

The Riemann Zeta Function on Arithmetic Progressions

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We prove asymptotic formulas for the first discrete moment of the Riemann zeta function on certain vertical arithmetic progressions inside the critical strip. The results give some heuristic arguments for a stochastic periodicity that we observed in the phase portrait of the zeta function.

1. INTRODUCTION

The Riemann zeta function $\zeta(s)$ is of special interest in number theory and complex analysis. The real zeros of $\zeta(s)$ are located at $s = -2n$, $n \in \mathbb{N}$, and are said to be trivial. All other zeros are called nontrivial and lie inside the critical strip $0 < \operatorname{Re} s < 1$. The famous Riemann hypothesis claims that all nontrivial zeros lie on the critical line $\operatorname{Re} s = 1/2$.

We are interested in the values of the zeta function on certain vertical arithmetic progressions. In [Putnam 54a, Putnam 54b], it was shown that there exists no infinite arithmetic progression of nontrivial zeros, and in [Frankenhuijsen 05], the author obtained bounds for the length of any hypothetical arithmetic progression. Asymptotic formulas proved for the second and fourth moments of the Riemann zeta function on arbitrary arithmetic progressions to the right of the critical line were proved in [Good 78]. Here we shall investigate the discrete first moments

$$A(s, d, M) := \frac{1}{M} \sum_{0 \leq m < M} \zeta(s + imd) \quad (1-1)$$

for positive real numbers d . It turns out that for special values of $d > 0$, the asymptotic behavior of $A(s, d, M)$ for $M \rightarrow \infty$ can be described explicitly.

Theorem 1.1. *Fix $s \in \mathbb{C} \setminus \{1\}$ with $0 < \sigma := \operatorname{Re} s \leq 1$, $t := \operatorname{Im} s \geq 0$, and let $d = 2\pi/\log \ell$, where $\ell \geq 2$ is an integer. Then for $M \rightarrow +\infty$,*

$$\frac{1}{M} \sum_{0 \leq m < M} \zeta(s + imd) = \frac{1}{1 - \ell^{-s}} + O(M^{-\sigma} \log M).$$

2000 AMS Subject Classification: Primary 11M06

Keywords: Riemann zeta function, value distribution, arithmetic progression



FIGURE 1. A phase portrait of $f(z) = z$ (color figure available online).

We would like to mention that our interest in sums of the Riemann zeta function on arithmetic progressions has also been motivated by *phase portraits*.

The phase portrait (or phase plot) of a meromorphic function $f : D \rightarrow \widehat{\mathbb{C}}$ visualizes f directly on its domain D by color-coding its phase $\psi := f/|f|$. In Figure 1, the standard hsv color scheme is used to encode the phase of all points in a square (with the origin at its center), which yields the phase portrait of $f(z) = z$.

For an introduction to phase plots and related phase diagrams we refer to [Wegert and Semmler 11] and [Wegert 10], respectively. Here we just point out that meromorphic functions can be reconstructed from their phase portraits uniquely up to a positive scaling factor. Figure 2 shows the phase plot of the Riemann zeta function in the square $-40 \leq \operatorname{Re} z \leq 10, -2 \leq \operatorname{Im} z \leq 48$.

Visual inspection of the phase plot of ζ in the critical strip almost immediately reveals a surprising “stochastic periodicity” of the phase $\psi = \zeta/|\zeta|$. This is illustrated in Figure 3, which shows 20 consecutive rectangles of width 1 and height 20 from the critical strip, with a small overlapping region between neighboring domains.

The eye-catching yellow diagonal stripes led us to the conjecture that the phase plot of zeta in the critical strip has a sort of stochastic period, and some heuristic arguments suggest that its length equals $2\pi/\log 2$. One option to make this statement more precise is to study mean values of the phase on arithmetic progressions,

$$\frac{1}{M} \sum_{0 \leq m < M} \frac{\zeta(s + imd)}{|\zeta(s + imd)|}. \tag{1-2}$$

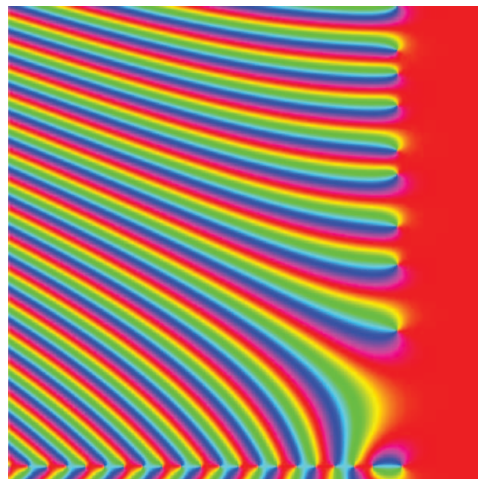


FIGURE 2. A phase portrait of the Riemann zeta function (color figure available online).

If the phase is randomly distributed (on the unit circle), these mean values should be almost independent of the choice of d and s for sufficiently large M . On the other hand, a periodic color pattern should manifest itself in the behavior of the mean values if d is in resonance with the period.

The investigation of (1-2) seems to be a hard problem, and we decided to omit the normalization of the summands, i.e., to investigate (1-1) instead of (1-2), hoping that the latter sums reflect the behavior of the former. This is, in a sense, confirmed by Theorem 1.1. If $d = 2\pi/\log \ell$ with $\ell \in \mathbb{Z}$ and $\ell \geq 2$, then there is indeed an asymptotic periodicity of $s \mapsto A(s, d, M)$ for large M . Moreover, among all values of ℓ , the strongest effect comes from $\ell = 2$, which coincides with our observations.

2. PROOF OF THE MAIN THEOREM

In what follows, we consider $d > 0$ and $s \in \mathbb{C}$ with $0 < \sigma := \operatorname{Re} s < 1, t := \operatorname{Im} s \geq 0$ as being fixed. The basic tool

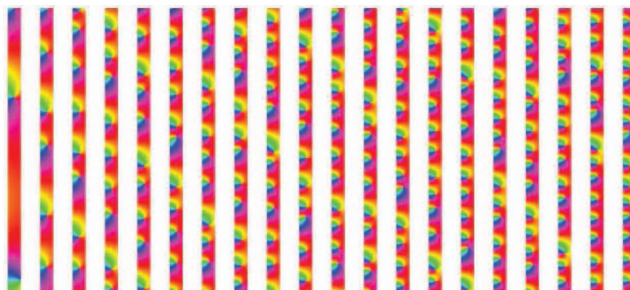


FIGURE 3. Phase plots of the Riemann zeta function along the critical strip (color figure available online).

in proving Theorem 1.1 is the representation

$$\zeta(s) = \frac{x^{1-s}}{s-1} + \sum_{n \leq x} n^{-s} + O(x^{-\sigma}), \quad (2-1)$$

valid for any s with $0 < \sigma \leq 2$ and $x \geq t/\pi$ (see [Ivić 85, Theorem 1.8]). Note that the $O(x^{-\sigma})$ term is uniform with respect to t and σ for $|t| \leq \pi x$ and $0 < \sigma_0 \leq \sigma \leq 2$. Evaluating the sums (1-1) with the asymptotic formula (2-1), we obtain

$$\begin{aligned} M \cdot A(s, d, M) &= \sum_{0 \leq m < M} \frac{x^{1-s-imd}}{s+imd-1} \\ &+ \sum_{0 \leq m < M} \sum_{n \leq x} n^{-s-imd} + O(Mx^{-\sigma}), \end{aligned} \quad (2-2)$$

where $x \geq (t + dM)/\pi$. A straightforward estimate yields

$$\begin{aligned} \left| \sum_{0 \leq m < M} \frac{x^{1-s-imd}}{s+imd-1} \right| &\leq x^{1-\sigma} \left(\frac{1}{\sigma} + \sum_{1 \leq m < M} \frac{1}{t+md} \right) \\ &= O(x^{1-\sigma} \log M). \end{aligned} \quad (2-3)$$

In the next step we fix $M, N \in \mathbb{N}$ and study the double sums

$$\Sigma_{N,M}(s) := \sum_{1 \leq n \leq N} \sum_{0 \leq m < M} n^{-s-imd}. \quad (2-4)$$

In order to estimate the inner sums, we introduce $\Delta := d/(2\pi)$ and remark that $n^{-id} = e^{2\pi i \alpha_n}$ or $n^{-id} = e^{-2\pi i \alpha_n}$, where $\alpha_n := \|\Delta \log n\|$ denotes the distance of $\Delta \log n$ to the integers. Setting $b := e^{1/\Delta}$, we have

$$\alpha_n = \min_{j \in \mathbb{Z}} |j - \Delta \log n| = \Delta \min_{j \in \mathbb{Z}} |\log(nb^{-j})|. \quad (2-5)$$

The outer sum is split into three parts by grouping the summands according to the values of α_n . A positive integer n is said to be *special* if it belongs to the set

$$\mathcal{S} := \left\{ n \in \mathbb{N} : \min_{j \in \mathbb{Z}} |n - b^j| < \frac{1}{2} \right\}.$$

The remaining integers in $\mathcal{N} := \mathbb{N} \setminus \mathcal{S}$ are called *normal*. The set \mathcal{S} is the disjoint union of \mathcal{S}_0 and \mathcal{S}_+ ,

$$\mathcal{S}_0 := \{n \in \mathcal{S} : \alpha_n = 0\}, \quad \mathcal{S}_+ := \{n \in \mathcal{S} : \alpha_n > 0\}.$$

Note that we always have $1 \in \mathcal{S}_0$. Further, $\mathcal{S}_0 \neq \{1\}$ if and only if $\Delta = k/\log l$ with $k, l \in \mathbb{Z}$ and $l \geq 2$, i.e., d belongs to the *exceptional set*

$$E := \left\{ d \in \mathbb{R}_+ : d = \frac{2k\pi}{\log l}, k, l \in \mathbb{Z}, l \geq 2 \right\}. \quad (2-6)$$

According to the decomposition of \mathbb{N} we have

$$\begin{aligned} \Sigma_{N,M}(s) &= \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_0}} \sum_{0 \leq m < M} \cdots + \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{N}}} \sum_{0 \leq m < M} \cdots \\ &+ \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_+}} \sum_{0 \leq m < M} \cdots. \end{aligned} \quad (2-7)$$

The first sum extends over those integers n with $\alpha_n = 0$ and can be computed explicitly. If $\mathcal{S}_0 = \{1\}$, then

$$\sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_0}} \sum_{0 \leq m < M} n^{-s-imd} = M. \quad (2-8)$$

If $\mathcal{S}_0 \neq \{1\}$, we denote by ℓ the smallest integer n with $n \geq 2$ and $\alpha_n = 0$. Then $\alpha_n = 0$ if and only if $n = \ell^j$ with $j \in \mathbb{N} \cup \{0\}$. Hence, with $J := \lfloor \log N / \log \ell \rfloor + 1$,

$$\begin{aligned} \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_0}} \sum_{0 \leq m < M} n^{-s-imd} &= M \sum_{\substack{1 \leq n \leq N \\ \alpha_n = 0}} n^{-s} \\ &= M \sum_{0 \leq j \leq J-1} \ell^{-js} = M \frac{1 - \ell^{-sJ}}{1 - \ell^{-s}} \\ &= \frac{M}{1 - \ell^{-s}} + O(MN^{-\sigma}). \end{aligned} \quad (2-9)$$

The estimations of the second and the third sum in (2-7) are based on the following lemma.

Lemma 2.1. Fix $\Delta := d/(2\pi)$ and $n \in \mathbb{N}$. If $\alpha_n = \|\Delta \log n\| \neq 0$, then for all $M \in \mathbb{N}$,

$$\left| \sum_{0 \leq m < M} e^{-imd \log n} \right| < \frac{1}{2\alpha_n}. \quad (2-10)$$

Proof. Recall that $e^{-id \log n} = e^{\pm 2\pi i \alpha_n}$. Since $0 < 2\pi \alpha_n \leq \pi$,/break it follows that $|1 - e^{\pm 2\pi i \alpha_n}| \geq 4\alpha_n$, and thus

$$\begin{aligned} \left| \sum_{0 \leq m < M} e^{-imd \log n} \right| &= \left| \sum_{0 \leq m < M} e^{\pm 2\pi i m \alpha_n} \right| \\ &= \left| \frac{1 - e^{\pm 2\pi i M \alpha_n}}{1 - e^{\pm 2\pi i \alpha_n}} \right| < \frac{2}{4\alpha_n}. \end{aligned}$$

□

The second sum in (2-7) is estimated using Lemma 2.1,

$$\left| \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{N}}} \sum_{0 \leq m < M} n^{-s-imd} \right| \leq \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{N}}} \frac{1}{2n^\sigma \alpha_n}. \quad (2-11)$$

Since $b = e^{1/\Delta} > 1$, it follows that every nonnegative integer n satisfies

$$b^j - \frac{1}{2} \leq n < b^{j+1} - \frac{1}{2}$$

for some natural number j . We denote by k_n^+ and k_n^- the nearest integers to $n - b^j$ and $n - b^{j+1}$, respectively,

$$k_n^+ := \lfloor n - b^j + 1/2 \rfloor, \quad k_n^- := \lfloor n - b^{j+1} + 1/2 \rfloor.$$

Then

$$0 \leq k_n^+ \leq b^{j+1} - b^j, \quad b^j - b^{j+1} - 1 \leq k_n^- \leq -1, \tag{2-12}$$

and n can be represented as

$$n = b^j + k_n^+ + \mu_n, \quad n = b^{j+1} + k_n^- + \nu_n, \tag{2-13}$$

with $-1/2 < \mu_n, \nu_n \leq 1/2$. To simplify notation we just write $\mu, \nu, k^-,$ and k^+ , respectively. From (2-5) we infer that $\alpha_n \log b$ is the minimum of the two numbers

$$\left| \log \left(1 + \frac{\mu + k^+}{b^j} \right) \right|, \quad \left| \log \left(1 + \frac{\nu + k^-}{b^{j+1}} \right) \right|.$$

Now it follows from (2-12) that

$$-\frac{1}{2b^j} \leq \frac{\mu + k^+}{b^j} < b, \quad -\frac{1}{2b^{j+1}} < \frac{\nu + k^-}{b^{j+1}} < 0,$$

and since $|\log(1+x)| \geq |x|/(1+a)$ for $-1 < x \leq a$, we have

$$\begin{aligned} \left| \log \left(1 + \frac{\mu + k^+}{b^j} \right) \right| &\geq \frac{|\mu + k^+|}{b^j(1+b)} \geq \frac{k^+}{2b^j(1+b)} \geq \frac{k^+}{4b^{j+1}}, \\ \left| \log \left(1 + \frac{\nu + k^-}{b^{j+1}} \right) \right| &\geq \frac{|\nu + k^-|}{b^{j+1}} \geq \frac{|k^-|}{2b^{j+1}}. \end{aligned}$$

Consequently,

$$\alpha_n \geq \frac{\min\{|k_n^-|, k_n^+\}}{4b^{j+1} \log b}. \tag{2-14}$$

As $n \in \mathbb{N}$ runs through the interval $[b^j - 1/2, b^{j+1} - 1/2)$, k_n^- attains every integer in $[b^j - b^{j+1} - 1, -1]$ at most once, and k_n^+ attains every integer in $[1, b^{j+1} - b^j]$ at most once.

Setting $J := \lfloor \log N / \log b \rfloor + 1$ and using (2-11), (2-14), and $n \geq b^j - 1/2 \geq b^j/2$, we estimate the second sum on the right-hand side of (2-7) for all $N \geq$

$\max\{3, b\}$,

$$\begin{aligned} &\left| \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{N}}} \sum_{0 \leq m < M} n^{-s-imd} \right| \\ &\leq \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{N}}} \frac{1}{2n^\sigma \alpha_n} \leq \sum_{0 \leq j < J} \sum_{1 \leq k < b^{j+1}} \frac{8b^{j+1} \log b}{b^{j\sigma} k} \\ &\leq 8b \log b \sum_{0 \leq j < J} b^{(1-\sigma)j} \sum_{1 \leq k < b^{j+1}} \frac{1}{k} \tag{2-15} \\ &\leq 8b \log b \sum_{0 \leq j < J} b^{(1-\sigma)j} (1 + (j+1) \log b) \\ &\leq \frac{8b \log b}{b^{1-\sigma} - 1} (b^{J(1-\sigma)} (J \log b + 1)) \\ &= O(N^{1-\sigma} \log N). \end{aligned}$$

Before we study the last sum of (2-7) over $n \in \mathcal{S}_+$, we summarize the results.

Lemma 2.2. *Let $d > 0$, $s = \sigma + it \neq 1$ with $0 < \sigma < 1$ and $t \geq 0$, and set*

$$\Sigma_{N,M}(s) := \sum_{1 \leq n \leq N} \sum_{0 \leq m < M} n^{-s-imd}.$$

(i) *If d does not belong to the exceptional set E defined in (2-6), then*

$$\begin{aligned} \Sigma_{N,M}(s) &= M + \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_+}} \sum_{0 \leq m < M} n^{-s-imd} \\ &\quad + O(N^{1-\sigma} \log N). \end{aligned}$$

(ii) *If $d \in E$ and $\ell \geq 2$ is the smallest integer with $d = 2k\pi / \log \ell$, then*

$$\begin{aligned} \Sigma_{N,M}(s) &= \frac{M}{1 - \ell^{-s}} + \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_+}} \sum_{0 \leq m < M} n^{-s-imd} \\ &\quad + O(N^{-\sigma} (M + N \log N)). \end{aligned}$$

(iii) *If $d = 2\pi / \log \ell$ with $\ell \in \mathbb{N}$ and $\ell \geq 2$, then*

$$\Sigma_{N,M}(s) = \frac{M}{1 - \ell^{-s}} + O(N^{-\sigma} (M + N \log N)).$$

Proof. The first two statements follow from (2-8), (2-9), and (2-15). If $d = 2\pi/\ell$, then $\Delta = 1/\log \ell$ and $\alpha_n = 0$ if and only if $n = \ell^p$ for some $p \in \mathbb{N}$. Moreover, if $\alpha_n \neq 0$, then $|n - \ell^j| \geq 1$ for all $j \in \mathbb{Z}$, i.e., \mathcal{S}_+ is void. \square

To complete the proof of Theorem 1.1 it suffices to choose $x = N = \lfloor Md + t \rfloor$ and to apply the asymptotic formula (2-1) to $\zeta(s + imd)$. This yields (2-2), and the desired result then follows from (2-3) and Lemma 2.2(iii).

In the cases that are not covered by Theorem 1.1, the behavior of the sum over $n \in \mathcal{S}_+$ is crucial. Again we have the estimate

$$\left| \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_+}} \sum_{0 \leq m < M} n^{-s-imd} \right| \leq \sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_+}} \frac{1}{2n^\sigma \alpha_n},$$

but even in the case that $\Delta = k/\log \ell$ with $k \geq 2$ (and the smallest possible value of ℓ), the sum on the right-hand side is difficult to estimate. If, for example, there exists a fixed value c such that $1 \leq |n^k - \ell^j| \leq c$ for an infinite number of positive integers n and j , then

$$\alpha_n \sim \frac{\Delta}{k} |\log(1 \pm c\ell^{-j})| \sim \frac{c\Delta}{k} \ell^{-j} \sim \frac{c\Delta}{k} n^k.$$

Consequently, if the considered sum admits an estimate

$$\sum_{\substack{1 \leq n \leq N \\ n \in \mathcal{S}_+}} \frac{1}{2n^\sigma \alpha_n} \leq CN^\beta,$$

then $\beta \geq k - \sigma$, which is useless for $k \geq 2$.

3. CONCLUDING REMARKS

The asymptotic formula of Theorem 1.1 holds in a wider range, namely the right half-plane $\operatorname{Re} s > 0$, provided that no term of the arithmetic progression $s + imd$ is identical to the pole of the zeta function at 1. Actually, in the half-plane $\operatorname{Re} s > 1$, where the zeta function is given by an absolutely convergent Dirichlet series,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

the asymptotic behavior of $A(s, d, M)$ can be found quite easily for all d in the exceptional set E . Using the same notation as above, we have, for some positive integer N ,

$$\begin{aligned} & \sum_{0 \leq m < M} \zeta(s + imd) \\ &= M \sum_{\substack{n \in \mathcal{S}_0 \\ n < N}} n^{-s} + \left\{ \sum_{\substack{n < N \\ n \notin \mathcal{S}_0}} + \sum_{\substack{n \geq N \\ n \notin \mathcal{S}_0}} \right\} n^{-s} \sum_{0 \leq m < M} e^{-imd \log n} \\ &= S_1 + S_2 + S_3, \end{aligned}$$

say. The first sum S_1 yields the main term as in the proof of Theorem 1.1. In view of Lemma 2.1, the sum over m in S_2 is bounded by $1/\min_{n < N} \alpha_n$, and thus S_2 is bounded by some constant independent of M . For S_3 we use the trivial bound to obtain

$$S_3 \ll M \sum_{n \geq N} n^{-\operatorname{Re} s} \ll MN^{1-\operatorname{Re} s}.$$

Since $\operatorname{Re} s > 1$, this implies

$$\frac{1}{M} \sum_{0 \leq m < M} \zeta(s + imd) = C(s, d) + o(1) \tag{3-1}$$

as $M \rightarrow \infty$, where $C(s, d) = (1 - \ell^{-s})^{-1}$ for $d \in E$, and $C(s, d) = 1$ otherwise. For $\operatorname{Re} s = 1$, the proof of Theorem 1.1 has to be modified in an obvious way.

As mentioned in the introduction, asymptotic formulas were obtained in [Good 78] for the second and fourth moments of the zeta function on arbitrary vertical arithmetic progressions for $\operatorname{Re} s > 1/2$, for instance,

$$\frac{1}{M} \sum_{0 \leq m < M} |\zeta(\sigma + imd)|^4 = \sum_{n=1}^{\infty} \frac{d(n)^2}{n^{2\sigma}} + o(1)$$

as $M \rightarrow \infty$, provided that d is not of the form $2\pi l(\log \frac{m}{n})^{-1}$ with integral $l \neq 0$ and positive integers $m \neq n$. Here the coefficients of the Dirichlet series on the right-hand side are defined by the divisor function $d(n) := \sum_{d|n} 1$; hence the series on the right does not converge for $\sigma = 1/2$. Good’s approach yields asymptotic formulas with remainder term if the common difference d of the arithmetic progression is of special Diophantine type. Applying the Cauchy–Schwarz inequality, we see that the results of [Good 78] can be used to show that the first moment $A(s, d, M)$ is bounded as $M \rightarrow \infty$ for every value of d , provided that $\sigma > 1/2$ (and $s + imd \neq 1$).

In contrast to Good’s results, Theorem 1.1 provides an asymptotic formula for the whole critical strip $0 < \operatorname{Re} s < 1$ for specific values of d .

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