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Discriminants, Symmetrized Graph Monomials, and Sums of Squares

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CONTENTS

In 1878, motivated by the requirements of the invariant theory of binary forms, J. J. Sylvester constructed, for every graph with possible multiple edges but without loops, its symmetrized graph monomial, which is a polynomial in the vertex labels of the original graph. We pose the question for which graphs this polynomial is nonnegative or a sum of squares. This problem is motivated by a recent conjecture of F. Sottile and E. Mukhin on the discriminant of the derivative of a univariate polynomial and by an interesting example of P. and A. Lax of a graph with four edges whose symmetrized graph monomial is nonnegative but not a sum of squares. We present detailed information about symmetrized graph monomials for graphs with four and six edges, obtained by computer calculations.

1. INTRODUCTION

In what follows, by a *graph* we will always mean a (directed or undirected) graph with (possibly) multiple edges but no loops. The classical construction of [Sylvester 78, Petersen 91] associates to an arbitrary directed loopless graph a symmetric polynomial as follows.

Definition 1.1. Let g be a directed graph with vertices $x_1, \dots, x_n$ and adjacency matrix (a_{ij}), where a_{ij} is the number of directed edges connecting x_i and x_j. Define its graph monomial P_g as

$$P_g(x_1, \dots, x_n) := \prod_{1 \le i,j \le n} (x_i - x_j)^{a_{ij}}.$$

The symmetrized graph monomial of g is defined as

$$\tilde{g}(\mathbf{x}) = \sum_{\sigma \in S_n} P_g(\sigma \mathbf{x}), \quad \mathbf{x} = x_1, \dots, x_n.$$

Observe that if the original g is undirected, one can still define $\tilde{g}$ up to a sign by choosing an arbitrary orientation of its edges. Symmetrized graph monomials are closely related to SL_2-invariants and covariants and were introduced in the 1870s in an attempt to find new tools

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in invariant theory. Namely, to obtain an SL_2 coinvariant from a given $\tilde{g}(\mathbf{x})$, we have to perform two standard operations. First, we express the symmetric polynomial $\tilde{g}(\mathbf{x})$ in n variables in terms of the elementary symmetric functions $e_1, \dots, e_n$ and obtain the polynomial $\hat{g}(e_1, \dots, e_n)$. Second, we perform the standard homogenization of a polynomial of a given degree d,

$$Q_g(a_0, a_1, \dots, a_n) := a_0^d \hat{g}\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right).$$

The following fundamental proposition apparently goes back to A. Cayley; see [Sabidussi 92a, Theorem 2.4].

Theorem 1.2.

(i) *If g is a d-regular graph with n vertices, then $Q_g(a_0, \dots, a_n)$ is either an SL_2 invariant of degree d in n variables, or it is identically zero.*

(ii) *Conversely, if $Q(a_0, \dots, a_n)$ is an SL_2 invariant of degree d and order n, then there exist d-regular graphs $g_1, \dots, g_r$ with n vertices and integers $\lambda_1, \dots, \lambda_r$ such that*

$$Q = \lambda_1 Q_{g_1} + \cdots + \lambda_r Q_{g_r}.$$

Remark 1.3. Recall that a graph is called *d-regular* if each of its vertices has valency d. Observe that if g is an arbitrary graph, then it is natural to interpret its polynomial $Q_g(a_0, \dots, a_n)$ as the SL_2 coinvariant.

The question about the kernel of the map sending g to $\tilde{g}(\mathbf{x})$ (or to Q_g) was discussed already by J. Petersen, who claimed that he had found a necessary and sufficient condition when g belongs to the kernel; see [Sabidussi 92a]. This claim turned out to be false. (An interesting correspondence among J. J. Sylvester, D. Hilbert, and F. Klein related to this topic can be found in [Sabidussi 92b].) The kernel of this map seems to be related to several open problems such as can be found in [Alon and Tarsi 92] and the Rota basis conjecture [Wild 94]. (We want to thank Professor A. Abdesselam for this valuable information; see [Abdesselam 11].)

In the present paper, we are interested in examples of graphs with a symmetrized graph monomial that is nonnegative or a sum of squares. Our interest in this matter has two sources.

The first is a recent conjecture of F. Sottile and E. Mukhin formulated at the AIM meeting "Algebraic Systems with Only Real Solutions" in October 2010. This conjecture is now settled; see [Sanyal et al. 12, Corollary 14].

Theorem 1.4. *The discriminant $\mathcal{D}_n$ of the derivative of a polynomial p of degree n is the sum of squares of polynomials in the differences of the roots of p.*

Based on our calculations and computer experiments, we propose the following extension and strengthening of Theorem 1.4. We call an arbitrary graph with all edges of even multiplicity a *square graph*. Observe that the symmetrized graph monomial of a square graph is obviously a sum of squares.

Conjecture 1.5. *For every nonnegative integer $0 \leq k \leq n-2$, the discriminant $\mathcal{D}_{n,k}$ of the kth derivative of a polynomial p of degree n is a finite positive linear combination of the symmetrized graph monomials, where all underlying graphs are square graphs with n vertices. The vertices $x_1, \dots, x_n$ are the roots of p. In other words, $\mathcal{D}_{n,k}$ lies in the convex cone spanned by the symmetrized graph monomials of the square graphs with n vertices and $\binom{n-k}{2}$ edges.*

Observe that $\deg \mathcal{D}_{n,k}$ is equal to $(n-k)(n-k-1)$ and is therefore even. The following examples support the above conjectures. Below we use the following convention. If a displayed graph has fewer than n vertices, then we always assume that it is appended by the required number of isolated vertices so that there are n vertices altogether.

Example 1.6. If $k = 0$, then $\mathcal{D}_{n,0}$ is proportional to $\tilde{g}$, where g is the complete graph on n vertices with all edges of multiplicity 2.

Example 1.7. For $k \geq 0$, the discriminant $\mathcal{D}_{k+2,k}$ equals

$$k!(k+1)! \sum_{1 \leq i < j \leq k+2} (x_i - x_j)^2.$$

In other words, $\mathcal{D}_{k+2,k} = \frac{(k+1)!}{2}\tilde{g}$, where the graph g is given in Figure 1 (appended with k isolated vertices).

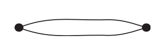

FIGURE 1. The graph g for the case $\mathcal{D}_{k+2,k}$.

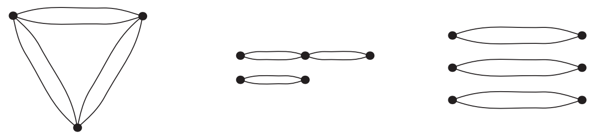

FIGURE 2. The graphs g_1, g_2, and g_3 for the case $\mathcal{D}_{k+3,k}$.

Example 1.8. For $k \geq 0$, we conjecture that the discriminant $\mathcal{D}_{k+3,k}$ equals

$$(k!)^3 \left[\frac{(k+1)^3(k+2)(k+6)}{72}\tilde{g}_1 + \frac{(k+1)^3 k(k+2)}{12}\tilde{g}_2 + \frac{(k-1)k(k+1)^2(k+2)(k-2)}{96}\tilde{g}_3 \right],$$

where the graphs g_1, g_2, and g_3 are given in Figure 2. (This claim has been verified for $k = 1, \dots, 12$.)

Example 1.9. The discriminant $\mathcal{D}_{5,1}$ is given by

$$\mathcal{D}_{5,1} = \frac{19}{6}\tilde{g}_1 + 14\tilde{g}_2 + 2\tilde{g}_3,$$

where g_1, g_2, g_3 are given in Figure 3.

Example 1.10. Finally,

$$\mathcal{D}_{6,2} = 19200\tilde{g}_1 + 960\tilde{g}_2 + 3480\tilde{g}_3 + 3240\tilde{g}_4 + \frac{3440}{3}\tilde{g}_5 + 2440\tilde{g}_6,$$

where $g_1, \dots, g_6$ are given in Figure 4. (Note that this representation as a sum of graphs is not unique.)

It is classically known that for any given number n of vertices and d edges, the linear span of the symmetrized graph monomials coming from all graphs with n vertices and d edges coincides with the linear space $\mathrm{PST}_{n,d}$ of all symmetric translation-invariant polynomials of degree d in n variables.

We say that a pair (n,d) is *stable* if $n \geq 2d$. For stable (n,d), we suggest a natural basis in $\mathrm{PST}_{n,d}$ of symmetrized graph monomials that seems to be new; see Proposition 2.10 and Corollary 2.11.

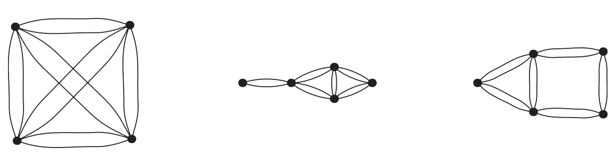

FIGURE 3. The graphs g_1, g_2, and g_3 for the case $\mathcal{D}_{5,1}$.

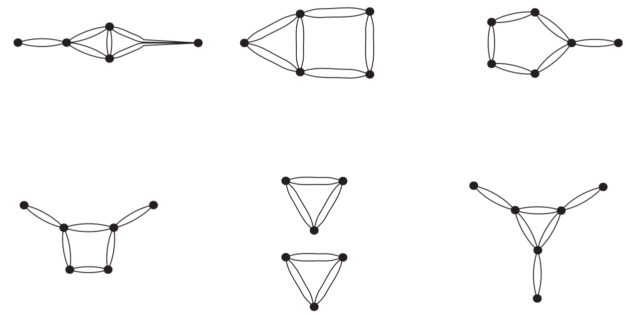

FIGURE 4. The graphs $g_1, \dots, g_6$ for the case $\mathcal{D}_{6,2}$.

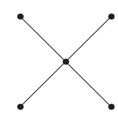

FIGURE 5. The Lax graph, i.e., the only four-edged graph that yields a nonnegative polynomial that is not SOS.

In the case of even degree, there is a second basis in $\mathrm{PST}_{n,d}$ of symmetrized graph monomials consisting of only square graphs; see Proposition 2.14 and Corollary 2.15.

The second motivation of the present study is an interesting example of a graph whose symmetrized graph monomial is nonnegative but not a sum of squares. Namely, the main result of [Lax and Lax 78] shows that $\tilde{g}$ for the graph given in Figure 5 has this property.

Finally, let us present our main computer-aided results regarding the case of graphs with four and six edges. Observe that there exist 23 graphs with four edges and 212 graphs with six edges. We say that two graphs are *equivalent* if their symmetrized graph monomials are nonvanishing identically and proportional. Note that two graphs do not need to be isomorphic to be equivalent; see, for example, the equivalence classes in Figure 6.

Proposition 1.11.

(i) *Ten graphs with four edges have identically vanishing symmetrized graph monomial.*

(ii) *The remaining* 13 *graphs are divided into four equivalence classes presented in Figure 6.*

(iii) *The first two classes contain square graphs, and thus their symmetrized monomials are nonnegative.*

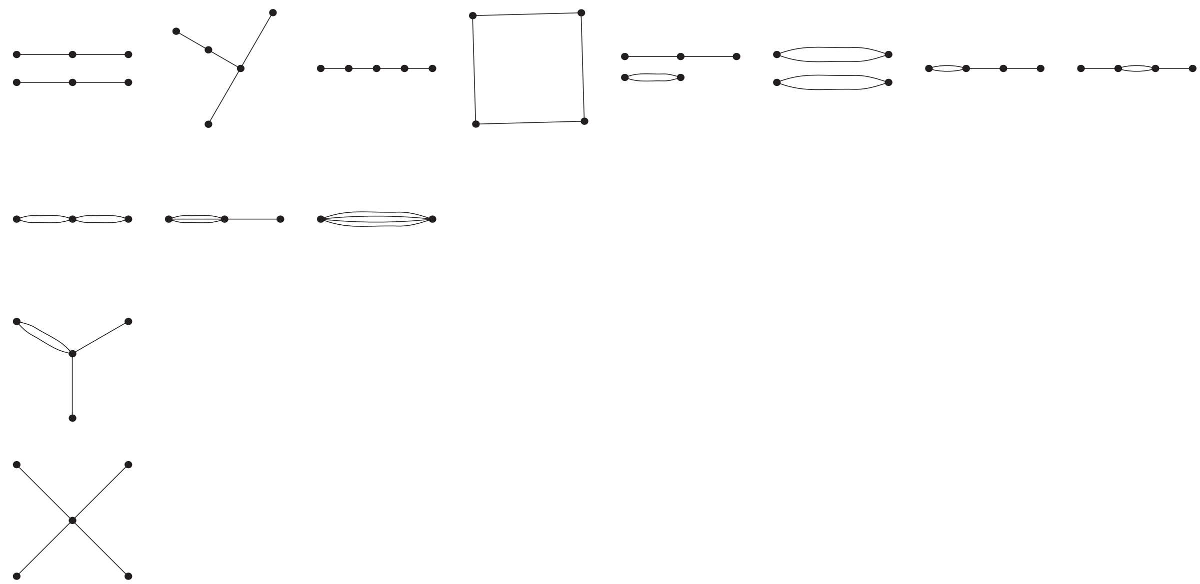

FIGURE 6. Four equivalence classes of the 13 graphs with four edges whose symmetrized graph monomials do not vanish identically.

(iv) *The third graph is nonnegative (as a positive linear combination of the Lax graph and a polynomial obtained from a square graph). Since it effectively depends only on three variables, it is SOS; see [Hilbert 93].*

(v) *The last graph is the Lax graph, which is thus the only nonnegative graph with four edges not being an SOS.*

Proposition 1.12.

(i) 102 *graphs with* 6 *edges have identically vanishing symmetrized graph monomial.*

(ii) *The remaining* 110 *graphs are divided into* 27 *equivalence classes.*

(iii) 12 *of these classes can be expressed as nonnegative linear combinations of square graphs, i.e., they lie in the convex cone spanned by the square graphs.*

(iv) *Of the remaining* 15 *classes, the symmetrized graph monomial of* 7 *of them changes sign.*

(v) *Of the remaining eight classes (which are presented in Figure 7) the first five are sums of squares. (Observe, however, that these symmetrized graph monomials do not lie in the convex cone spanned by the square graphs.)*

(vi) *The last three classes contain all nonnegative graphs with six edges, which are not SOS and therefore, give new examples of graphs à la Lax.*

Proving Proposition 1.11 is simply a matter of straightforward computation. Cases (i)–(iv) in Proposition 1.12 also follow from a longer calculation, by examining each of the 212 graphs. Proof of case (v) requires the notion of certificates.

It is well known that a polynomial is a sum of squares if and only if it can be represented as vQv^T, where Q is positive semidefinite and v a monomial vector. Such a representation is called a *certificate*. Certificates for the eight classes in Proposition 1.12, case (v), in the form of positive semidefinite matrices and corresponding monomial vectors are too large to be presented here and can be found in [Alexandersson 12]. The simplest certificate, for the third class, is given by the vector

$$\begin{aligned} v_3 = \{ & x_3x_4^2, x_3^2x_4, x_2x_4^2, x_2x_3x_4, x_2x_3^2, x_2^2x_4, x_2^2x_3, x_1x_4^2, \\ & x_1x_3x_4, x_1x_3^2, x_1x_2x_4, x_1x_2x_3, x_1x_2^2, x_1^2x_4, x_1^2x_3, \\ & x_1^2x_2 \} \end{aligned}$$

together with the positive semidefinite matrix Q_3, shown as Figure 8.

Case (vi) was analyzed with the Yalmip software, which provides a second kind of certificate that shows that the last three classes are not SOS.

Finally, observe that translation-invariant symmetric polynomials appeared also in the early 1970s in the study of integrable N-body problems in mathematical physics, particularly in the famous paper [Calogero 71]. A few much more recent papers related to the ring of such polynomials in connection with the investigation of

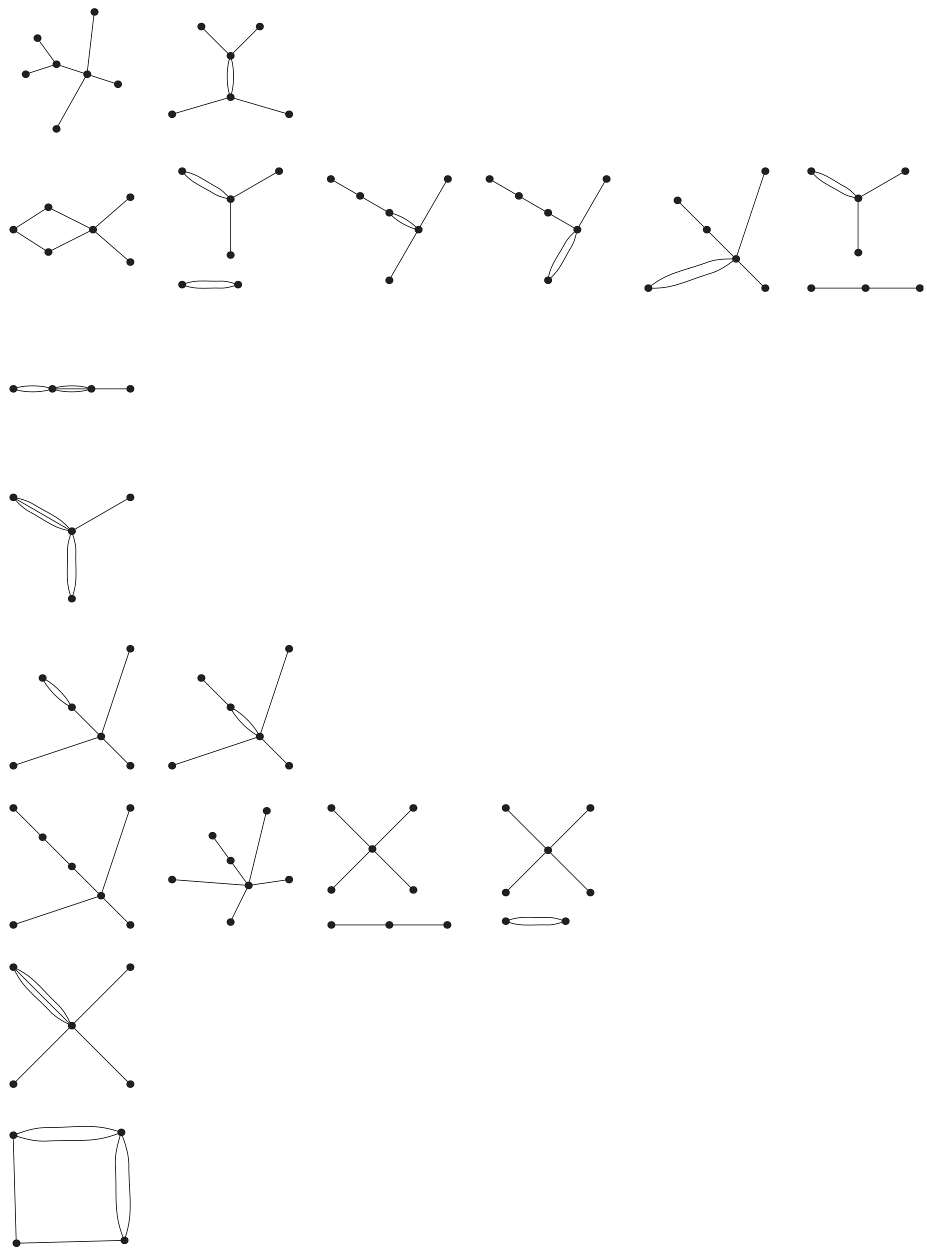

FIGURE 7. Eight equivalence classes of all nonnegative graphs with six edges.

multiparticle interactions and the quantum Hall effect have been published since then; see, e.g., [Simon et al. 12, Liptrap 10]. In particular, the ring structure and the dimensions of the homogeneous components of this ring were calculated. It was also shown [Simon et al. 12, Section IV] and [Liptrap 10] that the ring of translation-invariant symmetric polynomials (with integer coefficients) in $x_1, \dots, x_n$ is isomorphic as a graded ring to the polynomial ring $\mathbb{Z}[e_2, \dots, e_n]$, where e_i stands for the ith elementary symmetric function in $x_1 - x_{\text{avg}}, \dots, x_n - x_{\text{avg}}$ with $x_{\text{avg}} = \frac{1}{n}(x_1 + \cdots + x_n)$.

From this fact one can easily show that the dimension of its dth homogeneous component equals the number of distinct partitions of d in which each part is strictly bigger than 1 and the number of parts is at most n. Several natural linear bases have also been suggested for each such homogeneous component; see [Simon et al. 12, (29)] and [Liptrap 10]. It seems that the authors of the latter

$$
Q_3 = \begin{pmatrix}
10 & -6 & -5 & -4 & 3 & 3 & -1 & -5 & -4 & 3 & 8 & 0 & -2 & 3 & -1 & -2 \\
-6 & 10 & 3 & -4 & -5 & -1 & 3 & 3 & -4 & -5 & 0 & 8 & -2 & -1 & 3 & -2 \\
-5 & 3 & 10 & -4 & -1 & -6 & 3 & -5 & 8 & -2 & -4 & 0 & 3 & 3 & -2 & -1 \\
-4 & -4 & -4 & 24 & -4 & -4 & -4 & 8 & -8 & 8 & -8 & -8 & 8 & 0 & 0 & 0 \\
3 & -5 & -1 & -4 & 10 & 3 & -6 & -2 & 8 & -5 & 0 & -4 & 3 & -2 & 3 & -1 \\
3 & -1 & -6 & -4 & 3 & 10 & -5 & 3 & 0 & -2 & -4 & 8 & -5 & -1 & -2 & 3 \\
-1 & 3 & 3 & -4 & -6 & -5 & 10 & -2 & 0 & 3 & 8 & -4 & -5 & -2 & -1 & 3 \\
-5 & 3 & -5 & 8 & -2 & 3 & -2 & 10 & -4 & -1 & -4 & 0 & -1 & -6 & 3 & 3 \\
-4 & -4 & 8 & -8 & 8 & 0 & 0 & -4 & 24 & -4 & -8 & -8 & 0 & -4 & -4 & 8 \\
3 & -5 & -2 & 8 & -5 & -2 & 3 & -1 & -4 & 10 & 0 & -4 & -1 & 3 & -6 & 3 \\
8 & 0 & -4 & -8 & 0 & -4 & 8 & -4 & -8 & 0 & 24 & -8 & -4 & -4 & 8 & -4 \\
0 & 8 & 0 & -8 & -4 & 8 & -4 & 0 & -8 & -4 & -8 & 24 & -4 & 8 & -4 & -4 \\
-2 & -2 & 3 & 8 & 3 & -5 & -5 & -1 & 0 & -1 & -4 & -4 & 10 & 3 & 3 & -6 \\
3 & -1 & 3 & 0 & -2 & -1 & -2 & -6 & -4 & 3 & -4 & 8 & 3 & 10 & -5 & -5 \\
-1 & 3 & -2 & 0 & 3 & -2 & -1 & 3 & -4 & -6 & 8 & -4 & 3 & -5 & 10 & -5 \\
-2 & -2 & -1 & 0 & -1 & 3 & 3 & 3 & 8 & 3 & -4 & -4 & -6 & -5 & -5 & 10
\end{pmatrix}
$$

FIGURE 8. The positive semidefinite matrix Q_3.

papers were unaware of the mathematical developments in this field related to graphs.

2. SOME GENERALITIES ON SYMMETRIZED GRAPH MONOMIALS

We begin with a few definitions.

Definition 2.1. An *integer partition* of d is a d-tuple $(\alpha_1, \dots, \alpha_d)$ such that $\sum_i \alpha_i = d$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d \geq 0$.

Definition 2.2. Let g be a directed graph with d edges and n vertices $v_1, v_2, \dots, v_n$. Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be an integer partition of d. A *partition-coloring* of g with α is an assignment of colors to the edges and vertices of g satisfying the following conditions:

- For each color i, $1 \leq i \leq d$, we paint with the color i some vertex v_j and exactly α_i edges connected to v_j.
- Each edge of g is colored exactly once.

An edge is called *odd-colored* if it is directed toward a vertex with the same color. The coloring is said to be *negative* if there is an odd number of odd-colored edges in g, and *positive* otherwise.

Definition 2.3. Given a polynomial $P(\mathbf{x})$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we use the notation $\mathrm{Coeff}_\alpha(P(\mathbf{x}))$ to denote the coefficient in front of $\mathbf{x}^\alpha$ in $P(\mathbf{x})$.

Note that we may view α as a partition of the sum of the exponents.

Lemma 2.4. *Let g be a directed graph with d edges and vertices $v_1, v_2, \dots, v_n$. Then $\mathrm{Coeff}_\alpha(\tilde{g})$ is given by the difference of the numbers of positive and negative partition-colorings of g with α.*

Proof. See [Sabidussi 92a, Lemma 2.3]. $\square$

2.1. Bases for $\mathrm{PST}_{n,d}$

It is known that the dimension of $\mathrm{PST}_{n,d}$ with $n \geq 2d$ is given by the number of integer partitions of d in which each nonzero part is of size at least 2; see [Liptrap 10]. Such an integer partition will be called a 2-*partition*.

To each 2-partition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\alpha_i \neq 1$, we associate the graph b_α defined as follows. For each $\alpha_i \geq 2$, we have a connected component of b_α consisting of a root vertex, connected to α_i other vertices, with the edges directed away from the root vertex. Since α is an integer partition of d, it follows that b_α has exactly d edges. This type of graph will be called a *partition graph*.

The dimension of $\mathrm{PST}_{n,d}$ is independent of n (as long as $n \geq 2d$), and we deal only with homogeneous symmetric polynomials of degree d. Thus, each monomial is essentially determined only by the way the powers of the variables are partitioned. The variables themselves become unimportant, since every permutation of the variables is present. For example, the monomials x^3zw and xy^3w are always present simultaneously with the same coefficient, while x^3z^2 is different from the previous two.

2.2. Partition Graphs

Definition 2.5. Let $P(\mathbf{x})$ be a polynomial in $|\mathbf{x}|$ variables. We use the notation

$$\operatorname{Sym}_{(\mathbf{x}\cup\mathbf{y})} P = \sum_{(\tau_1,\tau_2,\ldots,\tau_n)\subseteq\mathbf{x}\cup\mathbf{y}} P(\tau_1,\tau_2,\ldots,\tau_n),$$

where we sum over all possible *permutations* and choices of n variables among the $|\mathbf{x}|+|\mathbf{y}|$ variables.

The following lemma is obviously true.

Lemma 2.6. *Let $P(\mathbf{x})$ be a polynomial. Then*

$$\operatorname{Sym}_{(\mathbf{x}\cup\mathbf{y})} P = \sum_{i=0}^{|\mathbf{y}|} \sum_{\substack{\sigma\subseteq\mathbf{y}\\|\sigma|=i}} \sum_{\substack{\tau\subseteq\mathbf{x}\\|\tau|=|\mathbf{x}|-i}} \operatorname{Sym}_{(\tau\cup\sigma)} P.$$

Here, the two inner sums denote choices of all subsets *of a certain size.*

Corollary 2.7. *If* $\operatorname{Sym}_{\mathbf{x}} P$ *is nonnegative, then* $\operatorname{Sym}_{(\mathbf{x}\cup\mathbf{y})} P$ *is nonnegative.*

Corollary 2.8. *If* $\operatorname{Sym}_{\mathbf{x}} P$ *is a sum of squares, then* $\operatorname{Sym}_{(\mathbf{x}\cup\mathbf{y})} P$ *is a sum of squares.*

Corollary 2.9. *If* $\sum_i \lambda_i \operatorname{Sym}_{\mathbf{x}} P_i = 0$*, then we have* $\sum_i \lambda_i \operatorname{Sym}_{(\mathbf{x}\cup\mathbf{y})} P_i = 0$.

We will use the notation that every symmetric polynomial $\tilde{g}$ associated with a graph on d edges is symmetrized over $2d$ variables. Corollary 2.9 says that if a relation holds for the symmetrizations in $2d$ variables, it will also hold for $2d+k$ variables ($k \geq 0$). Therefore, each relation derived in this section also holds for $2d+k$ variables.

Proposition 2.10. *Let b_α be a partition graph with d edges, $\alpha=(\alpha_1,\ldots,\alpha_d)$, and let $\beta=(\beta_1,\beta_2,\ldots,\beta_d)$ be a 2-partition.*
Then

$$\operatorname{Coeff}_\beta(\tilde{b_\alpha}) = \begin{cases} 0 & \text{if } \beta\neq\alpha,\\ \prod_{j=0}^{d}(\#\{i|\alpha_i=j\})! & \text{if } \beta=\alpha. \end{cases}$$

Proof. We will try to color the graph b_α with β. Since $\beta_i \neq 1$, we may only color the roots of b_α. Hence, all edges in each component of b_α must have the same color as the corresponding root. It is clear that such a coloring is impossible if $\alpha\neq\beta$. If $\alpha=\beta$, we see that each coloring has positive sign, since only roots are colored and all connected edges are directed outward.

The only difference between two colorings must be the assignment of the colors to the roots. Hence, components with the same size can permute colors, which yields

$$\prod_{j=0}^{d}(\#\{i|\alpha_i=j\})!$$

ways to color g with the partition $(\alpha_1,\ldots,\alpha_d)$. □

Corollary 2.11. *All partition graphs yield linearly independent polynomials, since each partition graph b_α contributes the unique monomial $\mathbf{x}^\alpha$. The number of partition graphs on d edges equals the dimension of $\mathrm{PST}_{d,n}$, and therefore, when $n\geq 2d$, they must span the entire vector space.*

2.3. Square Graphs

We will use the notation $\alpha=(\alpha_1,\ldots,\alpha_k \mid \alpha_{k+1},\ldots,\alpha_d)$ to denote a partition in which $\alpha_1,\ldots,\alpha_q$ are the odd parts in nonincreasing order, and $\alpha_{q+1},\ldots,\alpha_p$ are the even parts in nonincreasing order. (Note that this convention differs from the standard one for partitions.) As before, parts are allowed to be equal to zero, so that α can be used as a multi-index over d variables.

Now we define a second type of graph, which we associate with 2-partitions of even integers: Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_k \mid \alpha_{k+1},\ldots,\alpha_d)$, $\alpha_i\neq 1$, be a 2-partition of d. Since this is a partition of an even integer, k must be even.

For each even $\alpha_i\geq 2$, we have a connected component of h_α consisting of a root, connected to $\alpha_i/2$ other vertices, with the edges directed away from the root, and with multiplicity 2.

For each pair $\alpha_{2j-1},\alpha_{2j}$ of odd parts, $j=1,2,\ldots,k/2$, we have a connected component consisting of two roots v_{2j-1} and v_{2j} such that v_i is connected to $\lfloor\alpha_i/2\rfloor$ other vertices for $i=2j-1,2j$ with edges of multiplicity 2 and the roots are connected with a double edge. This type of component will be called a *glued component*.

Thus, each edge in h_α has multiplicity 2, and the number of edges, counting multiplicity, is d. This type of multigraph will be called a *partition square graph*. Note that all edges have even multiplicity, so $\tilde{h}_\alpha(\mathbf{x})$ is a sum of squares.

Lemma 2.12. *Let h_α be a partition square graph such that $\alpha=(\alpha_1,\ldots,\alpha_d)$. Then*

$$\operatorname{Coeff}_\alpha(\tilde{h}_\alpha) = (-1)^{\frac{1}{2}\#\{i|\alpha_i\equiv_2 1\}} 2^{\#\{i|\alpha_i=2\}} \prod_{j=0}^{n}(\#\{i \mid \alpha_i=j\})!.$$

Proof. Similarly to Proposition 2.10, it is clear that a coloring of h with p colors requires that each root be colored.

The root of a component with only two vertices is not uniquely determined, so we have $2^{\#\{i|\alpha_i=2\}}$ choices of the root.

It is clear that each glued component contributes exactly one odd edge for every coloring, and therefore the sign is the same for each coloring. The number of glued components is precisely $\frac{1}{2}\#\{i \mid \alpha_i \equiv_2 1\}$.

Lastly, we may permute the colors corresponding to the roots of the same degree. These observations together yield the formula

$$(-1)^{\frac{1}{2}\#\{i|\alpha_i\equiv_2 1\}} 2^{\#\{i|\alpha_i=2\}} \prod_{j=0}^{n} \#\{i \mid \alpha_i = j\}!,$$

which completes the proof. □

Define a total order on 2-partitions as follows:

Definition 2.13. Let $\alpha = (\alpha_1, \ldots, \alpha_k \mid \alpha_{k+1}, \ldots, \alpha_d)$ and $\alpha' = (\alpha'_1, \ldots, \alpha'_{k'} \mid \alpha'_{k'+1}, \ldots, \alpha_d)$ be 2-partitions. We say that $\alpha \prec \alpha'$ if $\alpha_i = \alpha'_i$ for $i = 1, \ldots, j-1$, $j \geq 1$ and one of the following holds:

- $\alpha_j > \alpha'_j$ and $\alpha_j \equiv \alpha'_j \mod 2$;
- α_j is odd and α'_j is even.

Proposition 2.14. *Let h_α be a square graph. Then we may write*

$$\tilde{h}_\alpha = \sum_\beta \lambda_\beta \tilde{b}_\beta, \quad b_\beta \text{ is a partition graph}, \qquad (2\text{–}1)$$

where $\lambda_\beta = 0$ if $\beta \prec \alpha$.

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_q \mid \alpha_{q+1} \ldots, \alpha_d)$ and let $\beta = (\beta_1, \ldots, \beta_r \mid \beta_{r+1}, \ldots, \beta_d)$, with $\beta \prec \alpha$. Consider equation (2–1) and apply Coeff_β to both sides. Lemma 2.10 implies

$$\mathrm{Coeff}_\beta(\tilde{h}_\alpha) = \lambda_\beta \cdot C_\beta, \quad \text{where } C_\beta > 0.$$

It suffices to show that there is no partition-coloring of h_α with β if $\beta \prec \alpha$, since this implies $\lambda_\beta = 0$.

We now have three cases to consider:

Cases 1 and 2: $\alpha_i = \beta_i$ for $i = 1, \ldots, j-1$ and $\beta_j > \alpha_j$, where α_j and β_j are either both odd or both even. We must color a root and β_j connected edges, since $\beta_j > \alpha_j \geq 2$. There is no vacant root in h_α with degree at least β_j, all such roots having already been colored with the colors $1, \ldots, j-1$. Hence a coloring is impossible in this case.

Case 3: $\alpha_i = \beta_i$ for $i = 1, \ldots, j-1$, β_j is odd and α_j is even. This condition implies that $q < r$.

Every component of h_α has an even number of edges, and only vertices of degree at least three can be colored with an odd color. Therefore, glued components must be colored with exactly zero or two odd colors, and nonglued components must have an even number of edges of each present color. This implies that a coloring is possible only if $r \leq q$, a contradiction.

Hence, there is no coloring of h_α with the colors given by β, and therefore $\mathrm{Coeff}_\beta(\tilde{h}_\alpha) = 0$, implying $\lambda_\beta = 0$. □

Corollary 2.15. *The polynomials obtained from the partition square graphs with d edges form a basis for $\mathrm{PST}_{d,n}$, for even d.*

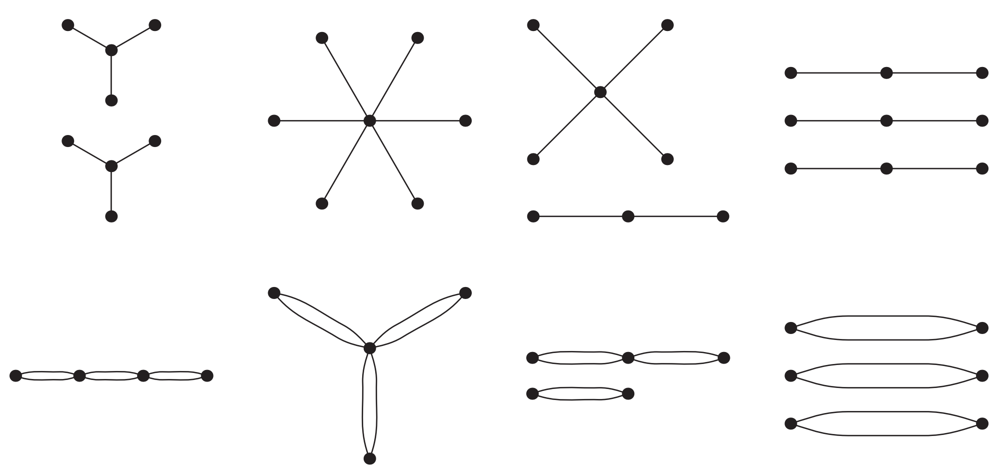

FIGURE 9. A base of partition graphs and a base of partition square graphs in the stable case with six edges.

Proof. Let $\alpha_1 \prec \cdots \prec \alpha_k$ be the 2-partitions of d. Since $\tilde{b}_{\alpha_1}, \ldots, \tilde{b}_{\alpha_k}$ is a basis, there is a uniquely determined matrix M such that

$$\left(\tilde{h}_{\alpha_1}, \ldots, \tilde{h}_{\alpha_k}\right)^T = M\left(\tilde{b}_{\alpha_1}, \ldots, \tilde{b}_{\alpha_k}\right)^T.$$

Lemma 2.14 implies that M is lower-triangular. Proposition 2.10 and Lemma 2.12 imply that the entry at (α_i, α_i) in M is given by

$$(-1)^{\frac{1}{2}\#\{j|\alpha_{ij}\equiv_2 1\}}2^{\#\{j|\alpha_{ij}=2\}},$$

which is nonzero. Hence M has an inverse, and the square graphs form a basis. See Figure 9 for an example of the two sets of bases for the case $d = 6$. □

3. FINAL REMARKS

Some obvious challenges related to this project are as follows.

1. Prove Conjecture 1.5.
2. Describe the boundary of the convex cone spanned by all square graphs with a given number of (double) edges and vertices.
3. Find more examples of graphs à la Lax.

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