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Semistable Vector Bundles and Tannaka Duality from a Computational Point of View

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CONTENTS

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We develop a semistability algorithm for vector bundles that are given as a kernel of a surjective morphism between splitting bundles on the projective space $\mathbb{P}^N$ over an algebraically closed field K. This class of bundles is a generalization of syzygy bundles. We show how to implement this algorithm in a computer algebra system. Further, we give applications, mainly concerning the computation of Tannaka dual groups of stable vector bundles of degree 0 on $\mathbb{P}^N$ and on certain smooth complete intersection curves. We also use our algorithm to close a case left open in a recent work of L. Costa, P. Macias Marques, and R. M. Miró-Roig regarding the stability of the syzygy bundle of general forms. Finally, we apply our algorithm to provide a computational approach to tight closure. All algorithms are implemented in the computer algebra system CoCoA.

1. INTRODUCTION

The notion of slope (semi)stability for vector bundles on a smooth projective variety over an algebraically closed field K, as introduced by D. Mumford in the case of curves and generalized by F. Takemoto to higher-dimensional varieties, is a very important tool in algebraic geometry. Unfortunately, for a concretely given vector bundle it is often very difficult to decide whether it is semistable or even stable. In this paper we develop an algorithm to determine computationally the semistability of certain vector bundles on the projective space $\mathbb{P}^N$. Throughout this paper we assume that $N \geq 2$, since for $N = 1$, by a theorem of A. Grothendieck, every vector bundle splits as a direct sum of line bundles. We restrict ourselves to vector bundles that are given as a kernel of a surjective morphism between splitting bundles, i.e., vector bundles $\mathcal{E}$ that sit in a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\varphi} \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0.$$

We call such bundles *kernel bundles*. For instance, by Horrocks's theorem , every nonsplit vector bundle on $\mathbb{P}^2$ admits such a presentation. The morphism φ that defines $\mathcal{E}$ is given by an $m \times n$ matrix $\mathcal{M} = (a_{ji})$, where the entries $a_{ji} \in R := K[X_0, \dots, X_N]$ are homogeneous polynomials of degree $b_j - a_j$. Special instances ($m = 1$ and $b_1 = 0$) of kernel bundles are the so-called *syzygy bundles* $\mathrm{Syz}(f_1, \dots, f_n)$ for R_+-primary homogeneous polynomials $f_1, \dots, f_n$ (i.e., $\sqrt{(f_1, \dots, f_n)} = R_+$), that is, a syzygy bundle has a presenting sequence

$$0 \longrightarrow \mathrm{Syz}(f_1, \dots, f_n) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^N}(-d_i) \overset{f_1,\dots,f_n}{\longrightarrow} \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0,$$

where $d_i = \deg(f_i)$. Due to their explicit nature, kernel bundles and syzygy bundles are suitable for direct computations, in particular using Gröbner-basis methods and combinatorics. But in general, not much is known about (semi)stability of kernel bundles or even syzygy bundles. One of the most important results in this direction, due to H. Brenner, is a combinatorial criterion for (semi)stability of syzygy bundles given by monomial families:

Theorem 1.1. (Brenner.) *Let K be a field, $R := K[X_0, \dots, X_N]$, and let $f_i = X^{\sigma_i}$ denote R_+-primary monomials of degree $d_i = |\sigma_i|$ in $K[X_0, \dots, X_N]$, $i = 1, \dots, n$. Suppose that for every subset $J \subseteq I := \{1, \dots, n\}$, $|J| \geq 2$, the inequality*

$$\frac{d_J - \sum_{i \in J} d_i}{|J| - 1} \leq \frac{-\sum_{i \in I} d_i}{n-1}$$

holds, where d_J is the degree of the greatest common factor of f_i, $i \in J$. Then the syzygy bundle $\mathrm{Syz}(f_1, \dots, f_n)$ is semistable (and stable if $<$ holds).

Proof. See [Brenner 08a, Corollary 6.4]. □

Another important theorem, due to G. Bohnhorst and H. Spindler, is a numerical (semi)stability criterion for kernel bundles of rank N on $\mathbb{P}^N$ in characteristic 0:

Theorem 1.2. (Bohnhorst–Spindler.) *Let $\mathcal{E}$ be a vector bundle of rank $N \geq 2$ on the projective space $\mathbb{P}^N$ over an algebraically closed field K of characteristic 0. Suppose there is a short exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{N+k} \mathcal{O}_{\mathbb{P}^N}(a_i) \longrightarrow \bigoplus_{j=1}^{k} \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0$$

such that $a_1 \geq \cdots \geq a_{N+k}$, $b_1 \geq \cdots \geq b_k$, and $b_j > a_j$ for $j = 1, \dots, k$. Then $\mathcal{E}$ is semistable (respectively stable) if and only if

$$a_{N+k} \geq \text{ (respectively } >) \ \mu(\mathcal{E}) = \frac{1}{N}\Big(\sum_{i=1}^{N+k} a_i - \sum_{j=1}^{k} b_j\Big).$$

Proof. This is [Bohnhorst and Spindler 92, Theorem 2.7] applied to the dual bundle $\mathcal{E}^*$. □

A general algorithm using Gröbner-basis methods (computation of syzygy modules) that detects semistability of syzygy bundles and its implementation by the first author was announced in [Brenner 08a, Remark 5.3]. In this article, we describe this algorithm more generally for kernel bundles and describe in detail how to implement it in a computer algebra system (this has been done concretely by the first author in CoCoA).[1] This semistability algorithm can be used as a tool to examine further problems regarding semi(stability) of vector bundles by providing interesting examples. We explain these applications in more detail in the sequel. The paper is organized as follows.

In Section 2, we recall a criterion due to H. J. Hoppe (see Proposition 2.1) that relates (semi)stability to global sections of exterior powers of a given vector bundle. In particular, we show that this result, originally formulated only in characteristic 0, holds in arbitrary characteristic. Hoppe's criterion is the key result for our algorithm.

In Section 3, we discuss some properties of kernel bundles and syzygy bundles on projective spaces. In particular, for these vector bundles we discuss necessary Bohnhorst/Spindler-like numerical conditions (compare Theorem 1.2 above) for semistability.

The actual semistability algorithm for kernel bundles and its implementation is explained in Section 4. Besides exterior powers, we also describe explicitly how to compute global sections of tensor products and symmetric powers of kernel bundles. These algorithms play an important role in our first application: the computation of Tannaka dual groups of polystable vector bundles $\mathcal{E}$ of degree 0 and rank r on $\mathbb{P}^N$ in characteristic 0.

Section 5 starts with a brief introduction to Tannaka duality. Roughly speaking, for a polystable vector bundle $\mathcal{E}$ of degree 0, one can find a semisimple algebraic group $G_{\mathcal{E}}$ and an equivalence of categories between the abelian tensor category generated by $\mathcal{E}$ and the category of finite-dimensional representations of $G_{\mathcal{E}}$. The algebraic group $G_{\mathcal{E}}$ is called the *Tannaka dual group* of $\mathcal{E}$. It was shown in [Kasprowitz 10, Lemma 4.4 and Proposition 5.3] that for stable vector bundles of degree 0 and rank r, as in Theorem 1.2, the almost simple components of the Tannaka dual group are of type A. Using a result in [Anisimov 11, Theorem 1], one can even show that it is always the group SL_r if it is almost simple. We explain how to compute the

[1] CoCoA is available at http://cocoa.dima.unige.it.

Tannaka dual group for an arbitrary stable kernel bundle of degree 0 on $\mathbb{P}^N$ and construct examples for low-rank syzygy bundles on $\mathbb{P}^2$ having the symplectic group Sp_r as Tannaka dual group.

Furthermore, we are interested in the behavior of the Tannaka dual group after restricting the bundles to smooth curves. In Section 6, we describe a method to construct for certain kernel bundles $\mathcal{E}$ on $\mathbb{P}^N$ a finite morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ such that the restriction of the pullback $f^*(\mathcal{E})$ to certain complete intersection curves of sufficiently large degree has the same Tannaka dual group as the vector bundle $f^*(\mathcal{E})$ on $\mathbb{P}^N$. We show that this works, for example, for the syzygy bundles constructed in Section 5. We would like to draw the reader's attention to the paper [Balaji 07], in which the author shows the existence of a rank-2 bundle $\mathcal{E}$ with $c_2(\mathcal{E}) \gg 0$ on a smooth surface X such that the restriction to a curve of genus > 1 has Tannaka dual group SL_2; see also [Balaji 09, Proposition 3]. Balaji's method is completely different from ours. He uses this result to show that the moduli space of stable principal H-bundles on X with large characteristic classes is nonempty, where H is any semisimple algebraic group ([Balaji 07, Chapter 7]).

In Section 7 we close a case left open in the paper [Costa et al. 10], where the authors show the stability of the generic syzygy bundle on $\mathbb{P}^2$ except for the bundle generated by five generic quadrics. We use the results obtained in Section 5 and construct an example for a stable syzygy bundle in this case, which gives the generic result via the openness of stability.

In the final section we provide another application of the semistability algorithm concerning the computation of tight/solid closure of homogeneous ideals in the coordinate ring of a smooth projective curve. This is possible due to the geometric approach to this topic developed by H. Brenner.

2. THEORETICAL BACKGROUND: HOPPE'S SEMISTABILITY CRITERION

We recall that a torsion-free sheaf $\mathcal{E}$ on a smooth projective variety X over an algebraically closed field K is *semistable* if for every coherent subsheaf $0 \neq \mathcal{F} \subset \mathcal{E}$, the inequality

$$\mu(\mathcal{F}) := \deg(\mathcal{F})/\operatorname{rk}(\mathcal{F}) \leq \deg(\mathcal{E})/\operatorname{rk}(\mathcal{E}) = \mu(E)$$

holds. The sheaf $\mathcal{E}$ is *stable* if the inequality is always strict. The degree of a sheaf $\mathcal{F}$ is defined using intersection theory and a fixed very ample invertible sheaf $\mathcal{O}_X(1)$ (which is also called a *polarization* of X) as $\deg(F) = \deg(c_1(\mathcal{F}).\mathcal{O}_X(1)^{\dim(X)-1})$. For every coherent torsion-free sheaf $\mathcal{E}$ there exists a unique filtration $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_t = \mathcal{E}$, called the *Harder–Narasimhan filtration*, such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and

$$\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \cdots > \mu(\mathcal{E}/\mathcal{E}_{t-1}).$$

The slopes $\mu(\mathcal{E}_1)$ and $\mu(\mathcal{E}/\mathcal{E}_{t-1})$ are also denoted by $\mu_{\max}(\mathcal{E})$ and $\mu_{\min}(\mathcal{E})$, respectively. If K is not algebraically closed, then we define the terms degree, semistable, etc., via the algebraic closure of K.

If the characteristic of the base field K is 0, it is well known that the tensor product $\mathcal{E} \otimes \mathcal{F}$ of two semistable vector bundles $\mathcal{E}$ and $\mathcal{F}$ on a smooth projective polarized variety $(X, \mathcal{O}_X(1))$ is again semistable, and this also holds for exterior powers and symmetric powers (cf. [Huybrechts and Lehn 97, Theorem 3.1.4 and Corollary 3.2.10]). This no longer holds in characteristic $p > 0$. This is due to the fact that the (absolute) *Frobenius morphism* $F : X \to X$ may destroy semistability. That is, the Frobenius pullback $F^*(\mathcal{E})$ of a semistable vector bundle $\mathcal{E}$ is in general not semistable; see, for instance, the example of Serre in [Hartshorne 71, Example 3.2].

In this paper, we are mainly interested in vector bundles on projective spaces. The following result is well known in characteristic 0 (see, for instance, [Brenner 08a, Proposition 5.1] or [Bohnhorst and Spindler 92, Proposition 1.1]). It gives an algorithmic criterion to check semistability of a vector bundle on $\mathbb{P}^N$ in terms of global sections of its exterior powers. It uses the trivial but useful fact (in particular for a computational approach to semistability) that a semistable vector bundle of negative degree (or slope) does not have any nontrivial global sections. Since the key idea goes back to H. J. Hoppe (see [Hoppe 84, Lemma 2.6]), this result is attributed to him. Hoppe's result is also true in positive characteristic due to the fact that semistability for vector bundles on a projective space behaves nicely with respect to tensor operations.

Proposition 2.1. (Hoppe.) *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^N$ over an algebraically closed field K. Then the following hold:*

1. *The bundle $\mathcal{E}$ is semistable if and only if for every $q < \operatorname{rk}(\mathcal{E})$ and every $k < -q\mu(\mathcal{E})$, there exists no nontrivial global section of $(\bigwedge^q \mathcal{E})(k)$.*
2. *If $\Gamma(\mathbb{P}^N, (\bigwedge^q \mathcal{E})(k)) = 0$ for every $q < \operatorname{rk}(\mathcal{E})$ and every $k \leq -q\mu(\mathcal{E})$, then $\mathcal{E}$ is stable.*

Proof. Since $\mu_{\max}(\Omega_{\mathbb{P}^N}) \leq 0$, a semistable vector bundle $\mathcal{E}$ on $\mathbb{P}^N$ is *strongly semistable* by [Mehta and Ramanathan 82a, Theorem 2.1] (we recall that this means that the Frobenius pullbacks $F^{e*}(\mathcal{E})$ are semistable for all $e \geq 0$). Hence it follows from [Ramanan and Ramanathan 84, Theorem 3.23] that $\bigwedge^q \mathcal{E}$ is also semistable for all $q \leq \operatorname{rk}(\mathcal{E})$. The first statement can now be proven as in [Brenner 08a, Proposition 5.1]. The second statement follows in the same way if we replace $<$ by $\leq$ appropriately. □

Remark 2.2. Let $(X, \mathcal{O}_X(1))$ be a polarized smooth projective variety of dimension $d \geq 1$ defined over an algebraically closed field of characteristic 0. For a semistable vector bundle $\mathcal{E}$ on X, the numerical condition on the exterior powers $\bigwedge^q \mathcal{E}$ in Proposition 2.1 is still fulfilled if we replace the degree bound for the global sections by $k < -q\mu(\mathcal{E})/\deg(\mathcal{O}_X(1))$ for every $1 \leq q < \operatorname{rk}(\mathcal{E})$. Hence, the numerical condition is, up to the factor $1/\deg(\mathcal{O}_X(1))$, always necessary for semistability.

If additionally, $\operatorname{Pic}(X) = \mathbb{Z}$, then the numerical criterion is again equivalent to the semistability of $\mathcal{E}$. Important examples of varieties with this property are general surfaces of degree ≥ 4 in $\mathbb{P}^3_{\mathbb{C}}$ (Noether's theorem) and (in arbitrary characteristic) complete intersections of dimension ≥ 3 in $\mathbb{P}^N$ (see [Hartshorne 70, Corollaries IV.3.2 and IV.4(i)]).

In positive characteristic (under the assumption $\operatorname{Pic}(X) = \mathbb{Z}$), the numerical condition on the exterior powers still implies semistability, but the equivalence in Proposition 2.1 holds only if every semistable vector bundle on X is strongly semistable (compare the proof of Proposition 2.1). Thus it is clear that Proposition 2.1 does not provide an algorithmic tool to detect semistability of vector bundles on curves. For algorithmic methods to determine semistability and strong semistability of vector bundles over an algebraic curve in positive characteristic; see [Kaid 09, Chapter 3].

Example 2.3. Let $F \in \mathbb{Z}[X_0, \ldots, X_N]$, $N \geq 4$, be a homogeneous polynomial of degree d such that the hypersurface $X := \operatorname{Proj}(\mathbb{Q}[X_0, \ldots, X_N]/(F))$ is smooth. By Remark 2.2, we have $\operatorname{Pic}(X) \cong \mathbb{Z}$, and thus Hoppe's criterion, Proposition 2.1, is applicable to determine semistability of vector bundles on X. Now we assume that $d = N + 1$. Then the canonical bundle $\omega_X \cong \mathcal{O}_X$ is trivial, which implies the semistability of the cotangent bundle Ω_X (see [Peternell 01, Theorem 3.1]). In particular, X is a Calabi–Yau variety. We consider X as the generic fiber $\mathcal{X}_0$ of the generically smooth projective morphism

$$\mathcal{X} := \operatorname{Proj}(\mathbb{Z}[X_0, \ldots, X_N]/(F)) \longrightarrow \operatorname{Spec} \mathbb{Z}$$

of relative dimension $N - 2$. Up to finitely many exceptions, the special fiber $\mathcal{X}_p$ over a prime number p is a smooth projective variety over the finite field $\mathbb{F}_p$ with $\operatorname{Pic}(\mathcal{X}_p) = \mathbb{Z}$. By the openness of semistability, the cotangent bundle $\Omega_{\mathcal{X}_p}$ of the special fiber $\mathcal{X}_p$ is semistable too for almost all prime numbers p. Since $\deg(\Omega_{\mathcal{X}_p}) = 0$, every semistable vector bundle is strongly semistable on $\mathcal{X}_p$ by [Mehta and Ramanathan 82a, Theorem 2.1]. Thus for $p \gg 0$, we can also use Proposition 2.1 to detect semistability of vector bundles on $\mathcal{X}_p$ (in positive characteristic).

3. SYZYGY BUNDLES AND KERNEL BUNDLES

In the remainder of this article, we restrict ourselves to vector bundles on $\mathbb{P}^N$, $N \geq 2$, that are kernels of surjective morphisms between splitting bundles, i.e., bundles sitting inside a short exact sequence of the form

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\varphi} \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0,$$

where $n \geq m$. The morphism φ is given by an $m \times n$ matrix $\mathcal{M} = (a_{ji})$, where the entries $a_{ji} \in R := K[X_0, \ldots, X_N]$ are homogeneous polynomials of degree $b_j - a_i$. In this paper, we call such a vector bundle a *kernel bundle*. Special instances of kernel bundles are *syzygy bundles* that correspond to the case $m = 1$ and $b_1 = 0$, i.e., a syzygy bundle $\operatorname{Syz}(f_1, \ldots, f_n)$ is given by a short exact sequence

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^N}(-d_i) \overset{f_1, \ldots, f_n}{\longrightarrow} \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0,$$

where $f_1, \ldots, f_n \in R = K[X_0, \ldots, X_N]$ are homogeneous polynomials of degree d_i, $i = 1, \ldots, n$. If one of the polynomials is constant, the syzygy bundle $\operatorname{Syz}(f_1, \ldots, f_n)$ is obviously split. To exclude this case, one often demands that the ideal $(f_1, \ldots, f_n)$ be R_+-primary, i.e., $\sqrt{(f_1, \ldots, f_n)} = R_+ = (X_0, \ldots, X_N)$. The most prominent example of a syzygy bundle is the *cotangent bundle* $\Omega_{\mathbb{P}^N} \cong \operatorname{Syz}(X_0, \ldots, X_N)$ of $\mathbb{P}^N$.

If $\mathcal{E}$ does not split as a direct sum of line bundles, then the dual bundle $\mathcal{E}^*$ of a kernel bundle has homological dimension 1, and therefore we obtain the inequality $\operatorname{rk}(\mathcal{E}) \geq N$ (see [Bohnhorst and Spindler 92, Corollary 1.7]). We can compute the topological invariants of a kernel bundle $\mathcal{E}$ from the presenting short exact sequence above. We have

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E})}{\operatorname{rk}(\mathcal{E})} = \frac{1}{n - m}\Big(\sum_{i=1}^{n} a_i - \sum_{j=1}^{m} b_j\Big).$$

Since the Chern polynomial is multiplicative on short exact sequences, it is also easy to compute higher Chern classes of kernel bundles.

In the sequel, we show that the twists $a_1, \dots, a_n$ and $b_1, \dots, b_m$ that occur in the presenting sequence of a kernel bundle have to fulfill a certain numerical condition that is necessary for semistability (stability). We remark that this condition is also necessary for the semistability (stability) of kernel bundles (and in particular of syzygy bundles) on arbitrary smooth projective varieties.

If $\mathcal{E}$ is a vector bundle on $\mathbb{P}^N$ of rank r, and $\mathcal{F} \subset \mathcal{E}$ a subsheaf of rank $r-1$, then the quotient $\mathcal{E}/\mathcal{F}$ is isomorphic outside codimension 2 to $\mathcal{O}_{\mathbb{P}^N}(\ell)$ for some $\ell \in \mathbb{Z}$. This is equivalent to a section $\mathcal{O}_{\mathbb{P}^N} \to \mathcal{E}^*(-\ell)$. For kernel bundles we are able to control such sections by an easy numerical condition. In particular, we can replace the condition on the global sections of the $(r-1)$th exterior power in Hoppe's criterion, Proposition 2.1, by this condition if the resolution of $\mathcal{E}^*$ is minimal. For the notion of minimality, see, for instance, [Ottaviani and Valles 06, Section 7.2].

Lemma 3.1. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^N$, $N \geq 2$, sitting in a short exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^N}(a_i) \longrightarrow \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0,$$

where $a_1 \geq a_2 \geq \cdots \geq a_n$. If

$$a_n \geq \text{(respectively} >\text{)}\ \mu(\mathcal{E}) = \frac{1}{n-m}\left(\sum_{i=1}^{n} a_i - \sum_{j=1}^{m} b_j\right),$$

then there are no mappings from $\mathcal{E}$ to line bundles that contradict the semistability (respectively stability) of $\mathcal{E}$. Moreover, if the dualized sequence is a minimal resolution for $\mathcal{E}^*$, then this numerical condition is necessary for semistability (stability).

Proof. We twist the short exact sequence presenting $\mathcal{E}$ with $\mathcal{O}_{\mathbb{P}^N}(\ell)$ and look at the dual sequence

$$0 \longrightarrow \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^N}(-\ell - b_j) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^N}(-\ell - a_i) \longrightarrow (\mathcal{E}(\ell))^* \cong \mathcal{E}^*(-\ell) \longrightarrow 0.$$

Since

$$H^1\left(\mathbb{P}^N, \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^N}(-\ell - b_j)\right) = 0$$

for all $\ell \in \mathbb{Z}$, every global section of $(\mathcal{E}(\ell))^*$ comes from $\Gamma(\mathbb{P}^N, \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^N}(-\ell - a_i))$. Consequently, for $\ell > -a_n$ there exists no nontrivial morphism $\mathcal{E}(\ell) \to \mathcal{O}_{\mathbb{P}^N}$. By assumption, we have $-a_n \leq \mu(\mathcal{E}^*)$. So we have $\Gamma(\mathbb{P}^N, \mathcal{E}^*(-\ell)) = 0$ for $\ell > \mu(\mathcal{E}^*)$. Hence there are no mappings to line bundles that contradict the semistability. Analogously, one obtains the corresponding statement for stability.

Next, we prove the remaining statement. Assume that

$$a_n < \text{(respectively} \leq\text{)}\ \mu(\mathcal{E}).$$

It follows from the assumption on the resolution of $\mathcal{E}^*$ that the mapping $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^N}(a_n)$ is nonzero. But such a morphism does not exist for $\mathcal{E}$ semistable or stable. □

It is easy to check that the subsheaf

$$\ker\left(\bigoplus_{i \neq n} \mathcal{O}_{\mathbb{P}^N}(a_i) \to \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^N}(b_i)\right)$$

destabilizes the kernel bundle $\mathcal{E}$ if $a_n < \mu(\mathcal{E})$ and the resolution of $\mathcal{E}^*$ is minimal.

In [Bohnhorst and Spindler 92, Proposition 2.3], the authors show that for a kernel bundle $\mathcal{E}$ of rank N on $\mathbb{P}^N$, the corresponding resolution of $\mathcal{E}^*$ is minimal if and only if $a_1 \geq \cdots \geq a_{N+k}$, $b_1 \geq \cdots \geq b_k$, and $b_j > a_j$ for $j = 1, \dots, k$. Hence, in characteristic 0, Theorem 1.2 shows that for kernel bundles of this type, the numerical condition of Lemma 3.1 is even sufficient for semistability (stability). For syzygy bundles, we give an easy characteristic-free proof of Theorem 1.2.

Proposition 3.2. *Let K be a field and let $f_1, \dots, f_{N+1} \in R = K[X_0, \dots, X_N]$ be homogeneous parameters of degree $1 \leq d_1 \leq \cdots \leq d_{N+1}$. If $d_1 + \cdots + d_N \geq (N-1)d_{N+1}$, then the syzygy bundle $\operatorname{Syz}(f_1, \dots, f_{N+1})$ is semistable on $\mathbb{P}^N$. If the inequality is strict, then $\operatorname{Syz}(f_1, \dots, f_{N+1})$ is a stable bundle.*

Proof. We use Proposition 2.1 to check the semistability of $\operatorname{Syz}(f_1, \dots, f_{N+1})$. For a subset $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, N+1\}$ we use the notation $d_I = \sum_{j=1}^{k} d_{i_j}$. We consider the Koszul complex

$$0 \longrightarrow \mathcal{F}_{N+1} \longrightarrow \mathcal{F}_N \longrightarrow \cdots \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0$$

on $\mathbb{P}^N$ associated with the parameters $f_1, \dots, f_{N+1}$, where

$$\mathcal{F}_k := \bigoplus_{|I|=k} \mathcal{O}_{\mathbb{P}^N}(-d_I)$$

for $k = 1, \dots, N+1$. We have to show for every $i < N$ and every

$$m < -i\mu(\mathrm{Syz}(f_1, \dots, f_{N+1})) = \frac{i \sum_{j=1}^{N+1} d_j}{N}$$

that

$$\Gamma(\mathbb{P}^N, \Big(\bigwedge^i \mathrm{Syz}\,(f_1, \dots, f_{N+1})\Big)(m)) = 0.$$

For every $1 \le i < N$ we have a surjection

$$\mathcal{F}_{i+1} \longrightarrow \bigwedge^i \mathrm{Syz}(f_1, \dots, f_n) \longrightarrow 0.$$

Since the Koszul complex of a regular sequence is also globally exact, every global section of $(\bigwedge^i \mathrm{Syz}(f_1, \dots, f_n))(m)$ comes from the bundle $\mathcal{F}_{i+1}(m)$. We have

$$\Gamma(\mathbb{P}^N, \mathcal{F}_{i+1}(m)) = 0$$

for $m < d_1 + d_2 + \dots + d_{i+1}$. The assumption

$$\sum_{i=1}^N d_i \ge (N-1)d_{N+1}$$

implies

$$(N-i)(d_1 + \dots + d_{i+1}) \ge i(d_{i+2} + \dots + d_{N+1}),$$

for $1 \le i \le N-1$ (for this easy computation, see [Brenner 08a, Corollary 2.4]). But this is equivalent to

$$N(d_1 + \dots + d_{i+1}) \ge i(d_1 + \dots + d_{N+1}),$$

$1 \le i \le N-1$, and we obtain the assertion. The remaining statement follows analogously. □

4. THE SEMISTABILITY ALGORITHM AND ITS IMPLEMENTATION

The aim of this section is to use Hoppe's semistability criterion, Proposition 2.1, to obtain a semistability algorithm for kernel bundles that can be implemented in a computer algebra system.[2]

The major advantage of kernel bundles compared to arbitrary vector bundles is that we can compute the global sections $\Gamma(\mathbb{P}^N, \bigwedge^q \mathcal{E})$ of their exterior powers in a way that is suitable for a computer algebra system. We do not claim that such a computational approach is impossible for other vector bundles, but at least it requires more technical effort. For a kernel bundle $\mathcal{E}$, we give (probably well-known) presentations of the tensor operations (tensor powers, exterior powers, and symmetric powers) as kernels of mappings between splitting bundles (but these mappings are in general not surjective). This enables us to compute the global sections of these vector bundles described by applying the left exact functor $\Gamma(\mathbb{P}^N, -)$. For our semistability algorithm, we require such a presentation only for the exterior powers, but we will need the other tensor operations in Section 5.

Proposition 4.1. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^N$ that sits in a short exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i) \overset{\mathcal{M}=(a_{ji})}{\longrightarrow} \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0.$$

Set $I := \{1, \dots, n\}$, $J := \{1, \dots, m\}$, and $Q := \{1, \dots, q\}$ for a fixed $q \in \mathbb{N}$.

The qth tensor product of $\mathcal{E}$ sits in the exact sequence

$$0 \longrightarrow \mathcal{E}^{\otimes q} \to \bigoplus_{\alpha \in I^q} \mathcal{O}_{\mathbb{P}^N}\Big(\sum_{p \in Q} a_{\alpha_p}\Big) \overset{\varphi_q}{\longrightarrow} \bigoplus_{\substack{(\beta,j,p) \\ \beta \in I^{q-1}, j \in J, p \in Q}} \mathcal{O}_{\mathbb{P}^N}\Big(\sum_{p \in Q - \{q\}} a_{\beta_p} + b_j\Big),$$

where the map φ_q is given by

$$e_\alpha \longmapsto \sum_{\substack{(j,p) \\ j \in J, p \in Q}} a_{j,\alpha_p} e_{(\alpha^{(p)},j,p)}.$$

Here $\alpha^{(p)}$ means the $(q-1)$-tuple α without the pth element.

The qth exterior power of $\mathcal{E}$, $1 \le q < n-m$, sits in the exact sequence

$$0 \longrightarrow \bigwedge^q \mathcal{E} \longrightarrow \bigoplus_{A \subseteq I, |A|=q} \mathcal{O}_{\mathbb{P}^N}\Big(\sum_{i \in A} a_i\Big) \overset{\varphi_q}{\longrightarrow} \bigoplus_{\substack{(B,j) \\ B \subseteq I, |B|=q-1, j \in J}} \mathcal{O}_{\mathbb{P}^N}\Big(\Big(\sum_{i \in B} a_i\Big) + b_j\Big),$$

where the map φ_q is given by

$$e_A \longmapsto \sum_{\substack{(i,j) \\ i \in A, j \in J}} \mathrm{sign}(i, A)\, a_{ji}\, e_{(A-\{i\},j)}.$$

We remark that the subset $A \subset I$ is supposed to have the induced order and

$$\mathrm{sign}(i, A) = \begin{cases} -1, & \text{if } i \text{ is an even element in } A, \\ 1, & \text{if } i \text{ is an odd element in } A. \end{cases}$$

[2] All implementations described in this section can be found at http://www2.math.uni-paderborn.de/people/ralf-kasprowitz/cocoa.html.

Let $\operatorname{char}(K) \nmid q$. *The* q*th symmetric power of* $\mathcal{E}$ *sits in the exact sequence*

$$0 \longrightarrow S^q(E) \longrightarrow \bigoplus_{\substack{i_1 \le \cdots \le i_q \\ i_k \in I}} \mathcal{O}_{\mathbb{P}^N}\Big(\sum_{k \in Q} a_{i_k}\Big) \xrightarrow{\varphi_q} \bigoplus_{\substack{i'_1 \le \cdots \le i'_{q-1}, j \\ i'_k \in I, j \in J}} \mathcal{O}_{\mathbb{P}^N}\Big(\sum_{k \in Q - \{q\}} a_{i'_k} + b_j\Big),$$

where the map φ_q *is given by*

$$e_{i_1 \le \cdots \le i_q} \longmapsto \sum_{\substack{i \in \{i_1, \ldots, i_q\} \\ j \in J}} a_{ji} e_{i_1 \le \cdots \le \hat{i} \le \cdots \le i_q, j}.$$

Here $\hat{i}$ *means that this element is omitted.*

Proof. This follows from standard results in multilinear algebra. □

Remark 4.2. The following presentation of exterior powers is not suitable for semistability algorithms. Let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a finitely generated graded R-module ($R = K[X_0, \ldots, X_N]$). If we fix homogeneous generators $g_1, \ldots, g_n$, we obtain a surjection

$$R^n \xrightarrow{f} M, \quad e_j \longmapsto g_j, \quad j = 1, \ldots, n.$$

From this map we can derive, for every positive integer $q \ge 1$, the well-known exact sequence

$$(\ker f) \otimes_R \bigwedge^{q-1} R^n \xrightarrow{\alpha} \bigwedge^q R^n \longrightarrow \bigwedge^q M \longrightarrow 0,$$

where the map α is given by

$$x \otimes (y_1 \wedge \cdots \wedge y_{q-1}) \longmapsto x \wedge y_1 \wedge \cdots \wedge y_{q-1}$$

(see [Scheja and Storch 88, §83, Aufgabe 26]). Fixing homogeneous generators of $\ker f$ gives a diagram

$$\begin{array}{ccccccc} 0 & & & & & & \\ \uparrow & & & & & & \\ (\ker f) \otimes_R \bigwedge^{q-1} R^n & \xrightarrow{\alpha} & \bigwedge^q R^n & \longrightarrow & \bigwedge^q M & \longrightarrow & 0 \\ \uparrow & \nearrow_{\bar{\alpha}} & & & & & \\ R^m \otimes_R \bigwedge^{q-1} R^n & & & & & & \end{array}$$

In this way, we obtain a presentation of $\bigwedge^q M$ as a cokernel of a map between free modules. Since all these mappings are graded, we have a corresponding sequence

$$\bigoplus_{i=1}^{\tilde{n}} \mathcal{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\bar{\alpha}} \bigoplus_{j=1}^{\tilde{m}} \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow \bigwedge^q \widetilde{M} \longrightarrow 0$$

of the associated coherent sheaves on $\mathbb{P}^N$ (with suitable twists). But the map $\bigoplus_{j=1}^{\tilde{m}} R_{b_j} \to \Gamma(\mathbb{P}^N, \bigwedge^q \widetilde{M})$ is in general not surjective. Hence this sequence cannot be the basis of an algorithmic approach. If the depth of M is at least 2, then this map is surjective if and only if

$$\bigwedge^q \Big(\Gamma\Big(\mathbb{P}^N, \widetilde{M}\Big)\Big) \to \Gamma\Big(\mathbb{P}^N, \bigwedge^q \widetilde{M}\Big)$$

is surjective (see [Eisenbud 95, Theorems A4.1 and A4.3]). An illustrative example is the following.

Example 4.3. We consider the syzygy bundle $\mathcal{S} := \operatorname{Syz}(X^3, Y^3, Z^3, XY^2Z^2)$ on $\mathbb{P}^2 = \operatorname{Proj} K[X, Y, Z]$ (see also [Brenner 08a, Example 7.4]) and use Brenner's criterion, Theorem 1.1. The slope of this bundle equals $-14/3 \approx -4.667$. For the subsheaves of rank 1 coming from two monomials, we have (we list only the combinations having a common factor)

$$\mu(\operatorname{Syz}(X^3, XY^2Z^2)) = 1 - 8 = -7,$$
$$\mu(\operatorname{Syz}(Y^3, XY^2Z^2)) = 2 - 8 = -6,$$
$$\mu(\operatorname{Syz}(Z^3, XY^2Z^2)) = 2 - 8 = -6.$$

Hence we see that the global sections of $\mathcal{S}$ do not contradict the semistability. But the monomial subfamily X^3, Y^3, Z^3 yields the rank-2 subbundle $\operatorname{Syz}(X^3, Y^3, Z^3) \subset \mathcal{S}$ of slope $-9/2 = -4.5$. Thus, $\mathcal{S}$ is not semistable with Harder–Narasimhan filtration

$$0 \longrightarrow \operatorname{Syz}(X^3, Y^3, Z^3) \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-5) \longrightarrow 0.$$

Moreover, the mapping $\bigwedge^2(\Gamma(\mathbb{P}^2, \mathcal{S})) \to \Gamma(\mathbb{P}^2, \bigwedge^2 \mathcal{S})$ is not surjective.

Since we assume that a vector bundle and its exterior powers are given as kernels of morphisms between splitting bundles, we have to know how to compute the kernel of an R-linear map $R^n \to R^m$ between finitely generated free modules over the polynomial ring R. The answer is given by the following well-known lemma, which shows that we can compute a minimal nontrivial global section with Gröbner bases.

Lemma 4.4. *Let* $R = K[X_0, \ldots, X_N]$ *be the polynomial ring over a field* K *and let* $\varphi : R^m \to R^n$ *be an* R*-linear map. Denote by* $e_1, \ldots, e_m$ *the standard basis vectors of* R^m*. With the notation* $w_j = \varphi(e_j)$, $j = 1, \ldots, m$, *we have*

$$\ker \varphi = \operatorname{Syz}_R(w_1, \ldots, w_m).$$

In other words, the kernel of φ *is the (*R*-)syzygy module of the columns of the matrix that describes* φ*.*

Proof. See [Kreuzer and Robbiano 00, Proposition 3.3.1(a)]. □

So we have accumulated the necessary technical tools to formulate an algorithm, based on Hoppe's criterion, Proposition 2.1, to determine semistability of a given kernel bundle on $\mathbb{P}^N$. Note that every instruction can be performed with any computer algebra system that is able to handle Gröbner-basis calculations. In our case, we implemented it in CoCoA.

Algorithm 4.5. Semistability of kernel bundles

Input: Two lists $[a_1,\dots,a_n]$, $[b_1,\dots,b_m]$ and a homogeneous $m\times n$ matrix $\mathcal{M}=(a_{ji})$ with no constant polynomial entries $a_{ji}\neq 0$ of degree b_j-a_i defining a kernel bundle

$$0\longrightarrow \mathcal{E}=\widetilde{\ker\mathcal{M}}\longrightarrow \bigoplus_{i=1}^{n}\mathcal{O}_{\mathbb{P}^N}(a_i)\xrightarrow{\mathcal{M}}\bigoplus_{j=1}^{m}\mathcal{O}_{\mathbb{P}^N}(b_j)\longrightarrow 0$$

with $a_1\geq a_2\geq\cdots\geq a_n$ (as usual, $\widetilde{\ker\mathcal{M}}$ denotes the sheaf associated with the graded R-module $\ker\mathcal{M}$).

Output: The decision whether $\mathcal{E}$ is semistable in terms of a Boolean value TRUE or FALSE respectively.

1. Compute the invariants $\operatorname{rk}(\mathcal{E})=n-m$, $\deg(\mathcal{E})=\sum_{i=1}^{n}a_i-\sum_{j=1}^{m}b_j$, and $\mu(\mathcal{E})=\frac{1}{n-m}\left(\sum_{i=1}^{n}a_i-\sum_{j=1}^{m}b_j\right)$.
2. If the slope condition $a_n\geq\mu(\mathcal{E})$ of Proposition 3.1 is fulfilled, then continue. Else return FALSE and terminate.
3. Set $q:=1$.
4. Construct the matrix $\mathcal{M}_q$ that describes the map φ_q in Proposition 4.1.
5. Compute the syzygy module S_q of the columns of $\mathcal{M}_q$.
6. Compute the initial degree $\alpha_q:=\min\{t:(S_q)_t\neq 0\}$ of the graded R-module S_q (i.e., α_q is the minimal twist ℓ such that $\Gamma(\mathbb{P}^N,(\bigwedge^q\mathcal{E})(\ell))\neq 0$).
7. If $\alpha_q<-q\mu(\mathcal{E})$, then return FALSE and terminate. Else set $q:=q+1$ and continue.
8. If $q<\operatorname{rk}(\mathcal{E})-1$, then go back to step (4). Else return TRUE and terminate.

By Horrocks's theorem (see [Okonek et al. 80, Theorem I.2.3.1]), every vector bundle $\mathcal{E}$ on the projective plane $\mathbb{P}^2$ that does not split as a direct sum of line bundles has homological dimension 1, that is, there exists a resolution

$$0\longrightarrow\mathcal{K}\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow 0$$

with splitting bundles $\mathcal{K}$ and $\mathcal{F}$. Since $\mathcal{E}$ is semistable if and only if $\mathcal{E}^*$ is semistable, we can dualize the short exact sequence and apply Algorithm 4.5 to the kernel bundle $\mathcal{E}^*$. Hence, our semistability algorithm is applicable to every (nonsplit) vector bundle on $\mathbb{P}^2$ and to vector bundles of homological dimension 1 on $\mathbb{P}^N$ in general.

Example 4.6. Let K be an arbitrary field. We consider the monomials

$$X^2,\ Y^2,\ XY,\ XZ,\ YZ\in R=K[X,Y,Z]$$

and the corresponding sheaf of syzygies

$$\mathcal{S}:=\operatorname{Syz}(X^2,Y^2,XY,XZ,YZ).$$

Is $\mathcal{S}$ a semistable sheaf? Since the ideal generated by these monomials is not R_+-primary, we can apply neither Theorem 1.1 nor (at first sight) Algorithm 4.5. We compute a resolution of $\mathcal{S}$ (for instance with CoCoA), namely,

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^2}(-4)^2\xrightarrow{\mathcal{A}}\mathcal{O}_{\mathbb{P}^2}(-3)^6\longrightarrow\mathcal{S}\longrightarrow 0,$$

where

$$\mathcal{A}=\begin{pmatrix} x & 0\\ -y & 0\\ -y & x\\ 0 & -y\\ -z & 0\\ 0 & z\end{pmatrix}.$$

Since $\mathcal{S}$ is a reflexive sheaf, it is locally free on $\mathbb{P}^2$ (cf. [Okonek et al. 80, Lemma 1.1.10]). So if we dualize the resolution, we obtain a short exact sequence

$$0\longrightarrow\mathcal{S}^*\longrightarrow\mathcal{O}_{\mathbb{P}^2}(3)^6\xrightarrow{\mathcal{A}^t}\mathcal{O}_{\mathbb{P}^2}(4)^2\longrightarrow 0,$$

i.e., $\mathcal{S}^*$ is a kernel bundle. Hence, we apply Algorithm 4.5 to $\mathcal{S}^*$ in order to obtain the answer to our question (we recall that $\mathcal{S}$ is semistable if and only if $\mathcal{S}^*$ is semistable). A CoCoA computation gives the following:

1. $H^0(\mathbb{P}^2,\mathcal{S}^*(m))=0$ for $m<-2$ and $-2\geq-\mu(\mathcal{S}^*)=-\frac{5}{2}$.
2. $H^0(\mathbb{P}^2,(\bigwedge^2(\mathcal{S}^*))(m))=0$ for $m<-5$ and $-5=-2\mu(\mathcal{S}^*)$. In particular, we obtain no information about stability.
3. The numerical condition of Proposition 3.1 is fulfilled. So there are no mappings $\mathcal{S}^*\to\mathcal{O}_{\mathbb{P}^2}(k)$ into line bundles that contradict the semistability.

Finally, we conclude via Proposition 2.1 that $\mathcal{S}^*$ is semistable and so is $\mathcal{S}$.

By applying restriction theorems, we can use Algorithm 4.5 to produce examples of semistable vector bundles on more complicated projective varieties.

There are famous restriction theorems by V. B. Mehta and A. Ramanathan (see [Mehta and Ramanathan 82b, Theorem 6.1]) and by H. Flenner (see [Flenner 84, Theorem 1.2]). The strongest restriction theorem is due to A. Langer. It holds in more general situations, but we give a formulation only for vector bundles on projective spaces. The theorem works in arbitrary characteristic and gives a degree bound for arbitrary smooth hypersurfaces in a projective space. It involves the *discriminant* $\Delta(\mathcal{E}) := 2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2$ of a locally free sheaf $\mathcal{E}$ of rank r, where $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ denote the first and second Chern classes respectively.

Theorem 4.7. (Langer.) *Let K be an algebraically closed field and let $\mathcal{E}$ be a stable coherent torsion-free sheaf of rank $r \geq 2$ on $\mathbb{P}^N$ and let $D \in |\mathcal{O}_{\mathbb{P}^N}(k)|$ be a smooth divisor such that $\mathcal{E}|_D$ is torsion-free. If*

$$k > \frac{r-1}{r}\Delta(\mathcal{E}) + \frac{1}{r(r-1)},$$

then the restriction $\mathcal{E}|_D$ is stable.

Proof. See [Langer 09, Theorem 2.19]. □

For a kernel bundle $\mathcal{E}$, it is easy to see that we have

$$c_2(\mathcal{E}) = \frac{1}{2}\Big(\Big(\sum_{i=1}^{n} a_i\Big)^2 - \sum_{i=1}^{n} a_i^2 + \Big(\sum_{j=1}^{m} b_j\Big)^2 + \sum_{j=1}^{m} b_j^2\Big) - \sum_{i,j} a_i b_j$$

and hence

$$\Delta(\mathcal{E}) = \Big(\sum_{i=1}^{n} a_i\Big)^2 + \Big(\sum_{j=1}^{m} b_j\Big)^2 - (n-m)\Big(\sum_{j=1}^{m} b_j^2 - \sum_{i=1}^{n} a_i^2\Big) - 2\sum_{i,j} a_i b_j.$$

5. TANNAKA DUALITY OF STABLE SYZYGY BUNDLES

As a first application of the algorithms described in the previous sections we will compute the Tannaka dual groups of certain stable syzygy bundles of degree 0 on the projective plane. We start with describing the setting. From now on, K denotes an algebraically closed field of characteristic 0, and X a smooth, irreducible, and projective variety over K. Furthermore, denote by $\mathfrak{B}_X$ the category of polystable vector bundles of degree 0 on X. We recall that a vector bundle is *polystable* if it is a direct sum of stable bundles of the same slope. It is well known that $\mathfrak{B}_X$ is an abelian tensor category that possesses the faithful fiber functor $\omega_x : \mathfrak{B}_X \to \mathrm{Vect}(K)$, where $\mathrm{Vect}(K)$ is the category of finite-dimensional K-vector spaces and ω_x maps a bundle E to its fiber $\mathcal{E}_x$ for a point $x \in X(K)$. In other words, it is a neutral Tannaka category, and hence there exist an affine group scheme G_X over K and an equivalence of categories

$$\mathfrak{B}_X \xrightarrow{\sim} \mathrm{Rep}_{G_X}(K).$$

For the theory of Tannaka categories, see, for example, [Deligne et al. 82]. We denote by $\mathfrak{B}_{\mathcal{E}}$ the Tannaka subcategory of $\mathfrak{B}_X$ generated by the vector bundle $\mathcal{E}$, and by $G_{\mathcal{E}}$, its Tannaka dual group. The group scheme G_X is proreductive and $G_{\mathcal{E}}$ is a reductive linear algebraic group (not necessarily connected). It is in a natural way an algebraic subgroup of $\mathrm{GL}_{\mathcal{E}_x}$. Furthermore, there is a faithfully flat morphism $G_X \to G_{\mathcal{E}}$. Since global sections of vector bundles in $\mathfrak{B}_X$ correspond to $G_{\mathcal{E}}$-invariant elements of the fiber, it follows from [Deligne et al. 82, Proposition 3.1] that the algebraic group $G_{\mathcal{E}}$ is uniquely determined by the global sections of $T^{r,s}(\mathcal{E}) := \mathcal{E}^{\otimes r} \otimes (\mathcal{E}^*)^{\otimes s}$ for $r, s \in \mathbb{N}$. It even suffices to know the global sections of $\mathcal{E}^{\otimes r}$ for $r \in \mathbb{N}$, since the dual of a stable bundle $\mathcal{E}$ occurs as a direct summand in some tensor power of $\mathcal{E}$.

Remark 5.1. The restriction to fields of characteristic 0 is essential, as the following example shows. It was communicated to us by H. Brenner. Let K be a field of positive characteristic p, $p \geq 3$. Consider the plane curve

$$C = V_+\left(X^{3p-1} + Y^{3p-1} + Z^{3p-1} + X^p Z^{2p-1}\right).$$

This curve is smooth by the Jacobian criterion. Now we look at the syzygy bundle $\mathcal{E} := \mathrm{Syz}(X^2, Y^2, Z^2)(3)$ on C of degree 0. Since $\mathcal{E}$ is stable on $\mathbb{P}^2$ by Proposition 3.2 and $p \geq 3$, it remains stable on C by Langer's restriction theorem, Theorem 4.7. The Frobenius pullback $F^*(\mathcal{E}) \cong \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$ has the nontrivial section $s := (ZX^{p-1}, ZY^{p-1}, Z^p + X^p)$ because we have the equation

$$\begin{aligned} &X^{2p} \cdot ZX^{p-1} + Y^{2p} \cdot ZY^{p-1} + Z^{2p} \cdot (Z^p + X^p) \\ &\quad = Z(X^{3p-1} + Y^{3p-1} + Z^{3p-1} + X^p Z^{2p-1}) = 0 \end{aligned}$$

on the curve. It is easy to see that s has no zeros on C and that there is no further nontrivial section of $F^*(\mathcal{E})$. Hence $F^*(\mathcal{E})$ is a nontrivial extension of the structure sheaf by itself and therefore not polystable. Since $F^*(\mathcal{S}) \subset S^p(\mathcal{E})$, it follows that $S^p\mathcal{E}$ is not polystable either. The same holds for $\mathcal{E}^{\otimes p}$, since $S^p\mathcal{E}$ is a quotient of the p-fold tensor product. So we see that $\mathfrak{B}_C^s$ is not a tensor category and in particular is not Tannakian.

Let us now consider stable bundles $\mathcal{E}$ of degree 0 on the projective space $\mathbb{P}^N$.

Lemma 5.2. *The Tannaka dual group $G_\mathcal{E}$ of a stable vector bundle $\mathcal{E}$ of degree 0 on $\mathbb{P}^N$ is a connected semisimple group.*

Proof. Suppose that the algebraic group $G_\mathcal{E}$ were not connected. Then the representations of the finite quotient $G_\mathcal{E}/G_\mathcal{E}^0$ would correspond to a subcategory of $\mathfrak{B}_\mathcal{E}$ containing nontrivial finite vector bundles; see [Nori 76, Lemma 3.1]. But the latter form, together with the obvious fiber functor, a neutral Tannaka category (see [Nori 76, Proposition 3.7]), and the K-valued points of the Tannaka dual group are well known to coincide with the étale fundamental group if the characteristic of the ground field is 0. Hence there are no nontrivial finite vector bundles on the projective space. Furthermore, the reductive group $G_\mathcal{E}$ does not have any nontrivial characters due to the fact that $\mathrm{Pic}(\mathbb{P}^N) = \mathbb{Z}$; hence it has to be semisimple. □

It can be shown that for stable vector bundles of degree 0 and rank r as in Theorem 1.2, the Tannaka dual group is the group SL_r if it is almost simple; see [Kasprowitz 10, Lemma 4.4 and Proposition 5.3] together with [Anisimov 11, Theorem 1]. One motivation for this paper was to construct examples of syzygy bundles on the projective space having a Tannaka dual group of type different from A. Note that one cannot simply try to guess an example for a syzygy bundle with group different from $\mathrm{SL}_r \subset \mathrm{GL}_r$; see Remark 5.8 below. Our idea is to exclude this case by constructing self-dual bundles. With the algorithmic methods described in Algorithm 4.5 and Proposition 4.1, it is in principle possible to compute the Tannaka dual group and its representation for an arbitrary stable kernel bundle of degree 0. However, the necessary computations grow very fast with the rank of the bundle, so we were able to handle syzygy bundles only up to rank 6 on $\mathbb{P}^2$. Furthermore, we found only syzygy bundles having the almost simple Tannaka dual group $\mathrm{Sp}_r \subset \mathrm{GL}_r$, where $r \in \{4, 6\}$. There are no stable self-dual syzygy bundles of odd rank on $\mathbb{P}^2$; see Corollary 5.4.

For stable rank-2 bundles of degree 0 on $\mathbb{P}^2$, there is only one possible Tannaka dual group, namely the 2-dimensional irreducible representation of SL_2. An example for this is the syzygy bundle $\mathrm{Syz}(X^2, Y^2, Z^2)(3)$. It is stable due to Theorem 1.2. To find higher-rank syzygy bundles whose Tannaka dual group is not the group SL_r with an r-dimensional representation, we will use the following simple lemma.

Lemma 5.3. *Let $f_1, \ldots, f_n \in R := K[X, Y, Z]$ be homogeneous polynomials such that the ideal $I := (f_1, \ldots, f_n)$ is R_+-primary and minimally generated by $f_1, \ldots, f_n$. Then $\mathcal{E} := \mathrm{Syz}(f_1, \ldots, f_n)$ is self-dual (up to a twist with a line bundle) if and only if R/I is Gorenstein.*

Proof. The minimal free resolution of R/I has length 3 and is self-dual up to twist since R/I is Gorenstein:

$$0 \longrightarrow R(-d) \xrightarrow{\varphi} \bigoplus_{i=1}^n R(-e_i) \longrightarrow \bigoplus_{i=1}^n R(-d_i) \xrightarrow{f_1,\ldots,f_n} R \longrightarrow R/I \longrightarrow 0,$$

and with $E := \ker(f_1, \ldots, f_n)$, we have $\mathrm{coker}(\varphi) = E(-d)^*$. In particular, there is an isomorphism $E \cong E(-d)^*$ with $\mathcal{E} = \widetilde{E}$. Conversely, one easily sees that if $\mathcal{E}$ is self-dual up to twist, then the minimal free resolution of R/I ends with a free module of rank 1; that is, R/I is Gorenstein. □

Corollary 5.4. *There are no self-dual (up to a twist with a line bundle) nonsplit syzygy bundles of odd rank on $\mathbb{P}^2$.*

Proof. This is [Buchsbaum and Eisenbud 77, Corollary 2.2], which says that the minimal number of generators of a Gorenstein ideal of grade 3 is odd. □

Corollary 5.5. *All stable syzygy bundles of degree 0 and odd rank less than or equal to 11 on the projective plane have a semisimple Tannaka dual group whose simple components are of type A.*

Proof. A table of representations of Lie algebras (see, for example, [McKay et al. 90]) shows that the smallest non-self-dual irreducible representation of a semisimple algebraic group with some simple component not of type A is the (up to duality) 12-dimensional representation of the semisimple group $\mathrm{SL}_3 \oplus \mathrm{Sp}_4$, which is the tensor product of the 3-dimensional irreducible representation of SL_3 with the 4-dimensional irreducible representation of Sp_4. □

Now looking at rank 4, what possibilities are there for the Lie algebra of the Tannaka dual group? There are three different self-dual and irreducible representations, where we use the notation of the tables of simple Lie algebras and their representations in [McKay et al. 90]: A_1 with highest weight 3, C_2 with highest weight $(1,0)$, and $A_1 \oplus A_1$ with highest weight $(1,1)$. Using the computer algebra package for Lie group computations LiE,[3] one finds that $\dim(\Gamma(\mathbb{P}^2, \mathcal{E}^{\otimes 4})) = \dim((\mathcal{E}_x^{\otimes 4})^{G_\mathcal{E}}) = 4$ in the cases of type A and $\dim(\Gamma(\mathbb{P}^2, \mathcal{E}^{\otimes 4})) = 3$ in the case of C_2. To find a bundle with Tannaka dual group of type C_2, we have to construct a Gorenstein ideal I with five minimal generators such that the associated syzygy bundle $\mathcal{E}$ has degree 0. Then we have to check its stability and the global sections of $\mathcal{E}^{\otimes 4}$.

To find a suitable Gorenstein ideal I, we consider the polynomial ring $R := K[X,Y,Z]$ as a module over itself by interpreting a polynomial in R as a differential operator acting on itself, e.g., $X \cdot f = \partial f / \partial X$. Choose a homogeneous polynomial f of degree r and define $I := \mathrm{Ann}_R(f) \subseteq R$. This ideal is Artinian and Gorenstein; see, for example, [Iarrobino and Kanev 99, Lemma 2.12].

Example 5.6. The Gorenstein ideal of the homogeneous form

$$f := X^2 + Y^2 + Z^2$$

is the ideal $I = (X^2 - Y^2, X^2 - Z^2, XY, XZ, YZ)$. If we pull back the associated syzygy bundle on $\mathbb{P}^2$ via the finite morphism $X \mapsto X^2$, $Y \mapsto Y^2$, $Z \mapsto Z^2$, we obtain the twisted syzygy bundle

$$\mathcal{E}(5) := \mathrm{Syz}\left(X^4 - Y^4, X^4 - Z^4, X^2Y^2, X^2Z^2, Y^2Z^2\right)(5).$$

It has degree 0, rank 4, and is self-dual by Lemma 5.3. We apply Algorithm 4.5 to $\mathcal{E}$ and obtain with the help of CoCoA:

1. $H^0(\mathbb{P}^2, \mathcal{E}(m)) = 0$ for $m \leq 5 = -\mu(\mathcal{E})$;
2. $H^0(\mathbb{P}^2, (\bigwedge^2 \mathcal{E})(m)) = 0$ for $m < 10 = -2\mu(\mathcal{E})$, $H^0(\mathbb{P}^2, (\bigwedge^2 \mathcal{E})(m)) \neq 0$ for $m = 10$;
3. $H^0(\mathbb{P}^2, (\bigwedge^3 \mathcal{E})(m)) = 0$ for $m \leq 15 = -3\mu(\mathcal{E})$ (this computation is, by Proposition 3.1, actually not necessary, since the degrees of the polynomials are constant).

Hence, we see that $\mathcal{E}$ (and all its twists) are semistable, but we get no information about stability because of item 2. Observe that Hoppe's criterion can never reveal stability of a self-dual bundle of rank 4, since in this case there must be nontrivial global sections of $\Lambda^2(\mathcal{E}(5))$; see again [McKay et al. 90].

Fortunately, the self-duality of the bundle allows us to prove its stability. By the computation above, we have only to consider subsheaves of rank 2 that may destroy the stability of $\mathcal{E}(5)$. Assume that $\mathcal{F} \subset \mathcal{E}(5)$ is a stable subsheaf of rank 2 and degree 0. Since we can pass over to the reflexive hull, and since we are working on $\mathbb{P}^2$, we may assume that $\mathcal{F}$ is locally free (see [Okonek et al. 80, Lemma 1.1.10]). In particular, we have

$$\mathcal{F} \cong \mathcal{F}^* \otimes \det(\mathcal{F}) \cong \mathcal{F}^* \otimes \mathcal{O}_{\mathbb{P}^2} \cong \mathcal{F}^*$$

by [Hartshorne 80, Proposition 1.10]. That is, the subsheaf $\mathcal{F}$ is self-dual too. So the composition of the morphisms

$$\mathcal{E}(5) \cong \mathcal{E}(5)^* \longrightarrow \mathcal{F}^* \cong \mathcal{F} \hookrightarrow \mathcal{E}(5)$$

yields an endomorphism of $\mathcal{E}(5)$ that is not a multiple of the identity. But a computation of global sections using the implementation of Proposition 4.1 yields that

$$h^0\left(\mathbb{P}^2, \mathrm{End}\left(\mathcal{E}(5)\right)\right) = h^0\left(\mathbb{P}^2, \mathcal{E}(5) \otimes \mathcal{E}(5)\right) = 1,$$

that is, the bundle $\mathcal{E}(5)$ is simple, and a morphism as above does not exist. Hence the bundle $\mathcal{E}(5)$ is stable. Finally, another computation shows that $h^0(\mathbb{P}^2, (\mathcal{E}(5)^{\otimes 4})) = 3$; hence the Tannaka dual group is in fact almost simple of type C_2, and the representation of its Lie algebra has highest weight $(1,0)$. It is well known that this corresponds to the irreducible and faithful representation $\mathrm{Sp}_4 \subset \mathrm{GL}_4$.

Example 5.7. The Gorenstein ideal associated with the homogeneous form $X^3Y + Y^3Z + Z^3X$ via the correspondence described above is the ideal

$$I := \big(X^3 - Y^2Z, Y^3 - XZ^2, X^2Y - Z^3, XY^2, YZ^2, X^2Z, XYZ\big).$$

We consider the same pullback as in the previous example to be able to twist the associated syzygy bundle to degree 0. The same computations as above show that the corresponding syzygy bundle $\mathcal{E}(7)$, defined as

$$\mathrm{Syz}\big(X^6 - Y^4Z^2, Y^6 - X^2Z^4, X^4Y^2 - Z^6, X^2Y^4, Y^2Z^4, X^4Z^2, X^2Y^2Z^2\big)(7)$$

is semistable of degree 0, where a subsheaf $\mathcal{F}$ that destroys stability has to be of rank $r = 2$ or $r = 4$. But since $\mathcal{E}(7)$ is self-dual, we can always assume $\mathcal{F}$ to be of rank 2 and obtain stability for the same reason as in the previous example, since the bundle again turns out to be simple. Again using LiE, we find that it is possible to determine the Lie algebra of the Tannaka dual group by computing $\dim(\mathcal{E}^{\otimes 4}) = 3$, which shows that it has to be simple of type C_3, with highest weight $(1,0,0)$. This corresponds to the faithful irreducible representation $\mathrm{Sp}_6 \subset \mathrm{GL}_6$.

[3] Available at http://young.sp2mi.univ-poitiers.fr/~marc/LiE/.

Remark 5.8. There is a good reason why one should expect to find the group Sp_r in these cases. It is well known that the moduli space of stable bundles of fixed rank and Chern classes exists as a quasiprojective variety; see, for example, [Huybrechts and Lehn 97, Theorem 4.3.4]. Let us denote by M the moduli space containing the syzygy bundle of Example 5.6 respectively of Example 5.7. Let $\mathcal{U}$ be the quasiuniversal bundle on $\mathbb{P}^2 \times M$; see [Huybrechts and Lehn 97, Chapter 4.6]. This means that the restriction of $\mathcal{U}$ to $\mathbb{P}^2 \times \{p\}$ is a finite product of copies of the stable bundle on $\mathbb{P}^2$ corresponding to the point p. Applying the semicontinuity theorem (e.g., [Hartshorne 77, Theorem 12.8]) for $\dim(H^0(\mathbb{P}^2 \times \{p\}, \mathcal{U}^{\otimes r}))$, one finds that the locus of the vector bundles having Tannaka dual group SL_r is open in M, since the dimension of $\Gamma(\mathbb{P}^2 \times \{p\}, \mathcal{U}^{\otimes r})$ is minimal in this case and strictly greater in all other cases. Hence for a generic choice of a stable syzygy bundle on $\mathbb{P}^2$ one expects to find the group SL_r as Tannaka dual group. The locus of self-dual bundles is closed in M (apply the semicontinuity theorem for $\mathcal{U} \otimes \mathcal{U}$), containing the bundles with Tannaka dual group Sp_r as an open locus (since the dimension of $\Gamma(\mathbb{P}^2 \times \{p\}, \mathcal{U}^{\otimes 4})$ is minimal for a self-dual stable bundle with this Tannaka dual group; see the discussion of the examples above). It follows that for a generic choice of a self-dual syzygy bundle one expects the group Sp_r as Tannaka dual group. It would be interesting to find a method for constructing stable syzygy bundles having a Tannaka dual group different from SL_r or Sp_r. Furthermore, one could try to determine the geometry of the Tannaka strata of the moduli spaces discussed above.

6. TANNAKA DUALITY OF STABLE BUNDLES RESTRICTED TO CURVES

We remain in the situation of Section 5. Recall that the ground field K is algebraically closed and of characteristic 0. Here we investigate the problem of the behavior of Tannaka dual groups after restricting a stable bundle of degree 0 on $\mathbb{P}^N$ to smooth connected curves X such that the restricted bundle is still stable. The main problem is that the connected Tannaka dual group of a stable vector bundle on $\mathbb{P}^N$ may become disconnected after restriction to a curve:

Lemma 6.1. *Let $X \subset \mathbb{P}^2$ be a connected smooth curve of genus greater than 1. Then there exists a stable bundle of degree 0 on $\mathbb{P}^2$ such that its restriction to X is again stable with nonconnected Tannaka dual group.*

Proof. Recall that a finite vector bundle is a vector bundle that is trivialized by a finite étale morphism. There is a one-to-one correspondence between finite vector bundles on X and representations of the étale fundamental group $\pi_1(X, x)$ having finite image. Now choose such an irreducible representation of dimension r with trivial determinant, which certainly exists if the genus of the curve is greater than 1. It is well known that the associated vector bundle E is stable of degree 0 and rank r, with Tannaka dual group equal to the image of the representation. Further, its determinant is the trivial bundle. The bundle $E^*(n)$ is generated by its global sections for $n \gg 0$. Furthermore, we may assume by [Brenner 06, Lemma 2.3] that already $r+1$ global sections generate $E^*(n)$. Hence one obtains a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-rn) \xrightarrow{\varphi} \mathcal{O}_X^{r+1} \longrightarrow E^*(n) \longrightarrow 0.$$

After twisting with $\mathcal{O}_X(-n)$, the dual of the morphism φ lifts to an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n)^{r+1} \xrightarrow{\varphi^*} \mathcal{O}_{\mathbb{P}^2}((r+1)n)$$

on $\mathbb{P}^2$, where $\mathcal{E}$ is a coherent sheaf with $\mathcal{E}|_X \cong E$, and the singularities of $\mathcal{E}$ are of codimension greater than 2 (see [Okonek et al. 80, Proposition II.1.1.6]); hence $\mathcal{E}$ is locally free and of course stable of degree 0. The Tannaka dual group of $\mathcal{E}$ is connected by Lemma 5.2, but the restriction of the bundle $\mathcal{E}$ to X has a finite Tannaka dual group. □

Hence we need a criterion for the Tannaka dual group to be connected after restricting the bundle $\mathcal{E}$ to the smooth and connected curve X. For the rest of this section we denote by p a prime number and by $\overline{\mathbb{Q}}_p$ an algebraic closure of the p-adic numbers with ring of integers $\mathfrak{o}$ and residue field $\kappa = \overline{\mathbb{F}}_p$. We call a finitely presented flat and proper scheme $\mathfrak{X}$ over $\mathfrak{o}$ together with an isomorphism $X \cong \mathfrak{X} \otimes_{\mathfrak{o}} \overline{\mathbb{Q}}_p$ a *model* of X. Note that any scheme $\mathfrak{X}$ over $\mathfrak{o}$ is the disjoint union of the generic fiber $\mathfrak{X} \otimes \overline{\mathbb{Q}}_p$, which is open in $\mathfrak{X}$, and the special fiber $\mathfrak{X} \otimes \overline{\mathbb{F}}_p$, which is closed. For the rest of this section, we set $K = \overline{\mathbb{Q}}_p$.

Theorem 6.2. *Let E be a vector bundle on the smooth, connected, and projective curve X over $\overline{\mathbb{Q}}_p$. If there exists a model $\mathfrak{X}$ of X together with a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ such that $E \cong \mathcal{E} \otimes_{\mathfrak{o}} \overline{\mathbb{Q}}_p$ and $\mathcal{E} \otimes_{\mathfrak{o}} \mathfrak{o}/p$ is a trivial bundle on the scheme $\mathfrak{X} \otimes_{\mathfrak{o}} \mathfrak{o}/p$, then E is semistable of degree 0 with connected Tannaka dual group G_E.*

Proof. The proof uses results from nonabelian p-adic Hodge theory. The semistability of the bundle is shown in [Deninger and Werner 05, Theorem 13]; the assertion that G_E is connected follows from [Kasprowitz 10, Theorem 3.12]. $\square$

Example 6.3. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^N$ sitting in the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{N+1} \mathcal{O}_{\mathbb{P}^N}(1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^N}(N+1) \longrightarrow 0,$$

where the morphism φ is defined by homogeneous polynomials

$$f_i := X_0^{i-1} X_1^{N-i+1} + p g_i, \quad i = 1, \dots, N+1,$$

with $g_i \in \mathfrak{o}[X_0, \dots, X_N]$. The vector bundle $\mathcal{E}$ is stable of degree 0 due to Theorem 1.2. If we consider $\mathcal{E}$ as a sheaf on $\mathbb{P}^N_{\mathfrak{o}}$, it is easy to see that it is a locally free sheaf outside the closed subset

$$S = \{[X_0; X_1; \dots; X_N] \ : \ X_0 = X_1 = 0\} \subset \mathbb{P}^N_\kappa \subset \mathbb{P}^N_{\mathfrak{o}}.$$

Then for every smooth connected curve $X \subset \mathbb{P}^N$ that has a model $\mathfrak{X} \subset \mathbb{P}^N_{\mathfrak{o}}$ such that the special fiber does not intersect the subspace S, the vector bundle $\mathcal{E}|_{\mathfrak{X}}$ has N linearly independent global sections modulo p and hence is trivial. If the curve X is the intersection of $N-1$ smooth divisors of degree $\gg 0$, the restriction theorem [Langer 09, Theorem 2.7] yields that the restriction E of $\mathcal{E}$ to X is a stable bundle, and it follows from Theorem 6.2 above that G_E is connected. See also [Kasprowitz 10, Example 4.6 and Remark 4.8]. A similar argument was used by H. Brenner to provide examples for stable vector bundles on p-adic curves with semistable reduction; see [Deninger and Werner 05, Remark p. 571].

In general, we cannot apply Theorem 6.2 to an arbitrary kernel bundle $\mathcal{E}$ on $\mathbb{P}^N$, but in many cases one can apply it to the pullback $\pi^*(\mathcal{E})$ with respect to a suitable chosen finite morphism $\pi : \mathbb{P}^N \to \mathbb{P}^N$ such that $\pi^*(\mathcal{E})$ is defined over $\mathfrak{o}$ and such that it is trivial modulo p outside a closed subset of codimension ≥ 2. Then for all curves X that have a model $\mathfrak{X} \subset \mathbb{P}^N_{\mathfrak{o}}$ whose special fiber does not intersect this closed subset, we have $G_{\pi^*(\mathcal{E})} = G_{\pi^*(\mathcal{E})|_X}$ if the curve X is a complete intersection of smooth divisors of sufficiently high degree. We will illustrate this method in the following examples.

Example 6.4. Consider the syzygy bundles

$$\mathcal{E}_1 = \mathrm{Syz}(X^4 - Y^4, X^4 - Z^4, X^2Y^2, X^2Z^2, Y^2Z^2)(5)$$

and

$$\mathcal{E}_2 = \mathrm{Syz}(X^6 - Y^4Z^2, Y^6 - X^2Z^4, X^4Y^2 - Z^6, X^2Y^4, Y^2Z^4, X^4Z^2, X^2Y^2Z^2)(7)$$

on $\mathbb{P}^2$ from Examples 5.6 and 5.7. To find a suitable finite morphism $\pi : \mathbb{P}^2 \to \mathbb{P}^2$ as explained above, we try to construct a nontrivial morphism $g : \mathbb{P}^1_{\mathbb{F}_p} \to \mathbb{P}^2_{\mathbb{F}_p}$ such that the pullback $g^*(\mathcal{E}_i \otimes \mathbb{F}_p)$ is the trivial bundle. Then we can choose a rational map $\pi : \mathbb{P}^2_{\mathbb{Z}_p} \dashrightarrow \mathbb{P}^2_{\mathbb{Z}_p}$ that is defined outside the point $[0;0;1] \in \mathbb{P}^2_{\mathbb{F}_p} \subset \mathbb{P}^2_{\mathbb{Z}_p}$ such that modulo p, there is the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2_{\mathbb{F}_p} \setminus \{[0;0;1]\} & \xrightarrow{\pi_{\mathbb{F}_p}} & \mathbb{P}^2_{\mathbb{F}_p} \\ \downarrow & \nearrow_{g} & \\ \mathbb{P}^1_{\mathbb{F}_p} & & \end{array}$$

where the vertical morphism is defined as $[X;Y;Z] \mapsto [X;Y]$. It is then clear that $\pi^*(\mathcal{E}_i)$ is modulo p the trivial bundle on the open subset $\mathbb{P}^2_{\mathbb{F}_p} \setminus \{[0;0;1]\}$. A computation of global sections (over $\mathbb{Q}$) with CoCoA shows that the morphism g can, for example, be chosen as $[X;Y] \mapsto [X;Y;2X+Y]$ if the prime p is odd. Then the rational map π can be chosen as $[X;Y;Z] \mapsto [X;Y;2X+Y+pZ]$. The restriction to the generic fiber gives the finite morphism $\pi : \mathbb{P}^2 \to \mathbb{P}^2$ we were looking for.

On the generic fiber the vector bundles $\pi^*(\mathcal{E}_i)$ are polystable, and one computes

$$\dim(\mathrm{End}(\pi^*(\mathcal{E}_i))) = h^0(\mathbb{P}^2, \pi^*(\mathcal{E}_i) \otimes \pi^*(\mathcal{E}_i)) = 1$$

using Proposition 4.1. It follows that they have to be stable. Let $\mathfrak{X} \subset \mathbb{P}^2_{\mathfrak{o}}$ be a model of a smooth connected curve $X \subset \mathbb{P}^2_{\overline{\mathbb{Q}}_p}$ such that the special fiber $\mathfrak{X}_{\overline{\mathbb{F}}_p}$ does not contain the point $[0;0;1]$. If the degree of the plane curve X is large enough, we may again use Theorem 4.7 and obtain that the restriction of the bundle $\pi^*(\mathcal{E}_i)$ is still a stable bundle. It follows from Theorem 6.2 that its Tannaka dual group is a connected semisimple group. Furthermore, we have $\Gamma(X, \pi^*(\mathcal{E}_i)^{\otimes 4}|_X) = \Gamma(\mathbb{P}^2_{\overline{\mathbb{Q}}_p}, f^*(\mathcal{E}_i)^{\otimes 4})$ for curves of sufficiently large degree. Hence the Tannaka dual groups satisfy $G_{\mathcal{E}_1|_X} = \mathrm{Sp}_4 \subset \mathrm{GL}_4$ and $G_{\mathcal{E}_2|_X} = \mathrm{Sp}_6 \subset \mathrm{GL}_6$.

We end this section with an example in which the morphism π has to be of degree greater than 1.

Example 6.5. The syzygy bundle

$$\mathcal{E} = \mathrm{Syz}(X^3, Y^3, Z^3, XYZ)(4)$$

on $\mathbb{P}^2$ is stable of degree 0 due to Theorem 1.1, with Tannaka dual group $G_{\mathcal{E}} = \mathrm{SL}_3 \subset \mathrm{GL}_3$ because of $\dim(\Gamma(\mathbb{P}^2, \mathcal{E}^{\otimes 3})) = 1$. The morphism $g : \mathbb{P}^1_{\mathbb{F}_p} \to \mathbb{P}^2_{\mathbb{F}_p}$ can be chosen as $[X;Y] \mapsto [X^2+Y^2;X^2;X^2+XY]$, and hence $\pi : \mathbb{P}^2_{\mathbb{Z}_p} \dashrightarrow \mathbb{P}^2_{\mathbb{Z}_p}$, for example, as $[X;Y;Z] \mapsto [X^2+Y^2;X^2;X^2+XY+pZ^2]$. The same computations and arguments as in Example 6.4 then show that $G_{\mathcal{E}|_X} = \mathrm{SL}_3$ for a plane curve X of sufficiently large degree.

It is natural to ask whether this method works for all semistable vector bundles of degree 0 on $\mathbb{P}^N$:

Question 6.6. Let $\mathcal{E}$ be a semistable vector bundle of degree 0 on $\mathbb{P}^N$. Are there always a finite morphism $\pi : \mathbb{P}^N \to \mathbb{P}^N$ and a model $\mathfrak{P}$ of $\mathbb{P}^N$ such that $\pi^*(\mathcal{E})$ lifts to a sheaf on $\mathfrak{P}$ that is modulo p a free sheaf outside a closed subset of codimension greater than or equal to 2?

7. THE STABILITY OF THE SYZYGY BUNDLE OF FIVE GENERIC QUADRICS

It is an open question whether for generic forms $f_1,\dots,f_n$ of degree $d_1,\dots,d_n$ in the polynomial ring $R = K[X_0,\dots,X_N]$ over an algebraically closed field K, the corresponding syzygy bundle is semistable or even stable. There is no chance if the d_i's do not satisfy the necessary degree condition of Proposition 3.1. Hence, the question makes sense only if the necessary condition on the degrees is fulfilled, e.g., if we consider forms of constant degree. Since semistability is an open property, it is enough to find a single R_+-primary family $g_1,\dots,g_n$ having the same degree configuration such that $\mathrm{Syz}(g_1,\dots,g_n)$ is semistable.

Via R_+-primary monomial families $f_i = X^{\sigma_i}$, $d_i = |\sigma_i|$, one can use Brenner's result, Theorem 1.1, to establish generic semistability in a combinatorial way by producing examples of monomial families with semistable syzygy bundle. This has been done recently in [Marques and Miró-Roig 11, Theorem 4.6], where the authors have proved the stability of the syzygy bundle $\mathrm{Syz}(f_1,\dots,f_n)$ on $\mathbb{P}^N$ for generic forms of degree d with $N+1 \le n \le \binom{d+N}{N}$, $(N,d,n) \ne (2,2,5)$. This extends [Costa et al. 10, Theorem 3.5], where only the case $N=2$ has been proven. The general result of [Marques and Miró-Roig 11] was obtained simultaneously and independently in [Coandă 09]. For the case $N=2$, $n=5$, and $d=2$, for which only semistability has been shown, Macias Marques asks the following question (see [Marques 09, Problem 2.9]).

Problem 7.1. (Macias Marques.) Is there a family of five quadratic homogeneous polynomials in $K[X_0,X_1,X_2]$ such that their syzygy bundle is stable?

Note that one cannot establish generic semistability via a monomial example, since for the only candidate we have

$$\mu(\mathrm{Syz}(X^2,Y^2,Z^2,XY,XZ)) = -\frac{5}{2} = \frac{1-6}{2} = \mu(\mathrm{Syz}(X^2,XY,XZ)).$$

We answer Macias Marques's question in the following proposition.

Proposition 7.2. *The syzygy bundle*

$$\mathrm{Syz}(X^2-Y^2, X^2-Z^2, XY, XZ, YZ)$$

is stable on $\mathbb{P}^2 = \mathrm{Proj}\, K[X,Y,Z]$. *Moreover, the syzygy bundle for five generic quadrics in* $K[X,Y,Z]$ *is stable on the projective plane.*

Proof. The syzygy bundle

$$\mathcal{S} = \mathrm{Syz}(X^4-Y^4, X^4-Z^4, X^2Y^2, X^2Z^2, Y^2Z^2),$$

which we have considered in Example 5.6, is the pullback of

$$\mathcal{E} := \mathrm{Syz}(X^2-Y^2, X^2-Z^2, XY, XZ, YZ)$$

under the finite morphism

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^2,\ X \longmapsto X^2,\ Y \longmapsto Y^2,\ Z \longmapsto Z^2.$$

Since $\mathcal{S}$ is a stable bundle, so is $\mathcal{E}$. The remaining statement follows from the openness of stability. $\square$

Remark 7.3. In [Coandă 09, Example 1.3], the author has independently proved the stability of the generic syzygy bundle for $(N,d,n) = (2,2,5)$. But his proof is more complicated and does not provide an explicit example of a family of five homogeneous quadrics in three variables.

Let $\mathcal{M}_{\mathbb{P}^N}(n-1,c_1,\dots,c_N)$ be the moduli space of stable vector bundles of rank $n-1$ and Chern classes $c_1,\dots,c_N$ on the projective space $\mathbb{P}^N$. Denote by $\mathcal{S}_{(N,n,d)} \subset \mathcal{M}_{\mathbb{P}^N}(n-1,c_1,\dots,c_N)$ the stratum of stable syzygy bundles $\mathcal{E}$ defined by the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^N}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0.$$

In his thesis [Marques 09], the author computes the dimension of the syzygy stratum and the codimension of its closure in the irreducible component of the moduli space. However, he could not give an answer for the case $N=2, d=2, n=5$ due to the lack of a stable syzygy bundle of five homogeneous quadrics. Our example also closes this gap. We recall that the moduli spaces of stable bundles on $\mathbb{P}^2$ with fixed invariants are irreducible and their dimensions are known; see, for example, [Ellingsrud 83].

Corollary 7.4. *The syzygy stratum* $\mathcal{S}_{(2,5,2)} \subset \mathcal{M}_{\mathbb{P}^2}(4,-10,40)$ *has dimension* 5. *In particular,* $\overline{\mathcal{S}}_{(2,5,2)} = \mathcal{M}_{\mathbb{P}^2}(4,-10,40)$.

Proof. See [Marques 09, Proposition 4.2 and Theorem 4.3], where the proof presented there works analogously for the case $N = 2$, $n = 5$, $d = 2$ due to Proposition 7.2. □

8. COMPUTING INCLUSION BOUNDS FOR TIGHT CLOSURE AND SOLID CLOSURE

Our semistability algorithm also has impact on certain ideal closure operations in commutative algebra due to a geometric interpretation by H. Brenner. We recall briefly the notions of *tight closure* and *solid closure*, where we restrict ourselves to the case in which the ring R under consideration is a Noetherian integral domain. For a detailed exposition of these closure operations and their background, see [Huneke 96].

Let

$$I = (f_1, \ldots, f_n) \subseteq R$$

be an ideal and $f \in R$. The R-algebra

$$A = R[T_1, \ldots, T_n]/(f_1T_1 + \cdots + f_nT_n + f)$$

is called the *forcing algebra* for the elements $f_1, \ldots, f_n, f \in R$. The element f belongs to the *solid closure*, which we denote by $I^\star$, if and only if the following holds: For every maximal ideal $\mathfrak{m}$ of R the top-dimensional local cohomology module $H^d_{\mathfrak{m}'}(A')$ does not vanish, where A' is the forcing algebra for the given data over the local complete domain $R' := \hat{R}_\mathfrak{m}$ and $d = \dim(R') = \mathrm{ht}(\mathfrak{m})$.

Now assume that R is of positive characteristic p (i.e., R contains a field of positive characteristic). Then the *tight closure* of I is defined as the ideal

$$\begin{aligned} I^* := \{ & f \in R : \text{ there exists } 0 \neq t \in R \text{ such that} \\ & tf^q \in I^{[q]} \text{ for all } q = p^e\}, \end{aligned}$$

where $I^{[q]} = (f_1^q, \ldots, f_n^q)$ denotes the extended ideal under the eth iteration of the Frobenius $F : R \to R$, $f \mapsto f^p$.

An important fact due to M. Hochster is that $I^* = I^\star$ holds in positive characteristic for a normal K-algebra R of finite type (in fact, this is true under weaker assumptions); see [Hochster 94, Theorem 8.6].

It follows already from the definitions of these closure operations that they are hard to compute. For a normal standard graded integral 2-dimensional algebra R over an algebraically closed field K, there is a well-developed theory by H. Brenner for solid closure and tight closure that connects these notions with (strong) semistability of the corresponding syzygy bundle $\mathrm{Syz}(f_1, \ldots, f_n)$ on the smooth projective curve $C = \mathrm{Proj}\, R$; see [Brenner 08b] for an excellent survey. This geometric approach combined with A. Langer's restriction theorem, Theorem 4.7 (or its more general formulation [Langer 09, Theorem 2.7]), enables us to use our semistability algorithm, Algorithm 4.5, to compute inclusion bounds for solid closure and tight closure in homogeneous coordinate rings of smooth projective curves, particularly plane curves, of sufficiently large degree. We recall that the syzygy bundle has to be stable in order that the restriction theorem be applicable. In characteristic 0, there is the following result for solid closure.

Theorem 8.1. (Brenner.) *Let K be an algebraically closed field of characteristic zero and let R be a normal standard graded K-domain of dimension two. Further, let $I = (f_1, \ldots, f_n)$ be an R_+-primary homogeneous ideal. If* $\mathrm{Syz}(f_1, \ldots, f_n)$ *is semistable on* $C = \mathrm{Proj}\, R$, *then*

$$I^\star = I + R_{\frac{d_1 + \cdots + d_n}{n-1}},$$

where $d_i = \deg(f_i)$ *for* $i = 1, \ldots, n$.

Proof. See the characteristic-zero version of [Brenner 08b, Theorem 6.4]. □

So Theorem 8.1 gives, for an element $f \in R_m$, an inclusion $f \in I^\star$ for $m \geq \frac{d_1 + \cdots + d_n}{n-1}$, and for $m < \frac{d_1 + \cdots + d_n}{n-1}$, the question whether f belongs to $I^\star$ reduces to an ideal membership test that is a well-known procedure in computational algebra (see, for instance, [Kreuzer and Robbiano 00, Proposition 2.4.10]).

The following theorem works in positive characteristic under the assumption that the syzygy bundle is strongly semistable. We recall that a vector bundle $\mathcal{E}$ is *strongly semistable* if for every $e \geq 0$, the Frobenius pullbacks $F^{e*}(\mathcal{E})$ are semistable. So our algorithmic methods for tight closure can be applied to homogeneous coordinate rings of the general plane curve of large degree via A. Langer's recent restriction theorem [Langer 10, Theorem 3.1].

Theorem 8.2. (Brenner.) *Let K be an algebraically closed field of characteristic $p > 0$ and let R be a normal standard graded K-domain of dimension two. Further, let $I = (f_1, \dots, f_n)$ be an R_+-primary homogeneous ideal such that $\mathrm{Syz}(f_1, \dots, f_n)$ is strongly semistable on $C = \mathrm{Proj}\, R$. Denote the genus of C by g. Then the following hold:*

1. *If $m \geq \frac{d_1+\cdots+d_n}{n-1}$, then $R_m \subseteq I^*$.*
2. *If $m < \frac{d_1+\cdots+d_n}{n-1}$ and $f \in R_m$, then $f \in I^*$ if and only if*
   (a) *$f^p \in I^{[p]} = (f_1^p, \dots, f_n^p)$ if $p > 4(g-1)(n-1)^3$ or*
   (b) *$f^q \in I^{[q]} = (f_1^q, \dots, f_n^q)$ for $q = p^e > 6g$ if $p < 4(g-1)(n-1)^3$.*

Proof. See the positive-characteristic version of [Brenner 08b, Theorem 6.4]. □

Remark 8.3. Let $\mathcal{C} = \mathrm{Proj}\, R \to \mathrm{Spec}\,\mathbb{Z}$ be a generically smooth projective relative curve and $I := (f_1, \dots, f_n)$ an R_+-primary ideal. In this situation, one can deduce tight-closure information of the reductions I_p in the fiber rings $R_p := R \otimes_{\mathbb{Z}} \mathbb{F}_p$ from semistability in characteristic 0. Let $\mathcal{S} := \mathrm{Syz}(f_1, \dots, f_n)$ denote the syzygy bundle on the total space $\mathcal{C}$. If $\mathcal{S}_0 := \mathcal{S}|_{\mathcal{C}_0}$ is semistable on the generic fiber $\mathcal{C}_0 := \mathcal{C} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Q}$ and $m > \frac{d_1+\cdots+d_n}{n-1}$, then $\mathcal{S}_0(m)$ has positive degree and is therefore ample (see [Hartshorne 71, Theorem 2.4]). Since ampleness is an open property, the reductions to positive characteristic $\mathcal{S}_p := \mathcal{S}|_{\mathcal{C}_p}$ and $(\mathcal{S}_p(m))^*$ are also ample on the special fibers $\mathcal{C}_p := \mathcal{C} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$ for almost all prime numbers $p \in \mathbb{Z}$. In this situation, Brenner's geometric approach also yields results on the tight closure of $I_p \subseteq R_p$ for $p \gg 0$; see [Brenner 08b, Section 4] for a detailed treatment of ampleness criteria for tight closure. In particular, by [Brenner 08b, Proposition 4.17], we have $(R_p)_m \subseteq I_p^*$ (in fact, $(R_p)_m$ already belongs to the *Frobenius closure* $I_p^F := \{f \in R_p : f^q \in I_p^{[q]}$ for some $q = p^e\} \subseteq I_p^*$).

Remark 8.4. What can be said in higher dimensions? As usual, we consider R_+-primary homogeneous polynomials $f_1, \dots, f_n$ in $P = K[X_0, \dots, X_N]$. We can compute a minimal graded free resolution $\mathfrak{F}_\bullet$ of the ideal $(f_1, \dots, f_n)$; see [Kreuzer and Robbiano 05, Section 4.8.B] for the computational background. Since the quotient $R = P/I$ is Artinian, the length of $\mathfrak{F}_\bullet$ equals $N+1$ by the Auslander–Buchsbaum formula. Consequently, the corresponding resolution of the associated sheaves on $\mathbb{P}^N$ gives a resolution

$$\mathfrak{F}_\bullet : 0 \longrightarrow \mathcal{F}_{N+1} \longrightarrow \mathcal{F}_N \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0$$

of the structure sheaf with splitting bundles $\mathcal{F}_i$, $i = 1, \dots, N+1$. Instead of looking at $\mathrm{Syz}(f_1, \dots, f_n) = \ker(\mathcal{F}_1 \to \mathcal{O}_{\mathbb{P}^N})$, we consider the bundle

$$\mathrm{Syz}_{N-1} := \mathrm{Syz}_{N-1}(f_1, \dots, f_n) := \mathrm{im}(\mathcal{F}_{N+1} \longrightarrow \mathcal{F}_N).$$

To check whether Syz_{N-1} is semistable, we can apply Algorithm 4.5 to its dual $(\mathrm{Syz}_{N-1})^*$, which is a kernel bundle. If the answer is positive, then we obtain an inclusion bound for the tight closure $(f_1, \dots, f_n)^*$ in the homogeneous coordinate ring R of a generic hyperplane $X \subset \mathbb{P}^N$ of sufficiently large degree. This works as follows. If we restrict the resolution $\mathfrak{F}_\bullet$ to X, we obtain an exact complex of sheaves on X, and the sheaf $\mathrm{Syz}_{N-1}|_X$ is strongly semistable for $k = \deg(X) \gg 0$ by the restriction theorem [Langer 10, Theorem 3.1]. Then Brenner's result [Brenner 05, Theorem 2.4] gives the inclusion bound $R_{\geq \nu} \subseteq (f_1, \dots, f_n)^*$, where

$$\nu := -\frac{\mu(\mathrm{Syz}_{N-1}|_X)}{\deg(X)}.$$

Note that we can compute all necessary invariants (rank, degree, discriminant) of Syz_{N-1} from the resolution $\mathfrak{F}_\bullet$.

We obtain the same inclusion bound if we restrict Syz_{N-1} to smooth hypersurfaces $X \subset \mathbb{P}^N$ for which every semistable bundle on X is strongly semistable; compare, for instance, Example 2.3. Here the degree bound for $\deg(X)$, which ensures the semistability of $\mathrm{Syz}_{N-1}|_X$, is given by Theorem 4.7.

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