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The Secant Conjecture in the Real Schubert Calculus

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CONTENTS

We formulate the secant conjecture, which is a generalization of the Shapiro conjecture for Grassmannians. It asserts that an intersection of Schubert varieties in a Grassmannian is transverse with all points real if the flags defining the Schubert varieties are secant along disjoint intervals of a rational normal curve. We present theoretical evidence for this conjecture as well as computational evidence obtained in over one terahertz-year of computing, and we discuss some of the phenomena we observed in our data.

1. INTRODUCTION

Some solutions to a system of real polynomial equations are real, and the rest occur in complex conjugate pairs. While the total number of solutions is determined by the structure of the equations, the number of real solutions depends rather subtly on the coefficients. Sometimes, there is finer information available in terms of upper bounds [Khovanskii 91, Bates et al. 07] or lower bounds [Eremenko and Gabrielov 02a, Soprunova and Sottile 06] on the number of real solutions. The Shapiro and secant conjectures assert the extreme situation of having only real solutions.

The Shapiro conjecture for Grassmannians posits that if the Wronskian of a vector space of univariate complex polynomials has only real roots, then that space is spanned by real polynomials. This striking instance of unexpected reality was proven in [Eremenko and Gabrielov 02b, Eremenko and Gabrielov 11] for two-dimensional spaces of polynomials, and the general case was established in [Mukhin et al. 09a, Mukhin et al. 09c]. While the statement concerns spaces of polynomials, or more generally the Schubert calculus on Grassmannians, its proofs use complex analysis [Eremenko and Gabrielov 02b, Eremenko and Gabrielov 11] and mathematical physics [Mukhin et al. 09a, Mukhin et al. 09c]. This story was described in [Sottile 10].

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The Shapiro conjecture first gained attention through partial results and computations [Sottile 00a, Verschelde 00], and further work [Sottile 00b] led to an extension that appears to hold for flag manifolds: the monotone conjecture. This extension was made in [Ruffo et al. 06], which also reported on partial results and experimental evidence. The monotone conjecture for a certain family of two-step flag manifolds was proved in [Eremenko et al. 06].

The result of [Eremenko et al. 06] was in fact a proof of reality in the Grassmannian of codimension-two planes for intersections of Schubert varieties defined with respect to certain disjoint secant flags. The *secant conjecture* postulates an extension of this result to all Grassmannians. We give the simplest open instance of the secant conjecture. Let $x_1, \ldots, x_6$ be indeterminates and consider the polynomial

$$f(s,t,u;x) := \det \begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ 1 & s & s^2 & s^3 & s^4 \\ 1 & t & t^2 & t^3 & t^4 \\ 1 & u & u^2 & u^3 & u^4 \end{pmatrix}, \quad (1\text{–}1)$$

which depends on parameters s, t, and u.

Conjecture 1.1. *Let $s_1 < t_1 < u_1 < s_2 < t_2 < \cdots < u_5 < s_6 < t_6 < u_6$ be real numbers. Then the system of polynomial equations*

$$f(s_i, t_i, u_i; x) = 0, \quad i = 1, \ldots, 6,$$

has five distinct solutions, and all of them are real.

Geometrically, the equation $f(s,t,u;x) = 0$ says that the 2-plane (spanned by the first two rows of the matrix in (1–1)) meets the 3-plane that is secant to the rational curve $\gamma : y \mapsto (1, y, y^2, y^3, y^4)$ at the points $\gamma(s), \gamma(t), \gamma(u)$. The hypotheses imply that each of the six 3-planes is secant to γ along an interval $[s_i, u_i]$, and these six intervals are disjoint. The conjecture asserts that all of the 2-planes meeting six 3-planes are real when the 3-planes are secant to the rational normal curve along disjoint intervals. This statement was true in each of the 285,502 instances we tested.

The purpose of this paper is to explain the secant conjecture and its relation to the other reality conjectures, to describe the data supporting it from a large computational experiment, and to highlight some other features in our data beyond the secant conjecture. These data may be viewed online.[1] We will assume some background on the Shapiro conjecture as described in the survey [Sottile 10] and paper [Ruffo et al. 06], and we will not describe the execution of the experiment, since the methods paper [Hillar et al. 10] presented the software framework we have developed for such distributed computational experiments.

This paper is organized as follows. In Section 2, we present the full secant conjecture, giving a history of its formulation. Section 3 presents some theoretical justification for the secant conjecture as well as a generalization based on limiting cases. In Section 4 we analyze the problem of lines meeting all possible configurations of four secant lines, giving conditions on the secant lines that imply that both solutions are real. Section 5 describes a statistic, the *overlap number*, that measures the extent of overlap among intervals of secancy. In Section 6 we explain the data from our experiment. About three-fourths of our over two billion computations did not directly test the secant conjecture, but rather tested geometric configurations that were close to those of the conjecture. Consequently, our data contain much more information than that in support of the secant conjecture, and we explore that information in the remaining sections. Section 7 discusses the lower bounds on the numbers of real solutions we typically observed for small overlap number, producing a striking *inner border* in the tabulation of our data. Finally, in Section 8, we discuss Schubert problems with provable lower bounds and gaps in their numbers of real solutions, a phenomenon we first noticed while trying to understand our data.

2. SCHUBERT CALCULUS AND THE SECANT CONJECTURE

In this section, we give background on the Schubert calculus necessary to state the secant conjecture, and then we state the equivalent dual cosecant conjecture.

2.1. Schubert Calculus

The Schubert calculus [Fulton 97, Fulton and Pragacz 98] involves problems of determining the linear spaces that have specified positions with respect to other, fixed (flags of), linear spaces. For example, what are the 3-planes in $\mathbb{C}^7$ meeting twelve given 4-planes nontrivially? (There are 462 [Schubert 86].) The specified positions are a

[1] Available at http://www.math.tamu.edu/~secant/secant/flagview.php.

Schubert problem, which determines the number of solutions. The actual solutions depend upon the linear spaces imposing the conditions, or *instance*, of the Schubert problem.

The *Grassmannian* $G(k,n)$ is the set of all k-dimensional linear subspaces of $\mathbb{C}^n$, which is an algebraic manifold of dimension $k(n-k)$. A *flag* $F_\bullet$ is a sequence of linear subspaces

$$F_\bullet \;:\; F_1 \subset F_2 \subset \cdots \subset F_n,$$

where $\dim F_i = i$. A *partition* $\lambda : (n-k) \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$ is a weakly decreasing sequence of integers. A fixed flag $F_\bullet$ and a partition λ define a *Schubert variety* $X_\lambda F_\bullet$,

$$X_\lambda F_\bullet := \{H \in G(k,n) \mid \dim H \cap F_{n-k+i-\lambda_i} \geq i \\ \text{for } i = 1, \ldots, k\},$$

which is a subvariety of codimension $|\lambda| := \lambda_1 + \cdots + \lambda_k$. Not every element of the flag is needed to define the Schubert variety.

A *Schubert problem* is a list $\lambda^1, \ldots, \lambda^m$ of partitions with $|\lambda^1| + \cdots + |\lambda^m| = k(n-k)$. For sufficiently general flags $F_\bullet^1, \ldots, F_\bullet^m$, the intersection

$$X_{\lambda^1} F_\bullet^1 \cap X_{\lambda^2} F_\bullet^2 \cap \cdots \cap X_{\lambda^m} F_\bullet^m$$

is transverse [Kleiman 74] and consists of a certain number, $d(\lambda^1, \ldots, \lambda^m)$, of points, which may be computed using algorithms in the Schubert calculus (see [Fulton 97,Kleiman and Laksov 72]). (*Transverse* means that at each point of the intersection, the annihilators of the tangent spaces to the Schubert varieties are in direct sum.)

We write a Schubert problem multiplicatively, $\lambda^1 \cdots \lambda^m = d(\lambda^1, \ldots, \lambda^m)$. For example, writing $\square$ for the partition $(1,0)$ with $|\square| = 1$, we have $\square \cdot \square \cdot \square \cdot \square \cdot \square \cdot \square = \square^6 = 5$ for the Schubert problem on $G(2,5)$ involving six partitions, each equal to $\square$. In this notation, Schubert's problem that we mentioned above is $\square^{12} = 462$ on $G(3,7)$.

A rational normal curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ is affinely equivalent to the *moment curve*

$$\gamma : t \;\longmapsto\; (1, t, t^2, \ldots, t^{n-1}).$$

The *osculating flag* $F_\bullet(t)$ has i-dimensional subspace the span of the first i derivatives $\gamma(t), \gamma'(t), \ldots, \gamma^{(i-1)}(t)$ of γ at t. We state a theorem of [Mukhin et al. 09a, Mukhin et al. 09c].

Theorem 2.1. (The Shapiro conjecture.) *For every Schubert problem $\lambda^1, \ldots, \lambda^m$ on a Grassmannian $G(k,n)$ and all distinct real numbers $t_1, \ldots, t_m$, the intersection*

$$X_{\lambda^1} F_\bullet(t_1) \cap X_{\lambda^2} F_\bullet(t_2) \cap \cdots \cap X_{\lambda^m} F_\bullet(t_m)$$

is transverse and consists of $d(\lambda^1, \ldots, \lambda^m)$ real points.

Transversality is unexpected, since osculating flags are not general.

The Shapiro conjecture concerns intersections of Schubert varieties given by flags osculating a rational normal curve, and in this form it makes sense for every flag manifold G/P. It was shown in [Purbhoo 10] that it holds for the orthogonal Grassmannians, but counterexamples are known for other flag manifolds. There is an appealing version of it—the monotone conjecture—that appears to hold for the classical flag variety [Ruffo et al. 06].

2.2. The Secant Conjecture

In [Eremenko et al. 06], the authors proved a generalization of the monotone conjecture for flags consisting of a codimension-two plane lying in a hyperplane, where it becomes a statement about real rational functions. Their theorem asserts that a Schubert problem on $G(n-2,n)$ has only real solutions if the flags satisfy a special property that we now describe. A flag $F_\bullet$ of linear subspaces is *secant along an interval* I of a rational normal curve γ if every subspace in the flag is spanned by its intersection with I. This means that there are distinct points $t_1, \ldots, t_{n-1} \in I$ such that for each $i = 1, \ldots, n-1$, the subspace F_i of the flag $F_\bullet$ is spanned by $\gamma(t_1), \ldots, \gamma(t_i)$.

Conjecture 2.2. (Secant conjecture.) *For every Schubert problem $\lambda^1, \ldots, \lambda^m$ on a Grassmannian $G(k,n)$ and all flags $F_\bullet^1, \ldots, F_\bullet^m$ that are secant to a rational normal curve γ along disjoint intervals, the intersection*

$$X_{\lambda^1} F_\bullet^1 \cap X_{\lambda^2} F_\bullet^2 \cap \cdots \cap X_{\lambda^m} F_\bullet^m$$

is transverse and consists of $d(\lambda^1, \ldots, \lambda^m)$ real points.

Conjecture 1.1 is the case of this secant conjecture for the Schubert problem $\square^6 = 5$ on $G(2,5)$. The Schubert variety $X_\square F_\bullet$ is

$$X_\square F_\bullet = \{H \in G(2,5) \mid \dim H \cap F_3 \geq 1\},$$

that is, the set of 2-planes meeting a fixed 3-plane nontrivially. Since F_4 and F_5 are irrelevant, we drop them from the flag and refer to F_3 and $X_\square F_3$. For every Schubert condition, there is a largest element of the flag imposing a relevant condition; call this the *relevant subspace*. The relevant subspace in this example is F_3.

For $s, t, u \in \mathbb{R}$, let $F_3(s,t,u)$ be the linear span of $\gamma(s), \gamma(t), \gamma(u)$, a 3-plane secant to γ with points $\gamma(s), \gamma(t), \gamma(u)$ of secancy. Thus, the condition $f(s,t,u;x) = 0$ of Conjecture 1.1 implies that the linear span H of the first two rows of the matrix in (1–1)—a general 2-plane in 5-space—meets the linear span $F_3(s,t,u)$ of the last three rows. Thus

$$f(s,t,u;x) = 0 \Longleftrightarrow H \in X_{\square} F_3(s,t,u).$$

Lastly, the condition on the ordering of the points s_i, t_i, u_i in Conjecture 1.1 implies that the six flags $F_3(s_i, t_i, u_i)$ are secant along disjoint intervals.

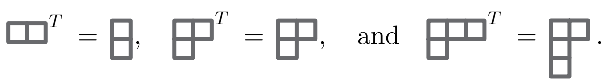
2.3. Grassmann Duality and the Cosecant Conjecture

Associating a linear subspace H of a vector space $V \simeq \mathbb{C}^n$ to its annihilator $\delta(H) := H^\perp \subset V^*$ induces an isomorphism $\delta : G(k,n) \to G(n-k,n)$ called *Grassmann duality.* This notion extends to flags, and the dual of an osculating flag is an osculating flag. Secancy is not preserved under duality. We next formulate the (equivalent) dual statement to the secant conjecture, which we call the cosecant conjecture.

Grassmann duality respects Schubert varieties. Given a flag $F_\bullet \subset \mathbb{C}^n$, let $F_\bullet^\perp$ be the flag whose i-dimensional subspace is $F_i^\perp := (F_{n-i})^\perp$. Then

$$\delta(X_\lambda F_\bullet) = X_{\lambda^T} F_\bullet^\perp,$$

where λ^T is the *conjugate partition* to λ. For example,



That is, if we represent λ by its *Young diagram*—a left-justified array of boxes with λ_i boxes in row i—then the diagram of λ^T is the matrix transpose of the diagram of λ.

If $\gamma(t) = (1, t, t^2, \ldots, t^{n-1})$ is the rational normal curve, then the dual of the family $F_{n-1}(t)$ of its osculating $(n-1)$-planes is a curve $\gamma^\perp(t) := (F_{n-1}(t))^\perp$, which is

$$\gamma^\perp(t) = \Big(\tbinom{n-1}{n-1}(-t)^{n-1}, \ldots, -\tbinom{n-1}{3}t^3, \tbinom{n-1}{2}t^2, - (n-1)t, 1\Big),$$

in the basis dual to the standard basis. Moreover, $(F_{n-k}(t))^\perp$ is the osculating k-plane to this dual rational normal curve $\gamma^\perp$ at the point $\gamma^\perp(t)$. Thus Grassmann duality preserves Schubert varieties given by flags osculating the rational normal curve, and the dual statement to Theorem 2.1 is simply itself.

This is, however, not the case for secant flags. The general secant $(n-1)$-plane

$$F_{n-1}(s_1, s_2, \ldots, s_{n-1}) = \text{span}\{\gamma(s_1), \gamma(s_2), \ldots, \gamma(s_{n-1})\},$$

secant to γ at the points $\gamma(s_1), \ldots, \gamma(s_{n-1})$, has dual space spanned by the vector

$$\big((-1)^{n-1} e_{n-1}, \ldots, -e_3, e_2, -e_1, 1\big),$$

where e_i is the ith elementary symmetric function in the parameters $s_1, \ldots, s_{n-1}$. This dual space is not secant to the dual rational normal curve $\gamma^\perp$.

In general, a *cosecant subspace* is a subspace that is dual to a secant subspace. If

$$F_k(s_1, s_2, \ldots, s_k) = \text{span}\{\gamma(s_1), \gamma(s_2), \ldots, \gamma(s_k)\},$$

then the corresponding cosecant subspace is

$$F_{n-1}^\perp(s_1) \cap F_{n-1}^\perp(s_2) \cap \cdots \cap F_{n-1}^\perp(s_k),$$

the intersection of k hyperplanes osculating the rational normal curve $\gamma^\perp$. A *cosecant flag* is a flag whose subspaces are cut out by hyperplanes osculating γ. It is *cosecant along an interval* of γ if these hyperplanes osculate γ at points of the interval.

Thus, under Grassmann duality, the secant conjecture for $G(n-k,n)$ becomes the following equivalent *cosecant conjecture* for $G(k,n)$.

Conjecture 2.3. (Cosecant conjecture.) *For every Schubert problem $\lambda^1, \ldots, \lambda^m$ on a Grassmannian $G(k,n)$ and all flags $F_\bullet^1, \ldots, F_\bullet^m$ that are cosecant to a rational normal curve γ along disjoint intervals, the intersection*

$$X_{\lambda^1} F_\bullet^1 \cap X_{\lambda^2} F_\bullet^2 \cap \cdots \cap X_{\lambda^m} F_\bullet^m$$

is transverse and consists of $d(\lambda^1, \ldots, \lambda^m)$ real points.

3. SOME SPECIAL CASES OF THE SECANT CONJECTURE

A degree of justification for posing the secant conjecture is provided by the history of its development from the Shapiro and monotone conjectures, since this shows its connection to proven results and established conjectures and its validity for $G(n-2,n)$ [Eremenko et al. 06]. Here, we give more concrete justifications, which include proofs in some special cases.

3.1. Arithmetic Progressions of Secancy

Fix a parameterization $\gamma : \mathbb{R} \to \mathbb{R}^n$ of a rational normal curve. For $t \in \mathbb{R}$ and $h > 0$, let $F_\bullet^h(t)$ be the flag whose

i-dimensional subspace is

$$F_i^h(t) := \text{span}\{\gamma(t), \gamma(t+h), \ldots, \gamma(t+(i-1)h)\},$$

which is spanned by an arithmetic progression of length i with step size h. Results in [Mukhin et al. 09b] imply the secant conjecture for the Schubert problem

$$\square^{k(n-k)} = [k(n-k)]! \frac{1!2! \cdots (k-1)!}{(n-k)! \cdots (n-2)!(n-1)!} \quad (3\text{–}1)$$

for such secant flags.

Let $\mathbb{C}_{n-1}[t]$ be the space of polynomials of degree at most $n-1$. The discrete Wronskian with step size h of polynomials $f_1, \ldots, f_k$ is the determinant

$$\mathrm{W}_h(f_1, f_2, \ldots, f_k) \quad (3\text{–}2)$$
$$:= \det \begin{pmatrix} f_1(t) & f_1(t+h) & \cdots & f_1(t+(k-1)h) \\ f_2(t) & f_2(t+h) & \cdots & f_2(t+(k-1)h) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(t) & f_k(t+h) & \cdots & f_k(t+(k-1)h) \end{pmatrix}.$$

For general $f_1, \ldots, f_k \in \mathbb{C}_{n-1}[t]$, this polynomial has degree $k(n-k)$. Up to a scalar, the polynomial W_h depends only on the linear span of the polynomials $f_1, \ldots, f_k$, giving a map

$$\mathrm{W}_h : G(k, \mathbb{C}_{n-1}[t]) \longrightarrow \mathbb{P}^{k(n-k)},$$

where $\mathbb{P}^{k(n-k)}$ is the projective space of polynomials of degree at most $k(n-k)$. It is shown in [Mukhin et al. 09b] that W_h is a finite map. It is a linear projection of the Grassmannian in its Plücker embedding, so the fiber over a general polynomial $w(t) \in \mathbb{P}^{k(n-k)}$ consists of $d(\square^{k(n-k)})$ reduced points, each of which is a space V of polynomials with discrete Wronskian $w(t)$. As a special case of [Mukhin et al. 09b, Theorem 2.1], we have the following statement.

Proposition 3.1. *Let $V \subset \mathbb{C}_{n-1}[t]$ be a k-dimensional space of polynomials whose discrete Wronskian $\mathrm{W}_h(V)$ has distinct real roots $z_1, \ldots, z_N$, each of multiplicity 1. If for all $i \neq j$, we have $|z_i - z_j| \geq h$, then the space V has a basis of real polynomials.*

Corollary 3.2. *Set $N := k(n-k)$ and suppose that $F_\bullet^h(z_1), \ldots, F_\bullet^h(z_N)$ are disjoint secant flags with $z_i + (n-1)h < z_{i+1}$ for each $i = 1, \ldots, N-1$. Then the intersection*

$$X_\square F_\bullet^h(z_1) \cap X_\square F_\bullet^h(z_2) \cap \cdots \cap X_\square F_\bullet^h(z_N) \quad (3\text{–}3)$$

in $G(n-k, n)$ is transverse with all points real.

Proof. We identify points in the intersection (3–3) with the fibers of the discrete Wronski map W_h over the polynomial $(t - z_1) \cdots (t - z_{k(n-k)})$, which will prove reality. Transversality follows by an argument in [Sottile 11, Chapter 13]: a finite analytic map between complex manifolds that has only real points in its fibers above an open set of real points is necessarily unramified over those points.

A polynomial of degree $n-1$ is the composition of the parameterization $\gamma : \mathbb{C} \to \mathbb{C}^n$ of the rational normal curve with a linear form $\mathbb{C}^n \to \mathbb{C}$. In this way, a subspace V of polynomials of dimension k corresponds to a surjective map $V : \mathbb{C}^n \to \mathbb{C}^k$. We will identify such a map with its kernel H, which is a point in $G(n-k, n)$.

The column space of the matrix in (3–2) is the image under V of the linearly independent vectors $\gamma(t), \gamma(t+h), \ldots, \gamma(t+(k-1)h)$. These vectors span $F_k^h(t)$. Thus the determinant $W_h(V)$ vanishes at a point t exactly when the map

$$V : F_k^h(t) \longrightarrow \mathbb{C}^k$$

does not have full rank, that is, when

$$\dim H \cap F_k^h(t) \geq 1,$$

which is equivalent to $H \in X_\square F_\bullet^h(t) \subset G(n-k, n)$.

It follows that points in the intersection (3–3) correspond to k-dimensional spaces of polynomials V with discrete Wronskian $(t - z_1) \cdots (t - z_{k(n-k)})$, and each of these are real, by Proposition 3.1. □

3.2. The Shapiro Conjecture Is the Limit of the Secant Conjecture

The osculating plane $F_i(s)$ is the unique i-dimensional plane having maximal order of contact with the rational normal curve γ at the point $\gamma(s)$. This implies that it is a limit of secant planes, and in fact, every limit of secant planes in which the points come together is an osculating plane.

Lemma 3.3. *Let $\{s_1^{(j)}, \ldots, s_i^{(j)}\}$ for $j = 1, 2, \ldots$ be a sequence of lists of i distinct complex numbers that all converge to the same number, $\lim_{j\to\infty} s_p^{(j)} = s$, for each $p = 1, \ldots, i$ and for some number s. Then*

$$\lim_{j\to\infty} \text{span}\left\{\gamma(s_1^{(j)}), \gamma(s_2^{(j)}), \ldots, \gamma(s_i^{(j)})\right\} = F_i(s).$$

Since transversality and reality are preserved under perturbation, we conclude that Theorem 2.1 is a limiting case of the secant conjecture. Conversely, Theorem 2.1 implies the following.

Theorem 3.4. *Let $\lambda^1, \dots, \lambda^m$ be a Schubert problem and let $t_1, \dots, t_m$ be distinct points of the rational normal curve γ. Then there exists an $\epsilon > 0$ such that if for each $i = 1, \dots, m$, $F^i_\bullet$ is a flag secant to γ along an interval of length ϵ containing t_i, then the intersection*

$$X_{\lambda^1} F^1_\bullet \cap X_{\lambda^2} F^2_\bullet \cap \cdots \cap X_{\lambda^m} F^m_\bullet \tag{3–4}$$

is transverse with all points real.

This implies that for generic secant flags $F^1_\bullet, \dots, F^m_\bullet$, the intersection (3–4) is transverse, which implies that secant flags are sufficiently general for the Schubert calculus. Furthermore, Theorem 3.4 reduces the secant conjecture (Conjecture 2.2) to its transversality statement.

3.3. Generalized Secant Conjecture

Theorem 3.4 suggests a conjecture involving flags that are intermediate between secant and osculating, and which includes the secant conjecture and Theorem 2.1 as special cases.

A *generalized secant subspace* to the rational normal curve γ is spanned by osculating subspaces of γ. This notion includes secant subspaces, for a one-dimensional subspace that osculates γ is simply one that is spanned by a point of γ. A flag $F_\bullet$ is *generalized secant* to γ if each of the linear spaces in $F_\bullet$ is a generalized secant subspace. A generalized secant flag is *secant along an interval* of γ if the osculating subspaces that span its linear spaces osculate γ at points of the interval.

Conjecture 3.5. (Generalized secant conjecture.) *For every Schubert problem $\lambda^1, \dots, \lambda^m$ on a Grassmannian $G(k, n)$ and all generalized secant flags $F^1_\bullet, \dots, F^m_\bullet$ that are secant to a rational normal curve γ along disjoint intervals, the intersection*

$$X_{\lambda^1} F^1_\bullet \cap X_{\lambda^2} F^2_\bullet \cap \cdots \cap X_{\lambda^m} F^m_\bullet$$

is transverse and consists of $d(\lambda^1, \dots, \lambda^m)$ real points.

This includes the secant conjecture as the case in which all of the flags are secant flags, but it also includes Theorem 2.1, which treats the case in which all flags are osculating. Many of the computations in our experiment tested instances of this conjecture in which one or two flags were osculating while the rest were secant flags. This choice was made to make the computation feasible for some Schubert problems.

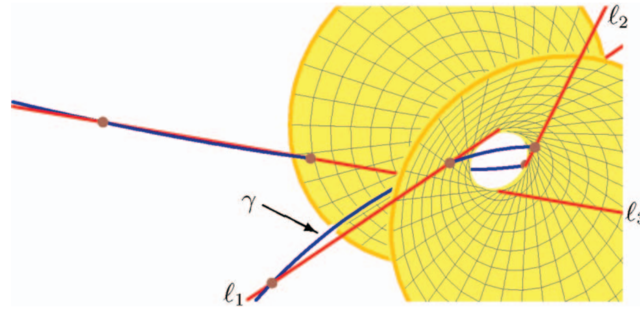


FIGURE 1. Quadric through three secant lines (color figure available online).

There are also a generalized cosecant conjecture and a corresponding version of Theorem 3.4, which we do not formulate.

4. THE PROBLEM OF FOUR SECANT LINES

We give an in-depth look at the Schubert problem $\square^4 = 2$ on $G(2, 4)$, where $\square$ denotes the Schubert condition that a two-plane in $\mathbb{C}^4$ meets a fixed two-plane nontrivially. Equivalently, $\square^4 = 2$ is the Schubert problem of lines in $\mathbb{P}^3$ that meet four fixed lines. Let $\gamma : \mathbb{R} \to \mathbb{P}^3$ be a rational normal curve. We consider the lines in $\mathbb{P}^3$ that meet four lines $\ell_1, \ell_2, \ell_3, \ell_4$ that are secant to γ.

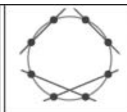
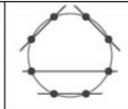
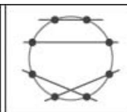
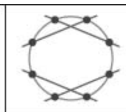
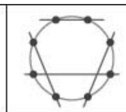
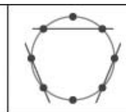
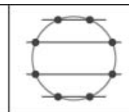
For $s_1, s_2 \in \mathbb{R}$, let $\ell(s_1, s_2)$ denote the secant line to γ through $\gamma(s_1)$, $\gamma(s_2)$. Given $s_1 < \cdots < s_8$, the secant conjecture (which is in this case a theorem of [Eremenko et al. 06]) asserts that both lines meeting the four fixed lines

$$\ell(s_1, s_2), \quad \ell(s_3, s_4), \quad \ell(s_5, s_6), \quad \ell(s_7, s_8)$$

are real. We investigate phenomena beyond the secant conjecture by letting ρ be a permutation of $\{1, \dots, 8\}$ and taking $\ell_1, \ell_2, \ell_3, \ell_4$ to be $\ell(s_{\rho(1)}, s_{\rho(2)}), \dots, \ell(s_{\rho(7)}, s_{\rho(8)})$.

There are 17 combinatorial configurations of four secant lines along $\gamma \simeq S^1$. These are indicated by the chord diagrams in Table 1, which shows the number of real solutions found when we computed 100,000 instances of each configuration. For most configurations, we observed only real solutions, and in only four configurations did we find any nonreal solutions. We will give a simple explanation of this observation.

Counting constants shows that there is a unique doubly ruled quadric surface Q that contains the lines ℓ_1, ℓ_2, and ℓ_3 in one ruling, as shown in Figure 1. The two lines of the second ruling of Q through the two points of intersection of ℓ_4 with Q are the solutions to the Schubert problem $\square^4 = 2$ for these four secant lines.

									
real	0			23723					29398
roots	2	100000	100000	76277	100000	100000	100000	100000	70602

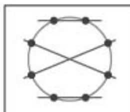
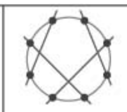
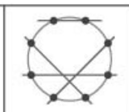
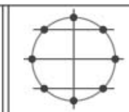
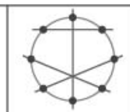
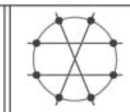
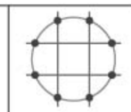
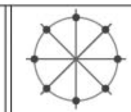
									
0				52395				65783	
2	100000	100000	100000	47605	100000	100000	100000	34217	100000

TABLE 1. Configurations of four secant lines with results of an experiment.

The quadric Q divides its complement in $\mathbb{R}\mathbb{P}^3$ into two connected components (the domains where the quadratic form is positive or negative), called the *sides* of Q. Three lines ℓ_1, ℓ_2, ℓ_3 give six points of secancy that are the intersections of γ with Q and that divide γ into six segments that alternate between the two sides of Q. If the fourth secant line ℓ_4 has its two points of secancy lying on opposite sides of Q, then ℓ_4 has a real intersection with Q, so that the Schubert problem has one (and hence two) real solutions. The points of secancy of ℓ_4 lie on opposite sides of Q if in the interval between the two points of secancy, the curve γ crosses Q an odd number of times. That is, the interval contains an odd number of points of secancy of the lines ℓ_1, ℓ_2, and ℓ_3.

This simple topological argument shows that if at least one of the four secant lines has such an odd interval of secancy, then the Schubert problem will have only real solutions, independently of the actual positions of the secant lines. Twelve of the seventeen configurations have at least one odd interval of secancy, and therefore will always give two real solutions. Four configurations with only even intervals of secancy were observed to have either zero or two real solutions. Only the configuration with disjoint intervals of secancy has even intervals of secancy and yet has only real solutions. This deeper fact was proven in [Eremenko et al. 06].

5. OVERLAP NUMBER

For most Schubert problems, the number of different configurations of secant flags is astronomical. Consider the problem $\square^4 \cdot \yng(3,1)^2 = 12$ on the Grassmannian of 3-planes in 7-space. The condition $\square$ has relevant subspace F_4, and the condition $\yng(3,1)$ has relevant subspace F_5. The resulting 26 points of secancy have at least

$$\left\lceil \binom{26}{4,4,4,4,5,5} \cdot \frac{1}{4!} \cdot \frac{1}{2!} \cdot \frac{1}{26} \cdot \frac{1}{2} \right\rceil = 3{,}381{,}948{,}761{,}563$$

combinatorially distinct configurations. To cope with this complexity, we introduce a statistic on these configurations—the overlap number—which is zero if and only if the flags are disjoint, and we tabulate the results of our experiment using this statistic.

In an instance of a Schubert problem $\lambda^1, \dots, \lambda^m$ with relevant subspaces of respective dimensions $i_1, \dots, i_m$, to define the relevant subspaces of the jth secant flag,

$$F_1^j \subsetneq F_2^j \subsetneq \cdots \subsetneq F_{i_j}^j,$$

we need a choice of an ordered set T_j of i_j points of γ. The overlap number measures how much these sets of points $T_1, \dots, T_m \subset \gamma$ overlap.

Let T be their union. Since γ is topologically a circle, removing a point $p \in \gamma \setminus T$, we may assume that $T_1, \dots, T_m \subset \mathbb{R}$. Each set T_j defines an interval I_j of $\mathbb{R}$, and we let o_j be the number of points of $T \setminus T_j$ lying in I_j. This sum $\Sigma := o_1 + \cdots + o_m$ depends on $p \in \gamma \setminus T$, and the *overlap number* is the minimum of these sums as p varies.

For example, consider a Schubert problem with relevant subspaces of dimensions 3, 2, and 2. Suppose that we have chosen seven points on γ in groups of 3, 2, and 2. This is represented schematically in the top part of Figure 2, in which γ is a circle, and the points in the sets T_1, T_2, and T_3 are represented by circles (●), squares (■), and triangles (▲), respectively. For each of three points p_1, p_2, and p_3 of γ, we compute the number o_i and their sum Σ, displaying the results in the table at the bottom of Figure 2. The minimum of the sum Σ for all choices of points is achieved by p_3.

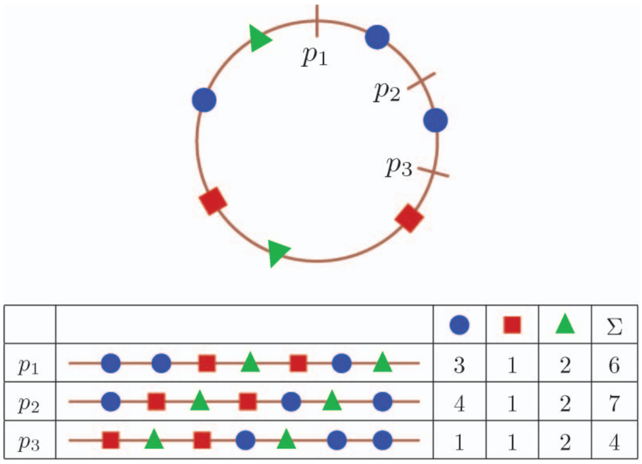


FIGURE 2. Computation of overlap number (color figure available online).

If one (or more) of the flags is osculating, we compute the overlap number by treating the point of osculation as a point with multiplicity equal to the dimension of the relevant subspace.

6. EXPERIMENTAL EVIDENCE FOR THE SECANT CONJECTURE

We tested the secant conjecture by conducting a massive experiment whose data are available online.[2] This experiment used symbolic exact arithmetic to compute the number of real solutions for specific instances of Schubert problems. These computations are possible because Schubert problems are readily modeled on a computer, and for those of moderate size, we may algorithmically determine the number of real solutions with software tools. Our experiment primarily used the mathematical software Singular [Decker et al. 10] and Maple (see [Hillar et al. 10] for further details about the implementation of the computations, including a comprehensive list of software tools used). If the software is reliably implemented, which we believe to be the case, then this computation provides a proof that the given instance has the computed number of real solutions. This procedure may be semiautomated and run on supercomputers (as described in [Hillar et al. 10]), which allows us to amass the considerable evidence we have collected in support of the secant conjecture.

$k\backslash n-k$	2	3	4	5	6
2	1	5	22	81	55
3	5	64	114	79	
4	22	107	67		
5	81				

TABLE 2. Schubert problems studied.

6.1. Experimental Data

Table 2 shows how many Schubert problems on each Grassmannian of k-planes in n-space had been studied when we halted the experiment on May 26, 2010. Our experiment not only tested the secant conjecture but also studied the relationship between the overlap number and the number of real solutions for many Schubert problems on small Grassmannians. We computed 2,058,810,000 instances of 703 Schubert problems. About one-fourth of these (498,737,669) were instances of the (generalized) secant conjecture, and the rest involved nondisjoint secant flags. The generalized secant conjecture held in every computed instance. The remaining 1,560,072,331 instances involved secant flags with some overlap in their intervals of secancy, measured by the overlap number.

The experiment computed Schubert problems using either zero, one, or two osculating flags, with the rest secant flags. In the online database `WWW_secant_Exp`,[3] this number of osculating flags determines the *computation type* which is 1, 2, or 3 for zero, one, or two osculating flags. The experiment used randomly chosen flags, which were generated using random generator seeds that are stored in our database, so that all computations are reproducible.

Table 3 shows part of the data we obtained testing the full secant conjecture for the Schubert problem $\square^4 \cdot \ydiagram{3,1}^2 = 12$ on $G(3,7)$.

We used 7.52 gigahertz-years to compute ten million instances of this Schubert problem, all involving secant flags. The rows are labeled with the even integers from 0 to 12, since the number of real solutions has the same parity as the number of complex solutions. The first column, with overlap number 0, represents tests of the secant conjecture. Since the only entry is in the row for 12 real solutions, the secant conjecture was verified in 2,320,873 instances. The column labeled overlap number 1 is empty, because flags for this problem cannot have

[2] Available at http://www.math.tamu.edu/~secant/secant/flagview.php.

[3] Available at http://www.math.tamu.edu/~secant/secant/flagview.php.

Real Solutions \ Overlap Number	0	1	2	3	4	5	6	$\cdots$	9	$\cdots$	Total
0	0	0	0	0	0	0	0	$\cdots$	1	$\cdots$	691
2	0	0	0	0	0	9	7	$\cdots$	8	$\cdots$	72857
4	0	0	0	0	79	917	1990	$\cdots$	524	$\cdots$	523362
6	0	0	0	814	5713	12550	18330	$\cdots$	4531	$\cdots$	1418911
8	0	0	0	635	4646	15947	17180	$\cdots$	6055	$\cdots$	1983639
10	0	0	0	1226	6912	18403	17236	$\cdots$	6801	$\cdots$	1649923
12	2320873	0	51120	99413	206398	203426	179955	$\cdots$	42883	$\cdots$	4350617
Total	2320873	0	51120	102088	223748	251252	234698	$\cdots$	60803	$\cdots$	10000000

TABLE 3. Experimental data for $\square^4 \cdot \yng(3,1)^2 = 12$ with all secant flags.

overlap number 1. Perhaps the most interesting feature is that for overlap number 2, all computed solutions were real, while for overlap number 3, at least six solutions were real, and for overlap number 4, at least four were real. It is only with overlap number 9 and above that we computed an instance with no real solutions.

We also computed 200 million instances of this same Schubert problem with four secant flags (for the Schubert variety $X_{\square}$) and two osculating flags (for the Schubert variety $X_{\yng(3,1)}$). These data are compiled in Table 4. This computation took 261 gigahertz-days—twenty times as many instances as Table 3 in about one-tenth of the time. This speedup occurs because using two osculating flags gives a formulation with only four variables instead of twelve. This computation tested the generalized secant conjecture; its computed instances form the first column. Since the only entry in that column is in the row for 12 real solutions, the generalized secant conjecture was verified in 49,743,228 instances. As with Table 3, there is visibly an inner border to these data, but for this computation there are instances with no real solutions starting with overlap number eight.

6.2. Computing Schubert Problems

A $k \times (n{-}k)$ matrix $X \in \mathbb{C}^{k\times(n-k)}$ determines a general point in $G(k,n)$, namely the row space H of the $k \times n$ matrix (also written H)

$$H := \big(I_k : X\big). \tag{6–1}$$

If we represent an i-plane F_i as the row space of an $i \times n$ matrix F_i of full rank, then

$$\dim H \cap F_i \geq j \iff \operatorname{rank}\begin{pmatrix} H \\ F_i \end{pmatrix} \leq k+i-j, \tag{6–2}$$

which is given by the vanishing of all $(k{+}i{-}j{+}1) \times (k{+}i{-}j{+}1)$ subdeterminants. We represent a flag $F_\bullet$ by a full-rank $n \times n$ matrix whose first i rows span F_i. Then (6–2) leads to equations for the Schubert variety $X_\lambda F_\bullet$ in the coordinate patch (6–1). In practice, we need only an $(n{-}k{+}i{-}\lambda_i) \times n$ matrix, where λ_i is the last nonzero part of λ.

To represent a secant i-plane, we use an $i \times n$ matrix $F_i(t_1,\dots,t_i)$ whose jth row is the vector $\gamma(t_j)$, where $t_1,\dots,t_i \in \mathbb{R}$, and $\gamma(t) = (1,t,\dots,t^{n-1})$ is the rational normal curve. Similarly, the i-plane $F_i(t)$ osculating γ at the point $\gamma(t)$ is represented by the $i \times n$ matrix whose jth row is $\gamma^{(j-1)}(t)$.

For example, Conjecture 1.1 involves the Schubert problem $\square^6 = 5$ on $G(2,5)$, where $\square$ is the Schubert condition of a 2-plane meeting a 3-plane. The solutions are 2-planes spanned by the first two rows of the matrix in (1–1). The last three rows in the matrix are the points $\gamma(s_i)$, $\gamma(t_i)$, $\gamma(u_i)$ that span the 3-plane of a secant flag.

We use Singular to compute an *eliminant* of the polynomial system modeling a given instance of the Schubert problem $\lambda^1,\dots,\lambda^m$. This is a univariate polynomial $f(x)$ whose roots are all the x-coordinates of solutions to the Schubert problem in the patch (6–1). (See, for example, [Cox et al. 05, Chapter 2].) By the shape lemma [Becker et al. 93], when the eliminant $f(x)$ has degree equal to $d(\lambda^1,\dots,\lambda^m)$ and is square-free, then the solutions to the Schubert problem are in one-to-one correspondence with the roots of the eliminant $f(x)$, with real roots corresponding to real solutions. We use Maple's `realroot` command to compute the number of real roots of the eliminant $f(x)$.

If the eliminant does not satisfy these hypotheses, then we compute an eliminant with respect to a different

Real Solutions	Overlap Number: 0	1	2	3	4	5	6	$\cdots$	Total
0	0	0	0	0	0	0	0	$\cdots$	13894
2	0	0	0	0	0	3799	19	$\cdots$	1357929
4	0	0	0	0	24756	93214	186521	$\cdots$	12146335
6	0	0	0	0	210843	495977	731938	$\cdots$	29925437
8	0	0	0	0	254875	640663	508884	$\cdots$	36708450
10	0	0	0	0	153520	442928	229530	$\cdots$	26500908
12	49743228	0	1171814	2324847	5900258	5944524	3971316	$\cdots$	93347047
Total	49743228	0	1171814	2324847	6544252	7621105	5628208	$\cdots$	200000000

TABLE 4. Experimental data for $\yng(1)^6 \cdot \yng(3,1)^2 = 12$ with two osculating flags.

coordinate of the patch (6–1). It is sometimes the case that no coordinate provides a satisfactory eliminant. This will occur if there is a solution with multiplicity greater than 1 (the Schubert varieties do not meet transversally) or if the coordinate patch does not contain all solutions. In general, it will occur when the computed instance lies in a *discriminant hypersurface* in the space of all instances. When developing and testing our software for this experiment, we observed that this situation was extremely rare, and it occurred only when the overlap number was positive and there were multiple solutions, which agrees with the transversality assertion in the secant conjecture. When our software detects that no coordinate provides a satisfactory eliminant, it deterministically perturbs the points of secancy, preserving the overlap number, and repeats this elimination procedure. This has always worked to give an eliminant satisfying the hypotheses.

As with Tables 3 and 4, working in a different set of local coordinates enables us to compute instances of the generalized secant conjecture (Conjecture 3.5) efficiently for one (and sometimes two) osculating flags. With one flag osculating at $\gamma(\infty)$, we may use local coordinates as described in [Ruffo et al. 06].

With two osculating flags, there is a smaller choice of local coordinates available. Suppose that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors corresponding to columns of our matrices. Then the flag $F_\bullet(\infty)$ osculating the rational normal curve γ at $\gamma(\infty) = \mathbf{e}_n$ and the flag $F_\bullet(0)$ osculating at $\gamma(0) = \mathbf{e}_1$ have

$$F_i(\infty) = \mathrm{span}\{\mathbf{e}_{n+1-i}, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_n\}$$

and

$$F_i(0) = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_i\}.$$

General points in $X_\lambda F_\bullet(\infty) \cap X_\mu F_\bullet(0)$ are represented by $k \times n$ matrices whose row i has a 1 in column $\lambda_{k+1-i} + i$, arbitrary entries in subsequent columns up to column $n-k-1+i-\mu_i$, and 0's elsewhere. Here is such a matrix with $k = 3$, $n = 8$, $\lambda = \yng(2,1)$, and $\mu = \yng(3,1)$:

$$\begin{pmatrix} 1 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & * \end{pmatrix}.$$

6.3. Numerical Experimentation

In [Hauenstein and Sottile 12], 25,000 instances of the Shapiro conjecture for the Schubert problem $\yng(2)^8 = 126$ were computed, and for each instance the software alphaCertified used 256-bit precision to softly certify that all solutions were real. (A *soft certificate* is one computed with floating-point arithmetic that would be rigorous if computed with exact rational arithmetic.) The solutions were computed using the software package Bertini [Bates et al. 06], which is based on numerical homotopy continuation [Sommese and Wampler 05]. Given a system of n polynomial equations in n unknowns, Smale's α-theory [Smale 86] gives algorithms for certifying that Newton iterations applied to an approximate solution will converge to a solution, and also may be used to certify that the solution is real. As explained in [Hauenstein and Sottile 12], this Schubert problem has such a formulation. These algorithms are implemented in alphaCertified [Hauenstein and Sottile 12].

7. LOWER BOUNDS AND INNER BORDERS

The most ubiquitous and enigmatic phenomenon that we have observed in our data is the apparent "inner border"

	0	1	2	3	4	5	6	$\cdots$	Total
0	0	0	0	0	0	0	0	$\cdots$	4272
2	0	0	0	0	0	0	0	$\cdots$	127217
4	0	0	0	0	693	1481	6660	$\cdots$	879658
6	0	0	0	0	224	510	2541	$\cdots$	2304233
8	0	0	0	0	526	939	3561	$\cdots$	2914837
10	0	0	0	0	1052	2074	6985	$\cdots$	2205198
12	0	0	0	0	1556	2595	7300	$\cdots$	1224667
14	3328772	0	60860	120625	310819	246910	237704	$\cdots$	5339918
Total	3328772	0	60860	120625	305870	254509	264751	$\cdots$	15000000

TABLE 5. Real solutions vs. overlap number for $\square^8 = 14$.

in many of the tables. Typically, we do not observe instances with zero or few real solutions when the overlap number is small. This is manifested by a prominent staircase separating observed pairs of the form (real solutions, overlap number) from unobserved pairs. This feature is clearly visible in Tables 3 and 4, and in Table 5 for the problem $\square^8 = 14$ in $G(2,6)$. There, it is only with overlap number 8 or larger that we observe instances with two real solutions; and with overlap number 16 or larger, instances with no real solutions. (These columns are not displayed for reasons of space.)

This problem involves 2-planes meeting eight secant 4-planes. There are over 10^{18} configurations of eight secant 4-planes, and hence it is impossible to systematically study all configurations as in Section 4. This is the case for most of the problems we studied. Because of the coarseness of our measure of overlap, we doubt whether it is possible to formulate a meaningful conjecture about this inner border based on our data. Nevertheless, we believe that this problem, like the problem of four lines, contains rich geometry, with certain configurations having a lower bound on the number of real solutions.

There are many meaningful polynomial systems and geometric problems having a nonzero lower bound on their number of real solutions. These include rational curves interpolating points on toric del Pezzo surfaces [Itenberg et al. 03, Itenberg et al. 04, Itenberg et al. 09, Mikhalkin 05, Welschinger 03], sparse polynomial systems from posets [Joswig and Witte 07, Soprunova and Sottile 06], and some lower bounds in the Schubert calculus [Azar and Gabrielov 11, Eremenko and Gabrielov 02a].

Lower bounds and inner borders were also observed in studying the monotone conjecture [Ruffo et al. 06, Section 3.2.2]. The original example of a lower bound appeared in [Eremenko and Gabrielov 02a]. The Wronskian of linearly independent polynomials $f_1(t), f_2(t), \ldots, f_k(t)$ of degree $n-1$,

$$\mathrm{W}(f_1, f_2, \ldots, f_k) := \det\begin{pmatrix} f_1(t) & f_1'(t) & f_1''(t) & \cdots & f_1^{(k-1)}(t) \\ f_2(t) & f_2'(t) & f_2''(t) & \cdots & f_2^{(k-1)}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_k(t) & f_k'(t) & f_k''(t) & \cdots & f_k^{(k-1)}(t) \end{pmatrix},$$

has degree $k(n-k)$, which gives a finite map $\mathrm{W} : G(k, \mathbb{C}_{n-1}[t]) \longrightarrow \mathbb{P}^{k(n-k)}$ with the general fiber consisting of $d(\square^{k(n-k)})$ (see (3–1)) linear spaces of polynomials. Theorem 2.1 implies that if $w(t)$ is a polynomial with $k(n-k)$ distinct real roots, then each of the $d(\square^{k(n-k)})$ points in the fiber of W over $w(t)$ is real. Eremenko and Gabrielov showed that if n is odd, there is a nontrivial lower bound on the number of real spaces of polynomials in the fiber of W over every polynomial $w(t)$ with real coefficients.

In [Azar and Gabrielov 11], the authors studied the problem $\square^{2n-4}$ in $G(n-2,n)$ of $(n-2)$-planes in $\mathbb{C}^n$ that meet one secant line and $2n-5$ tangent lines. When the interval of secancy contains no tangent points, this is an instance of the generalized secant conjecture (Conjecture 3.5). They establish lower bounds on the number of real solutions that depend on the configuration of the points of secancy and tangency.

8. GAPS

The Schubert problem $\boxplus^4 = 6$ on $G(4,8)$ involves 4-planes whose intersection with each of four general

	0	1	2	3	4	5	6	⋯	Total
0	0	0	0	0	0	0	0	0	0
2	0	0	0	1441	7730	14277	16636	⋯	147326
4	0	0	0	0	0	0	0	0	0
6	280304	0	13131	25708	62833	55919	57719	⋯	852674
Total	280304	0	13131	27149	70563	70196	74355	⋯	1000000

TABLE 6. Real solutions vs. overlap number for $W_{\boxplus}^4 = 6$.

4-planes is at least two-dimensional. We computed 1,000,000 instances of this problem, obtaining the results in Table 6. A system of real polynomial equations with six solutions can a priori have zero, two, four, or six real solutions; yet strikingly, this Schubert problem has only two or six real solutions, never zero or four.

Although our observations involved only secant flags, this phenomenon holds for any real flags. As we describe below, this follows from ideas in the discussion of this Schubert problem in [Vakil 06, Section 3.13]. (Vakil's discussion, however, focuses on explaining a different phenomenon, namely, Derksen's observation that the Galois group of this Schubert problem is *deficient*, i.e., smaller than the symmetric or alternating group.)

We consider the *auxiliary Schubert problem* $\square\!\square\!\square^{4} = 4$ on $G(2,8)$, counting 2-planes that meet four general 4-planes. Given 4-planes $W_1, \ldots, W_4$, let $P_1, \ldots, P_4$ be the 2-planes that meet them. Then the solutions to the original Schubert problem $W_{\boxplus}^4 = 6$ are precisely the six sums of the form $P_i + P_j$. Such a sum is real if and only if P_i and P_j are each real or if P_i and P_j are a pair of complex conjugate subspaces.

If the W_i are real, then there can be zero, one, or two complex conjugate pairs among the P_i. Then the number of solutions $P_i + P_j$ that are real is, respectively, 6, 2, and 2. This explains the observations in Table 6.

This is the first in a family of Schubert problems in $G(4, 2n)$ for $n \geq 4$ with such gaps in their numbers of real solutions. These involve enumerating the 4-planes that have at least a two-dimensional intersection with each of four general n-planes in $\mathbb{C}^{2n}$. For each, there is an auxiliary Schubert problem on $G(2, 2n)$ of 2-planes meeting four general n-planes. This will have n solutions, and the solutions to the original problem are 4-planes spanned by pairs of solutions to the auxiliary problem. The original problem will have $\binom{n}{2}$ solutions, corresponding to pairs of solutions to the auxiliary problem. A solution is real either when both elements of the pair are real or when the pair consists of complex conjugate solutions. We remark that the auxiliary problem may have any number r of real solutions, where $0 \leq r \leq n$ and $n-r$ is even; this may be deduced from the description of the Schubert problem in terms of elementary geometry given, for example, in [Sottile 97, Section 8.1]. These restrictions are identical to restrictions on the number of real quadratic factors of a general real polynomial of degree n, as in [Soprunova and Sottile 06, Theorem 7.8]. We summarize this discussion.

Theorem 8.1. *The Schubert problem of 4-planes that have at least a two-dimensional intersection with each of four general real n-planes in $\mathbb{C}^{2n}$ has $\binom{n}{2}$ solutions. The number of real solutions is*

$$\binom{r}{2} + c,$$

where the auxiliary problem of 2-planes meeting each of four general real n-planes in $\mathbb{C}^{2n}$ has r real solutions and c pairs of complex conjugate solutions and $r + 2c = n$.

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