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Self-Intersection Numbers of Curves in the Doubly Punctured Plane

Moira Chas and Anthony Phillips

CONTENTS

We address the problem of computing bounds for the self-intersection number (the minimum number of generic self-intersection points) of members of a free homotopy class of curves in the doubly punctured plane as a function of their combinatorial length L; this is the number of letters required for a minimal description of the class in terms of a set of standard generators of the fundamental group and their inverses. We prove that the self-intersection number is bounded above by $L^2/4 + L/2 - 1$, and that when L is even, this bound is sharp; in that case, there are exactly four distinct classes attaining that bound. For odd L we conjecture a smaller upper bound, $(L^2 - 1)/4$, and establish it in certain cases in which we show that it is sharp. Furthermore, for the doubly punctured plane, these self-intersection numbers are bounded *below*, by $L/2 - 1$ if L is even, and by $(L - 1)/2$ if L is odd. These bounds are sharp.

1. INTRODUCTION

By the doubly punctured plane we refer to the compact surface with boundary (familiarly known as the "pair of pants") obtained by removing, from a closed two-dimensional disk, two disjoint open disks. This work extends, to the doubly punctured plane, the research reported in [Chas and Phillips 10] for the punctured torus. In particular, it addresses the relation between the length and the self-intersection number (precise definitions below) of a free homotopy class of curves on that surface.

Like our previous work, this research was motivated by the results of experiments that used a JAVA program[1] based on the Cohen–Lustig algorithm [Cohen and Lustig 87] to tabulate self-intersection numbers for curves. Tables 1 through 4 display for each length $L \leq 19$ and for each possible self-intersection number s the number $N(L, s)$ of distinct free homotopy classes with those properties. The entries in that table show some patterns of potential mathematical interest:

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Keywords: Doubly punctured plane, thrice-punctured sphere, pair of pants, free homotopy classes of curves, self-intersection, combinatorial length

[1] Available at http://www.math.sunysb.edu/~moira/applets/intersectionApplet.html.

s \ L	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	**4**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	0	6	**4**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	0	0	8	10	**4**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
3	0	0	0	12	20	**12**	**4**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
4	0	0	0	0	20	34	**24**	**12**	**4**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
5	0	0	0	2	4	36	56	**40**	**24**	**12**	**4**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
6	0	0	0	0	4	26	72	92	**64**	**40**	**24**	**12**	**4**	**2**	**0**	**0**	**0**	**0**	**0**
7	0	0	0	0	0	16	76	156	168	**104**	**64**	**40**	**24**	**12**	**4**	**2**	**0**	**0**	**0**
8	0	0	0	0	0	0	44	158	284	292	**184**	**104**	**64**	**40**	**24**	**12**	**4**	**2**	**0**
9	0	0	0	0	0	4	16	110	292	460	464	**312**	**184**	**104**	**64**	**40**	**24**	**12**	**4**
10	0	0	0	0	0	0	16	104	280	528	712	690	**488**	**312**	**184**	**104**	**64**	**40**	**24**
11	0	0	0	0	0	2	4	80	320	660	960	1104	1012	**720**	**488**	**312**	**184**	**104**	**64**
12	0	0	0	0	0	0	4	40	268	742	1276	1636	1708	1474	**1048**	**720**	**488**	**312**	**184**
13	0	0	0	0	0	0	0	32	196	736	1564	2244	2596	2572	2152	**1516**	**1048**	**720**	**488**
14	0	0	0	0	0	0	0	4	132	678	1732	3004	3776	3978	3744	3096	**2200**	**1516**	**1048**
15	0	0	0	0	0	0	0	4	80	548	1756	3636	5340	6112	6020	5376	4368	**3152**	**2200**
16	0	0	0	0	0	0	0	0	48	412	1712	3996	6748	8886	9476	8898	7684	6100	4432
17	0	0	0	0	0	0	0	0	12	256	1388	4194	8084	11696	14004	14196	12852	10844	8512
18	0	0	0	0	0	0	0	0	12	182	1076	3888	8916	14738	19204	21328	20656	18232	15104
19	0	0	0	0	0	0	0	2	4	144	1044	3780	9432	17500	25084	30064	31596	29508	25448
20	0	0	0	0	0	0	0	0	8	66	776	3582	10156	20108	31572	40740	45332	45522	41436
21	0	0	0	0	0	0	0	0	0	48	528	2992	9932	22472	38264	53228	63312	66620	64220
22	0	0	0	0	0	0	0	0	0	12	376	2628	9536	24110	44796	66900	85076	94902	95548
23	0	0	0	0	0	0	0	0	0	8	200	2064	9240	25488	51860	82956	110832	130488	138248

TABLE 1. The number $N(L,s)$ of distinct free homotopy classes of curves on the doubly punctured plane with length L and self-intersection number s. Numbers satisfying $N(L,s) = N(L+2,s+1)$ appear in boldface.

s \ L	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
24	0	0	0	0	0	0	0	0	0	0	124	1432	7948	25644	57908	100112	142864	175704	194004
25	0	0	0	0	0	0	0	0	0	4	60	998	6356	24060	61128	115844	177808	231756	267224
26	0	0	0	0	0	0	0	0	0	0	44	714	5204	21896	61796	129328	213188	295870	359372
27	0	0	0	0	0	0	0	0	0	0	16	500	4308	20172	61920	140264	249536	367576	469716
28	0	0	0	0	0	0	0	0	0	0	16	336	3428	18062	60872	149232	285052	446248	599744
29	0	0	0	0	0	0	0	0	0	2	4	264	2740	15984	58588	154748	316472	527832	749448
30	0	0	0	0	0	0	0	0	0	0	4	108	1940	13620	55784	159048	345100	609806	911888
31	0	0	0	0	0	0	0	0	0	0	0	68	1332	11004	51164	160308	373028	696272	1091224
32	0	0	0	0	0	0	0	0	0	0	0	16	868	8688	46436	157696	394600	781908	1288952
33	0	0	0	0	0	0	0	0	0	0	0	12	460	6288	39616	151902	411128	862816	1495100
34	0	0	0	0	0	0	0	0	0	0	0	4	336	4822	32564	139528	415504	937274	1708340
35	0	0	0	0	0	0	0	0	0	0	0	8	196	3608	27712	127146	408748	992356	1920436
36	0	0	0	0	0	0	0	0	0	0	0	0	140	2482	22324	113778	397760	1028324	2118060
37	0	0	0	0	0	0	0	0	0	0	0	0	64	1844	18012	100648	378404	1048104	2289572
38	0	0	0	0	0	0	0	0	0	0	0	0	44	1232	14512	90036	358704	1056046	2434016
39	0	0	0	0	0	0	0	0	0	0	0	0	16	824	11168	77804	338312	1055532	2564276
40	0	0	0	0	0	0	0	0	0	0	0	0	16	522	8316	64984	310916	1039780	2672276
41	0	0	0	0	0	0	0	0	0	0	0	2	4	368	6060	53208	278732	1009028	2745164
42	0	0	0	0	0	0	0	0	0	0	0	0	4	162	4284	42652	245600	962960	2784956
43	0	0	0	0	0	0	0	0	0	0	0	0	0	108	3008	34100	215452	903024	2784508
44	0	0	0	0	0	0	0	0	0	0	0	0	0	32	2056	26964	187964	842192	2745352
45	0	0	0	0	0	0	0	0	0	0	0	0	0	32	1264	20116	157760	773248	2680744
46	0	0	0	0	0	0	0	0	0	0	0	0	0	12	888	15208	131076	694326	2578432

TABLE 2. Continuation of Table 1.

	L																		
s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
47	0	0	0	0	0	0	0	0	0	0	0	0	0	8	468	11008	107940	621280	2460680
48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	324	7770	85224	541228	2316356
49	0	0	0	0	0	0	0	0	0	0	0	0	0	4	176	5812	68696	466592	2137036
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	136	4036	54412	403350	1962436
51	0	0	0	0	0	0	0	0	0	0	0	0	0	0	76	2976	42644	343676	1786544
52	0	0	0	0	0	0	0	0	0	0	0	0	0	0	48	1944	33132	289832	1612560
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	1268	24740	240696	1437828
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16	752	18280	198072	1268644
55	0	0	0	0	0	0	0	0	0	0	0	0	0	2	4	540	13472	159848	1110444
56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	248	9528	126938	958972
57	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	176	6332	98240	814476
58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	76	4472	75678	683412
59	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	56	2860	57732	570396
60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16	1948	42804	467020
61	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16	1196	31704	378124
62	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	852	23636	306116
63	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	496	17344	245664
64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	376	12562	194208
65	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	188	9360	154000
66	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	124	6130	119244
67	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60	4252	91404
68	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	44	2704	68980
69	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16	1868	50952

TABLE 3. Continuation of Table 2.

	L																		
s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
70	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16	1154	37836
71	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	4	840	27392
72	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	418	19780
73	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	264	13272
74	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	108	9244
75	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	84	6212
76	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32	4432
77	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	36	2844
78	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	2036
79	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	1240
80	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	840
81	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	448
82	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	344
83	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	192
84	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	140
85	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	64
86	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	44
87	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16
88	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16
89	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	4
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4

TABLE 4. Continuation of Table 3.

1. For each length $L \leq 19$, the minimum and maximum self-intersection numbers vary as $L/2 - 1$ and $L^2/4 + L/2 - 1$ when L is even, and as $(L-1)/2$ and $(L^2 - 1)/4$ when L is odd.
2. For each $L \leq 19$, the function $N(L, s)$ appears to follow a normal distribution.
3. $N(L, s) = N(L + 2, s + 1)$ for $s \leq L - 3$ and $L \leq 19$. The relevant numbers are shown in boldface.

The goal of the present paper is to make item 1 into a theorem valid for all L. This goal is achieved for L even, and for many cases in which L is odd. See Theorems 1.3, 1.6, 1.7, and Conjecture 1.4 below. (Item 2 has been treated in the paper [Chas and Lalley 12]; item 3 is currently under study.)

Definition 1.1. The doubly punctured plane has fundamental group free on two generators; given a basis, say (a, b), a free homotopy class of curves on the surface can be uniquely represented as a reduced cyclic word in the symbols a, b, A, B (where A stands for a^{-1}, and B for b^{-1}). A *cyclic word* w is an equivalence class of words related by a cyclic permutation of their letters; we will write $w = \langle r_1 r_2 \dots r_n \rangle$, where the r_i are the letters of the word, and $\langle r_1 r_2 \dots r_n \rangle = \langle r_2 \dots r_n r_1 \rangle$, etc. *Reduced* means that the cyclic word contains no juxtapositions of a with A, or b with B. The *length* (with respect to the basis (a, b)) of a free homotopy class of curves is the number of letters occurring in the corresponding reduced cyclic word.

The *self-intersection number* of a free homotopy class of curves is the smallest number of self-intersections among all general-position curves in the class. (General position in this context means as usual that there are no tangencies or multiple intersections.) The self-intersection number is a property of the free homotopy class and hence of the corresponding reduced cyclic word w; we denote it by $\mathrm{SI}(w)$. Note that a word and its inverse have the same self-intersection number.

Remark 1.2. There are three natural generators a, b, c for the fundamental group of the (oriented) doubly punctured plane, corresponding to the three boundary components with their induced orientations; they satisfy the relation $abc = 1$; any two of them form a basis. The length of a free homotopy class of curves will depend to a certain extent on which basis is used for the computation, but the number $N(L, s)$ will not. See Figure 1.

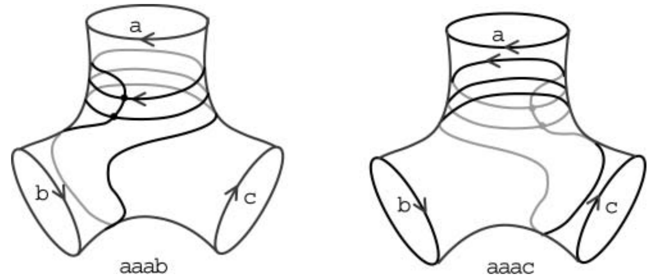

FIGURE 1. The curve γ corresponding to a word w in the (a, b) basis can be rotated (about a vertical axis in this image) to a curve γ' corresponding to the word w' in the (a, c) basis: each b (respectively B) has been substituted by c (respectively C). This defines a bijection $w \leftrightarrow w'$ between words of length L and self-intersection number s in the two bases. An analogous rotation relates calculations in the (a, c) and (b, c) bases. Note that in this example, γ itself ($aaab = aaC$) has different lengths in the two bases.

Theorem 1.3.

(1) *The self-intersection number for a reduced cyclic word of length L on the doubly punctured plane is bounded above by $L^2/4 + L/2 - 1$.*

(2) *If L is even, this bound is sharp: for $L \geq 4$ and even, the cyclic words realizing the maximal self-intersection number are (see Figure 2) $(aB)^{L/2}$ and $(Ab)^{L/2}$. For $L = 2$, they are aa, AA, bb, BB, aB, and Ab.*

(3) *If L is odd, the maximal self-intersection number of words of length L is at least $(L^2 - 1)/4$.*

Conjecture 1.4. *The maximal self-intersection number for a reduced cyclic word of odd length $L = 2k + 1$ in the doubly punctured plane is $(L^2 - 1)/4$; the words realizing the maximum have one of the four forms $\langle (aB)^k B \rangle$, $\langle a(aB)^k \rangle$, $\langle (Ab)^k b \rangle$, $\langle A(Ab)^k \rangle$.*

Definition 1.5. Any reduced cyclic word is either a pure power or may be written in the form $\left\langle \alpha_1^{a_1} \beta_1^{b_1} \dots \alpha_n^{a_n} \beta_n^{b_n} \right\rangle$, where $\alpha_i \in \{a, A\}$, $\beta_i \in \{b, B\}$, all a_i and b_i are positive, and $\sum_1^n (a_i + b_i) = L$, the length of the word. We will refer to each $\alpha_i^{a_i} \beta_i^{b_i}$ as an *$\alpha\beta$-block*, and to n as the word's *number of $\alpha\beta$-blocks.*

Theorem 1.6. *In the doubly punctured plane, consider a reduced cyclic word w of odd length L with n $\alpha\beta$-blocks. If $L > 3n$, if n is prime, or if n is a power of 2, then the self-intersection number of w satisfies $\mathrm{SI}(w) \leq \left(L^2 - 1\right)/4$. This bound is sharp.*

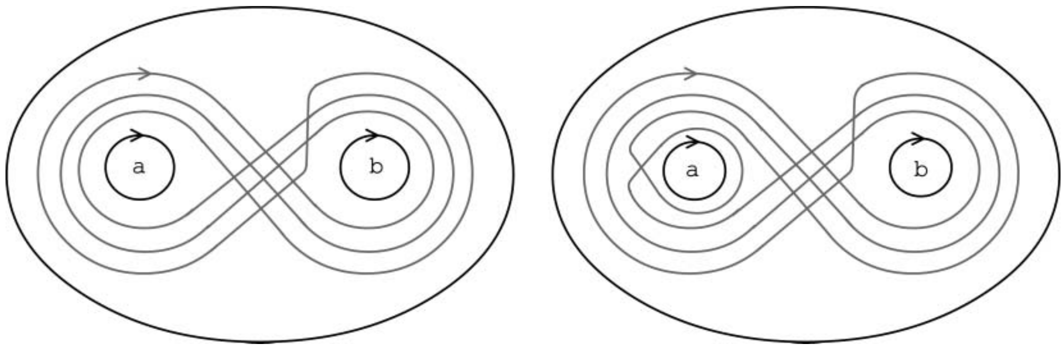

FIGURE 2. Left: curves of the form $\langle aBaBaB\rangle$ have maximum self-intersection number $L^2/4 + L/2 - 1$ for their length (Theorem 1.3). Right: curves of the form $\langle aaBaBaB\rangle$ have self-intersection number $(L^2 - 1)/4$; we conjecture (Conjecture 1.4) that this is maximal, and prove this conjecture in certain cases (Theorem 1.6).

It is elementary to show that the only simple closed curves on the doubly punctured plane correspond to the empty word and the words a, b, ab and their inverses. This generalizes to the statement that in the doubly punctured plane, self-intersection numbers of words are bounded *below.*

Theorem 1.7. *In the doubly punctured plane, curves in the free homotopy class represented by a reduced cyclic word of length L have at least* $L/2 - 1$ *self-intersections if L is even and* $(L-1)/2$ *self-intersections if L is odd. These bounds are achieved by* $(ab)^{L/2}$ *and* $(AB)^{L/2}$ *if L is even and by the four words* $a(ab)^{L-1/2}$*, etc. when L is odd.*

Corollary 1.8. *For any positive integer k, there are only finitely many free homotopy classes of curves on the doubly punctured torus with minimal self-intersection number k (since a curve with minimal self-intersection number k has combinatorial length at most* $2k+2$*).*

Remark 1.9. A surface of negative Euler characteristic that is not the doubly punctured plane has infinitely many homotopy classes of simple closed curves [Mirzakhani 08]. Since the $(k+1)$st power of a simple closed curve has self-intersection number k, it follows that for every k there are infinitely many distinct homotopy classes of curves with self-intersection number k. (A more elaborate argument using the mapping class group constructs, for each k, infinitely many distinct *primitive* classes (not a proper power of another class) with self-intersection number k.) So the doubly punctured plane is the unique surface of negative Euler characteristic satisfying Corollary 1.8.

1.1. Questions and Related Results

A free homotopy class of combinatorial length L in a surface with boundary can be represented by L chords in a fundamental polygon. Hence the maximal self-intersection number of a cyclic reduced word of length L is bounded above by $L(L-1)/2$. One may ask how closely the maximum can approach that bound.

We prove in [Chas and Phillips 10] that for the punctured torus, the maximal self-intersection number $\mathrm{SI}_{\max}(L)$ of a free homotopy class of combinatorial length L is equal to $(L^2-1)/4$ if L is even and to $(L-1)(L-3)/4$ if L is odd. This implies that the limit of $\mathrm{SI}_{\max}(L)/L^2$ is $1/4$ as L approaches infinity. Compare [Lalley 96]. The same limit holds for the doubly punctured plane (Theorem 1.3).

Our (limited) experiments do not suggest analogous polynomials for more general surfaces; but they do lead us to the following conjecture.

Conjecture 1.10. *Consider closed curves on a surface S with boundary, of Euler characteristic* χ*. Let* $\mathrm{SI}_{\max}(L)$ *be the maximum self-intersection number for all curves on S of combinatorial length L. Then*

$$\lim_{L\to\infty} \frac{\mathrm{SI}_{\max}(L)}{L^2} = \frac{\chi}{2\chi - 1}.$$

In particular, this limit approaches $1/2$ *as the Euler characteristic of the surface approaches infinity.*

The doubly punctured plane admits a hyperbolic metric making its boundary geodesic. An elementary argument shows that for curves on that surface, hyperbolic and combinatorial lengths are quasi-isometric. Some of our combinatorial results can be related in this way to

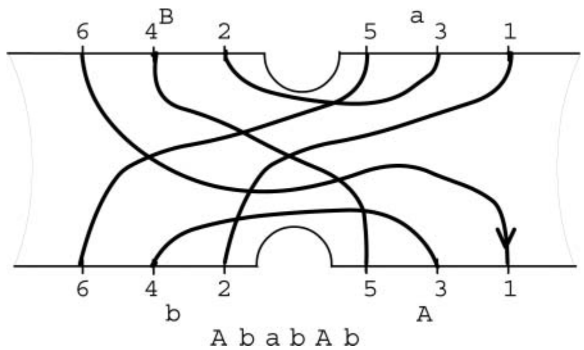

FIGURE 3. The skeleton curve $AbabAb$.

statements about intersection numbers and hyperbolic length.

It is proved in [Basmajian 93] that for a closed hyperbolic surface S, there exists a sequence M_k (for $k = 1, 2, 3, \ldots$) going to infinity such that if γ is a closed geodesic with self-intersection number k, then its geometric length is larger than M_k. For the doubly punctured plane, in terms of the combinatorial length, Theorem 1.3 (1) yields $M_k = \sqrt{5+4k} - 1$.

Question 1.11. Consider closed curves on a hyperbolic surface S (possibly closed). Let $\mathrm{SI}_{\max}(\ell)$ be the maximum self-intersection number for any curve of *hyperbolic* length at most ℓ. Does $\mathrm{SI}_{\max}(\ell)/\ell^2$ converge? And if so, to what limit?

2. A LINEAR MODEL

In this section we will need to distinguish between a cyclically reduced linear word w in the generators and their inverses, and the associated reduced cyclic word w. We introduce an algorithm for constructing from w a representative curve for w. An upper bound for the self-intersection numbers of these representatives may be easily estimated; taking the minimum of this bound over cyclic permutations of $\alpha\beta$-blocks will yield a useful upper bound for $\mathrm{SI}(w)$.

2.1. Skeleton Words

Given a cyclically reduced word

$$w = \left\langle \alpha_1^{a_1} \beta_1^{b_1} \ldots \alpha_n^{a_n} \beta_n^{b_n} \right\rangle,$$

where $\alpha_i = a$ or A, $\beta_i = b$ or B, all a_i, b_i are greater than 0, and the corresponding *skeleton word* is $w_S = \langle \alpha_1\beta_1 \ldots \alpha_n\beta_n \rangle$, a word of length $2n$, we now describe a systematic way of drawing a representative curve for w_S starting from one of its linear forms w_S, and for *thickening* this curve to a representative for w.

The skeleton-construction algorithm. (see Figures 3 and 4) Start by marking off n points along each of the edges of the fundamental domain; corresponding points on the a, A sides are numbered $1, 3, 5, \ldots, 2n-1$ starting from their common corner; and similarly, corresponding points on the b, B sides are numbered $2n, \ldots, 6, 4, 2$, the numbers decreasing away from the common corner.

If the first letter in w_S is a, draw a curve segment entering the a-side at 1, and one exiting the A-side at 1 (vice versa if the first letter is A). That segment is then extended to enter the b-side at 2 and exit the B-side at 2 if the next letter in w_S is b; vice versa if it is B. Continue in this way until the curve segment exiting the b (or B) side at $2n$ joins the initial curve segment drawn.

We will refer to a segment of type ab, ba, AB, BA as a *corner segment*, and one of type aB, Ab, bA, Ba as a *transversal*. Note that (as above) a skeleton word has even length $2n$ and therefore has $2n$ segments (counting the *bridging segment* consisting of the last letter and the first). The number of transversals must also be even, since if they are counted consecutively they go from lowercase to uppercase or vice versa, and the sequence (upper, lower, …) must end where it starts. It follows that the number of corners is also even.

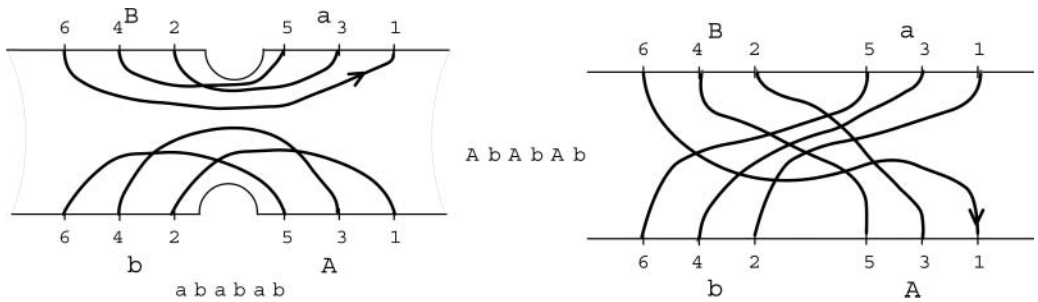

FIGURE 4. The skeleton curves $ababab$ and $AbAbAb$.

Proposition 2.1. *The self-intersection number of the representative of $(Ab)^n$ or $(aB)^n$ given by the curve-construction algorithm equals n^2+n-1.*

Proof. Consider $(Ab)^n$; see the right-hand picture of Figure 4. This curve has only transversals. There are n parallel segments of type Ab; they join $1, 3, \ldots, (2n-1)$ on the a-side to $2, 4, \ldots, 2n$ on the b-side. There are $n-1$ parallel segments of type bA, which join $2, 4, \ldots, 2n-2$ on the B-side to $3, 5, \ldots, 2n-1$ on the A-side. Each of these intersects all n of the Ab segments. Finally, the bridging bA segment joins $2n$ on the B-side to 1 on the A-side. This segment begins to the left of all the other segments and ends up on their right: it intersects all $2n-1$ of them. The total number of intersections is $n(n-1)+2n-1=n^2+n-1$. A symmetric argument handles $(aB)^n$. □

Proposition 2.2. *The self-intersection number of the representative of $(ab)^n$ given by the curve-construction algorithm equals $(n-1)^2$.*

Proof. See the left-hand picture in Figure 4. This curve has only corners. There are n segments of type ab, joining $1, 3, \ldots, 2n-1$ on the A-side to $2, 4, \ldots, 2n$ on the b-side. Since their endpoints interleave, each of these curves intersects all the others. There are $n-1$ segments of type ba, joining $2, 4, \ldots, 2n-2$ on the B-side to $3, 5, \ldots, 2n-1$ on the a-side. Again, each of these curves intersects all the others. Finally, the bridging ba segment joining $2n$ to 1 spans both endpoints of all the others and so intersects none of them. The total number of intersections is

$$\frac{1}{2}n(n-1)+\frac{1}{2}(n-1)(n-2)=(n-1)^2.$$

□

Proposition 2.3. *Let w be a skeleton word of length $2n$. The number of corner segments in w is even, as remarked above; write it as $2c$. Then the self-intersection number of w is bounded above by $n^2+n-1-2c$.*

Proof. Using Propositions 2.1 and 2.2, we can assume that w has both corner segments and transversals. We may then choose a linear representative w with the property that the bridging segment between the end of the word and the beginning is a transversal. Of the $2c$ corners; c will be on top for those of type AB or ba, and and c will be on the bottom for types ab and BA. An ab or AB corner segment joins a point numbered $2j-1$ to a point numbered $2j$ on the same side, top or bottom, as $2j-1$. It encloses segment endpoints $2j+1, 2j+3, \ldots, 2n-1, 2, 4, \ldots, 2j-2$, a total of $n-1$ endpoints; similarly, a ba or BA segment encloses $n-2$ endpoints. So there are at most $2c(n-1)-c(c-1)$ intersections involving corners, correcting for same-side corners having been counted twice. The $2n-2c$ transversals intersect each other just as in the pure-transversal case, producing $(n-c)^2+(n-c)-1$ intersections. The total number of intersections is therefore bounded by $n^2+n-1-2c$. Figure 3 shows the curve $AbabAb$ (here $n=3$, $c=1$) with eight self-intersections. □

2.2. Thickening a Skeleton; Proof of Theorem 1.3 (1), (2)

Once the skeleton curve corresponding to w_S is constructed, it may be *thickened* to produce a representative curve for w. The algorithm runs as follows.

The skeleton-thickening algorithm. (see Figure 5) Suppose for explicitness that w starts with A^{a_1}. The extra a_1-1 copies of A, inserted after the first one, correspond to segments entering the a-side (the first one at 1) and exiting the A-side (the last one at a point opposite the displaced entrance point of the first skeleton segment); the new segments are parallel. Similarly, the extra b_1-1 segments appear as parallel segments originating and ending near the two marks on the b and B sides; so there are no intersections between these segments and those in the first band. Proceeding in this manner, we introduce n nonintersecting bands of $a_1-1, b_1-1, a_2-1, \ldots, b_n-1$ parallel segments. New intersections occur between these bands and segments of the skeleton curve. The two outermost bands (corresponding to a_1 and b_n) are each intersected by one of the skeleton segments; the next inner bands (a_2 and b_{n-1}) each intersect three of the skeleton segments; …; the two innermost bands (a_n and b_1) each intersect $(2n-1)$ of the skeleton segments.

Adding these intersections to the bound on the self-intersections of the skeleton curve itself yields

$$\begin{aligned}\mathrm{SI}(w) \leq\ & (a_1+b_n-2)+3(a_2+b_{n-1}-2)+\cdots \\ & +(2n-1)(a_n+b_1-2)+n^2+n-1.\end{aligned}$$

Since $1+3+\cdots+(2n-1)=n^2$, we may repackage this expression as

$$\mathrm{SI}(w) \leq f(a_1, \ldots, a_n, b_1, \ldots, b_n)-n^2+n-1,$$

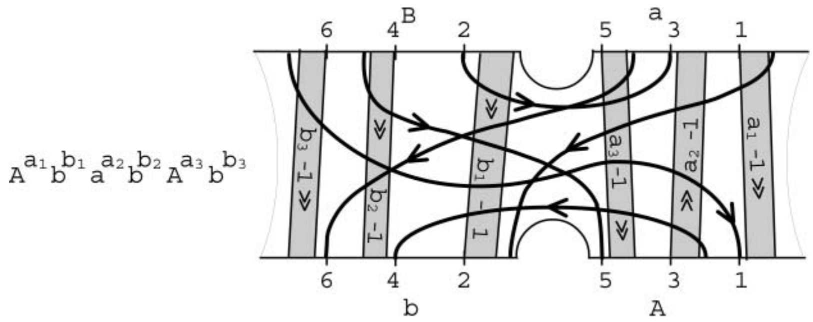

FIGURE 5. The skeleton curve $AbabAb$ thickened to represent the linear word $A^{a_1}b^{b_1}a^{a_2}b^{b_2}A^{a_3}b^{b_3}$. The gray bands represent the curve segments corresponding to the extra letters: a_1-1 copies of A, etc. Notice that the segments from the skeleton curve intersect the a_1 and b_3 bands once, the a_2 and b_2 bands three times, and the a_3 and b_1 bands five times.

where we define f by

$$\begin{aligned} f(a_1,\ldots,a_n,b_1,\ldots,b_n) &= (a_1+b_n)+3(a_2+b_{n-1})+\cdots \\ &\quad +(2n-1)(a_n+b_1). \end{aligned}$$

Applying the skeleton-thickening algorithm to the cyclic permutation

$$\alpha_1^{a_1}\beta_1^{b_1}\ldots\alpha_n^{a_n}\beta_n^{b_n} \to \alpha_2^{a_2}\beta_2^{b_2}\ldots\alpha_n^{a_n}\beta_n^{b_n}\alpha_1^{a_1}\beta_1^{b_1}$$

yields another curve representing the same word. There are n such permutations, leading to

$$\begin{aligned} \mathrm{SI}(w) \le &\left[\min_{i=0,\ldots,n-1} f\circ r^i(a_1,\ldots,a_n,b_1,\ldots,b_n)\right] \\ &-n^2+n-1, \end{aligned} \tag{2–1}$$

where r is the coordinate permutation

$$(a_1,\ldots,a_n,b_1,\ldots,b_n)\to(a_2,\ldots,a_n,a_1,b_2,\ldots,b_n,b_1).$$

Proposition 2.4. *Set* $L=a_1+\cdots+a_n+b_1+\cdots+b_n$. *Then* $\min_{i=0,\ldots,n-1} f\circ r^i(a_1,\ldots,a_n,b_1,\ldots,b_n)\le nL$.

Proof. We write

$$\begin{aligned} f(a_1,\ldots,b_n) &= (a_1+b_n)+3(a_2+b_{n-1})+\cdots \\ &\quad +(2n-1)(a_n+b_1), \\ f\circ r(a_1,\ldots,b_n) &= (a_2+b_1)+3(a_3+b_n)+\cdots \\ &\quad +(2n-1)(a_1+b_2), \\ &\cdots \\ f\circ r^{n-1}(a_1,\ldots,b_n) &= (a_n+b_{n-1})+3(a_1+b_{n-2})+\cdots \\ &\quad +(2n-1)(a_{n-1}+b_n). \end{aligned}$$

The average of these n functions is

$$\frac{1}{n}(L+3L+\cdots+(2n-1)L)=nL.$$

Since the minimum of n functions must be less than their average, the proposition follows. □

Proof of Theorem 1.3 (1) (2). We work with $w=\left\langle \alpha_1^{a_1}\beta_1^{b_1}\ldots\alpha_n^{a_n}\beta_n^{b_n}\right\rangle$. We have established that

$$\mathrm{SI}(w)\le \min_{i=0,\ldots,n-1} f\circ r^i(a_1,\ldots,a_n,b_1,\ldots,b_n)-n^2+n-1.$$

Using Proposition 2.4, we obtain

$$\mathrm{SI}(w)\le nL-n^2+n-1=-n^2+n(L+1)-1.$$

For a given L, this function has its real maximum at $n=(L+1)/2$. Since each $\alpha\beta$-block contains at least two letters, n must be less than or equal to $L/2$. So a bound on $\mathrm{SI}(w)$ is the value at $n=L/2$ (L even) or $n=(L-1)/2$ (L odd):

$$\mathrm{SI}(w)\le\begin{cases} L^2/4+L/2-1 & \text{if } L \text{ is even,}\\ L^2/4+L/2-7/4 & \text{if } L \text{ is odd.}\end{cases}$$

For L even, note (Proposition 2.1) that the skeleton words $w=(aB)^n$ and $w=(Ab)^n$ satisfy $\mathrm{SI}(w)=n^2+n-1=L^2/4+L/2-1$; so the bound for this case is sharp; furthermore, since words with $n=L/2$ must be skeleton words, it follows from Proposition 2.3 that these are the only words attaining the bound. □

Remark 2.5. For L odd, our numerical experiments (which go up to $L=20$) and the special cases we prove below have $\mathrm{SI}(w)\le(L^2-1)/4$, so the function constructed here does not give a sharp bound.

3. ODD-LENGTH WORDS

3.1. A Lower Bound for the Maximal Self-Intersection Number: Proof of Theorem 1.3 (3)

In this subsection we prove Theorem 1.3 (3), namely that the maximum self-intersection number for words of odd length L is at least $(L^2-1)/4$.

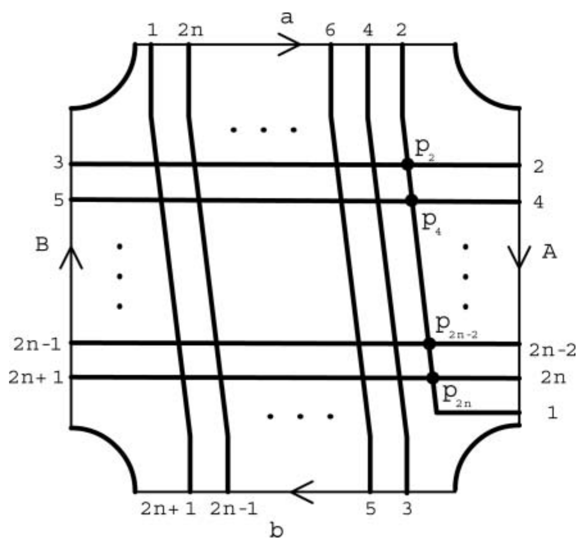

FIGURE 6. The curve $a(aB)^n$ represented in the fundamental domain for the doubly punctured disk.

We will show that the words of the form $a(aB)^{(L-1)/2}$ have self-intersection equal to $(L^2-1)/4$. Consider a representative of w as in Figure 6, where $n=(L-1)/2$. There is an $n \times n$ grid of intersection points in the center, plus the n additional intersections $p_2, \dots, p_{2n}$, a total of $n^2+n=(L^2-1)/4$. We need to check that none of these intersections spans a bigon (we know from [Hass and Scott 85] that this is the only way in which an intersection can be deformed away).

With notation from Figure 6, the only vertices that could be part of a bigon are those from which two segments exit along the same edge, i.e., $p_2, p_4, \dots, p_{2n}$. If we follow the segments from p_2 through edge A, they lead to 1 on edge A and $2n+1$ on edge b, so there is no bigon there; the segments from p_4 through edge A lead to 3, $2n+1$ on edge b, to 2, $2n$ on edge A, and then to 1 on edge A and $2n-1$ on edge b, so there is no bigon; etc. Finally, the segments from p_{2n} through edge A lead to $2n-1$, $2n+1$ on edge b and eventually to 1 on edge A and 3 on edge b: no bigon.

3.2. Preliminaries for Upper-Bound Calculation

In the analysis of self-intersections of odd-length words, the exact relation between L (the length of a word) and n (its number of $\alpha\beta$-blocks) becomes more important.

Proposition 3.1. *If a word w has length L and n $\alpha\beta$-blocks, with $L \geq 3n$, then $\mathrm{SI}(w) \leq (L^2-1)/4$. Note that by Theorem 1.3 (3), this estimate is sharp.*

Proof. As established in the previous section, (2–1), $\mathrm{SI}(w) \leq nL - n^2 + n - 1$.

The inequality $nL - n^2 + n - 1 \leq (L^2-1)/4$ is equivalent to $L^2 - 4nL + 4n^2 - 4n + 3 \geq 0$. As a function of L, this expression has two roots: $2n \pm \sqrt{4n-3}$. As soon as L is past the positive root, the inequality is satisfied.

If $n \geq 3$, then $L \geq 3n$ implies $L \geq 2n + \sqrt{4n-3}$.

If $n=2$, our inequality $\mathrm{SI}(w) \leq nL - n^2 + n - 1$ translates to $\mathrm{SI}(w) \leq 2L-3$, which is always less than $(L^2-1)/4$.

If $n=1$, our inequality becomes $\mathrm{SI}(w) \leq L-1$, which is less than $(L^2-1)/4$ as soon as $L \geq 3$. The only other possibility is $L=2$, an even length. $\square$

3.3. The Cases *n* prime or *n* a power of 2; Proof of Theorem 1.6

Other results for odd-length words require a more detailed analysis of the functions

$$f \circ r^i(a_1, \dots, a_n, b_1, \dots, b_n),$$

where we keep the notation of the previous section.

The proof of the following results is straightforward.

Lemma 3.2. *For a fixed $(a_1, \dots, a_n, b_1, \dots, b_n)$, set*

$$s_a = a_1 + \cdots + a_n, \quad s_b = b_1 + \cdots + b_n,$$
$$t_i = f \circ r^i(a_1, \dots, a_n, b_1, \dots, b_n).$$

Then

(i) $t_{i+1} - t_i = 2n(a_i - b_i) - 2(s_a - s_b)$.

(ii) $t_0 - t_{n-1} = 2n(a_n - b_n) - 2(s_a - s_b)$.

(iii) $t_{i+j} - t_i = 2n(a_i + \cdots + a_{i+j-1} - b_i - \cdots - b_{i+j-1}) - 2j(s_a - s_b)$.

In particular, if $t_i = t_{i+r}$ for some $r > 0$, then

$$n(a_1 - b_1 + a_2 - b_2 + \cdots + a_{i+r-1} - b_{i+r-1}) = r(s_a - s_b).$$

Lemma 3.3. *If n is prime and $L < 3n$, then all the numbers $t_0, \dots, t_{n-1}$ are distinct.*

Proof. By Lemma 3.2, if $t_i = t_{i+r}$ for some $r > 0$, then n must divide r or $s_a - s_b$. We will show that each is impossible. The first cannot happen because $r < n$. As for the second, observe that $s_a \geq n$ and $s_b \geq n$, and that their sum is $L < 3n$; so $s_a - s_b = s_a + s_b - 2s_b < 3n - 2n = n$. So n cannot divide $s_a - s_b$ either. $\square$

Lemma 3.4. *If n is a power of 2 and L is odd, then all the numbers $t_0, \dots, t_{n-1}$ are distinct.*

Proof. We argue as in Lemma 3.3. In this case, since $r < n$, it cannot be a multiple of n, so $s_a - s_b$ must be even. But $s_a - s_b$ is congruent modulo 2 to $s_a + s_b = L$, which is odd. □

Proposition 3.5. *If a word w of odd length L has a number of $\alpha\beta$-blocks that is prime or a power of two, then* $\mathrm{SI}(w) \le (L^2 - 2)/4$.

Proof. Let n be the number of $\alpha\beta$-blocks in w. By Lemmas 3.3 and 3.4, the numbers $t_0, \dots, t_{n-1}$ are all distinct; in fact (Lemma 3.2), their differences are all even, so any two of them must be at least two units apart. It follows that

$$\begin{aligned}\sum_{i=0}^{n-1} t_i &\ge \min t_i + (\min t_i + 2) + \cdots + (\min t_i + 2n - 2)\\ &= n \min t_i + n(n-1),\end{aligned}$$

so their average, which we calculated in the proof of Proposition 2.4 to be nL, is greater than or equal to $\min t_i + n - 1$, and so (using (2–1))

$$\mathrm{SI}(w) \le \min t_i - n^2 + n - 1 \le nL - n^2 = n(L-n) \le \frac{L^2}{4}.$$

Since L is odd and $\mathrm{SI}(w)$ is an integer, this means that

$$\mathrm{SI}(w) \le \frac{L^2 - 1}{4}.$$

□

Propositions 3.1 and 3.5 prove Theorem 1.6.

4. LOWER BOUNDS: PROOF OF THEOREM 1.7

Definition 4.1. A word in the generators of a surface group and their inverses is *positive* if no generator occurs along with its inverse. Note that a positive word is automatically cyclically reduced.

Notation 4.2. If w is a word in the alphabet $\{a, A, b, B\}$, we denote by $\alpha(w)$ (respectively $\beta(w)$) the total number of occurrences of a and A (respectively b and B).

Proposition 4.3. *For any reduced cyclic word w in the alphabet $\{a, A, b, B\}$, there is a positive cyclic word w' of the same length with $\alpha(w') = \alpha(w)$, $\beta(w') = \beta(w)$, and* $\mathrm{SI}(w') \le \mathrm{SI}(w)$.

Proof. We show how to change w into a word written with only a and b while controlling the self-intersection number. If all the letters in w are uppercase, take $w' = w^{-1}$. Otherwise, look in w for a maximal (cyclically) connected string of (one or more) uppercase letters. The letters at the ends of this string must be one of the pairs $(A, A), (A, B), (B, A), (B, B)$. In the case (B, B) (the other three cases admit a similar analysis), focus on that string and write

$$w = \left\langle x a^{a_1} B^{b_1} A^{a_2} B^{b_2} \dots A^{a_i} B^{b_i} a^{a_{i+1}} \right\rangle,$$

where x stands for the rest of the word.

Consider a representative of w with minimal self-intersection. In this representative consider the arcs corresponding to the segments aB (joining the last a of the a^{a_1}-block to the first B of B^{b_1}) and Ba (joining the last B in B^{b_i} to the first a in $a^{a_{i+1}}$). These two arcs intersect at a point p. Perform surgery around p in the following way: remove these two segments, and replace them with an ab and a ba respectively, using the same endpoints. This surgery links the arc $a^{a_{i+1}} x a^{a_1}$ to the arc $B^{b_1} A^{a_2} B^{b_2} \dots A^{a_i} B^{b_i}$ traversed in the opposite direction, i.e., gives a curve corresponding to the word

$$w' = \left\langle a^{a_{i+1}} x a^{a_1} (B^{b_1} A^{a_2} B^{b_2} \dots A^{a_i} B^{b_i})^{-1} \right\rangle.$$

This word has the same α and β values as w, has lost at least one self-intersection, and has strictly fewer uppercase letters than w. The process may be repeated until all uppercase letters have been eliminated. □

Proposition 4.4. *In any surface S with boundary, let w be a cyclically reduced word in the generators of $\pi_1 S$ that does not admit a simple representative curve. Then a linear word* w *representing w (notation from Section 2) can be written as the concatenation* $\mathsf{w} = \mathsf{u} \cdot \mathsf{v}$ *of two linear words in such a way that the associated cyclic words satisfy* $\mathrm{SI}(u) + \mathrm{SI}(v) + 1 \le \mathrm{SI}(w)$. *(Note that u and v are not necessarily cyclically reduced.)*

Proof. Consider a minimal representative of w drawn in the fundamental domain. It must have self-intersections; let p be one of them. Let $\mathsf{w} = x_1 x_2 \dots x_L$ (where $x_i \in \{a, A, b, B\}$) be a linear representative for w, and suppose that $x_i x_{i+1}$ and $x_j x_{j+1}$, with $i < j$, are the two segments intersecting at p (see Figure 7, where $x_i x_{i+1} = Ba$ and $x_j x_{j+1} = ba$). Set $\mathsf{u} = x_{j+1} \dots x_L x_1 x_2 \dots x_i$ and $\mathsf{v} = x_{i+1} \dots x_j$. (In case $i + 1 = j$, v is a single-letter word.) The cyclic words u and v together contain all the segments of w, except that $x_i x_{i+1}$ and $x_j x_{j+1}$ have been replaced by $x_i x_{j+1}$ and $x_j x_{i+1}$.

Furthermore, there is a one-to-one correspondence between the intersection points on $x_i x_{j+1} \cup x_j x_{i+1}$ and

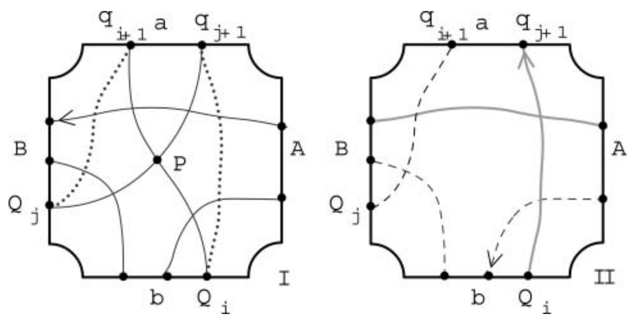

FIGURE 7. Splitting w as $\mathsf{u} \cdot \mathsf{v}$ does not add any new intersections, while the intersection corresponding to p is lost. This figure shows $\mathsf{w} = Babba$ (I) yielding $\mathsf{u} = aB$ and $\mathsf{v} = bba$ (II).

some subset of the intersection points on $x_i x_{i+1} \cup x_j x_{j+1}$. In fact, labeling the endpoints of the segment corresponding to $x_i x_{i+1}$ (respectively $x_j x_{j+1}$) as Q_i and q_{i+1} (respectively Q_j and q_{j+1}), as in Figure 7, observe that the segment corresponding to $x_i x_{j+1}$ and the broken arc $Q_i p q_{j+1}$ have the same endpoints, so any segment intersecting the first must intersect the second and therefore intersect part of $x_i x_{i+1} \cup x_j x_{j+1}$; similarly for $x_j x_{i+1}$ and $Q_j p q_{i+1}$ (compare Figure 7). Therefore, the change from w to $u \cup v$ does not add any new intersections, while the intersection corresponding to p is lost. Hence $\mathrm{SI}(u) + \mathrm{SI}(v) + 1 \leq \mathrm{SI}(w)$. □

The next lemma is needed in the proof of Proposition 4.6.

Lemma 4.5. *In the doubly punctured plane P, if a reduced nonempty word has a simple representative curve, then that curve is parallel to a boundary component. Thus with the notation of Figure 2, the only such words are a, b, ab, A, B, AB.*

Proof. Let γ be a simple essential curve in P. Since P is planar, $P \setminus \gamma$ has two connected components, P_1 and P_2. Since γ is essential, neither P_1 nor P_2 is contractible; hence their Euler characteristics satisfy $\chi(P_1) \leq 0$ and $\chi(P_2) \leq 0$. Since $\chi(P) = -1$ and $\chi(P) = \chi(P_1) + \chi(P_2)$, it follows that either $\chi(P_1) = 0$ or $\chi(P_2) = 0$. Hence, one of the two connected components is an annulus, which implies that γ is parallel to a boundary component, as desired. □

Proposition 4.6. *If w is a positive cyclic word representing a free homotopy class in the doubly punctured plane, then $\mathrm{SI}(w) \geq \alpha(w) - 1$ and $\mathrm{SI}(w) \geq \beta(w) - 1$.*

Proof. By Lemma 4.5, the only words corresponding to simple curves are a, b, ab and their inverses; for these, the statement holds. In particular, it holds for all words of length one. Suppose w is any other positive word. It has length L strictly greater than 1. We may suppose by induction that the statement holds for all words of length less than L. By Proposition 4.4, since the curve associated with w is nonsimple, the word w has a linear representative w that can be split as $\mathsf{u} \cdot \mathsf{v}$, so that the associated cyclic words satisfy $\mathrm{SI}(w) \geq \mathrm{SI}(u) + \mathrm{SI}(v) + 1$. Note that u and v have length strictly less than L; furthermore, since w is positive, so are u and v. Therefore by the induction hypothesis,

$$\mathrm{SI}(u) + \mathrm{SI}(v) + 1 \geq \alpha(u) - 1 + \alpha(v) - 1 + 1,$$

and so

$$\mathrm{SI}(w) \geq \alpha(u) + \alpha(v) - 1 = \alpha(w) - 1.$$

The β inequality is proved in the same way. □

Proof of Theorem 1.7. By Proposition 4.3, there is a positive word w' of length L such that $\alpha(w') = \alpha(w)$, $\beta(w') = \beta(w)$, and $\mathrm{SI}(w) \geq \mathrm{SI}(w')$. Then Proposition 4.6 yields $\mathrm{SI}(w') \geq \max\{\alpha(w), \beta(w)\} - 1$. Since $\alpha(w) + \beta(w) = L$, it follows that $\mathrm{SI}(w) \geq L/2 - 1$ if L is even and $\mathrm{SI}(w) \geq (L+1)/2 - 1 = (L-1)/2$ if L is odd. □

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of curves in surfaces. The program is currently available online.[2]

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Moira Chas, Department of Mathematics, Stony Brook University, Stony Brook NY 11794 (moira@math.sunysb.edu)

Anthony Phillips, Department of Mathematics, Stony Brook University, Stony Brook NY 11794 (tony@math.sunysb.edu)

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[2] At www.math.sunysb.edu/~moira/applets/chrisApplet.html.