

HIGHER-DIMENSIONAL ARITHMETIC USING p -ADIC ÉTALE TATE TWISTS

KANETOMO SATO

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Abstract

This paper is a survey on recent researches of the author and his recent joint work with Shuji Saito. We will explain how to construct p -adic étale Tate twists on regular arithmetic schemes with semistable reduction, and state some fundamental properties of those objects. We will also explain how to define cycle class maps from Chow groups to étale cohomology groups with coefficients in p -adic étale Tate twists and state injectivity and surjectivity results on those new cycle class maps.

Dedicated to V. Snaith on his 60th birthday

Introduction

In this paper, we like to survey recent results of the author [Sat2], [Sat3] and his joint work with Shuji Saito [SaSa] (cf. [JSS]). Let k be an algebraic number field, i.e., a finite field extension of \mathbb{Q} . Let A be the integer ring of k . According to the classical unramified class field theory, we have the following isomorphism of finite groups:

$$\rho_A : \text{Cl}(A) \xrightarrow{\cong} \pi^{\text{ab}}(\text{Spec}(A))^{\sim},$$

where $\text{Cl}(A)$ (resp. $\pi^{\text{ab}}(\text{Spec}(A))$) denotes the ideal class group of A (resp. étale abelian fundamental group of $\text{Spec}(A)$) and \sim means the quotient group corresponding to abelian coverings of $\text{Spec}(A)$ which splits completely at every real infinite place. This map is defined by sending the class of a prime ideal $\mathfrak{p} \subset A$ to the arithmetic Frobenius substitution of $\text{Spec}(A)$ at \mathfrak{p} . By works of Bloch [B1], Kato and Saito [KS], [Sa1], this isomorphism is generalized to an arbitrary noetherian regular scheme X which is proper flat of finite type over $\text{Spec}(A)$:

$$\rho_X : \text{CH}_0(X) \xrightarrow{\cong} \pi^{\text{ab}}(X)^{\sim},$$

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where $\mathrm{CH}_0(X)$ denotes the Chow group of 0-cycles on X modulo rational equivalence. In this paper we like to generalize this map, so called the *reciprocity map*, to an arbitrary intermediate codimension as an étale cycle class map by introducing p -adic étale coefficients which we call p -adic étale Tate twists (see (1.1), (4.1) below). We will state some injectivity and surjectivity results on those new cycle class maps (Corollary 4.3, Theorem 5.1). In our injectivity results, the unramified cohomology group $H_{\mathrm{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ for the function field $K := k(X)$ plays a key role, which generalizes the Tate-Shafarevich group of the Jacobian variety of X_k . We will state finiteness results on this group in §3, from which we will derive the injectivity results (see (4.2) below). Our surjectivity results, which actually concerns proper smooth or semistable families over local integer rings, is proved by using the arithmetic duality theorem stated in Theorem 2.2 (see §5 for more details). Using this surjectivity result, we will deduce the triviality of the group $H_{\mathrm{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ in some special cases (see Corollary 6.3 below).

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Notation

For a scheme X and an integer $q \geq 0$, X^q denotes the set of all points $x \in X$ of codimension q . Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology.

1. p -adic étale Tate twists

Let p be a rational prime number. Let A be a Dedekind ring whose fraction field has characteristic zero and which has a residue field of characteristic p . We always assume that

any residue field of A of characteristic p is perfect.

Let X be a noetherian regular scheme of pure-dimension which is flat of finite type over $B := \mathrm{Spec}(A)$ and satisfies the following condition:

(*) X is a smooth or semistable family around any fiber of X/B of characteristic p .

Let n and r be integers with $n \geq 0$ and $r \geq 1$.

(1.1) Construction. The p -adic étale Tate twist $\mathfrak{T}_r(n)_X$ is defined as an object $\mathcal{K} \in D^b(X_{\mathrm{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ satisfying the following four properties:

T1. There exists an isomorphism $t : \mathcal{K}|_V \simeq \mu_{p^r}^{\otimes n}$ with $V := X \otimes_A A[1/p]$.

T2. \mathcal{K} is concentrated in $[0, n]$.

T3. For a locally closed regular subscheme $i : Z \rightarrow X$ of characteristic p , we have a Gysin isomorphism:

$$W_r \Omega_{Z, \log}^{n-c}[-n-c] \xrightarrow{\simeq} \tau_{\leq n+c} Ri^! \mathcal{K} \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r \mathbb{Z}),$$

where c denotes $\text{codim}_X(Z)$ and $W_r \Omega_{Z, \log}^{n-c}$ means the zero sheaf in case $n < c$.

T4. Let y and x be points on X with $\text{ch}(x) = p$, $x \in \overline{\{y\}}$ and $\text{codim}_X(x) = \text{codim}_X(y) + 1$. Put $c := \text{codim}_X(x)$. Then the boundary map

$$\delta_{y,x}^{\text{loc}}(\mathcal{K}) : H_y^{n+c-1}(\text{Spec}(\mathcal{O}_{X,y}), \mathcal{K}) \longrightarrow H_x^{n+c}(\text{Spec}(\mathcal{O}_{X,x}), \mathcal{K})$$

in the localization theory in étale topology coincides with the following boundary map of Galois cohomology groups defined by Kato ([**KCT**], §1) via Gysin isomorphisms, up to a sign depending only on $(\text{ch}(y), c)$:

$$\partial_{y,x}^{\text{val}} : \left\{ \begin{array}{ll} H^{n-c+1}(y, \mu_{p^r}^{\otimes n-c+1}) & (\text{ch}(y) = 0) \\ H^0(y, W_r \Omega_{y, \log}^{n-c+1}) & (\text{ch}(y) = p) \end{array} \right\} \longrightarrow H^0(x, W_r \Omega_{x, \log}^{n-c}),$$

where the Gysin map for i_y in case $\text{ch}(y) = 0$ arises from the isomorphism t in **T1** and Deligne's cycle class in $H_y^{2c-2}(\text{Spec}(\mathcal{O}_{X,y}), \mu_{p^r}^{\otimes c-1})$.

The properties **T1–T3** are $\mathbb{Z}/p^r \mathbb{Z}$ -coefficient variants of Beilinson-Lichtenbaum axioms on conjectural étale motivic complexes ([**Be**], [**Li2**]). More precisely, **T2** is taken from the acyclicity axiom for $\Gamma(n)_X^{\text{ét}}$, **T1** comes from the axiom of Kummer theory for $\Gamma(n)_V^{\text{ét}}$, and **T3** is suggested by the purity axiom and the axiom of Kummer theory for $\Gamma(n-c)_Z^{\text{ét}}$. Although **T4** is not among Beilinson-Lichtenbaum axioms, it is a natural property to be satisfied. In the following theorem, ‘unique’ means ‘unique up to a unique isomorphism’ (see [**Sat3**], Theorem 0.2, Corollary 0.3):

Theorem 1.1. There exists a unique pair (\mathcal{K}, t) of an object $\mathcal{K} \in D^b(X_{\text{ét}}, \mathbb{Z}/p^r \mathbb{Z})$ and an isomorphism $t : \mathcal{K}|_V \simeq \mu_{p^r}^{\otimes n}$ that satisfies **T2–T4**.

Definition 1.2. For each $n \geq 0$ and $r > 0$, we fix a pair (\mathcal{K}, t) as in Theorem 1.1 and define $\mathfrak{T}_r(n)_X := \mathcal{K}$.

In case X/B is smooth, this object is already considered by Schneider [**Sch**]. We have $\mathfrak{T}_r(0) \simeq \mathbb{Z}/p^r \mathbb{Z}$ and $\mathfrak{T}_r(n) \simeq Rj_* \mu_{p^r}^{\otimes n}$ for $n > \dim(X)$.

Remark 1.3. Assume that A is local with residue field \mathbb{F} and $p \geq n + 2$. Let Y be the closed fiber of X/B and let ι (resp. j) be the closed (resp. open) immersion $Y \rightarrow X$ (resp. $V \rightarrow X$).

- (1) In case X/B is smooth, $\iota^* \mathfrak{T}_r(n)_X$ is canonically isomorphic to $R\epsilon_* S_r(n)$ (cf. [**K1**], [**Ku**]), where $S_r(n)$ denotes the syntomic complex of Fontaine-Messing [**FM**] on the crystalline site $(X_r/W_r)_{\text{cris}}$ with $X_r := X \otimes_A A/p^r A$ and $W_r := W_r(\mathbb{F})$, and ϵ denotes the natural continuous map $(X_r/W_r)_{\text{cris}} \rightarrow (X_r)_{\text{ét}}$ of sites. The morphism t corresponds to the Fontaine-Messing morphism $R\epsilon_* S_r(n) \rightarrow \tau_{\leq n} \iota^* Rj_* \mu_{p^r}^{\otimes n}$.

- (2) $\iota^*\mathfrak{T}_r(n)_X$ is *not* the log syntomic complex of Kato and Tsuji ([K2], [Ts1]) unless $n > \dim(X)$, because the latter object is isomorphic to $\tau_{\leq n}\iota^*Rj_*\mu_{p^r}^{\otimes n}$ by [Ts2]. Therefore $\mathfrak{T}_r(n)_X$ is a new object particularly on semistable families.

Let $\mathbb{Z}(n)_X^{\text{ét}}$ (resp. $\mathbb{Z}(n)_X^{\text{Zar}}$) be the étale sheafification (resp. the Zariski sheafification) of Bloch's cycle complex ([B2], [Le1]). By works of Levine ([Le1], [Le2]), these two objects are regarded as strong candidates for the motivic complexes $\Gamma(n)_X^{\text{ét}}$ and $\Gamma(n)_X^{\text{Zar}}$, respectively, so that we can expect the following conjecture (cf. [Sat3], Conjecture 0.4):

Conjecture 1.4. (1) *There exists a canonical isomorphism*

$$\mathbb{Z}(n)_X^{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\simeq} \mathfrak{T}_r(n)_X \quad \text{in } D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}).$$

- (2) *Let ε be the natural continuous map $X_{\text{ét}} \rightarrow X_{\text{Zar}}$ of sites. Then the morphism in (1) induces an isomorphism*

$$\mathbb{Z}(n)_X^{\text{Zar}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\simeq} \tau_{\leq n}R\varepsilon_*\mathfrak{T}_r(n)_X \quad \text{in } D^b(X_{\text{Zar}}, \mathbb{Z}/p^r\mathbb{Z}).$$

The case $n = 0$ is obvious. The case $n = 1$ holds true by the Kummer theory for \mathbb{G}_m (cf. [Sat3], Proposition 3.13):

$$\mathbb{G}_m \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z} \longrightarrow \mathfrak{T}_r(1)_X \quad \text{in } D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

and the isomorphisms $\mathbb{Z}(1)_X^{\text{ét}} \simeq \mathbb{G}_m[-1]$, $\mathbb{Z}(1)_X^{\text{Zar}} \simeq \varepsilon_*\mathbb{G}_m[-1]$ (cf. [Le2], Lemma 11.2), and $R^1\varepsilon_*\mathbb{G}_m = 0$ (Hilbert's theorem 90). For $n \geq 2$, by results of Geisser ([Ge], Theorems 1.2 (4), 1.3), Conjecture 1.4 (1) holds true in case X/B is smooth, under the Bloch-Kato conjecture on Galois symbol maps.

(1.2) *Product structure.* The following theorem plays key roles in later sections:

Theorem 1.5. *Let n and n' be integers at least 0. Then there exists a unique morphism*

$$\mathfrak{T}_r(n)_X \otimes^{\mathbb{L}} \mathfrak{T}_r(n')_X \longrightarrow \mathfrak{T}_r(n+n')_X \quad \text{in } D^-(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}) \quad (1.2.1)$$

that extends the natural map $\mu_{p^r}^{\otimes n} \otimes \mu_{p^r}^{\otimes n'} \rightarrow \mu_{p^r}^{\otimes n+n'}$ on $V_{\text{ét}}$.

Let $Y \subset X$ be the divisor defined by the radical ideal sheaf of $(p) \subset \mathcal{O}_X$, and let ι and j be as follows:

$$V = X[1/p] \xrightarrow{j} X \xleftarrow{\iota} Y.$$

To construct the morphism (1.2.1) in case $m \geq 1$ and $n \geq 1$, the author proved the following natural isomorphism of étale sheaves (cf. [Sat3], Theorem 2.7):

$$\iota^*\mathcal{H}^n(\mathfrak{T}_r(n)_X)/U^1M_r^n \simeq \text{Im}(d\log : (\mathbb{G}_{m,Y})^{\otimes n} \rightarrow \bigoplus_{y \in Y^0} \iota_{y*}W_r\Omega_{y,\log}^n) =: \lambda_{Y,r}^n, \quad (1.2.2)$$

where U^\bullet denotes the Bloch-Kato filtration on $M_r^n := \iota^*R^nj_*\mu_{p^r}^{\otimes n}$ (cf. [Sat3], Definition 2.2) and for $y \in Y$, ι_y denotes the natural map $y \rightarrow Y$ (in case Y is smooth, we have $\lambda_{Y,r}^n \simeq W_r\Omega_{Y,\log}^n$, and the isomorphism (1.2.2) is due to Bloch-Kato [BK1]). By (1.2.2), it turns out that the sheaf $\mathcal{H}^n(\mathfrak{T}_r(n)_X)$ is generated by $\iota_*(U^1M_r^n)$ and the image of $(\mathbb{G}_{m,X})^{\otimes n}$, so that the image of the natural cup-product map

$$\mathcal{H}^m(\mathfrak{T}_r(m)_X) \otimes \mathcal{H}^n(\mathfrak{T}_r(n)_X) \longrightarrow R^{m+n}j_*\mu_{p^r}^{\otimes m+n}$$

lies in $\mathcal{H}^{m+n}(\mathfrak{T}_r(m+n)_X)$. From this fact, we obtain the morphism (1.2.1) as the unique morphism extending the natural map $\mu_{p^r}^{\otimes m} \otimes \mu_{p^r}^{\otimes n} \rightarrow \mu_{p^r}^{\otimes m+n}$ on V (see [Sat3], Proposition 3.3 for details).

Remark 1.6. For $n \geq 0$, let $\nu_{Y,r}^n$ be the kernel of the following boundary map of logarithmic Hodge-Witt sheaves:

$$\bigoplus_{y \in Y^0} \iota_{y*} W_r \Omega_{y, \log}^n \longrightarrow \bigoplus_{x \in Y^1} \iota_{x*} W_r \Omega_{x, \log}^{n-1},$$

where for $x \in Y$, $\iota_x : x \rightarrow Y$ is as defined above. Then one can easily see that we have $\lambda_{Y,r}^n \subset \nu_{Y,r}^n$ in general, and that $\lambda_{Y,r}^n = \nu_{Y,r}^n = W_r \Omega_{Y, \log}^n$ in case Y is smooth. In case Y has normal crossings, the fundamental properties of $\nu_{Y,r}^n$ and $\lambda_{Y,r}^n$ are studied in [Sat2]. Moreover, we have the following isomorphism for $n \geq 1$:

$$M_r^n / \iota^* \mathcal{H}^n(\mathfrak{T}_r(n)_X) \simeq \nu_{Y,r}^{n-1} \tag{1.2.3}$$

(cf. [Sat3], Theorem 2.7; compare with (1.2.2)).

2. Arithmetic duality theorems

Let A be either a global integer ring or a p -adic local integer ring, and put $B := \text{Spec}(A)$. Let p and X/B be as in §1. In case A is global, for $\mathcal{L} \in D^+(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$, we define the m th étale cohomology group $H_c^m(X, \mathcal{L})$ with compact support to be $H_c^m(B, Rf_! \mathcal{L})$, where f denotes the structure morphism $X \rightarrow B$ and $H_c^*(B, \bullet)$ denotes the étale cohomology group with compact support of B (cf. [Mi2], II.2). In this section, we give the statements and an outline of the duality results that the author proved in [Sat3], §6 (see also [JSS] for other duality results).

(2.1) Duality theorems. We first assume that A is global. For $V = X[1/p]$, we have the following well-known pairing with $d := \dim(X)$, the absolute dimension of X :

$$H_c^i(V, \mu_{p^r}^{\otimes n}) \times H^{2d+1-i}(V, \mu_{p^r}^{\otimes d-n}) \longrightarrow H_c^{2d+1}(V, \mu_{p^r}^{\otimes d}) \xrightarrow{\text{tr}_V} \mathbb{Z}/p^r\mathbb{Z},$$

which is a non-degenerate pairing of finite groups by the Artin-Verdier duality ([AV], [Ma], [Mi2]; see also [Dn], [Sp]) and the relative Poincaré duality for regular schemes ([SGA4], XVIII, [Th], [Fu]).

Theorem 2.1. *Assume that A is global and that X is proper over B . Then:*

- (1) *There exists a canonical trace map $\text{tr}_X : H_c^{2d+1}(X, \mathfrak{T}_r(d)_X) \rightarrow \mathbb{Z}/p^r\mathbb{Z}$ extending tr_V . If X is connected, then tr_X is bijective.*
- (2) *For any integers n and i with $0 \leq n \leq d$, the natural pairing*

$$H_c^i(X, \mathfrak{T}_r(n)_X) \times H^{2d+1-i}(X, \mathfrak{T}_r(d-n)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}, \tag{2.1.1}$$

arising from (1.2.1) and tr_X , is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$ -modules.

By the non-degeneracy of the above pairing for V , Theorem 2.1 is reduced to the following duality theorem:

Theorem 2.2. *Assume that A is local and that X is proper over B . Then:*

- (1) *There exists a canonical trace map $\mathrm{tr}_{(X,Y)} : H_Y^{2d+1}(X, \mathfrak{T}_r(d)_X) \rightarrow \mathbb{Z}/p^r\mathbb{Z}$, which is bijective if X is connected.*
- (2) *For any integers n and i with $0 \leq n \leq d$, the natural pairing*

$$H_Y^i(X, \mathfrak{T}_r(n)_X) \times H^{2d+1-i}(X, \mathfrak{T}_r(d-n)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}, \quad (2.1.2)$$

arising from (1.2.1) and $\mathrm{tr}_{(X,Y)}$, is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$ -modules.

(2.2) Outline of Theorem 2.2. Without loss of generality, we may assume that X is integral and $A = \Gamma(X, \mathcal{O}_X)$. Let Y be the closed fiber of X/B . By a standard argument using the Bockstein triangle (cf. [Sat3], Proposition 3.5)

$$\mathfrak{T}_r(n)_X \longrightarrow \mathfrak{T}_1(n)_X \xrightarrow{\delta_{1,r-1}} \mathfrak{T}_{r-1}(n)_X[1] \longrightarrow \mathfrak{T}_r(n)_X[1],$$

we are reduced to the case $r = 1$, and then by a standard norm argument we are further reduced to the case where $\mathrm{Frac}(A)$ contains primitive p th roots of unity. By the isomorphisms (1.2.2) and (1.2.3), the non-degeneracy of (2.1.2) is reduced to that of the following two pairings:

$$H^i(Y, \nu_{Y,1}^t) \times H^{d-i}(Y, \lambda_{Y,1}^{d-1-t}) \longrightarrow \mathbb{Z}/p\mathbb{Z}, \quad (2.2.1)$$

$$H^i(Y, U^1 M_1^t) \times H^{d-1-i}(Y, U^1 M_1^{d+1-t}) \longrightarrow \mu, \quad (2.2.2)$$

where μ denotes the group of p th roots of unity in $\mathrm{Frac}(A)$; we need to consider several t 's to prove Theorem 2.2 for a single n . The non-degeneracy of (2.2.1) is due to Milne [Mil] in case Y is smooth, and proved by the author [Sat2] in case Y has normal crossings. The most delicate part of our duality is the non-degeneracy of (2.2.2). In fact, this pairing arises from the following morphism obtained by decomposing the morphism (1.2.1) with $(m, n) = (t, d + 1 - t)$:

$$U^1 M_1^t \otimes U^1 M_1^{d+1-t} \longrightarrow (\iota^* j_* \mu_p \otimes \nu_{Y,1}^{d-1})[1] \quad \text{in} \quad D^b(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z}),$$

where M_1^m and U^\bullet are as in (1.2). The author proved an explicit formula (cf. [Sat3], Theorem 5.6) to compute this morphism in terms of products of symbols and a simple connecting morphism in $D^b(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$. By that formula, the non-degeneracy of (2.2.2) is reduced to the Serre-Hartshorne duality for cohomology groups of Y with coefficients in differential modules.

3. Unramified cohomology and finiteness

In this and the later sections, we work with the following setting. Let k be either a local field or a global field. Let A be the integer ring of k . Let p be a prime number, and let X be a noetherian regular scheme which is proper flat of finite type over $B := \mathrm{Spec}(A)$. In case p is not invertible in A we always assume that X has good or semistable reduction at all closed points on B of characteristic p . In case p is

invertible in A (i.e., k is ℓ -adic local with $\ell \neq p$), $\mathfrak{T}_r(n)_X$ means the usual Tate twist $\mu_p^{\otimes n}$. Let K be the function field of X . We will often write X_k for $X \otimes_A k$.

(3.1) Unramified cohomology. We define the unramified cohomology group $H_{\text{ur}}^{n+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ as the kernel of the boundary map

$$H^{n+1}(\text{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n)) \longrightarrow \bigoplus_{y \in X^1} H_y^{n+2}(\text{Spec}(\mathcal{O}_{X,y}), \mathfrak{T}_\infty(n)_X|_{\text{Spec}(\mathcal{O}_{X,y})}),$$

where $\mathfrak{T}_\infty(n)_X$ denotes $\varinjlim_{r \geq 1} \mathfrak{T}_r(n)_X$. The y -component of the group on the right hand side is isomorphic to $H^n(y, \mathbb{Q}_p/\mathbb{Z}_p(n-1))$, if $\text{ch}(y) \neq p$. The following isomorphisms hold true:

$$H_{\text{ur}}^1(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) \simeq H^1(X, \mathbb{Q}_p/\mathbb{Z}_p) \quad \text{and} \quad H_{\text{ur}}^2(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \simeq \text{Br}(X)_{p\text{-tors}},$$

where $\text{Br}(X)$ denotes the Grothendieck-Brauer group $H^2(X, \mathbb{G}_m)$ (see [SaSa]). An intriguing question is whether $H_{\text{ur}}^{n+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is finite. In fact, this question is related to several significant theorems and conjectures in arithmetic geometry (see Remark 3.1 below). Moreover, the finiteness of $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ and its subgroup

$$H_{\text{ur}}^3(K, X_k; \mathbb{Q}_p/\mathbb{Z}_p(2)) :=$$

$$\text{Im}(H^3(X_k, \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow H^3(\text{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(2))) \cap H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

implies the injectivity of the cycle class map of X of codimension 2 and its torsion part, respectively (see §4 below).

Remark 3.1. (1) For $n = 0$, the quotient $H^1(X, \mathbb{Q}/\mathbb{Z})/H^1(B, \mathbb{Q}/\mathbb{Z})$ is finite by a theorem of Katz-Lang [KL]; in case k is global, $H_{\text{ur}}^1(K, \mathbb{Q}_p/\mathbb{Z}_p(0))$ is finite as well, because $H^1(B, \mathbb{Q}/\mathbb{Z})$ is finite.

(2) In case $n = 1$ and k is global, the finiteness of $H_{\text{ur}}^2(K, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is equivalent to the finiteness of $\text{Br}(X)_{p\text{-tors}}$.

(3) For $n = d := \dim(X)$, $H_{\text{ur}}^{d+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(d))$ is a group considered by Kato [KCT]. He conjectured that $H_{\text{ur}}^{d+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) = 0$ if $p \neq 2$ or k has no embedding into \mathbb{R} . This conjecture is viewed as a generalization of the classical Hasse principle on the Brauer groups of global and local fields. The case $d = 2$ is proved in [KCT], and the case $d = 3$ is proved in [JS].

(3.2) Finiteness. We state here the following results proved in [SaSa] (cf. loc. cit., Theorem 0.3, Corollary 0.4), where κ_{X_k} denotes the Kodaira dimension of X_k :

Theorem 3.2. *Assume $H^2(X_k, \mathcal{O}_{X_k}) = 0$. Then:*

- (1) $H_{\text{ur}}^3(K, X_k; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.
- (2) If k is ℓ -adic local with $\ell \neq p$ and X_k is a surface, then $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.
- (3) If k is p -adic local and X_k is a surface with $\kappa_{X_k} \leq 1$, then $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.
- (4) If k is global and X_k is a surface with $\kappa_{X_k} \leq 1$, then $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.

The conditions in this theorem comes from those in results of Bloch-Kas-Lieberman [BKL], Salberger [Sal], Colliot-Thélène–Raskind [CTR] and Saito-Sujatha [SaSu]. More precisely, we use p -adic Hodge Theory ([K2], [Ts1]) and arguments of the author in [Sat1], to reduce the problem to the above well-known facts. In case k is global, we need a local-global principle due to Jannsen ([J], §4, Theorem 4).

Remark 3.3. It would be a serious problem to prove the above finiteness theorems unconditionally (cf. [LS], [La2], [O]). In fact, the finiteness of $H_{\text{ur}}^3(K, X_k; \mathbb{Q}_p/\mathbb{Z}_p(2))$ and $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ in the general case is deduced from several well-known conjectures in arithmetic geometry which are known to hold true in the situations in Theorem 3.2. See [SaSa], (0.1) for precise statements and loc. cit., Chapter I for detailed proofs.

4. Cycle class maps and injectivity in codimension 2

Let the notation be as in §3. We state here the definition of cycle class maps and the injectivity results on the cycle class map of X of codimension 2 proved in [SaSa], Chapter I, which is deduced from the finiteness theorems mentioned in (3.2).

(4.1) *Definition of cycle class maps.* We define the cycle class map

$$\rho_{X,p^r}^n : \text{CH}^n(X)/p^r \longrightarrow H^{2n}(X, \mathfrak{T}_r(n)_X)$$

as follows. We have the localization spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in X^u} H_x^{u+v}(X, \mathfrak{T}_r(n)_X) \implies H^{u+v}(X, \mathfrak{T}_r(n)_X). \quad (4.1.1)$$

By the purity property **T3** in (1.1) and the absolute cohomological purity (cf. [Th], [Fu]), we have

$$E_1^{u,v} \simeq \bigoplus_{x \in X^u} H^{v-u}(x, \mathbb{Z}/p^r\mathbb{Z}(n-u)_x), \quad \text{if } v \leq n. \quad (4.1.2)$$

This implies that we have the edge homomorphism $E_2^{n,n} \rightarrow H^{2n}(X, \mathfrak{T}_r(n)_X)$ with

$$\begin{aligned} E_2^{n,n} &\simeq \text{Coker} \left(\bigoplus_{y \in X^{n-1}} H^1(y, \mathbb{Z}/p^r\mathbb{Z}(1)_y) \xrightarrow{\partial^{\text{val}}} \bigoplus_{x \in X^n} H^0(x, \mathbb{Z}/p^r\mathbb{Z}) \right) \\ &\simeq \text{CH}^n(X)/p^r, \end{aligned}$$

where ∂^{val} is the sum of boundary maps $\partial_{y,x}^{\text{val}}$ with $y \in X^{n-1}$ and $x \in X^n$ (cf. **T4** in (1.1)). Thus we obtain the map ρ_{X,p^r}^n and the following induced maps:

$$\begin{aligned} \rho_{X,p\text{-tors},r}^n &: \text{CH}^n(X)_{p\text{-tors}} \longrightarrow H^{2n}(X, \mathfrak{T}_r(n)_X), \\ \rho_{X,\mathbb{Z}_p}^n &: \text{CH}^n(X) \otimes \mathbb{Z}_p \longrightarrow H^{2n}(X, \mathfrak{T}_{\mathbb{Z}_p}(n)_X), \end{aligned}$$

where for an abelian group M , $M_{p\text{-tors}}$ denotes the p -primary torsion part, and $H^*(X, \mathfrak{T}_{\mathbb{Z}_p}(n)_X)$ is defined as $\varprojlim_{r \geq 1} H^*(X, \mathfrak{T}_r(n)_X)$.

Remark 4.1. In case A is global and X/B is proper, the map ρ_{X,\mathbb{Z}_p}^d with $d := \dim(X)$ coincides with the p -adic part of the reciprocity map ρ_X in the introduction (cf. Theorem 2.1).

(4.2) Injectivity in codimension 2. For an abelian group M , let M_{Div} (resp. $M_{p\text{-Div}}$) be the maximal divisible subgroup (resp. the maximal p -divisible subgroup). In the rest of this section, we restrict our attention to the case $n = 2$. In this case, we have the following proposition, which reduces the injectivity of cycle class maps to the finiteness of unramified cohomology groups (cf. [SaSa], Proposition 0.2):

Proposition 4.2. (1) Assume that $H_{\text{ur}}^3(K, X_k; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite. Then there exists a positive integer r_0 such that $\text{Ker}(\rho_{X, p\text{-tors}, r}^2) = (\text{CH}^2(X)_{p\text{-tors}})_{\text{Div}}$ for any $r \geq r_0$.

(2) Assume that $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite. Then we have

$$\text{Ker}(\rho_{X, \mathbb{Z}_p}^2) = (\text{CH}^2(X) \otimes \mathbb{Z}_p)_{\text{Div}},$$

and the kernel of the map

$$\rho' : \text{CH}^2(X) \longrightarrow H^4(X, \mathfrak{T}_{\mathbb{Z}_p}(2)_X)$$

induced by ρ_{X, \mathbb{Z}_p}^2 coincides with $\text{CH}^2(X)_{p\text{-Div}}$.

This proposition is proved by standard computations on the spectral sequence (4.1.1) together with a version of the commutative diagram of Colliot-Thélène–Sansuc–Soulé in [CTSS], §1 (cf. [SaSa] Chapter I, Lemma 3.4). By Theorem 3.2 and Proposition 4.2, we obtain the following corollary:

Corollary 4.3. Assume $H^2(X_k, \mathcal{O}_{X_k}) = 0$. Then:

- (1) $\text{CH}^2(X)_{p\text{-tors}}$ is finite and $\rho_{X, p\text{-tors}, r}^2$ is injective for a sufficiently large r .
- (2) If k is ℓ -adic local with $\ell \neq p$ and X_k is a surface, then $\text{Ker}(\rho_{X, \mathbb{Z}_p}^2)$ and $\text{Ker}(\rho')$ are uniquely p -divisible.
- (3) If k is p -adic local, and X_k is a surface with $\kappa_{X_k} \leq 1$, then ρ' is injective up to prime-to- p torsion elements.
- (4) If k is global and X_k is a surface with $\kappa_{X_k} \leq 1$, then ρ_{X, \mathbb{Z}_p}^2 is injective.

The finiteness part of Corollary 4.3 (1) is originally due to Salberger [Sal], Colliot-Thélène and Raskind [CTR].

Remark 4.4. If $H^2(X_k, \mathcal{O}_{X_k}) = 0$ and the Albanese variety of X_k has potentially good reduction at every closed point on B , then the torsion cycle class map $\rho_{X_k, p\text{-tors}, r}^2$ of X_k of codimension 2 is injective for a sufficiently large $r > 0$ [Sa4]. Moreover, if the Albanese variety does not have potentially good reduction, then we cannot expect the same injectivity any longer in general [PS], [Su]. However Corollary 4.3 shows that we recover satisfactory injectivity properties by considering the cycle class map of X (not of X_k).

5. Surjectivity of 1-cycle class map

Let the notation be as in §3. We explain here the surjectivity result on 1-cycle class maps proved in [SaSa], Chapter II, §5. Let d be the absolute dimension of X .

Theorem 5.1. *Assume that k is local. Then ρ_{X,p^r}^{d-1} is surjective for any $r \geq 1$.*

Note that this theorem is unconditional on X . We show an outline of a proof of Theorem 5.1. Let $Y \subset X$ be the closed fiber of X/B and take a dense open regular subset U of the reduced part of Y . Let ℓ be the residual characteristic of k . For $x \in X$, let A_x be the henselization of $\mathcal{O}_{X,x}$ at its maximal ideal. For a scheme Z , let $\text{Br}(Z)$ be the cohomological Brauer group $H^2(Z, \mathbb{G}_m)$. Let $(X_k)_0$ and $(Y)_0$ be the sets of all closed points on X_k and Y , respectively, and let $\text{sp} : (X_k)_0 \rightarrow (Y)_0$ be the specialization map of points. By Theorem 2.2, we have the natural isomorphism

$$a : H^{2d-2}(X, \mathfrak{T}_r(d-1)_X) \xrightarrow{\cong} H_Y^3(X, \mathfrak{T}_r(1)_X)^*,$$

where we put $M^* := \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$ for an abelian group M . A key step of our proof is to construct an injective map

$$\theta_{p^r} : H_Y^3(X, \mathfrak{T}_r(1)_X) \longrightarrow \prod_{x \in (U)_0} {}_{p^r}\text{Br}(A_x[1/\ell])$$

whose dual fits into the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{x \in (U)_0} \bigoplus_{v \in \text{sp}^{-1}(\{x\})} \mathbb{Z}/p^r\mathbb{Z} & \xrightarrow{(\psi_{p^r})^*} & \bigoplus_{x \in (U)_0} ({}_{p^r}\text{Br}(A_x[1/\ell]))^* \\ \downarrow & & (\theta_{p^r})^* \downarrow \\ \text{CH}^{d-1}(X)/p^r & \xrightarrow{a \circ \rho_{X,p^r}^{d-1}} & H_{Y,\text{ét}}^3(X, \mathfrak{T}_r(1)_X)^*, \end{array}$$

where $(\psi_{p^r})^*$ is the direct sum of the dual of the following map for $x \in (U)_0$:

$$\psi_{x,p^r} : {}_{p^r}\text{Br}(A_x[1/\ell]) \longrightarrow \prod_{v \in \text{sp}^{-1}(\{x\})} \mathbb{Z}/p^r\mathbb{Z}; \quad \omega \mapsto (\text{inv}_v(\omega|_v))_v$$

and inv_v denotes the invariant isomorphism ${}_{p^r}\text{Br}(v) \simeq \mathbb{Z}/p^r\mathbb{Z}$. More precisely, θ_{p^r} is defined as a natural restriction map and the commutativity of the above diagram is proved by a projection formula for p -adic étale Tate twists (cf. [Sat3], Corollary 4.10). By the above commutative diagram and the surjectivity of $(\theta_{p^r})^*$, Theorem 5.1 is reduced to the following local result:

Theorem 5.2. *ψ_{x,p^r} is injective for any $x \in (U)_0$ and $r \geq 1$.*

This theorem is proved by an induction argument on the dimension of local rings and the Brauer group theory of 2-dimensional local rings due to Saito [Sa2] (see [SaSa], Chapter II, §1, §2 for details).

Remark 5.3. Theorem 5.2 has an application to the Brauer-Manin pairing

$$\langle \cdot, \cdot \rangle : \text{CH}_0(X_k) \times \text{Br}(X_k) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

In fact, using Theorem 5.2, we proved the p -primary variant of results of Colliot-Thélène and Saito [CTS] (see [SaSa], Chapter II, §3, §4).

As a direct consequence of Corollary 4.3 (2) and Theorem 5.1, we obtain

Corollary 5.4. *Assume that k is p -adic local and that X_k is a surface with $H^2(X_k, \mathcal{O}_{X_k}) = 0$ and $\kappa_{X_k} \leq 1$. Then the cycle class map*

$$\text{CH}^2(X) \longrightarrow H^4(X, \mathfrak{T}_{\mathbb{Z}_p}(2)_X) \times \prod_{\ell \neq p} H^4(X, \mathbb{Z}_\ell(2))$$

is injective and has dense image with respect to the pro-finite topology on the group on the right hand side.

See [Ya] for a comparison fact between formal K -cohomology groups and syntomic cohomology groups.

Remark 5.5. In case k is global, the surjectivity of $\rho_{X,p}^{d-1}$ implies the p -adic part of the Hasse principle for 0-cycles on X_k up to the Brauer-Manin obstruction (cf. [Sa3], [CT1], [CT2], [SaSa], Theorem 0.13).

6. Triviality of unramified cohomology

The notation remains as in §5; we assume that k is local. In this section, we like to mention some consequences of Theorem 3.2 and Theorem 5.1. We first recall the following fact proved in [SaSa], Chapter I, Proposition 3.7:

Proposition 6.1. *Assume that $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite. Then the order of*

$$\text{Coker}(\rho_{X,\mathbb{Z}_p}^2 : \text{CH}^2(X) \otimes \mathbb{Z}_p \rightarrow H^4(X, \mathfrak{T}_{\mathbb{Z}_p}(2)_X))_{\text{tors}}$$

coincides with that of $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$.

This proposition is proved by standard arguments using the spectral sequence (4.1.1), and gives a generalization of the isomorphism

$$H_{\text{ur}}^2(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \simeq \text{Br}(X)_{p\text{-tors}}.$$

By Proposition 6.1 and Theorem 5.1, we obtain

Corollary 6.2. *Assume that $d = 3$ (i.e., X_k is a surface). Then $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite if and only if it is trivial.*

Hence by Theorem 3.2, we obtain the following corollary, which generalizes a result of Spiess ([Sp], §4) to more general situations:

Corollary 6.3. *Assume that $d = 3$ and that $H^2(X_k, \mathcal{O}_{X_k}) = 0$.*

(1) *If k is ℓ -adic local with $\ell \neq p$, then we have $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$.*

(2) *If k is p -adic local and $\kappa_{X_k} \leq 1$, then we have $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$.*

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Kanetomo Sato kanetomo@math.nagoya-u.ac.jp

Graduate school of Mathematics,
Nagoya University,
Furo-cho, Chikusa-ku,
Nagoya 464-8602,
Japan