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TOPOLOGY WITH MONOIDAL CONTROL

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(*communicated by Ralph Cohen*)

Abstract

This paper defines monoidal control, which is a specialization of control by entourage as presented in [**5**]. It is shown that on metric spaces monoidal control generalizes bounded control, and describes continuous control. A systematic way of obtaining results from bounded control, in the sense of [**1**], as results of monoidal-, and hence continouos-, control, is developed. Especially this provides versions of the Hurewicz and Whitehead theorems with monoidal control, thus simultaneously establishing them for continouos control over metric spaces.

1. Introduction

Classically there are two main approaches to topology with control, bounded and continuous control. The purpose of this paper is to give a description of a new controlled context termed monoidal control, which involves the most important examples of bounded control and a version of continuous control as special cases.
The material presented here hooks up on the founding paper [**1**] by Anderson and Munkholm on bounded control. We generalize and extend conceptually all of [**1**], actually we work through some of the most important notions and leave the rest as an excercise for the dedicated reader.
Monoidal structures are special cases of coarse structures as described in [**5**]. We investigate the categories of controlled spaces over some control space in the spirit of [**1**], whereas [**5**] concentrates on the control spaces carrying the entourage structures. The monoid approach was first described by Munkholm in a workshop lecture, which together with [**1**] and the Ph.D. thesis of Christensen, [**4**], forms a basis for this paper. The question of a structured connection between monoidal and bounded control, naturally arises. In fact we show that monoidal control notions are colimits of the corresponding bounded control notions. This is an interesting, and very general, new connection. This gives a machinery enabling the results of [**1**] to be

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systematically transferred to the corresponding results with monoidal control, and hence also with the version of continuous control that is covered. The machinery and explanations given, should enable the reader to transfer any result from [**1**] to monoidal context. As a main application we explicitly give Hurewicz and Whitehead theorems in the monoidal context.

2. Monoidal Posets Acting on Categories

In our setting a monoid is a poset $\mathcal{M}$, viewed as a category, also denoted $\mathcal{M}$, with objects the elements and morphisms given by the partial ordering. Furthermore $\mathcal{M}$ is equipped with an associative bifunctor (composition) $\bullet : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$, and has a unit u w.r.t. $\bullet$, which is also initial with respect to the partial ordering. Let hereafter $\mathcal{M}$ denote a monoid in the above sense, and let $\bullet$ be implicit in any juxtaposition of monoid elements. By specifying the object u of $\mathcal{M}$, we define a functor $u : [0] \to \mathcal{M}$, where $[0]$ denotes a category with one object and the corresponding identity morphism. The properties of a monoid are described by the commutativity of the following diagrams. Note that $\mathcal{M}\times\mathcal{M}\times\mathcal{M}$ is identified with both $(\mathcal{M}\times\mathcal{M})\times\mathcal{M}$ and $\mathcal{M}\times(\mathcal{M}\times\mathcal{M})$, and that $\mathcal{M}$ is identified with both $[0]\times\mathcal{M}$ and $\mathcal{M}\times[0]$.

$$\begin{tikzcd}
\mathcal{M}\times\mathcal{M}\times\mathcal{M} \arrow[r, "(1_{\mathcal{M}} \times \bullet)"] \arrow[d, "(\bullet \times 1_{\mathcal{M}})"'] & \mathcal{M}\times\mathcal{M} \arrow[d, "\bullet"] \\
\mathcal{M}\times\mathcal{M} \arrow[r, "\bullet"] & \mathcal{M}
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}
{[0]}\times\mathcal{M} \arrow[r, "u \times 1_{\mathcal{M}}"] \arrow[dr, "1_{\mathcal{M}}"'] & \mathcal{M}\times\mathcal{M} \arrow[d, "\bullet"] & \mathcal{M}\times[0] \arrow[l, "1_{\mathcal{M}} \times u"'] \arrow[dl, "1_{\mathcal{M}}"] \\
 & \mathcal{M} &
\end{tikzcd} \tag{2.2}$$

Definition 2.1. *$\mathcal{M}$ is said to act from the left on a category $\mathcal{C}$, if there is given a functor $\alpha : \mathcal{M}\times\mathcal{C} \to \mathcal{C}$ such that the usual (associativity and unit) diagrams of functors*

$$\begin{tikzcd}
\mathcal{M}\times\mathcal{M}\times\mathcal{C} \arrow[r, "1_{\mathcal{M}} \times \alpha"] \arrow[d, "(\bullet \times 1_{\mathcal{C}})"'] & \mathcal{M}\times\mathcal{C} \arrow[d, "\alpha"] \\
\mathcal{M}\times\mathcal{C} \arrow[r, "\alpha"'] & \mathcal{C}
\end{tikzcd} \tag{2.3}$$

$$\begin{tikzcd}
{[0]}\times\mathcal{C} = \mathcal{C} \arrow[r, "u \times 1_{\mathcal{C}}"] \arrow[dr, "1_{\mathcal{C}}"'] & \mathcal{M}\times\mathcal{C} \arrow[d, "\alpha"] \\
 & \mathcal{C}
\end{tikzcd} \tag{2.4}$$

commute.

Clearly any left $\mathcal{M}$ action on a category $\mathcal{C}$ induces a right $\mathcal{M}$ action on any functor category $\mathcal{A}^{\mathcal{C}}$.

3. The Category of Fractions induced by a Monoid Action

If $\mathcal{C}$ is a category with an action (wlog from the left) $\alpha : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$, then any element M in $\mathcal{M}$ defines a functor $M : \mathcal{C} \to \mathcal{C}$ which is $\alpha(M,\ -)$ on objects and $\alpha(M \leqslant M,\ -)$ on morphisms. Especially the minimal element u defines the identity functor $1_{\mathcal{C}}$.
For every M in $\mathcal{M}$, there exists a natural transformation $\tau^M : 1_{\mathcal{C}} \Rightarrow M$ such that, for any B in $|\mathcal{C}|$, $\tau^M_B = \alpha(u \leqslant M,\ 1_B)$. Remembering that a morphism φ can be written as $\alpha(u \leqslant u,\ \varphi)$, it is straightforward to check naturality. Depending on context we refer to either τ^M or τ^M_B as a delay map.

Proposition 3.1. *Let $\mathcal{C}$ be a category with action $\alpha : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$. Then*

$$\Sigma = \{\ \tau^M_B \ \mid\ M \in \mathcal{M}, B \in |\mathcal{C}|\ \} \tag{3.1}$$

admits a calculus of left fractions on $\mathcal{C}$.

Proof. We show that Σ has the four properties given in [**6**, page 258]. Σ contains all identities since $\tau^u_B = \alpha(u \leqslant u, id_B) = id_B$, for any B in $|\mathcal{C}|$.
Let $\tau^M_{\alpha(L,B)}, \tau^L_B \in \Sigma$ then the composite $\tau^M_{\alpha(L,B)} \circ \tau^L_B$ is defined and the second component is by definition $\alpha(u \leqslant L, id_B)$, for the first component we get :

$$\begin{aligned} \tau^M_{\alpha(L,B)} &= \alpha(u \leqslant M, id_{\alpha(L,B)}) \\ &= \alpha(u \leqslant M, \alpha(L \leqslant L, id_B)), \quad \text{by functoriality of } \alpha. \\ &= \alpha(L \leqslant ML, id_B), \qquad \text{by definition 2.1 } (ii) \end{aligned}$$

Hence the composite is $\alpha(u \leqslant ML, id_B)$ which is in Σ.
Let

$$D \xleftarrow{s} C \xrightarrow{f} E \tag{3.2}$$

be a diagram in $\mathcal{C}$ with s in Σ, say $s = \tau^M_C = \alpha(u \leqslant M, id_C)$. We have $C = \alpha(u, C)$ and $D = \alpha(M, C)$. Set $C' = \alpha(M, E)$ and $s' = \tau^M_E = \alpha(u \leqslant M, id_E)$ which is in Σ, and finally $f' = \alpha(M \leqslant M, f)$. Then the following diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "s"'] & E \arrow[d, "s'"] \\ D \arrow[r, "f'"'] & C' \end{tikzcd} \tag{3.3}$$

commutes by naturality of τ^M.
Let

$$\begin{tikzcd} \alpha(u, C) \arrow[r, "s"] & \alpha(M, C) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & E \end{tikzcd} \tag{3.4}$$

be a diagram in $\mathcal{C}$ with $s = \tau_C^M = \alpha(u \leqslant M, id_C)$ in Σ and $f \circ s = g \circ s$. We have to find an object E' in $|\mathcal{C}|$ and a morphism $t : E \to E'$ in Σ, such that $t \circ f = t \circ g$. The given equality of composites immediately gives :

$$\alpha(M \leqslant M,\ f \circ \alpha(u \leqslant M, id_C)) = \alpha(M \leqslant M, g \circ \alpha(u \leqslant M, id_C))$$

Computing the left hand side gives:

$$\begin{aligned}
&\alpha(M \leqslant M, f) \circ \alpha(M \leqslant M, \alpha(u \leqslant M, id_C)) \\
&\qquad = \alpha(M \leqslant M, f) \circ \alpha(M \leqslant M^2, id_C) \\
&\qquad = \alpha(M \leqslant M, f) \circ \alpha(u \leqslant M,\ \alpha(M \leqslant M, id_C)) \\
&\qquad = \alpha(M \leqslant M, f) \circ \alpha(u \leqslant M, id_{\alpha(M,C)}) \\
&\qquad = \alpha(u \leqslant M, f) = \tau_E^M \circ f
\end{aligned}$$

By symmetry the right hand side is $\tau_E^M \circ g$, thus with $t = \tau_E^M$ and $E' = \alpha(M, E)$, the desired result is obtained.

All in all Σ admits a calculus of left fractions on $\mathcal{C}$. □

Of course the above also holds for right actions, still giving a calculus of *left* fractions.

Note that given any morphism set, you can always form the category of fractions. But only when the used set allows a calculus of at least left fractions it is straightforward to transfer properties of $\mathcal{C}$ to $\mathcal{C}(\Sigma^{-1})$, see [**6**, p.19.5] and [**1**, I.2].

4. Categories with Endomorphism/Action

Here we develop some categorical tools for the comparison of bounded control and monoidal control.

Definition 4.1. *[**1**, p.3] A category with endomorphism is a triple $(\mathcal{B}, C, \tau)$, where $\mathcal{B}$ is a category, $C : \mathcal{B} \to \mathcal{B}$ is a functor and $\tau : 1_{\mathcal{B}} \Rightarrow C$ is a natural transformation, s.t. $C\tau = \tau C$, meaning that for all $B \in |\mathcal{B}|$, $C(\tau_B) = \tau_{C(B)}$.*

Proposition 4.2. *If $\mathcal{C}$ is a category with an $\mathcal{M}$- action, then for every M in $\mathcal{M}$, $(\mathcal{C}, M, \tau^M)$ is a category with endomorphism.*

The proof is by inspection. Note that induced actions on functor categories, give induced endomorphism structures in the sense of [**1**]. For brevity let $\mathcal{C}_M$ denote $(\mathcal{C}, M, \tau^M)$.

Definition 4.3. *[**1**, p.13] Let $(\mathcal{B}_i, C_i, \tau_i),\ i = 1, 2$, be categories with endomorphism. A functor $\varphi : \mathcal{B}_1 \to \mathcal{B}_2$ is said to be almost endomorphism preserving if for some $k \geqslant 0$, there exists a natural transformation $\psi : \varphi C_1 \Rightarrow C_2^k \varphi$ s.t. the diagram of natural transformations*

$$\begin{tikzcd} \varphi \arrow[r, "\varphi\tau_1"] \arrow[dr, "\tau_2^k\varphi"'] & \varphi C_1 \arrow[d, "\psi"] \\ & C_2^k\varphi \end{tikzcd} \tag{4.1}$$

commutes. Notice that for $B \in |\mathcal{B}|$, $(\varphi\tau_1)_B = \varphi((\tau_1)_B)$ and $(\tau_2^n\varphi)_B = (\tau_2^n)_{\varphi(B)}$.

Proposition 4.4. *Let $\mathcal{C}$ be a category with an $\mathcal{M}$ action α. If $M \leqslant N \in \mathcal{M}$, then the identity functor $1_{\mathcal{C}} : \mathcal{C}_M \to \mathcal{C}_N$ is almost endomorphism preserving.*

The proof is simply by the fact that $\alpha(M^n \leqslant N^n, id_B) \circ \tau_B^{M^n} = \tau_B^{N^n}$.
Let us recall some definitions from [**1**]. If $(\mathcal{B}_i, C_i, \tau_i),\ i = 1, 2$, are categories with endomorphism, [**1**, p.3], then a functor $\varphi : \mathcal{B}_1 \to \mathcal{B}_2$ is called endomorphism preserving, [**1**, lemma 5.2,p.27] if it is almost endomorphism preserving and there exist some l in $\mathbb{N}$ and a natural transformation $\omega : C_2\varphi \to \varphi C_1^l$ such that the diagram of natural transformations

$$\begin{tikzcd} \varphi \arrow[r, "\tau_2\varphi"] \arrow[dr, "\varphi\tau_1^l"'] & C_2\varphi \arrow[d, "\omega"] \\ & \varphi C_1^l \end{tikzcd} \tag{4.2}$$

commutes. It is obvious that a category with endomorphism $(\mathcal{B}, C, \tau)$, can be interpreted as a category with action by the cyclic monoid generated by C with the natural ordering. And of course any cyclic monoid action induces an endomorphism structure.
We define the concept of functors between categories with monoid actions being *action preserving*, which is a natural generalization of endomorphism preserving functors.

Definition 4.5. *Let $\alpha_i : \mathcal{M}_i \times \mathcal{B}_i \to \mathcal{B}_i,\ i = 1, 2$, be monoid actions on categories $\mathcal{B}_i,\ i = 1, 2$. Assume the existence of functors $\mu : \mathcal{M}_1 \to \mathcal{M}_2$ and $\nu : \mathcal{M}_2 \to \mathcal{M}_1$, which are composition and order preserving. Furthermore for any M in $\mathcal{M}_1$ and any N in $\mathcal{M}_2$, assume that $M \leqslant \nu\mu M$ and $N \leqslant \mu\nu N$. A functor $\varphi : \mathcal{B}_1 \to \mathcal{B}_2$ is called action preserving, with respect to μ and ν, if there exist binatural transformations*

- $\psi : \varphi\alpha_1 \to \alpha_2 \circ (\mu \times \varphi)$
- $\omega : \alpha_2 \circ (1_{\mathcal{M}_2} \times \varphi) \to \varphi\alpha_1 \circ (\nu \times 1_{\mathcal{B}_1})$

such that for all M in $\mathcal{M}_1$ and N in $\mathcal{M}_2$, the diagrams

$$
\begin{tikzcd}
\varphi \arrow[r, "\varphi\tau_1^M"] \arrow[dr, "\tau_2^{\mu(M)}\varphi"'] & \varphi\alpha_1(M, \_) \arrow[d, "\psi^M"] \\
& \alpha_2(\mu(M), \varphi(\_))
\end{tikzcd}
\qquad
\begin{tikzcd}
\varphi \arrow[r, "\tau_2^N\varphi"] \arrow[dr, "\varphi\tau_1^{\nu(N)}"'] & \alpha_2(N, \varphi(\_)) \arrow[d, "\omega^N"] \\
& \varphi\alpha_1(\nu(N), \_)
\end{tikzcd}
\tag{4.3}
$$

commute.

Remember $\alpha_1(M, \_)$, or similar, above, is the functor we normally just call M, here we write it out to emphasize the binaturality of ψ and ω. If a functor between categories with action (only) satisfies the leftmost diagram, we call the functor *almost action preserving.*
Given categories with endomorphism $(\mathcal{B}_i, C_i, \tau_i),\ i = 1, 2$, and an endomorphism preserving functor $\varphi : \mathcal{B}_\infty \to \mathcal{B}_\in$, with constants k and l respectively, we may define $\mu, \nu : \mathbb{N} \to \mathbb{N}$ for $n \in \mathbb{N}$, by $\mu(n) = kn$ and $\nu(n) = ln$ respectively. If we interpret the induced actions of the cyclic monoids generated by C_i on $\mathcal{B}_i$ as actions by $\mathbb{N}$, it is immediate that ψ of 4.1 and ω of 4.2, extend to $\overline{\psi}$ and $\overline{\omega}$ respectively, satisfying 4.3. So an endomorphism preserving functor is action preserving with respect to the canonically induced $\mathbb{N}$ actions. Generally in the case of cyclic monoid actions, all information can be obtained from the canonically induced endomorphism structure, especially 4.3 reduces to 4.1 and 4.2. So the concept of an action preserving functor is a true generalization of the concept of an endomorphism preserving functor.

5. Monoid Actions and Colimits

We now consider the category of fractions defined by the action of the cyclic submonoid defined by a single monoid element. This uses some results from [**1**], but the colimit results are entirely new.

Definition 5.1. *Let $\mathcal{C}$ be a category with an $\mathcal{M}$ action α. Let $M \in \mathcal{M}$. Set*

$$\Sigma_M = \{\ \tau_B^{M^n} : B \to M^n(B) \mid n \in \mathbb{Z}_+, B \in |\mathcal{C}|\ \} \tag{5.1}$$

Here $\tau_B^{M^n} = \alpha(u \leqslant M^n,\ id_B)$. Trivially $\tau_B^{M^n} = (\tau^M)_B^n$, so Σ_M is exactly the Σ defined by the category with endomorphism $\mathcal{C}_M$, see [**1**, p.5] and proposition 4.2.
For $M \leqslant N$ in $\mathcal{M}$ it follows from proposition 4.4 and [**1**, prop.I.3.2] that there exists a unique functor $\lambda_{M,N} : \mathcal{C}(\Sigma_M^{-1}) \to \mathcal{C}(\Sigma_N^{-1})$ such that the diagram

$$
\begin{tikzcd}
& \mathcal{C} \arrow[dl, "Q_M"'] \arrow[dr, "Q_N"] & \\
\mathcal{C}(\Sigma_M^{-1}) \arrow[rr, "\lambda_{M,N}"'] & & \mathcal{C}(\Sigma_N^{-1})
\end{tikzcd}
\tag{5.2}
$$

commutes, where Q_M and Q_N are the canonical functors. Note that $\lambda_{M,N}$ is the identity on objects. By uniqueness $\lambda_{M,N}$ followed by $\lambda_{N,K}$, where defined, is $\lambda_{M,K}$. Having an initial element and an ordering, $\mathcal{M}$ is directed. Hence any $\mathcal{M}$-action on any category $\mathcal{C}$, defines a direct system

$$D_{\mathcal{C}}^{\alpha} : \mathcal{M} \to \mathcal{C}at \tag{5.3}$$

with $D_{\mathcal{C}}^{\alpha}(M) = \mathcal{C}(\Sigma_M^{-1})$ and $D_{\mathcal{C}}^{\alpha}(M \leqslant N) = \lambda_{M,N}$.

Theorem 5.2. *If $\mathcal{C}$ is a category with an $\mathcal{M}$-action, and $D_{\mathcal{C}}^{\alpha}$ is the direct system defined by the action, then there exists an isomorphism*

$$\varinjlim D_{\mathcal{C}}^{\alpha} \to \mathcal{C}(\Sigma^{-1})$$

Proof. For $M \leqslant N$ in $\mathcal{M}$ we immediately have a commutative diagram

$$\begin{tikzcd}
 & \mathcal{C}(\Sigma^{-1}) & \\
 & \mathcal{C} \arrow[u, "Q"] \arrow[dl, "Q_M"'] \arrow[dr, "Q_N"] & \\
D_{\mathcal{C}}^{\alpha}(M) = \mathcal{C}(\Sigma_M^{-1}) \arrow[uur, bend left, "\psi_M"] \arrow[rr, "\lambda_{M,N}"'] \arrow[dr, "\beta_M"'] & & \mathcal{C}(\Sigma_N^{-1}) = D_{\mathcal{C}}^{\alpha}(N) \arrow[uul, bend right, "\psi_N"'] \arrow[dl, "\beta_N"] \\
 & \varinjlim D_{\mathcal{C}}^{\alpha} &
\end{tikzcd} \tag{5.4}$$

where ψ_M and ψ_N are uniquely determined by the universal properties of respectively Q_M and Q_N. The family $\{\psi_L |\ L \in \mathcal{M}\}$ is thus compatible with $D_{\mathcal{C}}^{\alpha}$, hence we get a unique $\psi : \varinjlim D_{\mathcal{C}}^{\alpha} \to \mathcal{C}(\Sigma^{-1})$, such that for any L in $\mathcal{M}$, $\psi\beta_L = \psi_L$. Note that, since we have a direct system, for any M and N in $\mathcal{M}$, $\beta_M Q_M = \beta_N Q_N$, thus we actually have a map $\hat{\rho} : \mathcal{C} \to \varinjlim D_{\mathcal{C}}^{\alpha}$, which for each M in $\mathcal{M}$ sends every $\sigma \in \Sigma_M$ into an isomorphism. Since $\Sigma = \bigcup_{L \in \mathcal{M}} \Sigma_L$, we can invoke the universal property of Q, which gives a unique map $\rho : \mathcal{C}(\Sigma^{-1}) \to \varinjlim D_{\mathcal{C}}^{\alpha}$, such that for any L in $\mathcal{M}$, $\rho Q = \beta_L Q_L = \hat{\rho}$. Collecting things, for arbitrary L in $\mathcal{M}$, we end up with a commutative diagram,

$$\begin{tikzcd}
\mathcal{C}(\Sigma^{-1}) \arrow[r, shift left, "\rho"] & \varinjlim D_{\mathcal{C}}^{\alpha} \arrow[l, shift left, "\psi"] \\
\mathcal{C} \arrow[u, "Q"] \arrow[r, "Q_L"'] & \mathcal{C}(\Sigma_L^{-1}) \arrow[u, "\beta_L"'] \arrow[ul, "\psi_L"]
\end{tikzcd} \tag{5.5}$$

on which a simple chase invoking all universal properties, shows that ψ, respectively ρ, is an isomorphism. □

Next we use the above colimit result to transfer an important result on functors induced by almost endomorphism preserving funtors, to the context of almost action preserving functors.

Proposition 5.3. *[**1**, prop.I.3.2] Let $\mathcal{B}_i$ be categories with monoid action, $\alpha_i : \mathcal{M}_i \times \mathcal{B}_i \to \mathcal{B}_i$, $(i = 1, 2)$. If F is an almost action preserving functor, then there* *exists a unique induced functor* $F_! : \mathcal{B}_1(\Sigma_1^{-1}) \to \mathcal{B}_2(\Sigma_2^{-1})$*, such that the diagram*

$$\begin{tikzcd} \mathcal{B}_1 \arrow[r, "F"] \arrow[d, "Q_1"'] & \mathcal{B}_2 \arrow[d, "Q_2"] \\ \mathcal{B}_1(\Sigma_1^{-1}) \arrow[r, "F_!"] & \mathcal{B}_2(\Sigma_2^{-1}) \end{tikzcd} \tag{5.6}$$

commutes.

Proof. Let $F : \mathcal{B}_1 \to \mathcal{B}_2$ be an almost action preserving functor, and M in $\mathcal{M}_1$ be arbitrary. Directly by definition 4.5, F is almost endomorphism preserving with respect to the endomorphism structures induced by M respectively $\mu(M)$, where μ is from the definition. Hence the definition of $Q_2^{\mu(M)}$ and [**1**, prop.I.3.2] provides a commutative diagram

$$\begin{tikzcd} \mathcal{B}_1 \arrow[r, "F"] \arrow[d, "Q_1^M"'] & \mathcal{B}_2 \arrow[d, "Q_2^{\mu(M)}"] \arrow[dr, "Q_2"] & \\ \mathcal{B}_1(\Sigma_M^{-1}) \arrow[r, "F_!^M"] & \mathcal{B}_2(\Sigma_{\mu(M)}^{-1}) \arrow[r] & \mathcal{B}_2(\Sigma_2^{-1}) \end{tikzcd} \tag{5.7}$$

By the commutative square $Q_2^{\mu(M)}F((\tau_1)_B^M)$ is an isomorphism for every M in $\mathcal{M}_1$, hence so is $Q_2F((\tau_1)_B^M)$ and the universal property of Q_1 thus tells us that there exists exactly one $F_!$ such that $Q_2F = F_!Q_1$. $\square$

6. Monoidal Control Spaces

Definition 6.1. *Let Z be a space. Denote by $\mathcal{E}(Z)$ the set of reflexive relations on Z, with monoid structure given by composition of relations, and partial order given by inclusions. A monoidal structure on Z, is a submonoid $\mathcal{M}$ of $\mathcal{E}(Z)$ with the following properties :*

- *i)* *$\mathcal{M}$ consists of the diagonal of $Z \times Z$, and open neighborhoods thereof. The diagonal plays the role of identity/initial element.*
- *ii)* *$\mathcal{M}$ is closed with respect to composition and inversion of relations.*
- *iii)* *For any M in $\mathcal{M}$ and any compact subset C of Z, $M(C)$ is relatively compact.*
- *iv)* $\bigcup_{M\in\mathcal{M}} M = Z \times Z$

Example 6.2. *Let (Z, d) be a proper metric space which is geosdesic in the sense of Ballmann, [**3**].*
For any $\epsilon > 0$, let $M_\epsilon = \{\ (x, y) \in Z \times Z \mid d(x, y) < \epsilon\ \}$. Then

$$\mathcal{M}_0 = \{\ M_\epsilon \mid \epsilon > 0\ \} \cup\ \{\delta(Z \times Z)\}$$

is a monoidal structure on Z.

It is straightforward to prove the following :

Proposition 6.3. *Let $\mathcal{M}$ be a monoidal structure on a space Z, then $\mathcal{M} - \{u\}$ is a coarse structure on Z, as defined in [**5**].*

In a private communication Pedersen told us that Higson, Pedersen and Roe, do not insist on entourages being open, even though they said so in [**5**]. If they did, then, in order to lift entourages, they would have to consider continuous, not just proper, control maps. This tells us that we can actually avoid removing u in the proposition above.

Definition 6.4. *A monoidal control space is a triple $(Z, P, \mathcal{M})$, where Z is a space, P is a subset of the family of all subsets of Z, containing all singleton sets, and not containing the empty set. And $\mathcal{M}$ is a monoidal structure on Z, acting by evaluation of relations on the category $\mathcal{P}$, which has objects the elements of P, and morphisms the respective inclusions.*

We call the pair $(P, \mathcal{M})$ a monoidal control structure on Z, see [**1**, p.41-42] for the corresponding bounded notions. Condition [**1**, 1.1(iv),p.42] in the definition of a boundedness control structure on Z, is replaced by the following simple fact.

Lemma 6.5. *Let $(Z, P, \mathcal{M})$ be a monoidal control space, then*

$$\forall K \in P \ \forall M \in \mathcal{M} \ \forall x \in Z : M(\{x\}) \cap K \neq \emptyset \Rightarrow x \in M^{-1}(K). \tag{6.1}$$

The proof is trivial.

Example 6.6. *Given a space Z with some monoidal structure $\mathcal{M}$, we can always obtain a monoidal control structure $(P, \mathcal{M})$ on Z, by taking P to be the whole powerset on Z, except for the empty set. More interestingly though, we can also take P to be the relatively compact non-empty subsets of Z. This structure is actually the right setting for doing mc algebraic topology.*

In [**1**, p.42] the notion of radius of a subset of a boundedness control space Z is defined. This is used to define the concept of a *bounded family* of subsets of Z, as follows : Let (Z, P, C) be a boundedness control space and $A \in \mathcal{A}$ then

$$\text{rad A } = \inf \{ \ n \in \mathbb{Z}_+ \cup \{\infty\} \mid A \subseteq C^n K_0 \text{ for some minimal } K_0 \in P \ \}$$

And a family $\mathcal{A}$ of subsets of Z is called bounded if the set $\{ \text{ rad A} \mid A \in \mathcal{A} \}$ is bounded in $\mathbb{Z}$. This is equivalent to the statement that there exists some $d \geqslant 0$ such that for all $A \in \mathcal{A}$ there exists some minimal $K_0 \in P$ with $A \subseteq C^d K_0$. The following generalizes this to the mc setting, where size is measured by monoid elements, and not by an integer radius.

Definition 6.7. *Let Z be a space with monoidal structure $\mathcal{M}$. A family $\mathcal{A}$ of subsets of Z is said to be $\mathcal{M}$-bounded if one of the following two equivalent statements hold:*

- $\exists M \in \mathcal{M} \ \forall A \in \mathcal{A} \ \exists x \in Z \ : \ A \subseteq M(\{x\})$
- $\exists M \in \mathcal{M} \ \forall A \in \mathcal{A} \ \forall a \in A \ : \ A \subseteq M(\{a\})$

Thus when reading [**1**] from a monoidal point of view, any occurrence of radius should be replaced by the concept of $\mathcal{M}$-boundedness, mutatis mutandis.

7. Monoidally Controlled Spaces

Definition 7.1. *Let Z be any space. A controlled space over Z is a pair (E, p), where E is some space and $p : E \to Z$ is a continuous map.*

Notice that in the setting of control by entourage, see [**5**], the corresponding map p is only demanded to be proper. Given a monoidal control space $(Z, P, \mathcal{M})$, any space controlled over Z is called an mc space over $(Z, P, \mathcal{M})$.

Definition 7.2. *Let (E, p) and (E', p') be mc spaces over a monoidal control space $(Z, P, \mathcal{M})$. An mc map $f : (E, p) \to (E', p')$ is a continuos map $f : E \to E'$ for which the following equivalent statements hold :*

- $\exists M \in \mathcal{M}\ \forall B \in \mathcal{P}(E) : p'(f(B)) \subseteq M(p(B))$.
- $\exists M \in \mathcal{M}\ \forall B \in P : f(p^{-1}(B)) \subseteq (p')^{-1}(M(B))$

We say that f is controlled by M.

It is immediate that the collection of mc spaces over $(Z, P, \mathcal{M})$, and mc maps between them, form a category. We denote this by $\mathcal{TOP}^{\mathcal{M}}/Z$.

Example 7.3. *As in example 6.2, let (Z, d) be a proper metric space which is geodesic, and P be the collection of open bounded subsets of Z. Take $\mathcal{M}$ to be the submoniod of $\mathcal{M}_0$ consisting of the elements $\{M_n \mid n \in \mathbb{N}\}$ together with the diagonal. By considering the usual, "blow up", action of $\mathbb{N} \cup \{0\}$ on the metric balls in Z, it is easily seen that $\mathcal{M}$ acts on $\mathcal{P}$ by evaluation of relations. This is the metric monoidal control structure on Z. Note that the action defines a function $C : P \to P$ which together with P gives the metric boundedness control structure on Z. The mc maps with respect to this control structure are called bounded, see also [**1**, example 1.3,p.43] where there is no obvious reason to stick with the metric balls. Notice that the condition A required in [**1**] is replaced by the more general requirement of the metric space being geodesic.*

An mc CW complex over $(Z, P, \mathcal{M})$ is a pair (E, p), where $(E, p) \in \mathcal{TOP}^{\mathcal{M}}/Z$, with E a *finite dimensional* CW complex, and such that the set $\{\ p(e) \mid \text{e a cell of E}\ \}$ is $\mathcal{M}$-bounded, see definition 6.7. We denote the subcategory of $\mathcal{TOP}^{\mathcal{M}}/Z$ consisting of mc CW complexes and mc maps between them by $\mathcal{CW}^{\mathcal{M}}/Z$. Any (E, p) in $\mathcal{CW}^{\mathcal{M}}/Z$, with the property that, for any K in P, $p^{-1}(K)$ is contained in a finite subcomplex of E, is called a finite mc CW complex . We denote the full subcategory of finite mc CW complexes by $\mathcal{CW}_f^{\mathcal{M}}/Z$.

8. Continuous Control, at infinity

As an example of the usefulness of monoidal control, we give an mc description of continuous control.
Let $(\overline{Z},\ Y)$ be a pair of compact Hausdorff spaces, with $Z = \overline{Z} - Y$, assuming Z is dense in $\overline{Z}$. Given a controlled space (E, p) over Z, set $\overline{E} = E \amalg Y$ and $\overline{p} = p \amalg 1_Y : \overline{E} \to \overline{Z}$. $\overline{E}$ is given the smallest topology such that:

i) $E \subseteq \overline{E}$ is open and inherits its original topology.

ii) $\overline{p}$ is continuous.

In other words $W \subseteq \overline{E}$ is open if and only if $W \cap$ E is open in E, and there exists some open subset U in $\overline{Z}$ s.t. $W = (W \cap E) \cup (\overline{p})^{-1}(U)$. This is sometimes called the *teardrop topology*, see [**2**, p.221]. Note that a function $g : \overline{E}_1 \to \overline{E}_2$ is continuous if and only if $g|E_1 : E_1 \to E_2$ and $\overline{p}_2 \circ \overline{g} : \overline{E}_1 \to \overline{Z}$ are continuous.

Definition 8.1. *Let (E, p) and (E', p') be controlled spaces over Z. A continuous map $f : E \to E'$ is called continuously controlled at infinity, if and only if $\overline{f} = f \amalg 1_Y : \overline{E} \to \overline{E}'$ is continuous.*

Let $\mathcal{CC}(\overline{Z},\ Y)$ denote the category of spaces controlled over Z and continuously controlled maps between them.

Remark 8.2. *The category $\mathcal{CC}(\overline{Z}, Y)$ is the subcategory of $\mathcal{TOP}^{cc}/\mathcal{LC}$ of [**2**, p.223], obtained by having fixed base space Z and fixed space at infinity Y.*

For the sake of comparison with other descriptions of continuous control, we give the following interpretation of continuous control via neighborhoods, without (the easy) proof.

Proposition 8.3. *Let $(E, p), (E', p') \in |\mathcal{CC}(\overline{Z},\ Y)|$ and let $f : E \to E'$ be continuous. Then $f \in \mathcal{CC}(\overline{Z},\ Y)((E, p),\ (E', p'))$ if and only if*

$$\forall y \in Y\ \forall U \subseteq \overline{Z}(\textit{nbh. of } y)\ \exists V \subseteq U(\textit{nbh. of } y)\ \forall e \in E : p(e) \in V \Rightarrow p' \circ f(e) \in U.$$

Next we give the monoid/entourage version of continuous control, this is parallel to [**5**].

Definition 8.4. *Let $(\overline{Z}, Y)$ be as above. Let $\mathcal{M}^{cc}$ be given as follows: A relation $R \in \mathcal{E}(Z)$ is in $\mathcal{M}^{cc}$ if it is either the diagonal, $\delta(Z \times Z)$ in $Z \times Z$, or an open proper neighborhood of $\delta(Z \times Z)$ satisfying the following characteristic property :*

$$\textit{closure}(R, \overline{Z} \times \overline{Z}) \cap (Y \times Y) = \delta(Y \times Y)$$

i.e. the closure at infinity of any $\mathcal{M}^{cc}$ element is the diagonal in $Y \times Y$. Here proper is in the sense of definition 6.1[(iii)].

It is obvious that $(\mathcal{M}^{cc}, \mathcal{P}(Z))$ is a monoidal structure on Z.

Remark 8.5. *We can generalize the setup further, see also [**5**, p.5]. If there is given some equivalence relation on Y, for example the diagonal as above, we may define a monoidal structure on Z, by requiring that the monoid elements, at infinity, close off to a subset of the graph of the relation.*

Proposition 8.6. *Let $(\overline{Z}, Y)$ be as above and metric. Then*

$$\mathcal{TOP}^{\mathcal{M}^{cc}}/Z((E_1, p_1), (E_2, p_2))$$

is precisely the set of continuously controlled maps from E_1 to E_2.

Proof. Assume $f \in \mathcal{TOP}^{\mathcal{M}^{cc}}/Z((E_1, p_1), (E_2,\ p_2))$ i.e.

$$\exists M \in \mathcal{M}^{cc}\ \forall e \in E_1 : (p_1(e), p_2(f(e))) \in M$$

Let $y \in Y$ be arbitrary and let $U_y \subseteq \overline{Z}$ be an arbitrary open neighborhood of y. Following [**5**, lemma 2.4] there exists an open neighborhood V_y of y in $\overline{Z}$, wlog assume $V_y \subseteq U_y$, such that for any e in E, $p_1(e) \in V_y$ implies that $f(p_2(e)) \in U_y$. Thus by proposition 8.3 f is continuously controlled. Assume $f \in \mathcal{CC}(\overline{Z}, Y)((E_1, p_1), (E_2,\ p_2))$ and that $f \notin \mathcal{TOP}^{\mathcal{M}^{cc}}/Z((E_1, p_1), (E_2,\ p_2))$ i.e. $\forall M \in \mathcal{M}^{cc}\ \exists e_M \in E_1 :$ $(p_1(e_M), p_2(f(e_M))) \notin M$. Note that for any N $\in \mathcal{M}^{cc}$ with N $\leqslant$ M we have that $(p_1(e_M), p_2(f(e_M))) \notin N$. Since $\mathcal{M}^{cc}$ is directed we obtain a net $(e_M)_{M \in \mathcal{M}^{cc}}$ in $\overline{E}_1$, and since $\overline{Z}$ is compact we can find a convergent subnet, $(p_1(e_{M'}))_{M' \in \mathcal{K}}$, of the p_1 image of this net in $\overline{Z}$. Since $Z \times Z$ is covered by $\mathcal{M}^{cc}$ and $\mathcal{K}$ is cofinal, the subnet cannot converge in Z without violating our assumption on f, hence $(p_1(e_{M'})) \rightarrow y \in Y$. By [**2**, remark 1.3,p.222] the preimage of this subnet is a convergent subnet, of the original net, in $\overline{E}_1$, also converging to $y \in Y$.
We will now construct a monoid element which controls f, hence contradicting the original assumption. Denote the distance from a point x in Z to Y by d(x,Y). This is welldefined since Y is compact. Let $B_r(x)$ denote the open ball in Z with center x and radius r. For $M \in \mathcal{K}$ set

- $r_1 = r_1(M) = \frac{1}{2} d(p_1(e_M), Y)$
- $r_2 = r_2(M) = \frac{1}{2} d(p_2(f(e_M)), Y)$
- $B_M = B_{r_1}(p_1(e_M)) \times B_{r_2}(p_2(f(e_M)))$.

Now for N in $\mathcal{M}^{cc} - \delta(Z \times Z)$ set $N_0 = N \cup \{\ B_M | M \in \mathcal{K}\ \}$

Clearly N_0 is open in $Z \times Z$ and contains the diagonal. For any compact $C \subseteq Z$ we can argue that only finitely many of the B_M, which all are relatively compact, can contribute to $N_0(C)$ and $N_0^{-1}(C)$ thus these are relatively compact.

We have, so far, shown that N_0 could be an element of some monoidal structure on Z, we need to show that it is actually in $\mathcal{M}^{cc}$.

Let $(x_n^1,\ x_n^2)$ be a sequence in N_0 that converges to a point $(y^1,\ y^2) \in Y \times Y$. If a subsequence is contained in N we are done. Thus let us assume that a subsequence is contained in the $\bigcup_{M \in \mathcal{M}} B_M$ part of N_0. The subsequence cannot be contained in any single B_M, since it would converge in the closure hereof which is disjoint from $Y \times Y$ by construction, we may even assume that each B_M contains at most one element of the subsequence. Since the radii of the B_M tend to zero as the centers converge, the subsequence converges to $(y,\ y)$ i.e. $(x_n^1,\ x_n^2)$ converges to $(y,\ y)$. As above we may write N_0^{-1} instead of N_0 and the same arguments hold.
All in all N_0 is in $\mathcal{M}^{cc}$ and this contradicts the construction of the original net, thus f is $\mathcal{M}^{cc}$ controlled. □

9. Comparison of Bounded- and Monoidal Control

In this section we show that given a monoidal control space $(Z, P, \mathcal{M})$ the symmetrized monoid elements induce boundedness control structures on Z in the sense of [**1**]. It is in this context that we, by the colimit result, theorem 5.2, say that monoidal control is a colimit of bounded control.

Theorem 9.1. *If $(Z, P, \mathcal{M})$ is a monoidal control space, where Z is a connected metric space, with proper metric $d : Z \times Z \to \mathbb{R}_+$, then for each symmetric $M \in \mathcal{M} - \{u\}$, (Z, M, P) is a boundedness control space.*

Proof. We have to check that for every symmetric M in $\mathcal{M}$, (P, M) defines a boundedness control structure on Z, as given in [**1**, definition II.1.1]. That is we check that:

i) P is a subposet of the power set on Z and $M : P \to P$ is an order preserving function such that for all $K \in P$, $K \subseteq M(K)$.

ii) For all $K \in P$, $Z = \bigcup_n M^n(K)$

iii) For all $K \in P$, there exists a minimal element $K_0 \in P$ with $K_0 \subseteq K$.

iv) There exists a function $\Theta : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that if $K_0 \in P$ is minimal, $L \in P$, and $M^n K_0 \cap L \neq \emptyset$, then $K_0 \subseteq M^{\Theta(n)}(L)$.

ad (i) P is by definition a subposet of $\mathcal{P}(Z)$ and clearly the function $K \mapsto M(K)$, is order preserving.

ad (ii) For any $x \in Z$ set $U_x = \bigcup_{n \in \mathbb{N}_0} M^n(x)$. Since $M^0(x) \subseteq M(x)$ and $M(V)$ is open for any $V \in P$, it follows that U_x is open. Let $x, y \in Z$ and $m, n \in \mathbb{N}_0$. If $M^n(y) \cap M^m(x) \neq \emptyset$, then by lemma 6.5 and symmetry of M, $y \in M^{m+n}(x)$ and $x \in M^{m+n}(y)$. Hence if $U_x \cap U_y \neq \emptyset$, then $U_x = U_y$.
Now assume there exists $x \in Z$ such that $U_x \neq Z$. Set $V_x = \bigcup_{y \in Z - U_x} U_y$. Then $U_x \cap V_x = \emptyset$, both are open and their union is Z. This contradicts the connectedness of Z, thus for all $x \in Z$, $U_x = Z$ proving (ii).

ad (iii) By definition the singleton sets are in P.

ad (iv) Let $\theta = id_{\mathbb{Z}_+}$, this works because for x in Z and L in P, we get that if $M^n(\{x\}) \cap L \neq \emptyset$ then, by lemma 6.5, x is in $(M^n)^{-1}(L)$, but by symmetry of M $(M^n)^{-1} = M^n$.

All in all (P, M) is a bc structure on Z. $\square$

For the connection between monoidal and bounded control we observe the following fact

Lemma 9.2. *For $(Z, P, \mathcal{M})$ a monoidal control space, the symmetrized elements*

$$\{ \; MM^{-1} \mid M \in \mathcal{M} \; \}$$

form a cofinal subset of $\mathcal{M}$.

Now if $(Z, P, \mathcal{M})$ is a monoidal control space, and M is in $\mathcal{M}$, then $\mathcal{TOP}^M/Z$ denotes the category of controlled spaces and maps between them controlled by the monoid generated by M, $\{\mathcal{TOP}^M/Z \mid M \in \mathcal{M}\}$ with inclusions $\iota_{M,N}$ for $M \leqslant N$ defines a direct system $\mathcal{M} \to \mathcal{C}at$. We get the following proposition almost immediately, the easy proof is left for the reader.

Proposition 9.3. *Let $(Z, P, \mathcal{M})$ be a monoidal control space. Then there exists an isomorphism of categories :*

$$\varinjlim_{M \in \mathcal{M}} \mathcal{TOP}^{MM^{-1}}/Z \to \mathcal{TOP}^{\mathcal{M}}/Z \tag{9.1}$$

The category with endomorphism determined by (MM^{-1}, P), following [**1**, p.42], is exactly the one determined by MM^{-1} as in 4.2. As we shall see later all interesting algebraic topology invariants of mc spaces, live in categories of fractions determined by the monoid action. Thus in the light of theorem 9.1, theorem 5.2 and proposition 9.3 we very concretely interpret monoidal control as the colimit of bounded control. This is, to our knowledge, the most explicit and general presentation of the connection between continuous control at infinity and bounded control. As we shall see later it enables us to take the results from bounded control, [**1**], and interpret them as results of monoidal, especially continuous, control.

10. Fragmented Spaces and Fragmentations

We recall definitions and results from [**1**, section II.2] and put them in a monoidal setting. The given proofs nicely demonstrate the transition from bc to mc context. Let $\mathcal{B}$ be a category with action $\alpha : \mathcal{M} \times \mathcal{B} \to \mathcal{B}$, and let Σ be the, by now, usual set of morphisms defined via this action. We call the category $\mathcal{TOP}^{\mathcal{B}}(\Sigma^{-1})$ the category of fragmented spaces over $\mathcal{B}$. By now we know a lot about the inner works of categories like this, and by section 5 we immediately conclude that it is the colimit of the categories of fragmented spaces $\mathcal{TOP}^{\mathcal{B}}(\Sigma_M^{-1})$ as defined in [**1**, p.47], where the subscript denotes that the endomorphism structure is induced by the element M of $\mathcal{M}$. Any $\underline{X}$ in $|\mathcal{TOP}^{\mathcal{B}}|$ is thus called a fragmented space, for $K \in |\mathcal{B}|$ we write $X_K = \underline{X}(K)$. It is convenient to think of $\underline{X}$ as the family $\{ \; X_K \mid K \in |\mathcal{B}| \; \}$ together with the family of maps $\{ \; \underline{X}(g) \mid g \in \mathcal{B}(K, L) \; \}$, corresponding to the fact that maps between fragmented spaces are natural transformations. If $\underline{X}$, $\underline{Y}$ are fragmented spaces, then a morphism $F : \underline{X} \to \underline{Y}$ in $\mathcal{TOP}^{B}(\Sigma^{-1})$, is represented by a natural transformation $f_M : \underline{X} \to \underline{Y}M$ in $\mathcal{TOP}^{\mathcal{B}}$, for some M in $\mathcal{M}$. In the above, when $\mathcal{TOP}$ is replaced by another category $\mathcal{C}$, the category $\mathcal{C}^{\mathcal{B}}$ is called the category of fragmented $\mathcal{C}$-objects over $\mathcal{B}$. Let $(Z, P, \mathcal{M})$ be a monoidal control space.

Definition 10.1. *[**1**, p.48] A fragmentation of $(E, p) \in \mathcal{TOP}^{\mathcal{M}}/Z$, is a functor $F : \mathcal{P} \to \mathcal{P}(E)$ such that $\cup_{K \in P} F(K) = E$.*

Here $\mathcal{P}$(E) denotes the subcategory of $\mathcal{TOP}$ consisting of all subsets of E and inclusions among them. Note that F is a fragmented space in the above sense.

Example 10.2. *Let $(E,p) \in |\mathcal{TOP}^{\mathcal{M}}/Z|$. Then $K \mapsto p^{-1}(K)$ defines a fragmentation of (E,p) called the inverse image fragmentation.*

Example 10.3. *Let $(E,p) \in |\mathcal{CW}^{\mathcal{M}}/Z|$. Then*

$$K \mapsto \textit{smallest subcomplex of } E \textit{ containing } p^{-1}(K)$$

defines a fragmentation of (E,p) called the smallest subcomplex fragmentation.

Definition 10.4. *[**1**, p.50] Let $(E,p) \in |\mathcal{TOP}^{\mathcal{M}}/Z|$ and let $F_1, F_2 : \mathcal{P} \to \mathcal{P}(E)$ be two fragmentations of (E,p). We say that F_1 and F_2 are equivalent if there exists $M, N \in \mathcal{M}$ such that for all $K \in P$, $F_1(K) \subseteq F_2(M(K))$ and $F_2(K) \subseteq F_1(N(K))$. (This is an equivalence relation).*

As an illustration of the transition between bc and mc contexts, especially concerning the radius notion, we present

Lemma 10.5. *[**1**, Lemma II.2.5] If (E,p) is an mc CW complex over some monoidal control space $(Z,P,\mathcal{M})$, then the inverse image fragmentation of (E,p) is equivalent to the smallest subcomplex fragmentation of (E,p).*

Proof. Let (E,p) be an mc space over $(Z,P,\mathcal{M})$ and, for K in P, let $K \to X_K$ be the smallest subcomplex fragmentation. By definition, for all K in P, $p^{-1}(K) \subseteq X_K$. Thus we only need to find some N in $\mathcal{M}$ such that, for all K in P, $X_K \subseteq p^{-1}(N(K))$. Let K be arbitrary in P. Following the proof of the bc version [**1**, p.51], an i-cell e^i is in X_K if and only if there exists a sequence of cells $e^i < e^{i(1)} < \ ... \ < e^{i(j)}$ where $i = i(0) < i(1) < \ ... \ < i(j)$, where $e^{i(j)} \cap p^{-1}(K) \neq \emptyset$, and where $e^{i(k)} < e^{i(k+1)}$, $k = 0, 1, ..., j-1$ means that $\overline{e}^{i(k)} \cap \overline{e}^{i(k+1)} \neq \emptyset$. Note that since (E,p) is an mc CW complex, there exists some M in $\mathcal{M}$ such that for every cell e of E, there exists an element z of Z, such that the image $p(e)$ is contained in $M(\{z\})$. Hence let z be in Z such that $p(e^{i(j)}) \subseteq M(\{z\})$, then $M(\{z\}) \cap K \neq \emptyset$, thus by the inversion lemma $z \in M^{-1}(K)$, whereby it follows that $p(e^{i(j)}) \subseteq MM^{-1}(K)$. Decreasing induction over the sequence of cells, gives that $e^i \subseteq p^{-1}((MM^{-1})^n(K))$, with $n = j+1 \leqslant dimE + 1$. Setting $N = (MM^{-1})^n$ every cell of X_K is contained in $p^{-1}(N(K))$, hence so is X_K, proving the lemma. □

Now we will consider categories of fragmented spaces induced by a monoidal control structure. We will need the following two results, which are parallel to results in [**1**].

Lemma 10.6. *[**1**, lemma II.2.6] Let $(Z,\mathcal{M},P)$ be a monoidal control space. Then*

- *The inverse image fragmentation defines a functor:*
$$Fr_1 : \mathcal{TOP}^{\mathcal{M}}/Z \to \mathcal{TOP}^{\mathcal{P}}(\Sigma^{-1})$$
- *The smallest subcomplex fragmentation defines a functor:*
$$Fr_2 : \mathcal{CW}_f^{\mathcal{M}}/Z \to \mathcal{CW}_f^{\mathcal{P}}(\Sigma^{-1})$$

Lemma 10.7. *[**1**, lemma II.2.7] The functors $Fr_1\Box$ and $\Box\, Fr_2$ are naturally equivalent. Here $\Box$ denotes the suitable forgetful functor.*

The proofs of these lemmata are direct translations of the bc versions. The first lemma is used to lift definitions from fragmented spaces to mc spaces, and the second lemma shows us that whatever we define using the first lemma, is independent of the actual CW-structure. If $(Z, P, \mathcal{M})$ is a monoidal control space, then the canonical universal cover of an mc space (E, p) is the universal cover of the fragmented space $Fr_1(E, p)$, details below. We could have used Fr_2, and if we do it will be clear from the context, thus $\mathcal{P}G_1(Fr_i(E,p)), i = (1, 2)$ is always denoted $\mathcal{P}G_1(E,p)$. Let $\underline{X}$ in $\mathcal{TOP}^{\mathcal{B}}(\Sigma^{-1})$ be a fragmented space over $\mathcal{B}$. Let $G_1(\underline{X})$ be the composite functor

$$\mathcal{B} \xrightarrow{\underline{X}} \mathcal{TOP} \xrightarrow{G_1} \mathcal{GPOID} \tag{10.1}$$

where G_1 is the fundamental groupoid functor. $\mathcal{B}G_1(\underline{X})$ is the wreath product $\mathcal{B} \int G_1(\underline{X})$, thus objects are pairs (K, x) where K is in $|\mathcal{B}|$ and x is in X_K, morphisms are also pairs

$$(i, \omega) : (K, x) \to (L, y) \tag{10.2}$$

where $i : K \to L$ is a morphism and ω is a homotopy class relative to endpoints of paths from y to $\underline{X}(i)(x)$ in X_L. Note that $\mathcal{B}G_1(\underline{X})$, in the obvious way, inherits an $\mathcal{M}$ action from the action on $\mathcal{B}$.
The universal cover of $\underline{X}$ is the fragmented space $\widetilde{\underline{X}} : \mathcal{B}G_1(\underline{X}) \to \mathcal{TOP}$ defined by setting

$$\widetilde{X}_{(K,x)} = P(X_K, x)/\sim$$

for $(K, x) \in |\mathcal{B}G_1(\underline{X})|$ and

$$\widetilde{\underline{X}}(i, \omega)(\alpha) = \omega * \underline{X}(i)(\alpha)$$

for $(i, \omega) : (K, x) \to (L, y)$ and $\alpha \in \widetilde{X}_{(K,x)}$. Here $P(X_K, x)/_\sim$ is the space of paths in X_K with initial point x, modulo the relation of homotopy relative to endpoints. We let ε_x denote the class of the constant path at x. Let $p_{(K,x)} : (\widetilde{X}_{(K,x)}, \varepsilon_x) \to (X_K, x)$ be the endpoint projection. When $\underline{X}$ is a fragmented CW complex, this is the universal cover of the component of X_K containing x. As in [**1**, p.184], an mc subspace (W, p) of (X, p) determines a sub fragmented space $\overline{W}$ of $\widetilde{X}$ by requiring that

$$\begin{tikzcd} \overline{W}(K,x) \arrow[r, "\subseteq"] \arrow[d] & \tilde{X}(K,x) \arrow[d, "p_{(K,x)}"] \\ W \cap X_K \arrow[r, "\subseteq"] & X_K \end{tikzcd} \tag{10.3}$$

be a pullback for any $K \in |\mathcal{B}|$.

11. Homology of Monoidally Controlled Spaces

Parallel to [**1**, Section II.3] we go via fragmented spaces to obtain the homology of mc spaces. Let $H_n : \mathcal{TOP} \to \mathcal{AB}$ denote the usual n'th singular homology functor. For any category $\mathcal{B}$, composition on the left with H_n induces a functor $H_{n_*} : \mathcal{TOP}^{\mathcal{B}} \to \mathcal{AB}^{\mathcal{B}}$. Given a category with action $\alpha : \mathcal{M} \times \mathcal{B} \to \mathcal{B}$, it is a straightforward inspection to see that, with respect to the induced actions on the two functor categories, H_{n_*} is almost action preserving. By proposition 5.3 there exists a unique functor

$$H_n^F : \mathcal{TOP}^{\mathcal{B}}(\Sigma^{-1}) \to \mathcal{AB}^{\mathcal{B}}(\Sigma^{-1}) \tag{11.1}$$

such that the diagram

$$\begin{tikzcd} \mathcal{TOP}^{\mathcal{B}} \arrow[r, "H_{n_*}"] \arrow[d, "Q"'] & \mathcal{AB}^{\mathcal{B}} \arrow[d, "Q"] \\ \mathcal{TOP}^{\mathcal{B}}(\Sigma^{-1}) \arrow[r, "H_n^F"'] & \mathcal{AB}^{\mathcal{B}}(\Sigma^{-1}) \end{tikzcd} \tag{11.2}$$

commutes. Details of the above in the bounded case are given in [**1**, lemma I.5.1]. H_n^F has the following concrete description :

- For any $\underline{E} \in \mathcal{TOP}^{\mathcal{B}}(\Sigma^{-1})$
  - For every $K \in |\mathcal{B}|$, $H_n^F(\underline{E})(K) = H_n(\underline{E}(K))$
  - For every $g \in \mathcal{B}(K, L)$, $H_n^F(\underline{E})(g) = \underline{E}(g)_*$
- For any $f \in \mathcal{TOP}^{\mathcal{B}}(\Sigma^{-1})(\underline{E}, \underline{Y})$, if $f_M : \underline{E} \to M\underline{Y}$ represents f with
  $$f_M = \{\ f_{MK} : \underline{E}(K) \to \underline{Y}(M(K)) \mid K \in |\mathcal{B}|\ \},$$
  then $H_n^F(f)$ is represented by
  $$f_{M_*} = \{\ f_{MK_*} : H_n(\underline{E}(K)) \to H_n(\underline{Y}(M(K))) \mid K \in |\mathcal{B}|\ \}.$$

As in [**1**] homology of pairs and triples of fragmented spaces follow the same lines, and we have the expected long exact sequences in homology. Furthermore, since homotopies of maps of fragmented spaces are defined fragmentwise, see [**1**, p.57], homotopy invariance is immediate. Notice that interchanging the n'th homology functor with the n'th chain functor changes nothing but notation in this setup, and we thereby get chain complexes of fragmented spaces.

Definition 11.1. *Let (E, p) be an mc space over a monoidal control space $(Z, P, \mathcal{M})$. The n'th mc homology of (E, p) is defined as:*

$$H_n^{\mathcal{M}}(E, p) = H_n^F(Fr_1(E, p)) \in \mathcal{AB}^{\mathcal{P}}(\Sigma^{-1}). \tag{11.3}$$

Thus, for any K in P, we have $H_n^{\mathcal{M}}(E, p)(K) = H_n(p^{-1}(K))$, and for any inclusion $\iota : K \to L$ in $\mathcal{P}$ we have $H_n^{\mathcal{M}}(E, p)(\iota) = \iota_* : H_n(p^{-1}(K)) \to H_n(p^{-1}(L))$. In [**1**, p.89 ff.] the n'th bc homology functor is defined as the composite

$$\mathcal{TOP}^c/Z \xrightarrow{Fr_1} \mathcal{TOP}^{\mathcal{P}}(\Sigma^{-1}) \xrightarrow{H_n^F} \mathcal{AB}^{\mathcal{P}}(\Sigma^{-1}) \tag{11.4}$$

Of course the n'th bc chain functor is defined similarly. Now by theorem 9.1 given a monoidal control space $(Z, P, \mathcal{M})$, for every $M \in \mathcal{M}$, we get a boundedness control space (Z, P, MM^{-1}), and the composite

$$\mathcal{TOP}^{MM^{-1}}/Z \xrightarrow{Fr_1^M} \mathcal{TOP}^{\mathcal{P}}(\Sigma^{-1}_{MM^{-1}}) \xrightarrow{H_n^{F^M}} \mathcal{AB}^{\mathcal{P}}(\Sigma^{-1}_{MM^{-1}}) \tag{11.5}$$

defines the n'th bc homology– (chain–) functor corresponding to this induced boundedness control structure. With theorems 5.2 and 9.3 in mind the colimit of this diagram is exactly the n'th mc homology of (E, p), with the concrete description given above. Notice that by the considerations of section 5 and [**1**, cor. I.4.3], all of the categories $\mathcal{AB}^{\mathcal{P}}(\Sigma_M^{-1})$ are abelian, hence by theorem 5.2 $\mathcal{AB}^{\mathcal{P}}(\Sigma^{-1})$ is abelian. Furthermore the mc equivalent of [**1**, corollary II.3.5] tells us that mc homology is the homology of the similarly defined mc chain complex.

12. Homotopy of Monoidally Controlled Spaces

Following [**1**] we define the homotopy of mc spaces via the homotopy of fragmented spaces. Let $(Z, \mathcal{M}, P)$ be a monoidal control space. All references to [**1**] refer to the bc case with boundedness control structure generated by some symmetrized monoid element, see theorem 9.1. Let $\mathcal{C}$ be the category $\mathcal{S}ets_*, \mathcal{G}p$ or $\mathcal{A}b$ depending on whether n is $0, 1$ or $\geqslant 2$ respectively. For (E, p) an mc space over Z, let $\underline{E}$ denote either the inverse image or the smallest subcomplex fragmentation induced by (E, p), according to circumstances. Define a functor

$$\pi_n(\underline{E}) \ : \ \mathcal{P}G_1(\underline{E}) \to \mathcal{C}$$

by

- $\pi_n(\underline{E})(A, x) = \pi_n(E_A, x)$, for $(A, x) \in |\mathcal{P}G_1(\underline{E})|$
- $\pi_n(\underline{E})(i, \omega) = \omega_* \circ (\underline{E}(i))_*$, for $(i, \omega) \in \mathcal{P}G_1(\underline{E})((A, x), (B, y))$

where ω_* is the change of basepoint isomorphism induced by conjugation with ω.

Definition 12.1. *[**1**, def. 4.1 p. 60] For $n \geqslant 0$ set*

$$\pi_n^F(\underline{E}) = Q(\pi_n(\underline{E})) \in |\mathcal{C}^{\mathcal{P}G_1(\underline{E})}(\Sigma^{-1})|$$

where Q is the canonical functor. $\pi_n^F(\underline{E})$ is called the n'th homotopy of the fragmented space $\underline{E}$.

Now G_1 is the fundamental groupoid functor, hence $Aut(A, x)$ in $\mathcal{P}G_1(\underline{E})$ is isomorphic to $\pi_1(E_A, x)$ whereby the latter gets an action on $\pi_n^F(\underline{E})$, hence the n'th homotopy of the fragmented space $\underline{E}$ gets a kind of $\mathcal{P}G_1(\underline{E})$ module structure, see [**1**, p.61]. Now we will need the notion of morphism induced by maps of fragmented spaces, furthermore these should turn out to be isomorphisms if the original maps are homotopy equivalences. There is a slight, but wellknown, problem here, the module structures do not match apriori. But as for the classical case, whenever there is a map $f : \underline{Y} \to \underline{E}$ of fragmented spaces we can give $\pi_n^F(\underline{E})$ a $\mathcal{P}G_1(\underline{Y})$ module structure, this is contained in the following.

Proposition 12.2. *[**1**, prop. 4.2 p. 61] Any map of fragmented spaces over $\mathcal{B}$, $f : \underline{Y} \to \underline{E}$, induces a functor*

$$f^! : \mathcal{C}^{\mathcal{B}G_1(\underline{E})}(\Sigma^{-1}) \to \mathcal{C}^{\mathcal{B}G_1(\underline{Y})}(\Sigma^{-1})$$

and a morphism

$$f_* : \pi_n^F(\underline{Y}) \to f^!\pi_n^F(\underline{E}).$$

The functor $f^!$ is unique up to a canonical natural equivalence, and, given $f^!$, f_* is unique.

The proof is mutatis mutandis (mm) the same as in [**1**]. For pairs and triples of fragmented spaces [**1**] gives us similar results, and it also provides exact sequences for both. Furthermore, by mm [**1**, cor.4.10, p.66], π_n^F is homotopy invariant. Now we define the n'th homotopy of an mc space.

Definition 12.3. *Let $(E,p) \in |\mathcal{TOP}^{\mathcal{M}}/Z|$. Define the n'th homotopy of (E,p) to be the object*

$$\pi_n^{\mathcal{M}}(E,p) = \pi_n^F(Fr_1(E,p)) \in |\mathcal{C}^{\mathcal{P}G_1(E,p)}(\Sigma^{-1})|$$

We are now able to give mc, and hence especially cc, versions of the Hurewicz and Whitehead theorems, demonstrating the usability of the colimit results. First however, we need a concrete description of the isomorphisms in categories of the type $\mathcal{C}^{\mathcal{B}}(\Sigma^{-1})$ where $\mathcal{M}$ acts on $\mathcal{B}$.

Lemma 12.4. *Let $\mathcal{M}$ be a monoid acting on a category $\mathcal{B}$, $\mathcal{C}$ be a category with zero object and $F \in |\mathcal{C}^{\mathcal{B}}(\Sigma^{-1})|$. Then $F = 0$ if and only if there exists an $N \in \mathcal{M}$ such that for all $B \in |\mathcal{B}|$, $F(\tau_B^N) : F(B) \to F(NB)$ is the zero morphism.*

The proof is exactly the same mm as the one given for the bc version [**1**, lemma I.4.5 p.23], and is left to the reader.

Definition 12.5. *[**1**, some of page 23] Let $\mathcal{C}$ be $R-mod$, $\mathcal{G}p$ or $\mathcal{S}ets_*$. Let $\sigma \in \mathcal{C}^{\mathcal{B}}(F,G)$, σ is :*

1. *Eventually monomorphic if, $\exists N \in \mathcal{M}\ \forall B \in |\mathcal{B}|\ \forall x_1, x_2 \in F(B) : \sigma_B(x_1) = \sigma_B(x_2) \Rightarrow \tau_B^N(x_1) = \tau_B^N(x_2)$*
2. *Eventually epimorphic if, $\exists N \in \mathcal{M}\ \forall B \in |\mathcal{B}|\ \forall x \in G(B) : \tau_B^N(x) \in Im\{\sigma_{N(B)} : F(NB) \to G(NB)\}$*

If $\sigma \in \mathcal{C}^{\mathcal{B}}(\Sigma^{-1})(F,G)$, we say that σ is eventually one or the other if it has a representative $\sigma_M : F \to MG$ which is eventually one or the other.

Directly from the characterization of isomorphisms in a category of fractions and [**1**, lemma I.4.6 p.24], using the fact that $R-mod$, $\mathcal{G}p$ and $\mathcal{S}ets_*$ are balanced [**6**, 5.3.3 p.34], we get the following lemma.

Lemma 12.6. *Let $\mathcal{C}$ be $R-mod$, $\mathcal{G}p$ or $\mathcal{S}ets_*$, $\sigma \in \mathcal{C}^{\mathcal{B}}(\Sigma^{-1})(F,G)$. Then σ is an isomorphism if and only if it is eventually monomorphic and eventually epimorphic.*

As an application of the lemmata above and earlier colimit results, we give mc versions of the absolute and relative Hurewicz theorem together with a Whitehead theorem. The corresponding bc versions are in [**1**, p.90 - 93].

Let $\rho : \mathcal{P}G_1(E,p) \to \mathcal{P}$ and $U : \mathcal{A}b \to \mathcal{G}p$ be the forgetful functors. Let

$$\varphi : \pi_1^{\mathcal{M}}(E,p) \to \rho^! U_! H_1^{\mathcal{M}}(E,p) \tag{12.1}$$

and

$$\varphi : \pi_n^{\mathcal{M}}(E,p) \to \rho^! H_n^{\mathcal{M}}(E,p),\ n \geqslant 2 \tag{12.2}$$

be induced by the Hurewicz map

$$\varphi_{(x,K)} : \pi_n(p^{-1}(K),x) \to H_n(p^{-1}(K)),\ n \geqslant 1.$$

Let

$$\psi : \rho_! \pi_1^{\mathcal{M}}(E,p) \to U_! H_1^{\mathcal{M}}(E,p) \tag{12.3}$$

and

$$\psi : \rho_! \pi_n^{\mathcal{M}}(E,p) \to H_n^{\mathcal{M}}(E,p),\ n \geqslant 2 \tag{12.4}$$

be the adjoints of the morphisms φ. Finally let ψ^{ab} be the image of ψ under the functor $()^{ab} : \mathcal{G}p^{\mathcal{P}}(\Sigma^{-1}) \to \mathcal{A}b^{\mathcal{P}}(\Sigma^{-1})$ induced by the abelianization functor $ab : \mathcal{G}p \to \mathcal{A}b$. Notice that viewing the categories of fractions with monoid action as colimits, see section 5, of the categories of fractions with monoid element induced boundedness control structures, almost trivially, gives that the above are the colimits of the corresponding bc versions denoted respectively $\rho^!_M$, $\rho^M_!$, $\iota^M \psi^{ab}_M$, $\iota^M \psi_M$ and $()^{ab}_M$, where the $M \in \mathcal{M}$ indicates that the boundedness control structure is induced by MM^{-1}. Here $\iota^M : \mathcal{C}^{\mathcal{P}}(\Sigma_M^{-1}) \to \mathcal{C}^{\mathcal{P}}(\Sigma^{-1})$ is given by the universal property of the canonical functor $Q_M : \mathcal{C}^{\mathcal{P}} \to \mathcal{C}^{\mathcal{P}}(\Sigma_M^{-1})$. Hence $\{\iota^M | M \in \mathcal{M}\}$ is the colimit cone associated with the colimit $\mathcal{C}^{\mathcal{P}}(\Sigma^{-1})$.

Theorem 12.7 (Absolute Hurewicz). *[**1**, thm.9.1,p.90] Let (E,p) be an mc space over $(Z,P,\mathcal{M})$. Then $\psi^{ab} : (\rho_! \pi_1^{\mathcal{M}}(E,p))^{ab} \to H_1^{\mathcal{M}}(E,p)$ is an isomorphism in $\mathcal{A}b^{\mathcal{P}}(\Sigma^{-1})$. If for some $n \geqslant 2$, $\pi_i^{\mathcal{M}}(E,p) = 0$ for $1 \leqslant i \leqslant n-1$, then $\psi : \rho_! \pi_n^{\mathcal{M}}(E,p) \to H_n^{\mathcal{M}}(E,p)$ is an isomorphism $\mathcal{A}b^{\mathcal{P}}(\Sigma^{-1})$. If (E,p) is simply connected i.e. $\pi_i^{\mathcal{M}}(E,p) = 0$, $i = 0,1$, and, for some $n \geqslant 2$, $H_i^{\mathcal{M}}(E,p) = 0$ for $1 \leqslant i \leqslant n-1$, then $\pi_i^{\mathcal{M}}(E,p) = 0$ for $1 \leqslant i \leqslant n-1$ and*

$$\varphi : \pi_n^{\mathcal{M}}(E,p) \to \rho^! H_n^{\mathcal{M}}(E,p) \tag{12.5}$$

is an isomorphism in $\mathcal{A}b^{\mathcal{P}G_1(E,p)}(\Sigma^{-1})$.

Proof. Let (E,p) be an mc space over Z. By [**1**, thm. 9.1 p.90],

$$\psi^{ab}_M : (\rho^M_! \pi_1^M(E,p))^{ab}_M \to H_1^M(E,p) \tag{12.6}$$

is an isomorphism for any $M \in \mathcal{M}$, hence the colimit in $\mathcal{A}b^{\mathcal{P}}(\Sigma^{-1})$ is an isomorphism.

Assume that, for some $n \geqslant 2$, $\pi_i^{\mathcal{M}}(E,p) = 0$ for $1 \leqslant i \leqslant n-1$, then by lemma 12.4 there exists $N \in \mathcal{M}$ such that for all $(K,x) \in |\mathcal{P}G_1(E,p)|$,

$$\pi_i^{\mathcal{M}}(E,p)(\tau^N_{(K,x)}) : \pi_i^{\mathcal{M}}(E,p)(K,x) \to \pi_i^{\mathcal{M}}(E,p)(NK,x) \tag{12.7}$$

is the zero morphism. Trivially for any $M \in \mathcal{M}$ with $M \geqslant N$, for all $(K,x) \in |\mathcal{P}G_1(E,p)|$, $\pi_i^{\mathcal{M}}(E,p)(\tau^M_{(K,x)})$ is the zero map. Especially $\pi_i^M(E,p) = 0$, for $1 \leqslant$

$i \leqslant n-1$, so by the bc Hurewicz theorem [**1**, thm. 9.1, p.90], $\psi_M : \rho_!^M \pi_n^M(E,p) \to H_n^M(E,p)$ is an isomorphism for all $M \geqslant N$. Thus the colimit $\psi : \rho_! \pi_n^{\mathcal{M}}(E,p) \to H_n^{\mathcal{M}}(E,p)$ is an isomorphism.

Assume $\pi_i^{\mathcal{M}}(E,p) = 0,\ i = 0,1$, and, for some $n \geqslant 2$, $H_i^{\mathcal{M}}(E,p) = 0,\ 1 \leqslant i \leqslant n-1$. Then again for some $N \in \mathcal{M},\ \pi_i^M(E,p) = 0,\ i = 0,1$, and $H_i^M(E,p) = 0,\ 1 \leqslant i \leqslant n-1$, for all $M \geqslant N$ in $\mathcal{M}$. Hence by the bc Hurewicz theorem, for such M,

$$\varphi_M : \pi_n^M(E,p) \to \rho^!_M H_n^M(E,p) \tag{12.8}$$

is an isomorphism and $\pi_i^M(E,p) = 0$ for $1 \leqslant i \leqslant n-1$, thus $\pi_i^{\mathcal{M}}(E,p) = 0$ for $1 \leqslant i \leqslant n-1$ and the colimit

$$\varphi : \pi_n^{\mathcal{M}}(E,p) \to \rho^! H_n^{\mathcal{M}}(E,p) \tag{12.9}$$

is an isomorphism. □

Definition 12.8. *[**1**, p.91] Let (E,p) and (Y,q) be mc spaces over $(Z,P,\mathcal{M})$, (Y,q) is coextensive with (E,p) if there exists an $M \in \mathcal{M}$ such that for every $K \in P$ if $p^{-1}(K) \neq \emptyset$, then $q^{-1}(M(K)) \neq \emptyset$ and if $q^{-1}(K) \neq \emptyset$, then $p^{-1}(M(K)) \neq \emptyset$.*

Note that mc coextensive implies bc coextensive (boundedness control structure induced by M) and vice versa. See [**1**, p.91] for the bc definition.

Theorem 12.9 (Relative Hurewicz). *[**1**, thm.9.2,p.91] Let (E,Y,p) be an mc pair over $(Z,P,\mathcal{M})$ and suppose that $\pi_i^{\mathcal{M}}(E,Y,p) = 0$ for $i = 0,1$. Then $\psi^{ab} : (\rho_! \pi_2^{\mathcal{M}}(E,Y,p))^{ab} \to H_2^{\mathcal{M}}(E,Y,p)$ is an isomorphism. In addition if, for some $n \geqslant 3,\ \pi_i^{\mathcal{M}}(E,Y,p) = 0$ for $i \leqslant n-1$, then $\psi : \rho_! \pi_n^{\mathcal{M}}(E,Y,p) \to H_n^{\mathcal{M}}(E,Y,p)$ is an isomorphism. If $(Y,p|Y)$ is coextensive with (E,p) and both mc spaces are simply connected, then $\varphi : \pi_2^{\mathcal{M}}(E,Y,p) \to \rho^! U_! H_2^{\mathcal{M}}(E,Y,p)$ is an isomorphism. Furthermore, if for some $n \geqslant 3,\ H_i^{\mathcal{M}}(E,Y,p) = 0$ for $i \leqslant n-1$, then $\pi_i^{\mathcal{M}}(E,Y,p) = 0$ for $i \leqslant n-1$ and $\varphi : \pi_n^{\mathcal{M}}(E,Y,p) \to \rho^! H_n^{\mathcal{M}}(E,Y,p)$ is an isomorphism.*

The proof is done in the same way as the absolute theorem above. Let $\mathcal{C}$ be one of the categories $\mathcal{S}ets_*$, $\mathcal{G}p$ or $\mathcal{A}b$ depending on context.

Theorem 12.10 (Whitehead). *[**1**, cor.10.4,p.93] Let $f : (E,p) \to (Y,q)$ be a morphism in $\mathcal{CW}^{\mathcal{M}}/Z$. Then f is a homotopy equivalence in $\mathcal{CW}^{\mathcal{M}}/Z$, if and only if (Y,q) is coextensive with (E,p) and for all $n \geqslant 0,\ f_* : \pi_n^{\mathcal{M}}(E,p) \to f^! \pi_n^{\mathcal{M}}(Y,q)$ is an isomorphism in $\mathcal{C}^{\mathcal{P}G_1(E,p)}(\Sigma^{-1})$.*

Proof. Let $f : (E,p) \to (Y,q)$ be an mc map controlled by $M \in \mathcal{M}$. For the only if part, assume that f has homotopy inverse $g : (Y,p) \to (E,q)$. We may wlog assume that g and the homotopies are controlled by $N \in \mathcal{M}$. Then f is a bc homotopy equivalence with boundedness control structure induced by any $L \in \mathcal{M}$ with $L \geqslant MN$. Hence by the bc Whitehead theorem [**1**, cor.10.4, p.93], (Y,q) is coextensive with (E,p) and $f_*^L : \pi_n^L(E,p) \to f^!_L \pi_n^L(Y,q)$ is an isomorphism for all $n \geqslant 0$ and all $L \geqslant MN$. Thus the colimit $f_* : \pi_n^{\mathcal{M}}(E,p) \to f^! \pi_n^{\mathcal{M}}(Y,q)$ is an isomorphism for all $n \geqslant 0$.

For the other direction, assume that (Y,q) is coextensive with (E,p) for some $N \in \mathcal{M}$, and that $f_* : \pi_n^{\mathcal{M}}(E,p) \to f^! \pi_n^{\mathcal{M}}(Y,q)$ is an isomorphism for all $n \geqslant 0$. By lemma

12.6 there exists $L \in \mathcal{M}$ (wlog $L \geqslant N$), such that $f_*^L : \pi_n^L(E,p) \to f_L^! \pi_n^L(Y,q)$ is an isomorphism in $\mathcal{C}^{\mathcal{P}G_1(E,p)}(\Sigma_L^{-1})$ for all $n \geqslant 0$, thus by the bc Whitehead theorem, f is a bc homotopy equivalence, where the boundedness control structure is induced by L. Hence f is an mc homotopy equivalence. □

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