Irrationality Measures of $\log 2$ and $\pi/\sqrt{3}$

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CONTENTS

Using a class of polynomials that generalizes Legendre polynomials, we unify previous works of E. A. Rukhadze, A. K. Dubitskas, M. Hata, D. V. and G. V. Chudnovsky about irrationality measures of $\log 2$ and $\pi/\sqrt{3}$.

1. INTRODUCTION

A usual way to estimate the quality of approximation of a real number x by a rational number p/q is to compare the quantity $|x-p/q|$ to negative powers of the denominator q. More precisely:

Definition 1.1. Let x and μ be real numbers. We say that μ is an *irrationality measure* of x if for all $\varepsilon > 0$ there exists $C(\varepsilon) \in \mathbb{R}^+$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{C(\varepsilon)}{q^{\mu+\varepsilon}}.$$

for any integer p and any positive integer q. If the constant $C(\varepsilon)$ can be computed effectively, we say that μ is an *effective irrationality measure* of x.

This article deals with irrationality measures of the transcendental numbers $\pi/\sqrt{3}$ and $\log(1-r/s)$, with $r \in \mathbb{N}$, $s \in \mathbb{Z} \setminus \{0\}$ and $r/s \in [-1, 1[$.

Rukhadze [1987], considering the class of polynomials

$$R_{n,m,m'}(z) = \frac{1}{(n-m')!}\left(z^{n+m-m'}(1-z)^n\right)^{(n-m')} = \sum_{j=0}^{n}(-1)^j \binom{n}{j}\binom{n+m-m'+j}{n-m'} z^{m+j},$$

where $n, m, m' \in \mathbb{N}$ and $n \geq m'$, obtained an irrationality measure of $\log 2$:

$$\mu(\log 2) = 3.893.$$

(If μ_0 is an irrationality measure of x we loosely write $\mu(x) = \mu_0$, though any real number exceeding

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μ_0 is also an irrationality measure of x.) In fact, Rukhadze's result gives the better value

$$\mu(\log 2) = 3.8913997\ldots,$$

but the numerical computations in her paper lacked sufficient accuracy. Dubitskas [1987] used the same polynomials to compute an irrationality measure of $\pi/\sqrt{3}$ equal to 5.52. With another class of approximants, M. Hata [1990] obtained

$$\mu(\log 2) = 3.8913997\ldots,$$
$$\mu(\pi/\sqrt{3}) = 5.0874626\ldots.$$

He used the polynomials $H_{n,m,m'}(z)$ defined by

$$\frac{1}{(n+m-m')!}\big(z^{n-m'}(1-z)^{n+m}\big)^{(n+m-m')} = \sum_{j=0}^{n}(-1)^{m+j}\binom{n+m}{m+j}\binom{n+m-m'+j}{n+m-m'}z^j,$$

with $n, m, m' \in \mathbb{N}$ and $n \geq m'$.

A. Heimonen, T. Matala-Aho and Keijo Väänänen [Heimonen et al. 1993; 1994] considered irrationality measures of values of Gauss hypergeometric functions

$${}_2F_1\left(\begin{matrix}1, b\\ c\end{matrix}; z\right) = \sum_{n=0}^{+\infty}\frac{(b)_n}{(c)_n}z^n,$$

where $b, c \neq 0, -1, -2, \ldots$ are rational parameters and

$$(b)_0 = 1,$$
$$(b)_n = b(b+1)(b+2)\cdots(b+n-1) \quad \text{for } n \in \mathbb{N}\setminus\{0\}.$$

The first of these papers used the polynomials $R_{n,m,m'}$ to compute irrationality measures of the numbers $\log(1-r/s)$ with $r \in \mathbb{N}$ and $s \in \mathbb{Z}\setminus\{0\}$ such that $r/s \in [-1, 1[$ — thus generalizing part of [Rukhadze 1987] — while the second paper considered Jacobi type polynomials to obtain irrationality measures of $\pi/\sqrt{3}$. In the last situation, the Jacobi type polynomials used were linked to the polynomials $H_{n,m,m'}$.

Finally, D. V. and G. V. Chudnovsky announced without proof [Chudnovsky and Chudnovsky 1993] that

$$\mu(\log 2) = 3.87 \quad \text{and} \quad \mu(\pi/\sqrt{3}) = 4.96,$$

these values being computed using the class of polynomials

$$C_{n,m}(z) = \frac{1}{m!}(L_n(z))^{(m)},$$

where

$$L_n(z) = \frac{1}{n!}\big(z^n(1-z)^n\big)^{(n)}$$

denotes the n-th Legendre type polynomial, used by F. Beukers [1979] to give another proof of the irrationality of $\zeta(3)$, and by K. Alladi and M. L. Robinson [1980] to compute irrationality measures of $\log 2$ and $\pi/\sqrt{3}$.

The polynomials $R_{n,m,m'}$, $H_{n,m,m'}$, and $C_{n,m}$ are subfamilies of a more general class of polynomials $P_{n,m,m'}$, defined for $n \in \mathbb{N}$, $m, m' \in \mathbb{Z}$ and $n \geq -\min(m, m', m+m')$ by setting $P_{n,m,m'}(z)$ equal to

$$\frac{1}{(n+m+m')!}\big(z^{n+m'}(1-z)^{n+m}\big)^{(n+m+m')} = \sum_{j=0}^{n}(-1)^{m+j}\binom{n+m}{m+j}\binom{n+m+m'+j}{n+m+m'}z^j,$$

We will call this family of approximants *generalized Legendre polynomials*, the substitution $m = m' = 0$ giving the Legendre type polynomials L_n.

If we suppose $m, m' \in \mathbb{N}$, we see that

$$R_{n,m,m'} = P_{n+m,-m,-m'}, \quad H_{n,m,m'} = P_{n,m,-m'},$$
$$C_{n,m} = \binom{n+m}{m}P_{n-m,m,m}.$$

The interest of these polynomials is that their content (that is, the greatest common divisor of their coefficients) is large. The authors quoted earlier studied their classes of polynomials only in the case $(n, m, m') = (an', bn', \pm bn')$ with a and $n' \in \mathbb{N}$ and $b \in \mathbb{Z}$. We generalize their works to the case $(n, m, m') = (an', bn', cn')$ with a and $n' \in \mathbb{N}$ and $b, c \in \mathbb{Z}$. Then we completely study the irrationality measures of $\log 2$ and $\pi/\sqrt{3}$ computable with the polynomials $P_{an,bn,cn}$, and we show the links between the results of [Rukhadze 1987; Dubitskas 1987; Hata 1990; Dubitskas 1993; Chudnovsky and Chudnovsky 1993; Heimonen et al. 1993; 1994]. In particular, we will examine the announcement made by the Chudnovskys.

Section 2 contains technical results needed to establish irrationality measures, and some properties of the polynomials $P_{an,bn,cn}$. In Sections 3 and 4, we compute irrationality measures of $\log 2$ and $\pi/\sqrt{3}$ respectively. In section 5, we modify slightly the polynomials $P_{an,bn,cn}$ and look at the consequences of this perturbation on the resulting values of irrationality measures of $\log 2$.

2. TECHNICAL LEMMAS

2A. Classical Results

To compute irrationality measures of a real number x we need good approximations of x. More precisely, consider a complex number x, a real number ρ, and two sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}+i\rho\mathbb{Z}$. For $n\in\mathbb{N}$, set $\varepsilon_n = q_n x - p_n$. Also set

$$\kappa_\rho = \min\{|z|; z\in\mathbb{Z}+i\rho\mathbb{Z}, z\neq 0\} = \begin{cases} |\rho| & \text{if } 0<|\rho|<1,\\ 1 & \text{if } \rho=0 \text{ or } |\rho|\geq 1.\end{cases}$$

The study of the asymptotic behaviour of q_n and ε_n allows us to obtain an irrationality measure of x, thanks to the following two lemmas. The first lemma is used to compute an irrationality measure of the numbers $\log(1-r/s)$ with $r/s\in\mathbb{Q}\cap[-1,1[$.

Lemma 2.1. *Suppose that*

$$\limsup_{n\to+\infty}\frac{1}{n}\log|q_n|\leq\sigma,\qquad \lim_{n\to+\infty}\frac{1}{n}\log|\varepsilon_n| = -\tau,$$

where σ and τ are positive real numbers. Then $1+\sigma/\tau$ is an effective irrationality measure of the real number x.

Proof. See [Chudnovsky 1982]. □

The following lemma will help us establish an irrationality measure of $\pi/\sqrt{3}$.

Lemma 2.2 [Hata 1990]. *Suppose that $q_n\neq 0$ for every $n\geq 0$ and that*

$$\limsup_{n\to+\infty}\frac{1}{n}\log|q_n|\leq\sigma,$$
$$\limsup_{n\to+\infty}\frac{1}{n}\log|\varepsilon_n|\leq-\tau$$

for some positive real numbers σ and τ. For a fixed positive integer M, for $n\geq 0$, let $V_n\subset\mathbb{C}$ be the set of the M complex numbers

$$\frac{p_n}{q_n},\frac{p_{n+1}}{q_{n+1}},\dots,\frac{p_{n+M-1}}{q_{n+M-1}}.$$

Suppose that, for each $n\geq 0$, either

(i) *V_n contains at least two distinct points, or*
(ii) *V_n consists of a single point $z_0\in\mathbb{C}\setminus\mathbb{Q}$.*

Then $1+\sigma/\tau$ is an effective irrationality measure of the real number x.

We now recall briefly two results necessary for the computation of the constants σ and τ of the two previous lemmas.

First, let d_n denote the least common multiple of $\{1,2,\dots,n\}$ for $n\in\mathbb{N}\setminus\{0\}$ and $\Theta(x)$ the sum

$$\sum_{\substack{p \text{ prime}\\ p\leq x}}\log p.$$

The prime number theorem gives

$$\lim_{n\to+\infty}\frac{\log d_n}{n}=1 \tag{2–1}$$

and

$$\lim_{x\to+\infty}\frac{\Theta(x)}{x}=1. \tag{2–2}$$

Finally, we recall a basic result of analysis. For every complex function integrable on a measurable set X, relative to a positive measure μ, we have

$$\lim_{n\to+\infty}\frac{1}{n}\log\int_X|f(x)|^n\,d\mu(x)=\log\|f\|_{\infty,X}, \tag{2–3}$$

where $\|f\|_{\infty,X}$ denotes the essential supremum of f over X.

2B. An Arithmetical Lemma

We will now study the content of the polynomials $P_{n,m,m'}$, by generalizing of [Hata 1990, Lemma 2.5]. For x a real number, $[x]$ will denote the integer part of x and $\{x\}$ its fractional part, that is, the real number $x-[x]$. The following result can be seen as a corollary of [Heimonen et al. 1993, Lemma 10].

Lemma 2.3. *Let λ,λ' be two real numbers such that $\min(\lambda,\lambda',\lambda+\lambda')>-1$, and let M and M' be two sequences of integers such that*

$$n\geq-\min(M(n),M'(n))$$

and

$$\max\{|\lambda n-M(n)|,\,|\lambda' n-M'(n)|\}\leq C_{\lambda,\lambda'},$$

where $C_{\lambda,\lambda'}$ is a nonnegative constant independent of n. Then there exists a common divisor $d_{\lambda,\lambda'}(n)$ of the coefficients of $P_{n,M(n),M'(n)}$ such that

$$e(\lambda,\lambda'):=\int_{E(\lambda,\lambda')}\frac{dx}{x^2}=\lim_{n\to+\infty}\frac{1}{n}\log d_{\lambda,\lambda'}(n),$$

where $E(\lambda,\lambda')$ is the open set

$$\{x>0: 0<\{x\}+\{\lambda' x\}-1<1-\{\lambda x\}<\{x\}\}.$$

Proof. Let $\delta_n = C_{\lambda,\lambda'}/\sqrt{(2+\lambda+\lambda')n+1}$. For n large enough, we have $\delta_n < \frac{1}{6}$. Define

$$E_n(\lambda,\lambda') = \bigl\{x>0 : 1-\min(\{\lambda' x\},\{\lambda x\})+\delta_n < \{x\} < 2-2\delta_n-\max(\{\lambda' x\}+\{\lambda x\},1)\bigr\},$$

and consider the three sequences $(k/\lambda)_{k\in\mathbb{N}}$, $(l/\lambda')_{l\in\mathbb{N}}$, and $(q)_{q\in\mathbb{N}}$. Sort the elements of these sequences in a nondecreasing order, obtaining an increasing sequence $(a_m)_{m\in\mathbb{N}}$ of real numbers. It is clear that the sequence $(E_n(\lambda,\lambda'))_{n\in\mathbb{N}}$ is nondecreasing and that $\bigcup_{n\in\mathbb{N}} E_n(\lambda,\lambda') = E(\lambda,\lambda')$.

Define H_n as the set of primes p such that $p \geq \sqrt{(2+\lambda+\lambda')n+1}$ and $n/p \in E_n(\lambda,\lambda')$. Let

$$d_{\lambda,\lambda'}(n) = \prod_{p\in H_n} p.$$

We next show that every element p of H_n divides the coefficients of the polynomial $P_{n,M(n),M'(n)}$, that is, for every integer j in the interval $[0,n]$, the prime p divides

$$Q(n,j) := \binom{n+M(n)}{M(n)+j}\binom{n+M(n)+M'(n)+j}{n+M(n)+M'(n)}.$$

Let j be such an integer. The p-adic valuation of $Q(n,j)$ is equal to

$$v_p(\theta) = [\alpha+\beta]-[\beta+\theta]-[\alpha-\theta] + [\alpha+\beta+\gamma+\theta]-[\alpha+\beta+\gamma], \tag{2–4}$$

where

$$\alpha = \Bigl\{\frac{n}{p}\Bigr\},\ \beta = \Bigl\{\frac{M(n)}{p}\Bigr\},\ \gamma = \Bigl\{\frac{M'(n)}{p}\Bigr\},\ \theta = \Bigl\{\frac{j}{p}\Bigr\}.$$

Since n/p belongs to $E_n(\lambda,\lambda')$, the numbers $\{\lambda n/p\}$ and $\{\lambda' n/p\}$ are in the interval $]3\delta_n,\, 1-3\delta_n[$. Moreover

$$\max\Bigl(\Bigl|\frac{\lambda n}{p}-\frac{M(n)}{p}\Bigr|,\Bigl|\frac{\lambda' n}{p}-\frac{M'(n)}{p}\Bigr|\Bigr) \leq \frac{C_{\lambda,\lambda'}}{p} \leq \frac{C_{\lambda,\lambda'}}{\sqrt{(2+\lambda+\lambda')n+1}} = \delta_n.$$

Hence, we deduce the inequalities

$$\Bigl|\Bigl\{\frac{\lambda n}{p}\Bigr\}-\beta\Bigr| \leq \delta_n \quad\text{and}\quad \Bigl|\Bigl\{\frac{\lambda' n}{p}\Bigr\}-\gamma\Bigr| \leq \delta_n.$$

It follows that

$$\alpha+\beta \geq \Bigl\{\frac{n}{p}\Bigr\}+\Bigl\{\frac{\lambda n}{p}\Bigr\}-\delta_n > 1,$$

$$\alpha+\beta+\gamma \leq \Bigl\{\frac{n}{p}\Bigr\}+\Bigl\{\frac{\lambda n}{p}\Bigr\}+\Bigl\{\frac{\lambda' n}{p}\Bigr\}+2\delta_n < 2,$$

$$\alpha+\gamma \geq \Bigl\{\frac{n}{p}\Bigr\}+\Bigl\{\frac{\lambda' n}{p}\Bigr\}-\delta_n > 1.$$

Therefore the right-hand side of (2–4) equals

$$-[\alpha-\theta]+[\alpha+\beta+\gamma+\theta]-[\beta+\theta] \geq 1,$$

since $[\alpha-\theta]\leq 0$ and $\alpha+\gamma>1$.

We now study the asymptotic behaviour of $d_{\lambda,\lambda',n}$. Let l be a fixed integer, arbitrarily large, and let $I =]a,b[$ be any connected component of $E_l(\lambda,\lambda')$. The interval I will be a subset of $E_n(\lambda,\lambda')$ for every integer $n\geq l$. If $n \geq (3+\lambda+\lambda')b^2$, every prime p in the interval $J_n =]n/b, n/a[$ belongs to H_n: indeed,

$$p^2 > \frac{n^2}{b^2} \geq (2+\lambda+\lambda')n+1$$

and $n/p \in]a,b[$. Hence we obtain the lower bound

$$\sum_{\substack{p \text{ prime} \\ p\in J_n}} \log p \geq \Bigl(\Theta\Bigl(\frac{n}{a}\Bigr)-\Theta\Bigl(\frac{n}{b}\Bigr)-\log\Bigl(\frac{n}{a}\Bigr)\Bigr).$$

Thus we have, thanks to (2–2),

$$\liminf_{n\to+\infty}\frac{1}{n}\sum_{\substack{p \text{ prime} \\ p\in J_n}} \log p \geq \Bigl(\frac{1}{a}-\frac{1}{b}\Bigr) = \int_I \frac{dx}{x^2},$$

which gives

$$\liminf_{n\to\infty}\frac{\log d_{\lambda,\lambda'}(n)}{n} \geq \int_{E_l(\lambda,\lambda')}\frac{dx}{x^2}.$$

When l tends to infinity, we get

$$\liminf_{n\to\infty}\frac{\log d_{\lambda,\lambda'}(n)}{n} \geq \int_{E(\lambda,\lambda')}\frac{dx}{x^2}.$$

The upper bound

$$\limsup_{n\to\infty}\frac{\log d_{\lambda,\lambda'}(n)}{n} \leq \int_{E(\lambda,\lambda')}\frac{dx}{x^2}$$

is obtained in a similar way, considering the set H'_n of primes p such that $p \geq \sqrt{(2+\lambda+\lambda')n+1}$ and $n/p\in E(\lambda,\lambda')$. □

Remark 2.4. The arithmetical gain provided by this lemma is easily seen to be optimal. Indeed, if v_p is given by (2–4), the condition $v_p(\theta) \geq 1$, checked for $\theta = 0$ and for $\theta = 1 - \beta$, induces the inequalities defining the set $E(\lambda, \lambda')$.

2C. An Algorithm for Computing the Integral $e(\lambda, \lambda')$

This section describes an algorithm that allows one to compute $\int_{E(\lambda,\lambda')} dx/x^2$ when λ and λ' are rational numbers satisfying the assumptions of Lemma 2.3. Let $\lambda = b/a$ and $\lambda' = c/a$, with a a positive integer and b, c two integers such that $a > -\min(b, c, b+c)$ and $abc \neq 0$. We have

$$E\Big(\frac{b}{a}, \frac{c}{a}\Big) = \Big\{x > 0 : 0 < \{x\} + \Big\{\frac{c}{a}x\Big\} - 1 < 1 - \Big\{\frac{b}{a}x\Big\} < \{x\}\Big\}.$$

Consider the three finite sequences

$$\Big(\frac{j}{a}\Big)_{j\in\mathbb{N}} \cap [0,1], \ \Big(\frac{k}{|b|}\Big)_{k\in\mathbb{N}} \cap [0,1], \ \Big(\frac{l}{|c|}\Big)_{l\in\mathbb{N}} \cap [0,1].$$

By sorting the elements of these sequences in increasing order we get a finite increasing sequence $(\alpha_m)_{m\in\mathbb{N}, 0\leq m\leq M(a,b,c)}$ of real numbers such that $\alpha_0 = 0$ and $\alpha_{M(a,b,c)} = 1$. From it, we build a covering of $\mathbb{R}_{>0}$ by a sequence of pairwise disjoint open intervals. The interest of this construction lies in that the intersection of $E(b/a, c/a)$ with every interval of the covering is an interval, on which the integration of the function $x \to 1/x^2$ will be very easy. Write

$$e\Big(\frac{b}{a}, \frac{c}{a}\Big) = \int_{E\left(\frac{b}{a}, \frac{c}{a}\right)} \frac{dx}{x^2} = \sum_{q\in\mathbb{N}} \sum_{m=0}^{M(a,b,c)-1} \int_{E(b/a, c/a)\cap]a(q+\alpha_m), a(q+\alpha_{m+1})[} \frac{dx}{x^2}.$$

From the form of the sequence $(\alpha_m)_{m\in\mathbb{N}, 0\leq m\leq M(a,b,c)}$ we know that there exist integers j, k and l such that

$$]a\alpha_m, a\alpha_{m+1}[\; = \;]j, j+1[\; \cap \; \Big]\frac{ak}{|b|}, \frac{a(k+1)}{|b|}\Big[\; \cap \; \Big]\frac{al}{|c|}, \frac{a(l+1)}{|c|}\Big[.$$

If we consider a real number $x \in]a\alpha_m, a\alpha_{m+1}[$, we know that $x \in]j, j+1[$,

$$\frac{|b|}{a}x \in]k, k+1[\quad \text{and} \quad \frac{|c|}{a}x \in]l, l+1[\,.$$

We deduce that

$$\{x\} = x - [x] = x - [a\alpha_m],$$
$$\Big\{\frac{b}{a}x\Big\} = \frac{b}{a}x - \Big[\frac{b}{a}x\Big] = \frac{b}{a}x - \inf\big([b\alpha_m], [b\alpha_{m+1}]\big),$$
$$\Big\{\frac{c}{a}x\Big\} = \frac{c}{a}x - \Big[\frac{c}{a}x\Big] = \frac{c}{a}x - \inf\big([c\alpha_m], [c\alpha_{m+1}]\big).$$

Because $x \in E(b/a, c/a)$, we have, by definition,

$$0 < \{x\} + \Big\{\frac{c}{a}x\Big\} - 1 < 1 - \Big\{\frac{b}{a}x\Big\} < \{x\}.$$

Then the following equivalent inequalities hold:

$$\begin{aligned} 0 &< \Big(1 + \frac{c}{a}\Big)x - \big([a\alpha_m] + \inf([c\alpha_m], [c\alpha_{m+1}])\big) - 1 \\ &< 1 - \frac{b}{a}x + \inf\big([b\alpha_m], [b\alpha_{m+1}]\big) \\ &< x - [a\alpha_m]. \end{aligned}$$

Thus we have $x > x_1(m)$, $x < x_2(m)$, and $x > x_3(m)$, where

$$x_1(m) := \frac{1 + \inf\big([b\alpha_m], [b\alpha_{m+1}]\big) + [a\alpha_m]}{1 + b/a},$$

$$x_2(m) := \frac{2 + [a\alpha_m] + \inf\big([b\alpha_m], [b\alpha_{m+1}]\big) + \inf\big([c\alpha_m], [c\alpha_{m+1}]\big)}{1 + b/a + c/a},$$

$$x_3(m) := \frac{1 + [a\alpha_m] + \inf\big([c\alpha_m], [c\alpha_{m+1}]\big)}{1 + c/a}.$$

Now put

$$y_1(m) = \max\Big(\frac{x_1(m)}{a}, \frac{x_3(m)}{a}, \alpha_m\Big),$$
$$y_2(m) = \min\Big(\alpha_{m+1}, \frac{x_2(m)}{a}\Big).$$

From the foregoing discussion and the a-periodicity of the functions considered in the set $E(b/a, c/a)$, it is obvious that

$$E\Big(\frac{b}{a}, \frac{c}{a}\Big) \cap]a(q+\alpha_m), a(q+\alpha_{m+1})[\; = \;]a(q+y_1(m)), a(q+y_2(m))[$$

for every m and q in $\mathbb{N}$, with $0 \leq m \leq M(a,b,c) - 1$. We now compute

$$\sum_{q\in\mathbb{N}} \int_{E(b/a,c/a)\cap]a(q+\alpha_m),a(q+\alpha_{m+1})[} \frac{dx}{x^2},$$

obtaining

$$\sum_{q\in\mathbb{N}} \int_{a(q+y_1(m))}^{a(q+y_2(m))} \frac{dx}{x^2} = \sum_{q\in\mathbb{N}} \left(\frac{1}{a(q+y_1(m))} - \frac{1}{a(q+y_2(m))}\right) = \frac{1}{a}\sum_{q\in\mathbb{N}} \left(\frac{1}{q+y_1(m)} - \frac{1}{q+y_2(m)}\right).$$

Let ψ denote the digamma function Γ'/Γ. Then, for any real numbers α, β with $0 < \alpha < \beta < 1$, we have

$$\sum_{q\in\mathbb{N}} \left(\frac{1}{q+\alpha} - \frac{1}{q+\beta}\right) = \psi(\beta) - \psi(\alpha); \qquad (2\text{–}5)$$

see [Bateman 1953, p. 15], for example. Hence we obtained the explicit formula

$$e\Big(\frac{b}{a}, \frac{c}{a}\Big) = \frac{1}{a} \sum_{m=0}^{M(a,b,c)-1} \big(\psi\left(y_2(m)\right) - \psi\left(y_1(m)\right)\big).$$

In the case $|b| = |c| = 1$, we can obtain a more practical formula, as in [Hata 1990]. In particular,

$$e\Big(\frac{1}{a}, \frac{1}{a}\Big) = \log\left(\frac{a+1}{(a+2)^{(a+2)/(2a+2)} a^{a/(2a+2)}}\right) + \frac{\pi}{2a+2}(\chi(a+2) - \chi(a))$$

and

$$e\Big(\frac{1}{a}, -\frac{1}{a}\Big) = \log\left(\frac{(a+1)^{(a+1)/(2a)}(a-1)^{(a-1)/(2a)}}{a}\right) + \frac{\pi}{2a}(\chi(a+1) - \chi(a-1)),$$

with

$$\chi(a) = \sum_{r=1}^{[a/2]} \cot\left(r\frac{\pi}{a}\right).$$

As explained in [Hata 1990], we can compute $\chi(a)$ for some small values of a (Table 1).

2D. Links Between the Contents of the Polynomials Used by Dubitskas, Hata, Rukhadze and Chudnovsky

We define two transformations ω_1 and ω_2 of $\mathbb{Z}^3$ by

$$\begin{aligned} \omega_1(a,b,c) &= (a+b, -b, c), \\ \omega_2(a,b,c) &= (a+c, b, -c), \end{aligned} \qquad (2\text{–}6)$$

for every $(a, b, c) \in \mathbb{Z}^3$. These transformations generate a group isomorphic to the group $D_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Now, we suppose that $a \in \mathbb{N} \setminus \{0\}$ and $b, c \in \mathbb{Z} \setminus \{0\}$ are such that $a > -\min(b, c, b+c)$, and we put

$$I(a,b,c) = a\, e\left(\frac{b}{a}, \frac{c}{a}\right). \qquad (2\text{–}7)$$

Then:

Lemma 2.5. *Let $a \in \mathbb{N}\setminus\{0\}$ and $b, c \in \mathbb{Z}\setminus\{0\}$ be such that $a > -\min(b, c, b+c)$. Then*

$$I(a,b,c) - I(\omega_i(a,b,c)) = \log\left(\frac{(a+b)^{a+b}(a+c)^{a+c}}{a^a(a+b+c)^{a+b+c}}\right)$$

for $i = 1, 2$.

In particular, if we compose ω_1 and ω_2, we get

$$a\, e\Big(\frac{b}{a}, \frac{c}{a}\Big) = (a+b+c)\, e\Big(-\frac{b}{a+b+c}, -\frac{c}{a+b+c}\Big).$$

This last relation means that the arithmetical gain obtained from the polynomials proposed by D. V. and G. V. Chudnovsky can be obtained from those considered by Rukhadze and Dubitskas.

Proof. By definition,

$$I(a,b,c) = a \int_{E\left(\frac{b}{a}, \frac{c}{a}\right)} \frac{dx}{x^2}.$$

We make the change of variable $x = ay$. The last relation becomes

$$I(a,b,c) = \int_{F(a,b,c)} \frac{dy}{y^2}$$

with $F(a, b, c)$ given by

$$\{y > 0 : 0 < \{ay\} + \{cy\} - 1 < 1 - \{by\} < \{ay\}\}.$$

a	1	2	3	4	5	6	8	10	12
$\chi(a)$	0	0	$1/\sqrt{3}$	1	$\sqrt{2+2/\sqrt{5}}$	$4/\sqrt{3}$	$1+2\sqrt{2}$	$4\sqrt{1+2\sqrt{5}}$	$5+4/\sqrt{3}$

TABLE 1. First values of χ.

First, we study the quantity

$$I(a,b,c)-I(\omega_1(a,b,c)) = \int_{F(a,b,c)} \frac{dy}{y^2} - \int_{F(\omega_1(a,b,c))} \frac{dy}{y^2}.$$

The conditions for $y>0$ to belong to $F(a,b,c)$ can be rewritten as

$$\left.\begin{aligned} 1 &< \{ay\}+\{by\}, \\ 1 &< \{ay\}+\{cy\}, \\ 1 &< \{ay\}+\{by\}+\{cy\} < 2. \end{aligned}\right\} \tag{2–8}$$

Therefore the conditions for y to be in $F(\omega_1(a,b,c))$ are

$$\begin{aligned} 1 &< \{(a+b)y\}+\{-by\}, \\ 1 &< \{(a+b)y\}+\{cy\}, \\ 1 &< \{(a+b)y\}+\{-by\}+\{cy\} < 2. \end{aligned}$$

The first of these inequalities implies $\{-by\} \neq 0$, and so $\{-by\} = 1-\{by\}$. The first and third inequalities then become respectively

$$\{by\} < \{(a+b)y\} \tag{2–9}$$

and

$$\{(a+b)y\} - \{by\} + \{cy\} < 1.$$

We have $\{(a+b)y\} = \{ay\}+\{by\}-1$ or $\{(a+b)y\} = \{ay\}+\{by\}$; the first case is excluded because of (2–9). Thus the conditions for $y>0$ to belong to $F(\omega_1(a,b,c))$ reduce to

$$\begin{aligned} \{ay\}+\{by\} &< 1, \\ \{ay\}+\{cy\} &< 1, \\ 1 < \{ay\}+\{by\}+\{cy\} &< 2. \end{aligned}$$

Introduce the notation

$$\begin{aligned} g_1(y) &= \{ay\}+\{by\}, \\ g_2(y) &= \{ay\}+\{cy\}, \\ g_3(y) &= \{ay\}+\{by\}+\{cy\}, \end{aligned}$$

noting that the sets $\{g_i = j\}$ for $i=1,2,3$ and $j=0,1,2$ are of measure zero. For E a set, let χ_E denote the characteristic function of E, and for notational simplicity write $\chi_{\{P\}}$ if E is defined by condition P (that is, $E=\{P\}$). Then

$$\begin{aligned} \int_{F(a,b,c)} \frac{dy}{y^2} &= \int_0^{+\infty} \left(\chi_{1<g_1}\chi_{1<g_2}\chi_{1<g_3<2}\right)(y)\frac{dy}{y^2} \\ &= \int_0^{+\infty} \left(\chi_{1<g_1}\chi_{1<g_3<2} - \chi_{1<g_1}\chi_{g_2<1}\chi_{1<g_3<2}\right)(y)\frac{dy}{y^2} \end{aligned}$$

and

$$\begin{aligned} \int_{F(\omega_1(a,b,c))} \frac{dy}{y^2} &= \int_0^{+\infty} \left(\chi_{g_1<1}\chi_{g_2<1}\chi_{1<g_3<2}\right)(y)\frac{dy}{y^2} \\ &= \int_0^{+\infty} \left(\chi_{g_2<1}\chi_{1<g_3<2} - \chi_{1<g_1}\chi_{g_2<1}\chi_{1<g_3<2}\right)(y)\frac{dy}{y^2}. \end{aligned}$$

It follows that

$$\begin{aligned} I(a,b,c) &- I(\omega_1(a,b,c)) \\ &= \int_0^{+\infty} \left(\chi_{1<g_1}\chi_{1<g_3<2} - \chi_{g_2<1}\chi_{1<g_3<2}\right)(y)\frac{dy}{y^2}. \end{aligned}$$

But

$$\chi_{1<g_1}\chi_{1<g_3<2} = \chi_{1<g_1} - \chi_{2\le g_3},$$

and also

$$\begin{aligned} \chi_{g_2<1}\chi_{1<g_3<2} &= \chi_{1<g_3<2} - \chi_{1\le g_2}\chi_{1<g_3<2} \\ &= \chi_{1<g_3<2} - \chi_{1\le g_2} + \chi_{2\le g_3}. \end{aligned}$$

Now, for almost all $y \in \mathbb{R}_+$,

$$\begin{aligned} \chi_{1<g_1}(y) &= \{ay\}+\{by\}-\{(a+b)y\}, \\ \chi_{1\le g_2}(y) &= \{ay\}+\{cy\}-\{(a+c)y\}, \\ \chi_{1<g_3<2}(y)+2\chi_{2\le g_3}(y) &= \{ay\}+\{by\}+\{cy\} \\ &\quad - \{(a+b+c)y\}. \end{aligned}$$

This gives

$$\begin{aligned} I(a,b,c) &- I(\omega_1(a,b,c)) \\ &= \int_0^{\infty} \left(\chi_{1<g_1}+\chi_{1\le g_2}-\chi_{1<g_3<2}-2\chi_{2\le g_3}\right)(y)\frac{dy}{y^2} \\ &= \int_0^{\infty} \frac{\{(a+b)y\}+\{(a+c)y\}-\{ay\}-\{(a+b+c)y\}}{y^2}\,dy \\ &= \int_0^{\infty} \frac{\{(a+b)y\}-(a+b)\{y\}}{y^2}\,dy \\ &\quad + \int_0^{\infty} \frac{\{(a+c)y\}-(a+c)\{y\}}{y^2}\,dy \\ &\quad - \int_0^{\infty} \frac{\{ay\}-a\{y\}}{y^2}\,dy \\ &\quad - \int_0^{\infty} \frac{\{(a+b+c)y\}-(a+b+c)\{y\}}{y^2}\,dy. \end{aligned} \tag{2–10}$$

Take $k \in \mathbb{N}\setminus\{0\}$. We will compute the integral

$$J_k = \int_0^{\infty} \frac{\{ky\}-k\{y\}}{y^2}\,dy,$$

by considering the decomposition

$$J_k = \sum_{q\in\mathbb{N}} \sum_{j=0}^{k-1} \int_{q+j/k}^{q+(j+1)/k} \frac{\{ky\}-k\{y\}}{y^2}\,dy.$$

With the change of variable $y = q + (j+\theta)/k$, we obtain

$$J_k = \sum_{q\in\mathbb{N}} \sum_{j=0}^{k-1} \int_0^1 \frac{-j}{\big(q+(j+\theta)/k\big)^2}\frac{d\theta}{k}$$
$$= \sum_{q\in\mathbb{N}} \sum_{j=1}^{k-1} \Big(\frac{j}{q+(j+1)/k} - \frac{j}{q+j/k}\Big)$$
$$= -\sum_{j=1}^{k-1} j\Big(\psi\Big(\frac{j+1}{k}\Big) - \psi\Big(\frac{j}{k}\Big)\Big),$$

the last equality coming from (2–5). Thus, for every $k \in \mathbb{N}\setminus\{0\}$,

$$J_k = \sum_{j=1}^{k-1} \psi\Big(\frac{j}{k}\Big) - (k-1)\psi(1).$$

Now we recall a consequence of the Gauss and Legendre multiplication formula [Bateman 1953, p. 16]. For $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and for $k \in \mathbb{N}\setminus\{0\}$, we have

$$\psi(kz) = \frac{1}{k}\sum_{j=0}^{k-1} \psi\Big(z + \frac{j}{k}\Big) + \log k.$$

If we give z the value $1/k$, we obtain $J_k = k\log k$. Returning to the relation (2–10), it now becomes

$$\begin{aligned} I(a,b,c) &- I(\omega_1(a,b,c)) \\ &= (a{+}b)\log(a{+}b) + (a{+}c)\log(a{+}c) \\ &\qquad - a\log a - (a{+}b{+}c)\log(a{+}b{+}c), \end{aligned} \tag{2–11}$$

which was the expected result.

It remains to prove that this relation is also satisfied if we replace ω_1 by ω_2. If a_1, b_1, c_1 are elements of $\mathbb{Z}$ satisfying the hypotheses of the lemma, we have $F(a_1,b_1,c_1) = F(a_1,c_1,b_1)$ (see the conditions (2–8)), so $I(a_1,b_1,c_1) = I(a_1,c_1,b_1)$ and

$$I(a,b,c) - I(\omega_2(a,b,c)) = I(a,c,b) - I(\omega_1(a,c,b)).$$

We finish the proof by noticing that the relation (2–11) is symmetric in b and c. □

3. COMPUTATION OF IRRATIONALITY MEASURES OF log 2

We now give the formulas necessary to compute an irrationality measure of $\log 2$. Let us recall that, for $n \in \mathbb{N}$, the notation $\mathbb{Z}_n[X]$ refers to the set of the polynomials with coefficients in $\mathbb{Z}$ and degree $\le n$. We consider, for $n \in \mathbb{N}$, the polynomial

$$F_{a,b,c,n}(X) = \frac{1}{d_{b/a,c/a}(an)} P_{an,bn,cn}(X) \in \mathbb{Z}_{an}[X],$$

where a is a positive integer and b, c are integers such that $a > -\min(b,c,b{+}c)$ and $abc \ne 0$. In the following, we shall write it $F_n(X)$. Morevover, we introduce the integers

$$\begin{aligned} \beta &= \max(0,b), \quad \gamma = \max(0,c), \\ \Delta &= (2b+c)^2 + 8a(a+b+c). \end{aligned} \tag{3–1}$$

Recall that d_{an} denotes the l.c.m. of $\{1,\dots,an\}$. We have

$$J_n = \int_0^1 \frac{F_n(x) - F_n(-1)}{1+x}\,dx \in \frac{\mathbb{Z}}{d_{an}}$$

and

$$\begin{aligned} J_n &+ F_n(-1)\log 2 \\ &= \int_0^1 \frac{F_n(x)}{1+x}\,dx \\ &= \int_0^1 Q_n(x)F_n(x)\,dx \\ &\qquad + \frac{(-1)^{\beta n}}{2^{\gamma n}} \int_0^1 \frac{x^{\beta n}(1-x)^{\gamma n} F_n(x)}{1+x}\,dx, \end{aligned}$$

where

$$Q_n(X) = \frac{1 - 2^{-\gamma n}(-X)^{\beta n}(1-X)^{\gamma n}}{1+X} \in \frac{1}{2^{\gamma n}}\mathbb{Z}_{(\beta+\gamma)n-1}[X].$$

We put, for $n \in \mathbb{N}$,

$$\begin{aligned} q_n &= 2^{\gamma n} d_{(a+\beta+\gamma)n} F_n(-1), \\ p_n &= 2^{\gamma n} d_{(a+\beta+\gamma)n} \int_0^1 Q_n(x)F_n(x)\,dx \\ &\qquad - 2^{\gamma n} d_{(a+\beta+\gamma)n} J_n, \\ \varepsilon_n &= (-1)^{\beta n} d_{(a+\beta+\gamma)n} \int_0^1 \frac{x^{\beta n}(1-x)^{\gamma n}F_n(x)}{1+x}\,dx, \end{aligned}$$

so that $\varepsilon_n = q_n\log 2 - p_n$ with $p_n, q_n \in \mathbb{Z}$.

We study the asymptotic behaviour of the sequence (q_n). Let C_t be the circle of radius t centered at -1. The Cauchy formula gives

$$q_n = \frac{d_{(a+\beta+\gamma)n}2^{\gamma n}}{d_{b/a,c/a}(an)} \oint_{C_t} \frac{z^{(a+c)n}(1-z)^{(a+b)n}}{(z+1)^{(a+b+c)n+1}} \frac{dz}{2i\pi}$$
$$= \frac{d_{(a+\beta+\gamma)n}2^{\gamma n}}{d_{b/a,c/a}(an)} \times \int_0^1 \frac{\left(-1+te^{2i\pi\theta}\right)^{(a+c)n}\left(2-te^{2i\pi\theta}\right)^{(a+b)n}}{\left(te^{2i\pi\theta}\right)^{(a+b+c)n}}\, d\theta,$$

for every $t > 0$. Hence

$$|q_n| \le \frac{d_{(a+\beta+\gamma)n}2^{\gamma n}}{d_{b/a,c/a}(an)} t^{-(a+b+c)n}(1+t)^{(a+c)n}(2+t)^{(a+b)n}$$

for every $t > 0$. Then, thanks to the equality (2–1) and to Lemma 2.3, it follows that

$$\limsup_{n\to+\infty} \frac{\log|q_n|}{n} \le a+\beta+\gamma(1+\log 2) - ae\Big(\frac{b}{a},\frac{c}{a}\Big) + \min_{t>0}\log(t^{-(a+b+c)}(1+t)^{a+c}(2+t)^{a+b}) = \sigma_{a,b,c}(\log 2).$$

The function defined over $\mathbb{R}\setminus\{0,-1,-2\}$ by

$$u_{a,b,c}(t) = \log\left(\frac{|1+t|^{a+c}|2+t|^{a+b}}{|t|^{a+b+c}}\right)$$

reaches its minimum at $t_0(a,b,c) = \big(2b+c+\sqrt{\Delta}\big)/2a$, with Δ given by (3–1). We have

$$\begin{aligned} u_{a,b,c}(t_0(a,b,c)) &= -a\log(2a) + (a+c)\log\big(2(a+b)+c+\sqrt{\Delta}\big) \\ &\quad + (a+b)\log\big(2(2a+b)+c+\sqrt{\Delta}\big) \\ &\quad - (a+b+c)\log\big(2b+c+\sqrt{\Delta}\big). \end{aligned} \tag{3–2}$$

We call $u_{a,b,c}(t_0(a,b,c))$ the analytic component of $\sigma_{a,b,c}(\log 2)$.

It remains to study the behaviour of the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. If we make $(a+b+c)n$-fold partial integrations, we find

$$\varepsilon_n = \frac{(-1)^{\beta n}d_{(a+\beta+\gamma)n}}{d_{b/a,c/a}(an)((a+b+c)n)!} \times \int_0^1 x^{(a+c)n}(1-x)^{(a+b)n}\left(\frac{x^{\beta n}(1-x)^{\gamma n}}{1+x}\right)^{(a+b+c)n} dx$$
$$= \frac{2^{\gamma n}d_{(a+\beta+\gamma)n}}{d_{b/a,c/a}(an)} \int_0^1 \frac{x^{(a+c)n}(1-x)^{(a+b)n}}{(1+x)^{(a+b+c)n+1}}\, dx,$$

since

$$\frac{x^{\beta n}}{1+x}(1-x)^{\gamma n} = \frac{(-1)^{\beta n}2^{\gamma n}}{1+x} - (-1)^{\beta n}2^{\gamma n}Q_n(x).$$

Equalities (2–1) and (2–3) and Lemma 2.3 yield

$$\lim_{n\to+\infty}\frac{1}{n}\log|\varepsilon_n| = a+\beta+\gamma(1+\log 2) - ae\Big(\frac{b}{a},\frac{c}{a}\Big) + \max_{0\le x\le 1}\log\left(\frac{x^{a+c}(1-x)^{a+b}}{(1+x)^{a+b+c}}\right) = -\tau_{a,b,c}(\log 2).$$

The function defined over $\mathbb{R}\setminus\{0,1,-1\}$ by

$$v_{a,b,c}(x) = \log\left(\frac{|x|^{a+c}|1-x|^{a+b}}{|1+x|^{a+b+c}}\right) = u_{a,b,c}(-1-x)$$

reaches its maximum at

$$x_0(a,b,c) = \frac{\sqrt{(2a+2b+c)^2+4a(a+c)} - 2(a+b) - c}{2a},$$

and its value at this point is

$$\begin{aligned} v(x_0(a,b,c)) &= -a\log(2a) + (a+c)\log\big(-c-2(a+b)+\sqrt{\Delta}\big) \\ &\quad + (a+b)\log\big(2(2a+b)+c-\sqrt{\Delta}\big) \\ &\quad - (a+b+c)\log\big(-c-2b+\sqrt{\Delta}\big). \end{aligned}$$

We know, from Lemma 2.1, that

$$1 + \frac{\sigma_{a,b,c}(\log 2)}{\tau_{a,b,c}(\log 2)}$$

is an irrationality measure of $\log 2$.

Remark 3.1. The saddle method (see [Dieudonné 1968] for instance) allows us to prove that the analytic component in $\sigma_{a,b,c}(\log 2)$ is the best possible.

Remark 3.2. The computations that we have just done can be adapted very easily to the numbers $\log(1-r/s)$ with $r\in\mathbb{N}$ and $s\in\mathbb{Z}\setminus\{0\}$ such that $r/s\in[-1,1[$. In particular, it allows one to recover Theorem 2 of [Heimonen et al. 1993]. Actually, we obtain

$$\sigma_{a,b,c}(\log(1-r/s)) = a(1+\log|r|) + \beta(1+\log|s|) + \gamma(1+\log|r-s|) - ae\Big(\frac{b}{a},\frac{c}{a}\Big) + \min_{t>0}\log\left(\frac{(t+|s/r|)^{a+c}\,(t+|1-s/r|)^{a+b}}{t^{a+b+c}}\right)$$

and

$$\begin{aligned}&\tau_{a,b,c}(\log(1-r/s))\\&\quad=a(1+\log|r|)+\beta(1+\log|s|)+\gamma(1+\log|r-s|)\\&\quad-a\,e\Big(\frac{b}{a},\frac{c}{a}\Big)+\max_{0\le x\le 1}\log\bigg(\frac{x^{a+c}(1-x)^{a+b}}{(x-s/r)^{a+b+c}}\bigg).\end{aligned}$$

3A. Links Between the Irrationality Measures of $\log 2$ Given by the Polynomials of Rukhadze, Dubitskas, Hata and Chudnovsky

Let ω_1 and ω_2 be as in formulas (2–6). We now show that every irrationality measure of $\log 2$ obtained from the polynomials of D. V. and G. V. Chudnovsky can also be found using those of Rukhadze and those of Hata.

Lemma 3.3. *Let $a \in \mathbb{N}\setminus\{0\}$ and $b,c\in\mathbb{Z}\setminus\{0\}$ be such that $a > -\min(b,c,b{+}c)$. Then*

$$\mu_{\omega_i(a,b,c)}(\log 2)=\mu_{(a,b,c)}(\log 2)$$

for $i = 1, 2$.

Proof. We first show that

$$\sigma_{\omega_1(a,b,c)}(\log 2)=\sigma_{a,b,c}(\log 2).$$

Recall that Δ denotes $(c{+}2b)^2+8a(a{+}b{+}c)$, which is invariant under the action of ω_1. We use (3–2) to compute $\min_{r>0}u_{a,b,c}(r)-\min_{r>0}u_{\omega_1(a,b,c)}(r)$:

$$\begin{aligned}&u_{a,b,c}(r_0(a,b,c))-u_{\omega_1(a,b,c)}(r_0(\omega_1(a,b,c)))\\&\quad=-a\log(2a)+(a{+}b)\log(2(a{+}b))\\&\qquad+(a{+}b)\log\big(2(2a{+}b){+}c+\sqrt{\Delta}\,\big)\\&\qquad-a\log\big(2(2a{+}b){+}c{+}\sqrt{\Delta}\,\big)\\&\qquad+(a{+}c)\log\big(2(a{+}b){+}c{+}\sqrt{\Delta}\,\big)\\&\qquad+(a{+}c)\log\big(-2b{+}c{+}\sqrt{\Delta}\,\big)\\&\qquad-(a{+}b{+}c)\log\big(2b{+}c{+}\sqrt{\Delta}\,\big)\\&\qquad-(a{+}b{+}c)\log\big(2a+c{+}\sqrt{\Delta}\,\big)\\&\quad=b\log 2-a\log a+(a{+}b)\log(a{+}b)\\&\qquad+b\log\big(2(2a{+}b){+}c{+}\sqrt{\Delta}\,\big)\\&\qquad-(a{+}b{+}c)\log\big(2(a{+}b{+}c)\big(2(2a{+}b){+}c{+}\sqrt{\Delta}\,\big)\big)\\&\qquad+(a{+}c)\log\big(2(a{+}c)\big(2(2a{+}b){+}c+\sqrt{\Delta}\,\big)\big)\\&\quad=-a\log a+(a{+}b)\log(a{+}b)\\&\qquad-(a{+}b{+}c)\log(a{+}b{+}c)+(a{+}c)\log(a{+}c).\end{aligned}$$

Hence, using Lemma 2.5, we deduce

$$\begin{aligned}&u_{a,b,c}(r_0(a,b,c))-I(a,b,c)\\&\qquad=u_{\omega_1(a,b,c)}(r_0(\omega_1(a,b,c)))-I(\omega_1(a,b,c)),\end{aligned}$$

where $I(a,b,c)$ is given by (2–7). Recall that, for any triple (a,b,c) of integers satisfying the conditions of the lemma, we have

$$\begin{aligned}\sigma_{a,b,c}(\log 2)=a+\max(0,b)+\max(0,c)(1{+}\log 2)\\-I(a,b,c)+u_{a,b,c}(r_0(a,b,c)).\end{aligned}$$

It easily follows that

$$\sigma_{a,b,c}(\log 2)=\sigma_{\omega_1(a,b,c)}(\log 2).$$

We can similarly show that

$$\tau_{a,b,c}(\log 2)=\tau_{\omega_1(a,b,c)}(\log 2).$$

In the same way, we prove

$$\mu_{\omega_2(a,b,c)}(\log 2)=\mu_{(a,b,c)}(\log 2).\qquad\square$$

Remark 3.4. Here again, the computations are the same for the numbers $\log(1-r/s)$ with $r\in\mathbb{N}$ and $s\in\mathbb{Z}\setminus\{0\}$ such that $r/s\in[-1,1[$. Thus, we obtain

$$\mu_{\omega_i(a,b,c)}(\log(1-r/s))=\mu_{(a,b,c)}(\log(1-r/s))$$

for $i=1,2$, with $a\in\mathbb{N}\setminus\{0\}$ and $b,c\in\mathbb{Z}\setminus\{0\}$ such that $a>-\min(b,c,b+c)$. Such a result is mentioned in the final remark of [Viola 1997], where one can find a simpler approach for getting irrationality measures of $\log(1-r/s)$ using the Euler–Pochhammer integral representation of the hypergeometric function ${}_2F_1$, a method inspired by the one developed in [Rhin and Viola 1996].

3B. Figures and Results

In the preceding section, we established the formulas

$$\begin{aligned}&\sigma_{a,b,c}(\log 2)\\&\quad=a+\beta+\gamma(1{+}\log 2)-a\,e\Big(\frac{b}{a},\frac{c}{a}\Big)-a\log(2a)\\&\qquad+(a{+}c)\log\big(2(a{+}b){+}c{+}\sqrt{\Delta}\,\big)\\&\qquad+(a{+}b)\log\big(2(2a{+}b){+}c{+}\sqrt{\Delta}\,\big)\\&\qquad-(a{+}b{+}c)\log\big(2b{+}c{+}\sqrt{\Delta}\,\big)\end{aligned}\tag{3–3}$$

and

$$\begin{aligned}\tau_{a,b,c}(\log 2) &= a\,e\Big(\frac{b}{a},\frac{c}{a}\Big) - a - \beta - \gamma(1+\log 2) + a\log(2a)\\ &\quad - (a+c)\log\big(-c-2(a+b)+\sqrt{\Delta}\,\big)\\ &\quad - (a+b)\log\big(2(2a+b)+c-\sqrt{\Delta}\,\big)\\ &\quad + (a+b+c)\log\big(-c-2b+\sqrt{\Delta}\,\big) \qquad (3\text{–}4)\end{aligned}$$

with $a \in \mathbb{N}\setminus\{0\}$ and $b, c \in \mathbb{Z}$ such that

$$a > -\min(b, c, b+c)$$

and with β, γ and Δ as in (3–1).

Using the integration algorithm described in Section 2C, we can compute explicitly the irrationality measure of $\log 2$ associated to the choice of parameters (a, b, c) (which choice determines the class of approximants of the kind $P_{n,m,m'}$ that is going to be used). So, we wrote a program which works under GP, the calculator of the PARI system [Batut et al. 1999]. It associates to every suitable triple (a, b, c) an irrationality measure of $\log 2$ computed numerically to the desired accuracy. On the other hand, looking at formulas (3–3) and (3–4), it is easy to notice that

$$\mu_{(ka,kb,kc)}(\log 2) = \mu_{(a,b,c)}(\log 2)$$

for all $a, k \in \mathbb{N}$ and $b, c \in \mathbb{Z}$ such that

$$a > -\min(b, c, b+c).$$

This allows us to restrict our parametrization to pairs of rational numbers of the form $(a/b, a/c)$, with a, b, c relatively prime integers. From now on, we write $\mu_{(a,b,c)}(\log 2)$ or $\mu_{a/b,a/c}(\log 2)$ interchangeably to denote the irrationality measure of $\log 2$ associated to the triple (a, b, c).

Thanks to Lemma 3.3, we can restrict our study to the rational points $(x, y) = (a/b, a/c)$ with $a, b, c \in \mathbb{N}\setminus\{0\}$. Inspecting this quadrant of the plane, we noticed that the most interesting area was around the point $(6, 6)$. Hence we made a more precise study of the irrationality measures obtained near this point, sampling on the grid $\mathcal{M}_{6,6}$ consisting of points of the form

$$\Big(6+\frac{j}{25}+\frac{k}{50},\ 6+\frac{j}{25}-\frac{k}{50}\Big)_{j,k\in\mathbb{Z},\,-25\le j\le 25,\,-50\le k\le 50}.$$

This gave Figure 1 (left), which represents the surface interpolating the points

$$(x,\ y,\ \mu_{x,y}(\log 2))_{(x,y)\in\mathcal{M}_{6,6}}.$$

We can clearly see on this surface a crease, corresponding to values obtained along the half-line $\{y = x,\ x > 0\}$. Restricting our attention to this

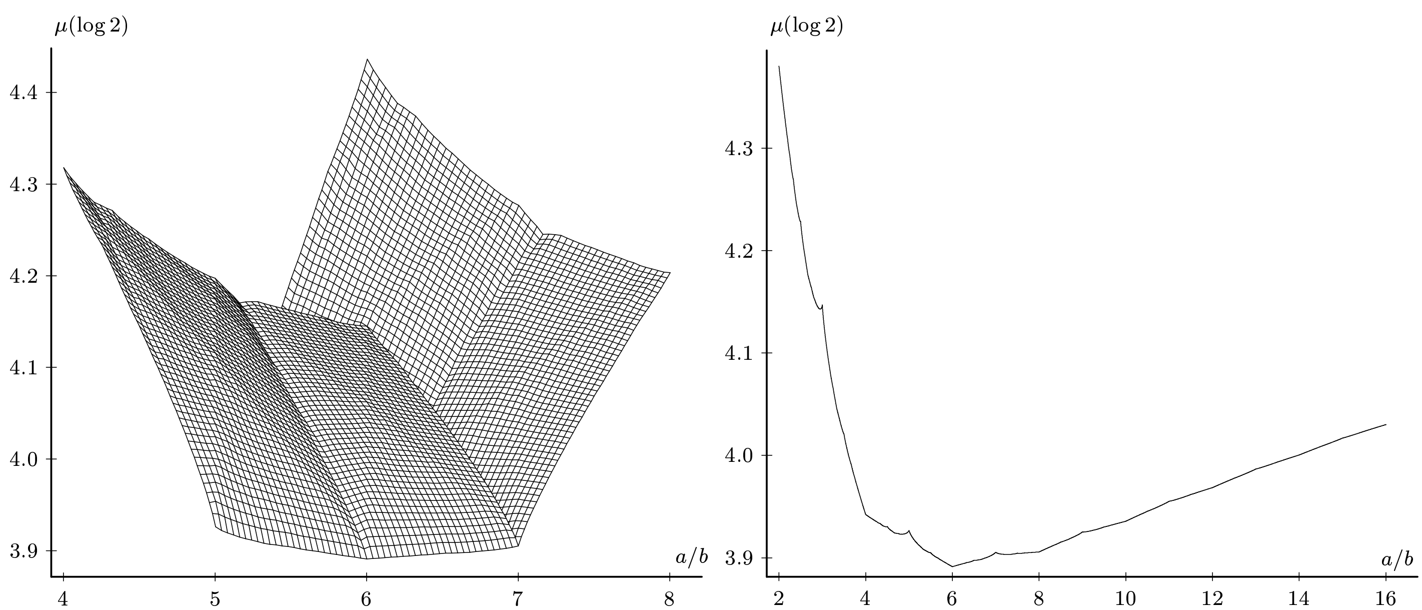

FIGURE 1. Irrationality measures of $\log 2$ around the triple $(a, b, c) = (6, 1, 1)$. In the three-dimensional plot on the left, the horizontal coordinates are a/b and (coming out of the plane of the paper) a/c; the height gives the associated irrationality measure. On the right we restrict attention to the line $a/b = a/c =: x$, plotting x horizontally and the associated irrationality measure vertically.

crease, we obtain Figure 1 (right), a two-dimensional plot of $\mu_{x,x}(\log 2)$ vs. x sampled at the points

$$x_j = 2 + j/14000,$$

for $0 \le j \le 14000$. The apparent minimum on this curve is reached at $a/b = 6$.

Finally, the best irrationality measures found correspond to the choices of parameters $(8,-1,-1)$, $(7,1,-1)$, $(6,1,1)$ and $(7,-1,1)$. The first of these triples corresponds to the choice of parameters made by E. A. Rukhadze [1987]. Hence

$$\mu_{8,-1,-1}(\log 2) = 1 + \frac{\log(2\alpha)+\beta}{\log\alpha - \beta}$$

with

$$\alpha = 2^{-11}(153333125 + 7734633\sqrt{393}),$$
$$\beta = 8 - \pi\Big(\frac{1}{2} - \sqrt{2} - \frac{2}{\sqrt{3}}\Big).$$

Numerically,

$$\mu_{8,-1,-1}(\log 2) = 3,8913997\ldots.$$

The triple $(7,1,-1)$ considered by Hata [1990] gives the same irrationality measure, and we also find this value if we choose $(6,1,1)$ and $(7,-1,1)$ (which may be predicted from Lemma 3.3). The choice $(6,1,1)$ correspond to the polynomials used by the Chudnovskys, namely

$$C_{6n,n}(z) = \frac{1}{n!}\frac{d^n}{dz^n}L_{6n}(z) = \binom{7n}{n}P_{6n,n,n}(z),$$

where as before L_n is the n-th Legendre type polynomial. (In [Chudnovsky and Chudnovsky 1993] it is said that the use of a simple one-dimensional parametrization and the computation of the content of these polynomials yield an irrationality measure of $3.87\ldots$, but we were unable to duplicate this value using the same methods, which simply yield the result of Rukhadze and Hata. In fact, Remarks 2.4 and 3.1 tell us that the arithmetic component $e\,(b/a, c/a)$ and the analytic component in $\sigma_{a,b,c}(\log 2)$ are optimal. An improvement might conceivably arise from replacing the l.c.m. $d_{(a+\beta+\gamma)n}$, introduced to ensure that $p_n, q_n \in \mathbb{Z}$, with something smaller; but this is not done in the Chudnovskys' paper in question.)

4. COMPUTATION OF IRRATIONALITY MEASURES OF $\pi/\sqrt{3}$

We use again the class of polynomials

$$F_n(X) = \frac{1}{d_{b/a,c/a}(an)}P_{an,bn,cn}(X) \in \mathbb{Z}_{an}[X]$$

with $a \in \mathbb{N}\setminus\{0\}$ and $b, c \in \mathbb{Z}\setminus\{0\}$ such that $a > -\min(b,c,b{+}c)$.

Then

$$J_n = \int_0^1 \frac{F_n(x) - F_n\big(e^{i\pi/3}\big)}{x - e^{i\pi/3}}\,dx \in \frac{1}{d_{an}}(\mathbb{Z} + j\mathbb{Z})$$

where $j = e^{2i\pi/3}$; also

$$J_n + F_n\big(e^{i\pi/3}\big)\frac{i\pi}{3} = \int_0^1 \frac{F_n(x)}{x - e^{i\pi/3}}\,dx = \int_0^1 Q_n(x)F_n(x)\,dx + v_n,$$

where we've put

$$v_n = \frac{(e^{-i\pi/3})^{\beta n}}{(1-e^{i\pi/3})^{\gamma n}}\int_0^1 \frac{x^{\beta n}(1{-}x)^{\gamma n}F_n(x)}{x - e^{i\pi/3}}\,dx,$$

$$Q_n(X) = \frac{1-(-j)^{(\beta-\gamma)n}X^{\beta n}(1{-}X)^{\gamma n}}{X - e^{i\pi/3}} \in (\mathbb{Z}+j\mathbb{Z})_{(\beta+\gamma)n-1}[X],$$

with $\beta = \max(0,b)$ and $\gamma = \max(0,c)$. Setting

$$q_n = i\sqrt{3}\,d_{(a+\beta+\gamma)n}F_n\big(e^{i\pi/3}\big),$$
$$p_n = 3\,d_{(a+\beta+\gamma)n}\left(\int_0^1 Q_n(x)F_n(x)\,dx - J_n\right),$$
$$\varepsilon_n = 3\,d_{(a+\beta+\gamma)n}v_n,$$

for $n \in \mathbb{N}$, we obtain $\varepsilon_n = q_n\pi/\sqrt{3} - p_n$, with $p_n, q_n \in \mathbb{Z} + j\mathbb{Z}$.

We now check the asymptotic behaviour of the sequence $(q_n)_{n\in\mathbb{N}}$. The Cauchy formula gives us

$$q_n = i\sqrt{3}\,\frac{d_{(a+\beta+\gamma)n}}{d_{b/a,c/a}(an)}\oint_C \frac{z^{(a+c)n}(1-z)^{(a+b)n}}{(z - e^{i\pi/3})^{(a+b+c)n+1}}\frac{dz}{2i\pi},$$

where C is the unit circle centered at $e^{i\pi/3}$. Putting $z = e^{i\pi/3}(1 - e^{2i\theta})$, we have

$$|q_n| \le \sqrt{3}\,\frac{d_{(a+\beta+\gamma)n}2^{(2a+b+c)n}}{d_{b/a,c/a}(an)} \times \max_{0\le\theta\le\pi}\left(\sin\theta^{(a+c)n}\left|\sin\Big(\theta - \frac{\pi}{6}\Big)\right|^{(a+b)n}\right).$$

Hence we deduce

$$\limsup_{n\to+\infty} \frac{\log|q_n|}{n} \le a+\beta+\gamma+(2a+b+c)\log 2 - a\,e\Big(\frac{b}{a},\frac{c}{a}\Big) + \max_{0\le\theta\le\pi} g_{a,b,c}(\theta) = \sigma_{a,b,c}\big(\pi/\sqrt{3}\big),$$

the function $g_{a,b,c}$ being defined by

$$g_{a,b,c}(\theta) = (a+c)\log|\sin\theta| + (a+b)\log\left|\sin\left(\theta-\frac{\pi}{6}\right)\right|.$$

On the interval $[0,\pi]$, the derivative of this function is cancelled at the points

$$\theta_0(a,b,c) = \frac{\pi}{12} + \frac{1}{2}\arcsin\left(\frac{c-b}{2(2a+b+c)}\right) \qquad (4\text{–}1)$$

and

$$\theta_1(a,b,c) = \frac{7\pi}{12} - \frac{1}{2}\arcsin\left(\frac{c-b}{2(2a+b+c)}\right).$$

Indeed, since $a > -\min(b,c,b+c)$, we easily see that $(c-b)/2(2a+b+c)$ belongs to $]-1/2,1/2[$. Therefore, the real number $\arcsin((c-b)/(2a+b+c))$ lies in $]-\pi/6,\pi/6[$. From this, we deduce that $\theta_0(a,b,c)$ lies in $]0,\pi/6[$ and $\theta_1(a,b,c)$ in $]\pi/2,2\pi/3[$.

It remains to study the behaviour of the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. We make $(a+b+c)n$-fold partial integrations. We find

$$\varepsilon_n = \frac{3d_{(a+\beta+\gamma)n}\,(e^{-i\pi/3})^{\beta n}}{(1-e^{i\pi/3})^{\gamma n} d_{b/a,c/a}(an)} \int_0^1 \frac{x^{(a+c)n}(1-x)^{(a+b)n}}{((a+b+c)n)!} \times \left(\frac{x^{\beta n}(1-x)^{\gamma n}}{x-e^{i\pi/3}}\right)^{((a+b+c)n)} dx = \frac{3\,d_{(a+\beta+\gamma)n}}{d_{b/a,c/a}(an)} \int_0^1 \frac{x^{(a+c)n}(1-x)^{(a+b)n}}{(x-e^{i\pi/3})^{(a+b+c)n+1}}\,dx,$$

since $Q_n(x)$ has degree $(\beta+\gamma)n-1$ and since $\beta+\gamma \le a+b+c$ (recall that $a > -\min(b,c,b+c)$).

From this we deduce, replacing the initial integration path by the smallest arc of circle centered in $e^{i\pi/3}$ joining 0 and 1,

$$\lim_{n\to+\infty} \frac{1}{n}\log|\varepsilon_n| \le a+\beta+\gamma+(2a+b+c)\log 2 - a\,e\Big(\frac{b}{a},\frac{c}{a}\Big) + \max_{0\le\theta\le\pi/6} g_{a,b,c}(\theta) = -\tau_{a,b,c}\big(\pi/\sqrt{3}\big).$$

The maximum of $g_{a,b,c}$ on the interval $[0,\pi/6]$ is reached at $\theta_0(a,b,c)$.

We prove, exactly in the same way as in the end of the proof of [Hata 1990, Theorem 4.2], that we can apply Lemma 2.2. Then we know that $1+\sigma_{a,b,c}\big(\pi/\sqrt{3}\big)/\tau_{a,b,c}\big(\pi/\sqrt{3}\big)$ is an irrationality measure of $\pi/\sqrt{3}$.

Remark 4.1. The saddle method [Dieudonné 1968] ensures that the analytic components

$$\max_{0\le\theta\le\pi} g_{a,b,c}(\theta) \quad \text{and} \quad \max_{0\le\theta\le\pi/6} g_{a,b,c}(\theta)$$

in the upper bounds for $\limsup_{n\to\infty}\log|q_n|/n$ and $\limsup_{n\to\infty}\log|\varepsilon_n|/n$, respectively, are best possible.

4A. Links Between the Irrationality Measures of $\pi/\sqrt{3}$ Given by the Polynomials of Rukhadze, Dubitskas, Hata and Chudnovsky

We now prove a result that will allow us to restrict the field of parameters (a,b,c) to study.

Lemma 4.2. *Let $a\in\mathbb{N}\setminus\{0\}$ and $b,c\in\mathbb{Z}\setminus\{0\}$ be such that $a > -\min(b,c,b+c)$. Then*

$$\mu_{(a,b,c)}\big(\pi/\sqrt{3}\big) = \mu_{\omega_1\omega_2(a,b,c)}\big(\pi/\sqrt{3}\big) = \mu_{(a+b+c,-b,-c)}\big(\pi/\sqrt{3}\big).$$

This means in particular that every measure of irrationality of $\pi/\sqrt{3}$ obtained from the polynomials of D. V. and G. V. Chudnovsky can also be found with those used by Dubitskas.

Proof. We clearly have

$$\theta_0(a+b+c,-b,-c) = \frac{\pi}{6} - \theta_0(a,b,c),$$
$$\theta_1(a+b+c,-b,-c) = \pi + \frac{\pi}{6} - \theta_1(a,b,c).$$

From this we deduce

$$g_{a+b+c,-b,-c}(\theta_i(a+b+c,-b,-c)) = g_{a,b,c}(\theta_i(a,b,c))$$

for $i=0,1$. Then the formulas

$$\sigma_{a,b,c}\big(\pi/\sqrt{3}\big) = a+\beta+\gamma+(2a+b+c)\log 2 - a\,e\Big(\frac{b}{a},\frac{c}{a}\Big) + \max_{0\le\theta\le\pi} g_{a,b,c}(\theta),$$
$$\tau_{a,b,c}\big(\pi/\sqrt{3}\big) = -a-\beta-\gamma-(2a+b+c)\log 2 + a\,e\Big(\frac{b}{a},\frac{c}{a}\Big) - \max_{0\le\theta\le\pi/6} g_{a,b,c}(\theta)$$

and Lemma 2.5 give us the equalities

$$\sigma_{a+b+c,-b,-c}\big(\pi/\sqrt{3}\big) = \sigma_{a,b,c}\big(\pi/\sqrt{3}\big),$$
$$\tau_{a+b+c,-b,-c}\big(\pi/\sqrt{3}\big) = \tau_{a,b,c}\big(\pi/\sqrt{3}\big). \qquad \square$$

4B. Figures and Results

We can explicitly compute the irrationality measure of $\pi/\sqrt{3}$ associated to a choice of parameters (a,b,c). This involves a simple adaptation of the GP program written for $\log 2$. The slight difficulty represented by the computation of the integral $e(b/a, c/a)$ has already been dealt with, so we only have to enter the formulas giving the maxima of $g_{a,b,c}(\theta)$ over $[0,\pi]$ and over $[0,\pi/6]$. This program gives an irrationality measure of $\pi/\sqrt{3}$ computed numerically to the desired accuracy from the parameters (a,b,c).

If we look at the expressions of $\sigma_{a,b,c}\left(\pi/\sqrt{3}\right)$ and $\tau_{a,b,c}\left(\pi/\sqrt{3}\right)$, we see that

$$\mu_{(ka,kb,kc)}\left(\pi/\sqrt{3}\right) = \mu_{(a,b,c)}\left(\pi/\sqrt{3}\right)$$

for $a,k \in \mathbb{N}$ and $b,c \in \mathbb{Z}$ with $a > -\min(b,c,b+c)$. Together with Lemma 4.2, this allows us to work not with triples of integers (a,b,c) but with pairs of rational numbers that can be written in the form $(a/b, a/c)$, with $a,b \in \mathbb{N}\setminus\{0\}$ and $c \in \mathbb{Z}\setminus\{0\}$ such that $a > -c$ and a,b,c relatively prime. Here again we denote by $\mu_{(a,b,c)}\left(\pi/\sqrt{3}\right)$ or $\mu_{a/b,a/c}(\pi/\sqrt{3})$ interchangeably the irrationality measure of $\pi/\sqrt{3}$ associated to the triple (a,b,c).

An examination of these two quadrants of the plane suggested that the most interesting areas are around the points $(4,4)$ and $(5,-5)$. Thus, we sampled the irrationality measure at the points of the grid $\mathcal{M}_{4,4}$ around $(4,4)$ defined by

$$\left(4+\frac{j}{25}+\frac{k}{50},\ 4+\frac{j}{25}-\frac{k}{50}\right)_{j,k\in\mathbb{Z},\,-25\le j\le 25,\,-50\le k\le 50},$$

and at points in the grid $\mathcal{M}_{5,-5}$ around $(5,-5)$ defined by

$$\left(5+\frac{j}{50}+\frac{k}{25},\ -5+\frac{j}{50}-\frac{k}{25}\right)_{j,k\in\mathbb{Z},\,-50\le j\le 50,\,-25\le k\le 25}.$$

This yields the left portions of Figures 2 and 3. Here again, the best values seem to lie along the half-lines of equations $\{y=x,\ x>0\}$ and $\{y=-x,\ x>0\}$, respectively.

Restricting our attention to these lines, we plot in Figure 2 (right) the value of $\mu_{x,x}(\pi/\sqrt{3})$ versus x sampled at the points

$$x_j = 2 + j/14000, \quad \text{for } 0 \le j \le 14000,$$

and in Figure 3 (right) we plot $\mu_{x,-x}(\pi/\sqrt{3})$ versus x sampled at the points

$$x_j = 2 + j/14000, \quad \text{for } 1000 \le j \le 14000.$$

The minima on these curves appear to be at the points $x = 4$ and -5, respectively, corresponding to $(a/b, a/c) = (4,4)$ and $(5,-5)$.

The best irrationality measures that we can find are associated to the choices of parameters $(5,1,-1)$

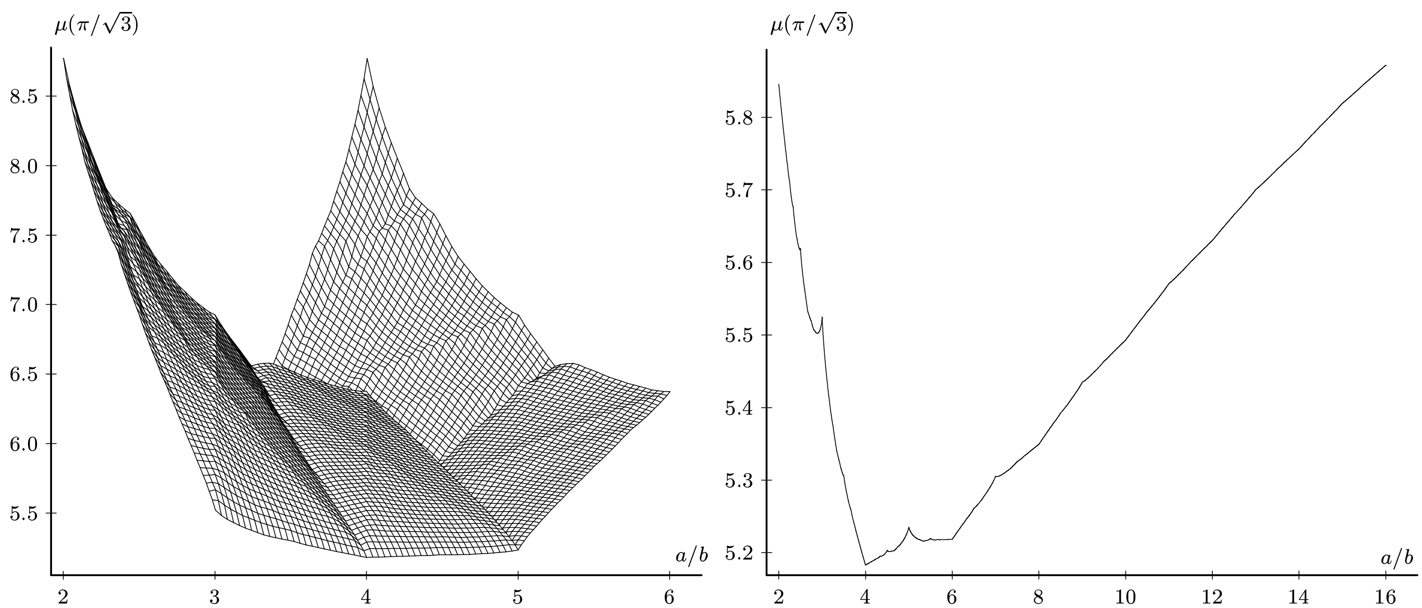

FIGURE 2. Irrationality measures of $\pi/\sqrt{3}$ around the triple $(a,b,c) = (4,1,1)$. The plotting conventions are the same as in Figure 1.

and $(5,-1,1)$. The first of these was used by Hata [1990], who obtained the value

$$\mu_{5,1,-1}\big(\pi/\sqrt{3}\big) = 1 + \frac{\log\alpha_1 + \beta_1}{\log\alpha_1 - \beta_1} = 5.0874625\ldots,$$

where

$$\alpha_1 = \tfrac{1}{4}(1327 + 231\sqrt{33}),$$
$$\beta_1 = 6 - \pi\Big(\frac{2\sqrt{3}}{3} - \frac{1}{2}\Big).$$

Dubitskas [1987] obtained 5.516 and later [1993] the better value 5.2, using the choice of parameters $(6,-1,-1)$. The best value we can obtain with this triple is

$$\mu_{6,-1,-1} = 1 + \frac{\log\alpha_2 + \beta_2}{\log\alpha_2 - \beta_2} = 5.18300789\ldots,$$

with

$$\alpha_2 = 362 + 209\sqrt{3},$$
$$\beta_2 = 6 - \pi\Big(\frac{2\sqrt{3}}{3} - \frac{1}{2}\Big) - \log(2^{-7}3^{-3}5^{5}).$$

D. V. and G. V. Chudnovsky [1993] report obtaining the value 4.96, still considering a simple one-dimensional parametrization of the polynomials $P_{n,m,m}$ (m positive) and computing the content of these polynomials. The best value we were able to find with these restrictions is the one associated to the triple (4,1,1), which according to Lemma 4.2, gives the same value as the choice of parameters of Dubitskas. Here also, Remarks 2.4 and 4.1 tell us that the arithmetic component $e\,(b/a, c/a)$ and the analytic components in $\sigma_{a,b,c}\,(\pi/\sqrt{3})$ and $\tau_{a,b,c}\,(\pi/\sqrt{3})$ are optimal. The only room for improvement would lie in replacing the l.c.m. $d_{(a+\beta+\gamma)n}$ by something smaller.

Since then, Hata [1993] has obtained by other techniques the value 4.6016, which stands as the lowest known irrationality measure of $\pi/\sqrt{3}$.

5. AN ATTEMPT AT ANALYTIC IMPROVEMENT

One can try to get lower irrationality measures in several ways. The first would be a sharpening of Lemmas 2.1 and 2.2, whose proofs are very easy. A second way would be to get directly the p-adic valuation of the numbers p_n and q_n involved in the good approximations $\varepsilon_n = q_n x - p_n$ instead of computing only the p-adic valuation of the greatest common divisor of the numbers

$$\binom{(a+b)n}{bn+j}\binom{(a+b+c)n+j}{(a+b+c)n},$$

for $0 \le j \le an$. Another possibility would be, as already said, to replace the lowest common multiple $d_{(a+\beta+\gamma)n}$ involved in the expression of p_n and q_n by something smaller. A fourth way is to search for new classes of approximants. We worked in this

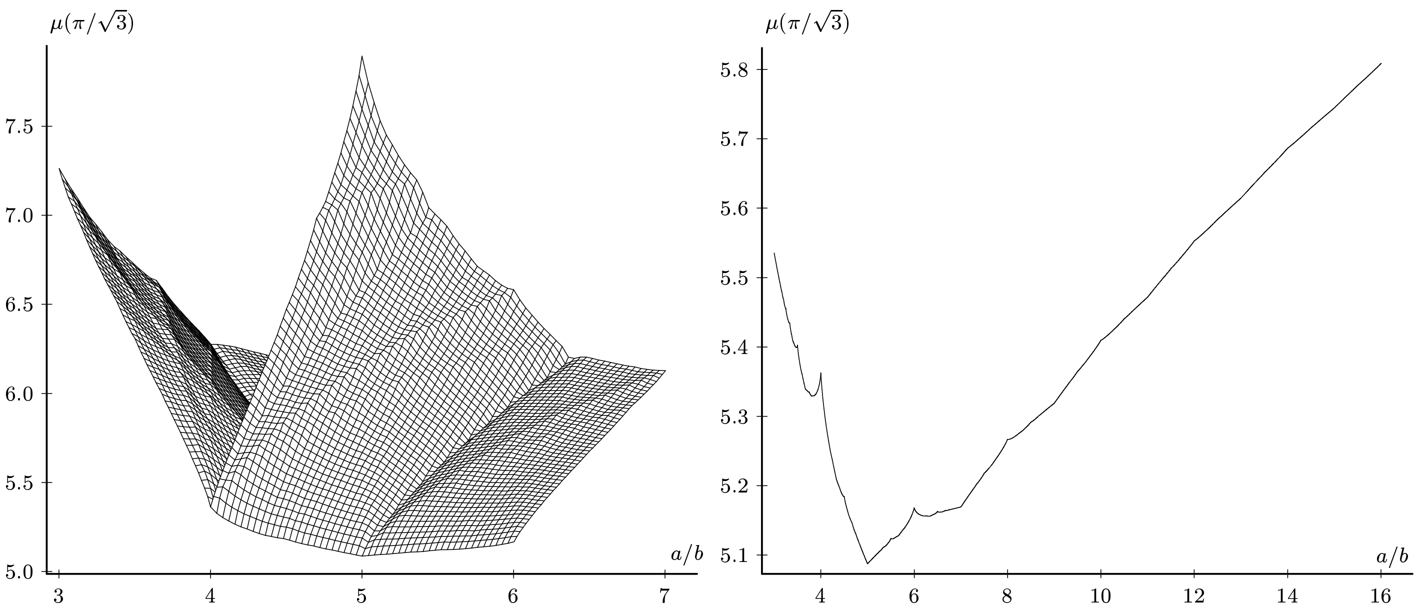

FIGURE 3. Irrationality measures of log 2 around the triple $(a,b,c) = (5,1,-1)$. The conventions are the same as in Figure 1, except that the plane plot is the restriction to the line $a/b = -a/c =: x$, rather than $a/b = a/c$.

direction. In connection with $\log 2$, we considered the polynomials

$$U_{n,m,m',s,t}(z) = \frac{1}{(2n+2m+m')!} \times \Big((z(1-z))^{2n+m} (s-tz)^{m'} \Big)^{(2n+2m+m')},$$

with $n, m' \in \mathbb{N}$ and $m, s, t \in \mathbb{Z}$ such that $2n \geq -\min(m, 2m+m')$, which can be written

$$U_{n,m,m',s,t}(z) = \sum_{k_1=0}^{2n} \sum_{k_2=-\min(k_1,m')}^{0} \Big((-1)^{k_1+k_2+m+m'} \times t^{k_2+m'} s^{-k_2} C_{n,m,m',k_1,k_2} z^{k_1+k_2} \Big),$$

where

$$C_{n,m,m',k_1,k_2} = \binom{2n+m}{k_1+m} \binom{m'}{k_2+m'} \binom{2n+2m+m'+k_1+k_2}{2n+2m+m'}.$$

Putting $m' = 0$, we see that this class generalizes the polynomials $P_{2n,m,m}$. We used it in order to improve the analytic behaviour of the polynomials $P_{2n,m,m}$, i.e., the analytic components which appear in $\sigma_{a,b,c}(\log 2)$ and $\tau_{a,b,c}(\log 2)$ (see (3–3) and (3–4)). The role of the factors $(s - tz)^{m'}$ is to decrease the size of the minimum of the function involved in $\tau_{a,b,c}(\log 2)$ in order to increase it. As in Section 2, we looked in particular at the asymptotic behaviour of the content of the coefficients C_{n,m,m',k_1,k_2}. We put

$$f(\lambda, \lambda') := \int_{F_1(\lambda,\lambda') \cup F_2(\lambda,\lambda')} \frac{dx}{x^2},$$

where

$$F_1(\lambda, \lambda') = \big\{ x > 0 : \{x\} < \tfrac{1}{2}, \; 1+\{\lambda x\} < 2\{x\}+2\{\lambda x\} < 2-\{\lambda' x\} \big\}$$

and

$$F_2(\lambda, \lambda') = \big\{ x > 0 : 2+\{\lambda x\} < 2\{x\}+2\{\lambda x\} < 3-\{\lambda' x\} \big\}.$$

We obtained

$$\sigma_{a,b,c,s,t}(\log 2) = 2a + (\beta+\gamma)(2+\log 2) - a\, f\Big(\frac{b}{a}, \frac{c}{a}\Big) + \min_{r>0} u_{a,b,c,s,t}(r),$$

where $u_{a,b,c,s,t} : \mathbb{R} \setminus \{0, -1, -2, -|s+t|/|t|\} \to \mathbb{R}$ is defined by

$$u_{a,b,c,s,t}(r) = \log \frac{\big|(1+r)(2+r)\big|^{2a+b} \big|(|s+t| + |t| r)\big|^{c}}{|r|^{2a+2b+c}},$$

and

$$-\tau_{a,b,c,s,t}(\log 2) = 2a + \beta(2+\log 2) + c(2+\log 2) - a\, f\Big(\frac{b}{a}, \frac{c}{a}\Big) + \max_{0 \leq x \leq 1} v_{a,b,c,s,t}(x),$$

where $v_{a,b,c,s,t} : \mathbb{R} \setminus \{0, 1, -1, s/t\} \to \mathbb{R}$ is defined by

$$v_{a,b,c,s,t}(x) = \log \frac{\big|x(1-x)\big|^{2a+b} |s-tx|^{c}}{|1+x|^{2(a+b)+c}}.$$

Let $x_0(a,b)$ denote the point in $[0,1]$ where the function involved in $\tau_{a,b,c}(\log 2)$ reaches its maximum. The pairs of integers (s,t) that we use are the numerator and the denominator of the convergents of the real number $x_0(a,b)$. The improvement of the analytic components are then clear for some pairs (s,t) but the results we get are worse than in the previous situation because of the very bad asymptotic behaviour of the content of the numbers C_{n,m,m',k_1,k_2}. To illustrate this, consider the class $U_{3an,an,cn,s,t}(X)$ with $a, c, n \in \mathbb{N}$ and $s, t \in \mathbb{Z}$. When $c = 0$, we recover the polynomials $P_{6n,n,n}$, from which we could establish an irrationality measure numerically equal to $3.8913997\ldots$. The value of $x_0(3,1)$ is

$$\frac{\sqrt{393} - 15}{12} = 0.4020189\ldots.$$

The pairs (s,t) used are the numerator and the denominator of the first convergents of $x_0(3,1)$. Tables 2–4 list data about irrationality measures obtained from the polynomials $U_{3an,an,cn,s,t}(X)$ using these pairs (s,t).

s	t	$c/a =$ 10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1	2	4.4945	3.9656	3.9008	3.8925	3.891536
2	5	4.5214	3.9556	3.8985	3.8922	3.891501
39	97	5.3634	4.0117	3.9037	3.8926	3.891534
80	199	5.6156	4.0258	3.9050	3.8928	3.891547
199	495	5.9720	4.0438	3.9067	3.8929	3.891563
478	1189	6.3601	4.0613	3.9083	3.8931	3.891756

TABLE 2. Values of $\mu_{3a,a,c,s,t}(\log 2)$, obtained using the polynomials $U_{3an,an,cn,s,t}(X)$.

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s	t	$c/a =$ 10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1	2	4.4030	4.2465	4.2291	4.2273	4.2271
2	5	4.4152	4.2582	4.2316	4.2278	4.2272
39	97	4.1157	4.2276	4.2283	4.2273	4.2271
80	199	4.0439	4.2204	4.2276	4.2273	4.2271
199	495	3.9527	4.0158	4.2267	4.2272	4.2271
478	1189	3.8651	4.2025	4.2258	4.2271	4.2271

TABLE 3. Absolute values of the analytic component of $\tau_{3a,a,c,s,t}(\log 2)$ (that is, $\max_{0\le x\le 1} v_{3a,a,c,s,t}(x)/a$). The sign of this expression is negative in all cases and has been omitted for brevity.

$c/a =$ 1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
0.2043	0.3251	0.3739	0.3839	0.3855	0.3857

TABLE 4. Arithmetic component of $\tau_{3a,a,c,s,t}(\log 2)$ (that is, $f\left(\frac{1}{3}, c/a\right)/a$).

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