{(i-1)} \sum_j C_{n,1+r-i,j} u^j. \tag{29}$$

For the Betti-numbers we have that

$$\beta_q = \sum_{i=1}^{r} C_{n,1+r-i} \binom{i-1}{q}. \tag{30}$$

In [**6, 1**] it is shown that if $S = K[x_1, \ldots, x_r]$ and I is a stable monomial ideal in S, then S/I is a Golod ring. Hence, from a result of Golod [**4**] (see also [**5**]), it follows that

$$P(\mathrm{Tor}_*^{S/I}(K, K), t) = \frac{(1+t)^r}{1 - t^2 P(\mathrm{Tor}_*^S(I, K), t)} \tag{31}$$

Regarding S as an $\mathbb{N}$-graded ring, one can show that in fact

$$P(\mathrm{Tor}_*^{S/I}(K, K), t, u) = \frac{(1+ut)^r}{1 - t^2 P(\mathrm{Tor}_*^S(I, K), t, u)} \tag{32}$$

The following theorem is an immediate consequence:

Theorem 4.3 (Herzog-Aramova, Peeva). *Let $S = K[x_1, \ldots, x_r]$, and suppose that I is a stable monomial ideal in S. Put*

$$b_{i,d} = \#\{\, x \in G(I) | \max(x) = i,\ |x| = d \,\}$$
$$b_i = \#\{\, x \in G(I) | \max(x) = i \,\}$$

Then, for $R = S/I$, we have that

$$P(\mathrm{Tor}_*^R(K,K),t) = \frac{(1+t)^r}{1 - t^2 \sum_{i=1}^r (1+t)^{(i-1)} \sum_j b_i} \tag{33}$$

$$P(\mathrm{Tor}_*^R(K,K),t,u) = \frac{(1+t)^r}{1 - t^2 \sum_{i=1}^r (1+tu)^{(i-1)} \sum_j b_{i,j} u^j} \tag{34}$$

Specialising to the case of A_n, we obtain:

Corollary 4.4. *Let $n \in \mathbb{N}^+$, and let $r = r(n)$. Regard A_n as a naturally graded K-algebra, with each x_i given weight 1, and regard K as a cyclic A-module. Then*

$$P(\mathrm{Tor}_*^{A_n}(K,K),t) = \frac{(1+t)^r}{1 - t^2 \sum_{i=1}^r (1+t)^{(i-1)} C_{n,r-i+1}} \tag{35}$$

$$P(\mathrm{Tor}_*^{A_n}(K,K),t,u) = \frac{(1+ut)^r}{1 - t^2 \left(\sum_{i=1}^r \left((1+tu)^{(i-1)} \sum_j C_{n,r-i+1,j} u^j \right) \right)} \tag{36}$$

Part IV of the main theorem is now proved.

Example 4.5. We consider the case $n = 5$, then $r = r(n) = 3$, so $S = K[x_1, x_2, x_3]$ and $I = I_5 = (x_1^3, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)$. We get that $C_{5,1} = 3$, $C_{5,2} = 2, C_{5,3} = 1$. According to our formulas[1] we have

$$P_I^S(t) = 1 + 2(1+t) + 3(1+t)^2 = 6 + 8t + 3t^2$$
$$P_K^{S/I} = \frac{(1+t)^r}{1 - t^2 P_I^S(t)} = \frac{1}{1-3t}$$

When we consider the grading by total degree, we have that $C_{5,1,2} = 2$, $C_{5,1,3} = 1$, $C_{5,2,2} = 2$, $C_{5,3,2} = 1$. Hence, our formulas yield

$$\begin{aligned} P_I^S(t,u) &= u^2 + 2u^2(1+t) + (2u^2 + u^3)(1+t)^2 \\ &= 5u^2 + u^3 + (6u^2 + 2u^3)t + \left(2\,u^2 + u^3\right) t^2 \end{aligned}$$
$$P_K^{S/I}(t,u) = -\frac{1+tu}{u^3t^2 + 2\,t^2u^2 + 2\,tu - 1}$$

We list the first few Poincaré-Betti series $P(\mathrm{Tor}_*^{A_n}(K,K),t,u)$ in table 3.

Conjecture 4.6. *$P(\mathrm{Tor}_*^{A_n}(K,K),t) = -\frac{(1+t)^{\ell_1(n)}}{q_n(t)}$, $q_n(t) = \sum_{i=0}^{\ell_2(n)} h_i(n)t^i$, with*

1. *$q_n(-1) \neq 0$,*
2. *$\ell_1(n)$ is the number of odd primes p such that $p^2 \leqslant n$,*
3. *$\ell_2(n) = \ell_1(n) + 1$,*
4. *$h_0(n) = -1$,*
5. *$h_1(n) = r(n) - \ell_1(n)$,*
6. *$h_{\ell_2(n)}(n) = C_{n,1} = \lceil n/2 \rceil$.*

5. Acknowledgements

I am indebted to Johan Andersson for suggesting the idea of studying the homological properties of the truncations Γ_n. I thank the referee for suggesting a simplified proof of parts of Theorem 3.4.

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[1] Here, we have used the abbreviation $P_I^S(t) = P(\mathrm{Tor}_*^S(I,K),t)$, we will also write $P_K^{S/I}(t) = P(\mathrm{Tor}_*^{S/I}(K,K),t)$ et cetera.

n	*Graded*	*Non − graded*
2	$-(tu-1)^{-1}$	$-(t-1)^{-1}$
3	$-(2\,tu-1)^{-1}$	$-(2\,t-1)^{-1}$
4	$-\frac{1+tu}{(u^3+u^2)t^2+tu-1}$	$-(2\,t-1)^{-1}$
5	$-\frac{1+tu}{(u^3+2\,u^2)t^2+2\,tu-1}$	$-(3\,t-1)^{-1}$
6	$-\frac{1+tu}{(2\,u^3+u^2)t^2+2\,tu-1}$	$-(3\,t-1)^{-1}$
7	$-\frac{1+tu}{(2\,u^3+2\,u^2)t^2+3\,tu-1}$	$-(4\,t-1)^{-1}$
8	$-\frac{1+tu}{(u^4+u^3+2\,u^2)t^2+3\,tu-1}$	$-(4\,t-1)^{-1}$
9	$-\frac{1+2\,tu+t^2u^2}{(u^5+2\,u^4+2\,u^3)t^3+(u^4+3\,u^3+4\,u^2)t^2+2\,tu-1}$	$-\frac{1+t}{5\,t^2+3\,t-1}$
10	$-\frac{1+2\,tu+t^2u^2}{(u^5+3\,u^4+u^3)t^3+(u^4+4\,u^3+3\,u^2)t^2+2\,tu-1}$	$-\frac{1+t}{5\,t^2+3\,t-1}$
11	$-\frac{1+2\,tu+t^2u^2}{(u^5+3\,u^4+2\,u^3)t^3+(u^4+4\,u^3+5\,u^2)t^2+3\,tu-1}$	$-\frac{1+t}{6\,t^2+4\,t-1}$
12	$-\frac{1+2\,tu+t^2u^2}{(2\,u^5+2\,u^4+2\,u^3)t^3+(2\,u^4+3\,u^3+5\,u^2)t^2+3\,tu-1}$	$-\frac{1+t}{6\,t^2+4\,t-1}$
13	$-\frac{1+2\,tu+t^2u^2}{(2\,u^5+2\,u^4+3\,u^3)t^3+(2\,u^4+3\,u^3+7\,u^2)t^2+4\,tu-1}$	$-\frac{1+t}{7\,t^2+5\,t-1}$
14	$-\frac{1+2\,tu+t^2u^2}{(2\,u^5+3\,u^4+2\,u^3)t^3+(2\,u^4+4\,u^3+6\,u^2)t^2+4\,tu-1}$	$-\frac{1+t}{7\,t^2+5\,t-1}$
15	$-\frac{1+2\,tu+t^2u^2}{(2\,u^5+4\,u^4+2\,u^3)t^3+(2\,u^4+6\,u^3+5\,u^2)t^2+4\,tu-1}$	$-\frac{1+t}{8\,t^2+5\,t-1}$
16	$-\frac{1+2\,tu+t^2u^2}{(u^6+u^5+4\,u^4+2\,u^3)t^3+(u^5+u^4+6\,u^3+5\,u^2)t^2+4\,tu-1}$	$-\frac{1+t}{8\,t^2+5\,t-1}$
17	$-\frac{1+2\,tu+t^2u^2}{(u^6+u^5+4\,u^4+3\,u^3)t^3+(u^5+u^4+6\,u^3+7\,u^2)t^2+5\,tu-1}$	$-\frac{1+t}{9\,t^2+6\,t-1}$
18	$-\frac{1+2\,tu+t^2u^2}{(u^6+2\,u^5+3\,u^4+3\,u^3)t^3+(u^5+2\,u^4+5\,u^3+7\,u^2)t^2+5\,tu-1}$	$-\frac{1+t}{9\,t^2+6\,t-1}$
19	$-\frac{1+2\,tu+t^2u^2}{(u^6+2\,u^5+3\,u^4+4\,u^3)t^3+(u^5+2\,u^4+5\,u^3+9\,u^2)t^2+6\,tu-1}$	$-\frac{1+t}{10\,t^2+7\,t-1}$
20	$-\frac{1+2\,tu+t^2u^2}{(u^6+3\,u^5+2\,u^4+4\,u^3)t^3+(u^5+3\,u^4+4\,u^3+9\,u^2)t^2+6\,tu-1}$	$-\frac{1+t}{10\,t^2+7\,t-1}$
21	$-\frac{(1+tu)^2}{t^3u^6+3\,u^5t^3+t^2u^5+3\,t^3u^4+3\,u^4t^2+4\,t^3u^3+6\,t^2u^3+8\,t^2u^2+6\,tu-1}$	$-\frac{1+t}{11\,t^2+7\,t-1}$
22	$-\frac{(1+tu)^2}{t^3u^6+3\,u^5t^3+t^2u^5+4\,t^3u^4+3\,u^4t^2+3\,t^3u^3+7\,t^2u^3+7\,t^2u^2+6\,tu-1}$	$-\frac{1+t}{11\,t^2+7\,t-1}$
23	$-\frac{(1+tu)^2}{t^3u^6+3\,u^5t^3+t^2u^5+4\,t^3u^4+3\,u^4t^2+4\,t^3u^3+7\,t^2u^3+9\,t^2u^2+7\,tu-1}$	$-\frac{1+t}{12\,t^2+8\,t-1}$
24	$-\frac{(1+tu)^2}{2\,t^3u^6+2\,u^5t^3+2\,t^2u^5+4\,t^3u^4+2\,u^4t^2+4\,t^3u^3+7\,t^2u^3+9\,t^2u^2+7\,tu-1}$	$-\frac{1+t}{12\,t^2+8\,t-1}$
25	$-\frac{(1+tu)^3}{q(t,u)}$	$-\frac{(1+t)^2}{13\,t^3+22\,t^2+7\,t-1}$

Figure 3: Graded and non-graded Poincaré-Betti series of the minimal free resolution of K over A_n.

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