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Toward a Salmon Conjecture

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CONTENTS

Methods from numerical algebraic geometry are applied in combination with techniques from classical representation theory to show that the variety of $3 \times 3 \times 4$ tensors of border rank 4 is cut out by polynomials of degree 6 and 9. Combined with results of Landsberg and Manivel, this furnishes a computational solution of an open problem in algebraic statistics, namely, the set-theoretic version of Allman's salmon conjecture for $4 \times 4 \times 4$ tensors of border rank 4. A proof without numerical computation was given recently by Friedland and Gross.

1. INTRODUCTION

In 2007, E. Allman offered a prize of Alaskan salmon to anyone who could find the defining ideal of the following secant variety:

$$\sigma_4\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right)$$

[Allman 10]. Recall that if A, B, C are vector spaces, then the Segre product is defined by the following embedding into the tensor product:

$$\begin{aligned} \mathrm{Seg}\colon \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C &\to \mathbb{P}(A \otimes B \otimes C), \\ ([a],[b],[c]) &\mapsto [a \otimes b \otimes c]. \end{aligned}$$

Further recall that if $X \subset \mathbb{P}^N$ is a variety, then the k-secant variety of X, denoted by $\sigma_k(X) \subset \mathbb{P}^N$, is the Zariski closure of all points on secant $\mathbb{P}^{k-1}$'s to X. For simplicity, we will drop the reference to the Segre embedding and write $\sigma_k(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ for the secant variety to the Segre product.

Secant varieties have been studied classically, but we have a renewed interest in their study because of the salmon prize and other related recent works on the subject (see [Allman and Rhodes 03, Landsberg and Manivel 08, Catalisano et al. 08, Landsberg 08, Allman and Rhodes 08, Friedland 10, Landsberg and Weyman 07, Sidman and Sullivant 09]).

Allman's ideal-theoretic question is still open. Our main result is Theorem 3.10, in which we give a geometric argument (relying on [Landsberg and Manivel 08, Corollary 5.6] and the recent correction of the proof in

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[Friedland 10]) combined with a calculation using numerical algebraic geometry to show that up to high numerical accuracy, $\sigma_4\left(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3\right)$ is cut out set-theoretically by 1728 equations in degree 5, 1000 equations in degree 6, and 8000 equations in degree 9.

Even though these dimensions are large, we show that in each degree, the large space of polynomials can be constructed from a small number of representatives via substitutions (see Remarks 2.2, 3.2, and 3.5). Theorem 3.10 solves the set-theoretic version of Allman's question (up to high numerical accuracy), uses equations of lower degree than Friedland's solution [Friedland 10], and gives evidence for a conjecture to the ideal-theoretical question asked by Allman.

Remark 1.1. After the first version of this article appeared on arXiv, Friedland and Gross proved Theorem 3.10 without relying on numerical methods [Friedland and Gross 11]. See Example 3.6.

One practical interest of the secant variety $\sigma_4\left(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3\right)$ is in phylogenetics, where the secant variety is associated with the statistical model for evolution called the mixture model of independence models [Allman and Rhodes 03, Allman and Rhodes 08]. The main motivation to study this particular model is that [Allman and Rhodes 08, Theorem 11] shows that finding the polynomial invariants for this small evolutionary tree would provide all polynomial invariants for the statistical model for *any* binary evolutionary tree with *any* number of states.

Note that in this paper we work exclusively over the complex numbers. However, in phylogenetics, one is often interested in studying models restricted to the real numbers, the positive real numbers, or the probability simplex. Since equations for a given model considered over the complex numbers also provide equations for the restricted model, it is natural to start with the complex setting and then study the additional necessary equations and inequalities imposed by the given restriction. We leave this further study to other works.

While Allman asks for the generators of the defining ideal of the secant variety, a collection of set-theoretic defining equations provides a necessary and sufficient test on the model for membership. Very recently, it was proved in [Friedland 10] (without a computer) that a set of polynomials in degrees 5, 9, and 16 defines $\sigma_4\left(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3\right)$ set-theoretically. Indeed, Friedland's set of polynomials does (in theory) allow one to test whether a given set of data fits the model. Because it uses polynomials in smaller degree, Theorem 3.10 provides a more efficient practical membership test for the model.

On the other hand, Casanellas and Fernandez-Sanchez have studied more practical issues regarding phylogenetic tree construction using algebraic methods [Casanellas and Fernandez-Sanchez 09]. In particular, they point out that for phylogenetic tree reconstruction, the equations coming from the edges of the tree (minors of flattenings below) seem to be more relevant than the equations coming from vertices (the equations of degrees 5 and 9 are examples of such).

Our equations in degree 6 are not in the ideal of the equations in degree 5; thus they are nontrivial generators in the ideal, and Friedland's result cannot be a set of minimal generators of the ideal. We have not found any such obstructions to our result holding ideal-theoretically, and this leads to a salmon conjecture that the ideal-theoretic version of Theorem 3.10 also holds.

This work was begun in October 2008 when Bernd Sturmfels asked for a Macaulay2-readable file of the degree-6 polynomials in the ideal of $\sigma_4(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3)$. Proposition 2.1 is a representation-theoretic description of these polynomials and corrects minor errors in [Landsberg and Manivel 04, Proposition 6.3] and [Landsberg and Manivel 08, Remark 5.7]. In Section 2 we give a brief overview of how these polynomials were constructed from their representation-theoretic description. These equations and other ancillary materials for this paper are available in the ancillary materials that accompany the arXiv version of this paper or by contacting either author.

At the December 2008 MSRI workshop on algebraic statistics, Oeding presented Conjecture 3.8, which, when combined with an argument of Landsberg and Manivel, implies our main result. This argument is discussed in Section 3. The missing ingredient for the conjecture was to understand the zero-set of the degree-6 polynomials. Shortly after this workshop, Oeding asked for help from Bates and the Bertini team.

The two authors worked together to get the correct mixture of initial input and computing strategies in order to find a computation that would finish in a reasonable amount of time. Finally, on July 12, 2010, a computation that had taken approximately two weeks on eight processors (two 2.66-GHz quad-core Xeon 5410s set up as one head processor and seven worker processors) finished, providing a numerical proof to Conjecture 3.8. Because our calculations use numerical approximations, we say that the proof holds up to high numerical accuracy.

In Section 4, we discuss our computational methods and the reliability of this result.

2. SYMMETRY AND THE EQUATIONS IN DEGREE 6

In this section we recall well-known facts about the variety and equations we are studying. The main purpose is to set up notation. The reader who is unfamiliar with these concepts may consult [Fulton and Harris 91], or for a more detailed account related to secant varieties, see [Landsberg and Manivel 04, Landsberg and Manivel 08, Landsberg and Weyman 07] or the upcoming [Landsberg 11].

Let A, B, C be vector spaces of dimensions a, b, c respectively. The symmetry group of $\sigma_r\left(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C\right)$ is the change of coordinates in each factor $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ (or when $A \cong B \cong C$ there is an additional symmetric group $\mathfrak{S}_3$ acting, and the symmetry group is $(\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)) \ltimes \mathfrak{S}_3$). Therefore we can use tools from representation theory to aid in our search for defining equations. Since much of this work has already been done, we only describe the equations relevant for our application.

The module $S^d(A^* \otimes B^* \otimes C^*)$ of degree-d homogeneous polynomials on $A \otimes B \otimes C$ has an *isotypic decomposition* (see [Landsberg and Manivel 04, Proposition 4.1])

$$S^d(A^* \otimes B^* \otimes C^*) = \bigoplus_{|\pi_1|=|\pi_2|=|\pi_3|=d} \left(S_{\pi_1} A^* \otimes S_{\pi_2} B^* \otimes S_{\pi_3} C^*\right)^{\oplus m_{\pi_1,\pi_2,\pi_3}},$$

where the π_i are partitions of d, and the multiplicity m_{π_1,π_2,π_3} is the dimension of the highest-weight space that can be computed via characters. The modules

$$\left(S_{\pi_1} A^* \otimes S_{\pi_2} B^* \otimes S_{\pi_3} C^*\right)^{m_{\pi_1,\pi_2,\pi_2}}$$

are called *isotypic components*, and the individual modules $S_{\pi_1} A^* \otimes S_{\pi_2} B^* \otimes S_{\pi_3} C^*$ are irreducible $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$-modules, sometimes called *Schur modules*.

The ideal of any $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$-invariant variety in $\mathbb{P}(A \otimes B \otimes C)$ consists of a subset of the modules occurring in the isotypic decomposition. If X is a projective variety, let $\mathcal{I}_s(X)$ denote the ideal of homogeneous degree-s polynomials in the ideal of X.

In general, if X is any variety with ideal generated in degree 2 (of which the Segre variety is an example), then $\mathcal{I}_s(\sigma_k(X)) = 0$ for $s \leq k$ (see [Landsberg and Manivel 04, Corollary 3.2]), and in particular, $\mathcal{I}_s(\sigma_4(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C)) = 0$ for $s \leq 4$. Also, one can calculate (by checking every irreducible module of degree-5 polynomials) that

$$\mathcal{I}_5\left(\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)\right) = 0.$$

In addition, we have found the following result.

Proposition 2.1. *Let $A \cong B \cong \mathbb{C}^3$, $C \cong \mathbb{C}^4$, and let M_6 denote the module $S_{2,2,2} A^* \otimes S_{2,2,2} B^* \otimes S_{3,1,1,1} C^*$. Then $M_6 = \mathcal{I}_6\left(\sigma_4\left(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C\right)\right)$ as $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$-modules.*

Proof. The module M_6 was found by following the ideal membership test described in [Landsberg and Manivel 04]. We repeated the procedure outlined there as follows. We first decomposed $S^6(A^* \otimes B^* \otimes C^*)$ into its isotypic decomposition. This plethysm calculation can be done in one line using the program LiE [Van Leeuwen et al. 92] as

```
plethysm([6],[1,0,1,0,1,0,0],A2A2A3)
```

or by using the procedure `mults`, which we implemented in Maple and can be found in the file `iso_mults.mw`, available with our ancillary materials. Next we computed a basis of the highest-weight space for each isotypic component. We implemented in Maple a standard algorithm to compute a basis of the highest-weight space in the image of the relevant Schur functors associated to each module.

This implementation is in the file called `poly_make_algo.mw`, which also may be found with our ancillary materials. A detailed exposition of this concept may be found below.

We applied this algorithm to each module in the isotypic decomposition, and we checked by direct evaluation to see whether any linear subspace of the highest-weight space of an isotypic component vanished on the variety. The only module that passed this test was M_6, which occurs with multiplicity 1 in $S^6(A^* \otimes B^* \otimes C^*)$. $\square$

We note that there was some confusion between the statements and proofs in the preprint and the print versions of [Landsberg and Manivel 04, Proposition 6.3] as well as in the statement [Landsberg and Manivel 08, Remark 5.7], and we believe that Proposition 2.1 corrects this confusion.

The module $S_{2,2,2}\mathbb{C}^3$ is one-dimensional, and as a vector space, the module $S_{3,1,1,1}\mathbb{C}^4$ is isomorphic to $S^2\mathbb{C}^4$, which is 10-dimensional. Our construction produces a basis of the module M_6 consisting of 10 polynomials that also correspond to the 10 semistandard fillings (strictly increasing in the columns and nondecreasing in the rows) of the tableau of shape $(3, 1, 1, 1)$ with the numbers $1, 2, 3, 4$. We list these fillings below. The basis of polynomials is contained in the file `deg_6_salmon.txt`, which is available with our ancillary materials, as mentioned above.

Here is a brief overview of an algorithm to construct the polynomials in $S_{\pi_1} A^* \otimes S_{\pi_2} B^* \otimes S_{\pi_3} C^*$. While this algorithm is based on classical methods, we refer the reader to the works [Landsberg 11, Oeding 09, Oeding 11], which use similar language, for more details. We point out that the complexity of any algorithm to compute polynomials from Schur modules will depend on dimension and degree. This piece-by-piece algorithm attempts work with the smallest-dimensional space possible at each step, thus reducing the complexity and increasing the chances that the computation will finish in a reasonable amount of time.

For concreteness, we fix the degree $d = 6$ and describe the algorithm that produces the highest-weight vector (and a weight basis) of the module $S_{2,2,2}A^* \otimes S_{2,2,2}B^* \otimes S_{3,1,1,1}C^*$, with $\dim(A) = \dim(B) = 3$ and $\dim(C) = 4$. The input to the algorithm is the fillings of the tableau of shapes π_1, π_2, π_3. The first step is to construct a highest-weight vector in $A^{\otimes 6} \otimes B^{\otimes 6} \otimes C^{\otimes 6}$. For this, we work one vector space at a time. Suppose a_1, a_2, a_3 is a basis of A^*. Then, $a_1 \otimes a_1 \otimes a_2 \otimes a_2 \otimes a_3 \otimes a_3$ is a *source weight vector* for the partition $(2, 2, 2)$, and can be represented by the Young tableau

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}$$

In general, source vectors for $S_{\pi_1} A^*$ correspond to Young tableaux of shape π_1 filled with the numbers $1, \ldots, \dim(A^*)$.

The Young symmetrizer

$$\begin{aligned} Y_{\pi_1} : {} & A^* \otimes A^* \otimes A^* \otimes A^* \otimes A^* \otimes A^* \\ & \to A^* \otimes A^* \otimes A^* \otimes A^* \otimes A^* \otimes A^* \end{aligned}$$

is the map that skew-symmetrizes the vector spaces A^* in positions corresponding to the columns of the filling associated with π_1 and then symmetrizes the vector spaces corresponding to the rows of the filling associated with π_1.

The Young symmetrizers do not change the weight of a vector, and in particular, if the source vector is of the highest possible weight for the tableau of shape π_1, then the image in $S_{\pi_1} A \subset (A^*)^{\otimes 6}$ will have highest weight. We perform the analogous construction in the B^* and C^* factors and take the tensor product of the resulting highest-weight vectors.

The resulting vector we have constructed is in $S_{\pi_1} A^* \otimes S_{\pi_2} B^* \otimes S_{\pi_3} C^*$. However, it is embedded in $(A^*)^{\otimes 6} \otimes (B^*)^{\otimes 6} \otimes (C^*)^{\otimes 6}$. The final step is to perform the reordering isomorphism

$$(A^*)^{\otimes 6} \otimes (B^*)^{\otimes 6} \otimes (C^*)^{\otimes 6} \to (A^* \otimes B^* \otimes C^*)^{\otimes 6},$$

and then symmetrize the result to arrive at a polynomial in $S^6(A^* \otimes B^* \otimes C^*)$.

The symmetrization map is also defined by the fillings of the Young tableau; namely, a given indecomposable vector in $S_{\pi_1} A^* \otimes S_{\pi_2} B^* \otimes S_{\pi_3} C^*$ will be the tensor product of six vectors from A^*, six from B^*, and six from C^*, from which we extract six triples consisting of one vector from each of A^*, B^*, and C^*, where the triples are determined by those with matching labels in the three fillings of the Young tableau. The symmetrization is then found simply by taking the symmetric product of the resulting six triples. (One must make judicious choices in fillings so that the result of this construction is nonzero. However, an in-depth description of how to find good fillings is beyond the scope of this work.)

We computed the highest-weight vector in $S_{2,2,2}A^* \otimes S_{2,2,2}B^* \otimes S_{3,1,1,1}C^*$ using the fixed fillings

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

for π_1, π_2, and π_3 respectively to define the Young symmetrizers. We paired these partitions with source vector

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}$$

for both of $S_{2,2,2}A^*$ and $S_{2,2,2}B^*$ and the source vector

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

for $S_{3,1,1,1}C^*$. To produce a basis of $S_{2,2,2}A^* \otimes S_{2,2,2}B^* \otimes S_{3,1,1,1}C^*$ consisting of ten weight vectors, one can use the same Young symmetrizer and source vectors for $S_{2,2,2}A^*$ and $S_{2,2,2}B^*$, but let the source vector for $S_{3,1,1,1}C^*$ be each of the following:

$$\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&4\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&3\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&4&4\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}$$

Observe that up to renaming the numbers, the fillings for π_3 can be divided into two classes, depending on whether the last two numbers in the first row are equal. The four fillings of the first class (with the last two numbers in the first row equal) correspond to polynomials with 936 terms, whereas the six fillings of the second class correspond to polynomials with 576 terms.

Denote by $p_{i,j,k}$, $1 \leq i, j \leq 3$ and $1 \leq k \leq 4$, a basis of $A^* \otimes B^* \otimes C^* \cong \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$. Then define the swap $p_{i,j,k} \leftrightarrow p_{i,j,l}$ for fixed k, l and for all $1 \leq i, j \leq 3$. Up to sign, this swap takes the polynomial associated with the filling

$$\begin{array}{|c|c|c|}\hline 1&k&k\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}$$

to the polynomial associated with the filling

$$\begin{array}{|c|c|c|}\hline 1&l&l\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}$$

and if m is different from k and l, the swap takes the polynomial associated with the filling

$$\begin{array}{|c|c|c|}\hline 1&k&m\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}$$

to the one associated with

$$\begin{array}{|c|c|c|}\hline 1&l&m\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}$$

This additional symmetry could be useful for the Bertini computation. However, our computation finished without the need to implement this symmetry, so we did not use it. We hope to exploit this for future work.

These fillings produce homogeneous polynomials that are, moreover, homogeneous in multidegree. In general, the *multidegree* of a monomial is a collection of vectors

$$\Big[\, [l_1^A, l_2^A, l_3^A]\,,\ [l_1^B, l_2^B, l_3^B]\,,\ [l_1^C, l_2^C, l_3^C, l_4^C] \,\Big],$$

and is defined on a single variable $x_{i,j,k}$ by the rule that $l_{i'}^A$ is 0 (respectively 1) for $x_{i,j,k}$ if $i \neq i'$ (respectively $i = i'$); $l_{j'}^B$ and $l_{k'}^C$ are defined similarly. The multidegree is then defined for monomials by declaring it to be additive over products of variables.

For example, the following is a sampling of terms in the highest-weight polynomial corresponding to the filling

$$\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1} \multicolumn{1}{|c|}{3}\\ \cline{1-1} \multicolumn{1}{|c|}{4}\\ \cline{1-1}\end{array}$$

$$\cdots - x_{321}x_{113}x_{211}x_{221}x_{134}x_{332} - x_{321}x_{122}x_{231}^2x_{313}x_{114} + x_{211}x_{312}x_{131}x_{121}x_{334}x_{223} + \cdots,$$

and one finds that this polynomial has multidegree $\big[[2,2,2],[2,2,2],[3,1,1,1]\big]$.

Remark 2.2. Note that when $a = b = 3$ and $c = 4$, $S_{2,2,2}A^* \otimes S_{2,2,2}B^* \otimes S_{3,1,1,1}C^*$ is 10-dimensional. When $a = b = c = 4$, the dimension of $S_{2,2,2}A^* \otimes S_{2,2,2}B^* \otimes S_{3,1,1,1}C^*$ increases to 1000. However, the basis of this larger space can still be constructed from the two polynomials that have 576 and 936 monomials via the type of swap of variables described above for the index k in p_{ijk}, but also allowing similar swaps for each of the indices i and j.

3. GEOMETRIC TECHNIQUES FOR SECANT VARIETIES

Suppose $A' \subset A$, $B' \subset B$, and $C' \subset C$. Landsberg and Manivel have shown how to take equations on $\sigma_r(\mathbb{P}A' \times \mathbb{P}B' \times \mathbb{P}C')$ to equations on $\sigma_r(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. They call this procedure *inheritance* [Landsberg and Manivel 04, Proposition 4.4].

Subspace varieties contain tensors that can be written using fewer variables. More specifically,

$$\begin{aligned}&\mathrm{Sub}_{a',b',c'}(A \otimes B \otimes C)\\ &\quad := \big\{[T] \in \mathbb{P}(A \otimes B \otimes C) \mid \exists \mathbb{C}^{a'} \subseteq A, \mathbb{C}^{b'} \subseteq B,\\ &\qquad \mathbb{C}^{c'} \subseteq C, \text{ with } [T] \in \mathbb{P}(\mathbb{C}^{a'} \otimes \mathbb{C}^{b'} \otimes \mathbb{C}^{c'})\big\}.\end{aligned}$$

Landsberg and Weyman have shown that $\mathrm{Sub}_{a',b',c'}(A \otimes B \otimes C)$ is normal with rational singularities, and the ideal is generated by minors of flattenings [Landsberg and Weyman 07, Theorem 3.1].

Recall that a flattening of a 3-tensor in $A \otimes B \otimes C$ is the choice to view it as a matrix in $A \otimes (B \otimes C)$, $B \otimes (A \otimes C)$ or $(A \otimes B) \otimes C$.

The subspace varieties are important in light of equations because of the fact that

$$\mathrm{Sub}_{r,r,r}(A \otimes B \otimes C) \supseteq \sigma_r(\mathbb{P}\,A \times \mathbb{P}\,B \times \mathbb{P}\,C),$$

and therefore when nontrivial, the ideal of $\mathrm{Sub}_{r,r,r}$ gives equations of σ_r. There is an easy test for a module to be in the ideal of a subspace variety, namely $S_{\pi_1} A^* \otimes S_{\pi_2} B^* \otimes S_{\pi_3} C^*$ is in the ideal of $\mathrm{Sub}_{a',b',c'}(A \otimes B \otimes C)$ if and only if at least one of the following holds: $\#(\pi_1) > a'$, $\#(\pi_2) > b'$, $\#(\pi_3) > c'$, where $\#(\cdot)$ is the number of parts of the partition.

Landsberg and Manivel made an important reduction for the salmon problem, which we record here. Friedland pointed out that their proof contained an error, which he corrected in [Friedland 10]. Let a, b, c respectively denote the dimensions of A, B, C.

Theorem 3.1. (Landsberg–Manivel, Friedland.) *As sets, for* $a, b, c \geq 3$, $\sigma_4\left(\mathbb{P}^{\,a-1} \times \mathbb{P}^{\,b-1} \times \mathbb{P}^{\,c-1}\right)$ *is the zero-set of the union of the following:*

(i) *Strassen's commutation conditions,*

$$\begin{aligned} M_5 := {} & S_{(3,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(2,1,1,1)}C^* \\ & \oplus S_{(2,1,1,1)}A^* \otimes S_{(3,1,1)}B^* \otimes S_{(2,1,1,1)}C^* \\ & \oplus S_{(2,1,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(3,1,1)}C^*, \end{aligned}$$

(ii) *equations inherited from* $\sigma_4\left(\mathbb{P}^{\,2} \times \mathbb{P}^{\,2} \times \mathbb{P}^{\,3}\right)$,

(iii) *modules in* $S^5(A^* \otimes B^* \otimes C^*)$ *containing a* $\bigwedge^5$, *i.e., equations for* $\mathrm{Sub}_{4,4,4}$.

Note that when $a = b = c = 4$, the third set of equations is trivial. The key point is that we will have a complete description of the set-theoretic defining equations of $\sigma_4(\mathbb{P}^{\,3} \times \mathbb{P}^{\,3} \times \mathbb{P}^{\,3})$ as soon as we have the equations of $\sigma_4(\mathbb{P}^{\,2} \times \mathbb{P}^{\,2} \times \mathbb{P}^{\,3})$.

Remark 3.2. The equations in degree 5 as well as equations in degree 9 inherited from $\sigma_4\left(\mathbb{P}^{\,2} \times \mathbb{P}^{\,2} \times \mathbb{P}^{\,3}\right)$ were found in [Strassen 83] and were described in terms of certain commutation conditions. See also [Landsberg and Manivel 04] and [Allman and Rhodes 03]. Later, [Landsberg and Manivel 08] reinterpreted these conditions from the geometric and representation-theoretic point of view and provided generalizations in this language. These equations were further studied in [Friedland 10]. In [Sturmfels 09], one finds a nice description of these equations requiring only basic linear algebra. Analogous to our description of the equations in degree 6, here we give the representation-theoretic description of the polynomials of degree 5.

Note also that when $a = b = c = 4$, M_5 is a 1728-dimensional irreducible G-module, for

$$G = (\mathrm{GL}(4) \times \mathrm{GL}(4) \times \mathrm{GL}(4)) \ltimes \mathfrak{S}_3.$$

A natural basis of M_5 can be constructed as in the previous section. For this we need to give the fillings and source vectors for the triple of Young diagrams corresponding to the partitions $(3,1,1), (2,1,1,1), (2,1,1,1)$. The fillings we chose for constructing the Young symmetrizer are

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$$

We note that up to permutation, there is just one equivalence class for the source vectors for $(2,1,1,1)$ with representative

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$$

There are three equivalence classes for the source vectors for $(3,1,1)$ with representatives

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$$

Therefore, to construct representatives for a basis of $S_{(2,1,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(3,1,1)}C^*$, we fix the representative filling for $(2,1,1,1)$ in both instances, and we let the filling for $(3,1,1)$ vary over the three representatives. Thus we construct three polynomials, one for each representative filling of the diagram for $(3,1,1)$, and respectively, these polynomials have 180, 360, and 540 monomials. A basis of polynomials for one of the three isomorphic modules in M_5 is contained in the file `deg_5_salmon.txt` with our ancillary materials. After these three polynomials have been constructed, the rest of the polynomials in the basis of M_5 can be constructed by the substitutions and swaps of variables as mentioned above (see the discussion above Remark 2.2).

Another important result for the salmon problem is from Strassen, which has been reinterpreted in representation-theoretic language in [Landsberg and Manivel 08].

Theorem 3.3. [Strassen 83] *The ideal of the hypersurface* $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^{26}$ *is generated in degree* 9 *by a nonzero vector in the* 1*-dimensional module*

$$S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3.$$

Let M_9 denote the inherited module $S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^4$. Inheritance implies that $M_9 \in \mathcal{I}(\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$.

Remark 3.4. Suppose $[T] \in \mathbb{P}(A \otimes B \otimes C)$, with $\dim(A) = 3$. Then write $T = a_1 \otimes T_1 + a_2 \otimes T_2 + a_3 \otimes T_3$, where the T_i are $b \times c$ matrices in $B \otimes C$ and the a_i are a basis of A.

Strassen described his equation in degree 9 as follows. On an open set, one may assume that T_1 is invertible. Then consider the polynomial

$$\det(T_1)^2 \det(T_2 T_1^{-1} T_3 - T_3 T_1^{-1} T_2).$$

He showed that this polynomial is irreducible, of degree 9, and vanishes on $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.

A useful reformulation by Ottaviani of Strassen's equation is the following (see [Landsberg and Ottaviani 10, Ottaviani 07]). As before, write $T = a_1 \otimes T_1 + a_2 \otimes T_2 + a_3 \otimes T_3$. Here one does not require any of the slices T_1, T_2, T_3 to be invertible. Construct the block matrix

$$\psi_T = \begin{pmatrix} 0 & T_3 & -T_2 \\ -T_3 & 0 & T_1 \\ T_2 & -T_1 & 0 \end{pmatrix}. \qquad (3\text{–}1)$$

One checks that ψ_T is linear in T, and that if $[T] \in \mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, then $\mathrm{rank}(\psi_T) = 2$. Therefore, if $[T]$ is a general point in $\sigma_k(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, it can be written as the sum of k points on $\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, so $\mathrm{rank}(\psi_T) \leq 2k$ by the subadditivity of matrix rank. In particular, in the case $\dim(A) = \dim(B) = \dim(C) = 3$, the 9×9 determinant $\det(\psi_T)$ gives a nontrivial equation for $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, which is also Strassen's equation. This polynomial has 9216 monomials. Note that ψ_T is not a skew-symmetric matrix unless the matrices T_i are symmetric. Otherwise, any odd-sized determinant would vanish identically.

Remark 3.5. In the case that $a = b = 3$ and $c = 4$, as a vector space, M_9 is isomorphic to $S^3\mathbb{C}^4$, so $\dim(M_9) = 20$. When the highest-weight vector of a module has a determinantal representation (as in the case of M_9), it is typically much faster to compute a basis of the module from the highest-weight vector using lowering operators. (Lowering operators are standard in the theory of Lie algebras, but are not the focus of this work. We refer the interested reader to [Oeding 08, Section 3.4] for an explicit treatment of this method.) Using this method, we found that the natural basis of M_9 consists of polynomials with 9,216 or 25,488 or 43,668 monomials.

This basis is a 23-MB text file of polynomials, too large to include with our ancillary files due to the restrictions of arXiv, but may be obtained from either author. As in Remarks 2.2 and 3.2, these polynomials can be associated with representative polynomials, depending on fillings. In the A- and B-factors, the diagram for $(3,3,3)$ can have only one semistandard filling, namely

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&3\\\hline\end{array}$$

In the C-factor, there are three classes of fillings, namely

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&3\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&4\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline 3&4&4\\\hline\end{array}$$

These fillings yield the representative polynomials consisting of 9,216 or 25,488 or 43,668 monomials respectively. The rest of the polynomials in a basis of M_9 can be constructed by the substitutions and swaps described in our treatment of M_6 (see the discussion above Remark 2.2).

Alternatively, a basis of M_9 can be constructed via Ottaviani's formulation. It is derived from the condition that the now 9×12 matrix appearing in (3–1) have rank 8 or less. However, the space of 9×9 minors of ψ_T is no longer irreducible when $a = b = 3$ and $c = 4$. Namely, the space of 9×9 minors of the 9×12 matrix ψ_T is the following representation:

$$\begin{aligned} &S_{3,3,3}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,3}C^* \\ &\quad \oplus S_{4,3,2}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,2,1}C^* \\ &\quad \oplus S_{5,2,2}A^* \otimes S_{3,3,3}B^* \otimes S_{3,2,2,2}C^*. \end{aligned}$$

There are three equivalence classes of maximal minors of ψ_T depending only on the column index I of the maximal minor of $\Delta_I(\psi_T)$. Let $P = (P_1, P_2, P_3)$ be the partition of the set $\{1, \dots, 12\}$ into three sets $P_1 = \{1,2,3,4\}$, $P_2 = \{5,6,7,8\}$, $P_3 = \{9,10,11,12\}$. The representation $S_{3,3,3}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,3}C^*$ is associated with the minors $\Delta_I(\psi_T)$ such that $|I \cap P_i| = 3$ for $i = 1,2,3$. This condition precisely forces the minor of ψ_T to be constructed with 3×3 submatrices of T_1, T_2, and T_3.

The representation $S_{4,3,2}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,2,1}C^*$ is associated with the minors $\Delta_I(\psi_T)$ such that $|I \cap P_1| = 4$, $|I \cap P_2| = 3$, $|I \cap P_3| = 2$.

The representation $S_{5,2,2}A^* \otimes S_{3,3,3}B^* \otimes S_{3,2,2,2}C^*$ is associated with the minors $\Delta_I(\psi_T)$ such that $|I \cap P_1| = 4$, $|I \cap P_2| = 4$, $|I \cap P_3| = 1$.

Note that the symmetry implied by the fact that A and B have the same dimension allows us to reverse the roles of A and B to find two more modules in the ideal; namely, the two modules $S_{3,3,3}A^* \otimes S_{4,3,2}B^* \otimes S_{3,3,2,1}C^*$ and $S_{3,3,3}A^* \otimes S_{5,2,2}B^* \otimes S_{3,2,2,2}C^*$ must also vanish on $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.

While we have described five modules of degree-9 equations that vanish on $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, we use only the module $M_9 = S_{3,3,3}A^* \otimes S_{3,3,3}B^* \otimes S_{3,3,3}C^*$ along with M_6 described above for our set-theoretic defining equations. We can conclude that $\langle M_9 \rangle \not\subset \langle M_6 \rangle$ by analyzing the shapes of the partitions involved. More specifically, in the C-factor the partition $(3,3,3)$ has only three parts, but if $S_{\pi_1}A^* \otimes S_{\pi_2}B^* \otimes S_{\pi_3}C^*$ is a module in the ideal generated by M_6, then π_3 must have at least four parts.

However, this argument fails for the other four degree-9 modules, so it is possible that these equations are in the ideal generated by M_6. Moreover, our set-theoretic result implies that it must be the case that the other degree-9 modules are in the ideal generated by M_6 (up to high numerical accuracy).

Example 3.6. [Friedland 10] Friedland has shown that the known equations in degree 9 are insufficient for defining $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ set-theoretically when $\dim(A) \geq 3$, $\dim(B) \geq 3$, and $\dim(C) \geq 4$. We thank J. M. Landsberg for the following clarification of Friedland's example. Consider the point

$$\begin{aligned} P = (a_1 \otimes b_1 + a_2 \otimes b_2) \otimes c_1 + (a_1 \otimes b_1 + a_2 \otimes b_3) \otimes c_2 \\ + (a_1 \otimes b_1 + a_3 \otimes b_2) \otimes c_3 \\ + (a_1 \otimes b_1 + a_3 \otimes b_3) \otimes c_4. \end{aligned}$$

The span of $\{a_1, a_2, a_3\} \subset A$ and the span of $\{b_1, b_2, b_3\} \subset B$ are both no more than 3-dimensional, so P is a zero of M_5, since the representations $S_{\pi_1}A^* \otimes S_{\pi_2}B^* \otimes S_{\pi_3}C^*$ in M_5 each have either $|\pi_1| = 4$ or $|\pi_2| = 4$, and therefore the respective Schur functor S_{π_i} with $|\pi_i| = 4$ will annihilate a 3-dimensional subspace.

One finds that $\psi_T(P)$ has rank 8, and therefore P is a zero of M_9. However, P is not a point of $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. This geometric argument implies that more polynomials are needed than just the degree-5 and -9 equations. For this, Friedland produces equations of degree 16 that do not vanish on P.

On the other hand, P is not in the zero set of M_6, so M_6 is sufficient to rule out the possibility of points of the same form as P to have border rank 4.

Therefore, one could repeat Friedland's proof, modifying the argument where he uses degree-16 equations with these degree-6 equations and thus obtain a new result, and a computer-free proof of Theorem 3.10.

Remark 3.7. To construct a basis of the 8000-dimensional space $S_{(3,3,3)}\mathbb{C}^4 \otimes S_{(3,3,3)}\mathbb{C}^4 \otimes S_{(3,3,3)}\mathbb{C}^4$, one can repeat the lowering operator procedure. Since these polynomials are very complicated, our experience is that in practice, one should use the degree-9 equations in their determinantal form. In particular, to check whether a point z vanishes on all of the polynomials in $S_{(3,3,3)}\mathbb{C}^4 \otimes S_{(3,3,3)}\mathbb{C}^4 \otimes S_{(3,3,3)}\mathbb{C}^4$, it is more efficient first to construct the matrix in (3–1) for the point z and check that the determinant vanishes. Then repeat this test for all allowable changes of coordinates.

In other words, for every $g \in \mathrm{GL}(4) \times \mathrm{GL}(4) \times \mathrm{GL}(4)$, construct the matrix in (3–1) for $g \cdot z$ and check that the determinant still vanishes. (This is sufficient because our module is the span of the orbit of a single polynomial.) Moreover, if one wants only a quick check that z is in the zero-set with high probability, it suffices to check that $g \cdot z$ is in the zero-set for a random g. In this quick test, a nonvanishing result is certain, but vanishing must be reverified with an exact (nonrandomized) test.

Since $(2,2,2)$ has three parts, and $(3,1,1,1)$ has four parts, M_6 must vanish on the subspace varieties $\mathrm{Sub}_{2,3,4} \cup \mathrm{Sub}_{3,2,4} \cup \mathrm{Sub}_{3,3,3}$. Also, note that two of these subspace varieties are already contained in the secant variety, namely $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right) \supset \mathrm{Sub}_{2,3,4} \cup \mathrm{Sub}_{3,2,4}$. Indeed, if $x \in \mathrm{Sub}_{2,3,4}$, then there exists $A' \subset A$ such that $\dim(A') = 2$ and $x \in \mathbb{P}(A' \otimes B \otimes C)$. But in this case,

$$\begin{aligned} \mathbb{P}(A' \otimes B \otimes C) &= \sigma_4(\mathbb{P}A' \times \mathbb{P}B \times \mathbb{P}C) \\ &\subset \sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C). \end{aligned}$$

The same argument is repeated for $\mathrm{Sub}_{3,2,4}$.

If M is a set of polynomials, let $\mathcal{V}(M)$ denote the zero-set of M. Based on the above evidence, we make the following conjecture:

Conjecture 3.8. *As sets,*

$$\begin{aligned} &\mathcal{V}(S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(3,1,1,1)}\mathbb{C}^4) \\ &\quad = \sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right) \cup \mathrm{Sub}_{3,3,3}. \end{aligned}$$

Computation 4.1 below verifies that Conjecture 3.8 is true up to high numerical accuracy.

Theorem 3.9. (Corollary to Computation 4.1.) *Let* $A \cong \mathbb{C}^3$, $B \cong \mathbb{C}^3$, $C \cong \mathbb{C}^4$. *Up to high numerical accuracy,* *the secant variety $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is defined set-theoretically by*

$$M_6 = S_{(2,2,2)}A^* \otimes S_{(2,2,2)}B^* \otimes S_{(3,1,1,1)}C^*,$$
$$M_9 = S_{(3,3,3)}A^* \otimes S_{(3,3,3)}B^* \otimes S_{(3,3,3)}C^*.$$

Proof. By Proposition 2.1 and by Strassen's theorem (Theorem 3.3) combined with inheritance, we know that both M_6 and M_9 are in the ideal of $\sigma_4(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$. So we know that

$$\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathcal{V}(M_6 \oplus M_9).$$

For the other inclusion, select a point z in the common zero locus of M_6 and M_9. Since $z \in \mathcal{V}(M_6)$, Conjecture 3.8 says that either z is on the secant variety, in which case we are done, or z is on the subspace variety. In the latter case, let $C' \subset C$ be a 3-dimensional vector space such that $z \in \mathbb{P}(A \otimes B \otimes C')$. Then z is a zero of $M_9 = S_{(3,3,3)}A^* \otimes S_{(3,3,3)}B^* \otimes S_{(3,3,3)}C^*$, and therefore is also a zero of the polynomials in the restriction $S_{(3,3,3)}A^* \otimes S_{(3,3,3)}B^* \otimes S_{(3,3,3)}C'^*$. So by Strassen's theorem,

$$z \in \sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C') \cong \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2).$$

We are finished, because we have the obvious inclusion

$$\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C') \subset \sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C).$$

□

We used numerical algebraic geometry, specifically Bertini, to compute the decomposition of the zero-set $\mathcal{V}(M_6)$ into irreducible varieties. We outline this computation in the next section. However, if one were to prove Conjecture 3.8, then the qualifier "with high numerical accuracy" could be removed from the statement of Theorem 3.9.

Recall that the Landsberg–Manivel–Friedland theorem cited above, Theorem 3.1, said that set-theoretic defining equations of $\sigma_4(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1})$ with $a, b, c \geq 3$ will be known as soon as set-theoretic defining equations of $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$ are known, and this is the content of Theorem 3.9. Therefore, we can restate the immediate consequence of combining Theorem 3.1 with our computations.

Theorem 3.10. *As sets, for $a, b, c \geq 3$, up to high numerical accuracy, $\sigma_4(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1})$ is the zero-set of the following:*

(i) *Strassen's commutation conditions,*

$$M_5 := S_{(3,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(2,1,1,1)}C^* \oplus S_{(2,1,1,1)}A^* \otimes S_{(3,1,1)}B^* \otimes S_{(2,1,1,1)}C^* \oplus S_{(2,1,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(3,1,1)}C^*,$$

(ii) *equations inherited from $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$,*

$$M_6 = S_{(2,2,2)}A^* \otimes S_{(2,2,2)}B^* \otimes S_{(3,1,1,1)}C^*,$$
$$M_9 = S_{(3,3,3)}A^* \otimes S_{(3,3,3)}B^* \otimes S_{(3,3,3)}C^*,$$

(iii) *modules in $S^5(A^* \otimes B^* \otimes C^*)$ containing a $\bigwedge^5$, i.e., equations for* $\mathrm{Sub}_{4,4,4}$.

Remark 3.11. As mentioned in the introduction, the qualifier "up to high numerical accuracy" can be removed if one uses Friedland's argument [Friedland 10] modified by our computations, as mentioned in Example 3.6. See [Friedland and Gross 11].

4. RESULTS USING NUMERICAL ALGEBRAIC GEOMETRY

In this section, we provide a brief overview of the basic methods of numerical algebraic geometry; references for further details are provided. We then describe the results of the run establishing the main result of this article and conclude with a short discussion regarding the reliability of numerical algebraic geometry methods and, more to the point, the reliability of this result.

4.1. Brief Overview of Numerical Algebraic Geometry Methods

Given generators of an ideal of $\mathbb{C}[x_1, \dots, x_N]$, the methods of numerical algebraic geometry will produce a *numerical irreducible decomposition* for the associated variety $X \subset \mathbb{C}^N$. In particular, for each irreducible component Z of X, these methods will produce $\deg Z$ numerical approximations (to any number of digits) of generic points on Z. The end result is a catalog of all irreducible components of X, each indicated by a set of *witness points* on the component (together referred to as a *witness set* for the component), its dimension, and its degree.

The core method of numerical algebraic geometry is *homotopy continuation*, a method for approximating the complex zero-dimensional solution set of a polynomial system. The basic idea of homotopy continuation is to cast the given polynomial system F as a member of a parameterized family of polynomial systems, one of which, G, has known solutions or is otherwise easily solved. If done correctly, the solutions of G will vary continuously to those of F as the parameters are varied appropriately. By tracking these paths numerically (using predictor–corrector methods), one will arrive at numerical approximations of all complex zero-dimensional solutions of F. There have been many technical advances in this area that contribute heavily to the reliability of these methods. See [Sommese and Wampler 05, Li 03] for general references and [Bates et al. 08, Bates et al. 09] regarding the use of adaptive precision methods for added reliability.

Pairing homotopy continuation with the use of hyperplane sections, monodromy, and a few other methods described fully in [Sommese and Wampler 05] yields the numerical irreducible decomposition. Briefly, a d-dimensional irreducible algebraic set in $\mathbb{C}^N$ will intersect a generic codimension-d linear space in a set of points. This statement about genericity (along with similar assumptions of genericity throughout numerical algebraic geometry) is the reason for referring to these methods as *probability-one* methods, as described further below.

The computation of a numerical irreducible decomposition begins by searching for codimension-one irreducible components (by adding $N-1$ linear polynomials to the set of generators and solving for zero-dimensional components via homotopy continuation), followed by codimension-two components, and so on. Once this sweep through all possible dimensions has been completed, we have a superset of the desired numerical irreducible decomposition, since a linear variety of codimension d will intersect any component of dimension *d or higher*.

Sommese, Verschelde, and Wampler (and others) have developed methods for removing points in the "wrong dimension," i.e., those discovered while searching for components in dimension d that actually lie on higher-dimensional components, called *junk points*. They have also developed algorithms for performing pure-dimensional decompositions to yield witness sets on each irreducible component (instead of the initially found witness sets for the union of all equidimensional irreducible components). See [Sommese and Wampler 05] for further details.

There are three main software packages in this field: Bertini [Bates et al. 10b], HOM4PS-2.0 [Lee et al. 10], and PHCpack [Verschelde 10]. Each package has various advantages over the others [Bates et al. 10a]. Since Bertini is typically the most efficient package for large parallel positive-dimensional problems as well as the package with the most reliability and precision features, we used Bertini in our computations for this article. In fairness, it should also be noted that Bates is a Bertini developer.

4.2. Numerical Results for the Salmon Problem

Computation 4.1. Up to ten digits of accuracy, the zero-set of the ten polynomials in a basis of M_6 (defined above) has precisely two irreducible components. One, in dimension 31, has degree 345. The other, in dimension 29, has degree 84.

Indeed, $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$ is nondefective and has dimension 31 [Abo et al. 09, Theorem 4.6]. It is also straightforward to check that $\mathrm{Sub}_{3,3,3}$ has dimension 29, and by the pigeonhole principle, these must be our components in the zero-set of M_6. Though these dimensions are sufficient information to identify our varieties, as additional information we find that this secant variety has degree 345 and the subspace variety has degree 84. The degrees of subspace varieties are well known in general. However, we were unable to find a previous result about the degree of this secant variety.

Proof. The conclusion comes from the results of a calculation in Bertini using approximately two weeks of computing time on eight processors, using tight controls including small tracking and final tolerances (10^{-10} or smaller), adaptive precision numerical methods, and a variety of checks and error controls built into Bertini (such as checking at $t = 0.1$ that no paths have crossed). The output of our computation is included in the files `main_data.txt` and `screen_out.txt`, which may be obtained with the other ancillary materials as mentioned above. □

Remark 4.2. After the submission of the first draft of this paper, Friedland provided a counterexample to [Landsberg and Manivel 08, Proposition 5.4], thus invalidating the proof of [Landsberg and Manivel 08, Corollary 5.6]. Because we quote this result in the present paper, we tried to use numerical methods to find out what could be true. In particular, we tried to find a corrected statement for [Landsberg and Manivel 08, Proposition 5.4] by computing the zero-set of $S_{(3,1,1)}A^* \otimes S_{(2,1,1,1)}B^* \otimes S_{(2,1,1,1)}C^*$ using Bertini. This proved to be a very expensive computation. This module, in its smallest form, has a basis of 96 degree-5 polynomials in 48 variables (see the file `deg_5_salmon.txt` in our ancillary files).

After one month of computational time on 72 processors, we had only completed the first 10 codimensions, tracking up to two million paths for each codimension. An easy geometric argument implies that the smallest possible component has projective dimension 8, indicating that we were very far from completing the computation. In the meantime, a second version of [Friedland 10] appeared, with a corrected proof for [Landsberg and Manivel 08, Corollary 5.6]. Due to limited computational resources and time, we decided to abandon further computation and accept the computer-free proof of [Landsberg and Manivel 08, Corollary 5.6] in [Friedland 10].

4.3. Reliability of this Result

Can Computation 4.1 be accepted as absolute proof? No, unfortunately, it may not. However, this numerical computation gives extremely strong evidence that Computation 4.1 is indeed true even without the phrase "with high numerical accuracy."

There are two types of approximations that are used in order to compute the numerical decomposition of a zero-set. One is the choice of a set of random hyperplanes that cut the space and allow one to look for zero-dimensional solutions to a set of equations. The other type of approximation is the numerical homotopy continuation method, which actually searches for the zero-dimensional solutions.

The choice of random hyperplanes amounts to the choice of random numbers from a Zariski-open dense set S of some parameter space rather than choosing some set of points in the complement of S. Since the complement of S is an algebraic set, we know that it must have positive codimension, making it a set of measure zero for any reasonable choice of measure. Thus, the set of hyperplanes that fail in that they would cause us to miss a component in the zero-set has measure zero, and we say that the choice of hyperplanes will yield the correct result with probability one.

The second type of approximation that is done in this type of computation is the heart of Bertini and is thoroughly described in [Sommese and Wampler 05]. Bertini allows one to set desired accuracy to arbitrary levels, and any computational errors (such as path crossing) are reported. Further, Bertini has additional features, such as adaptive-precision path-tracking, that increase security [Bates et al. 08, Bates et al. 09].

The run for this article used a special equation-by-equation algorithm called regeneration [Hauenstein et al. 11]. The run required the following of more than 200,000 paths, and there were no path failures and no crossed paths detected. In addition, there were no errors in the monodromy or trace test procedures. The numerical output of our run is contained in the files `main_data.txt` and `screen_out.txt` with our ancillary materials.

We cannot conclude with unquestionable certainty that Computation 4.1 holds unconditionally, but we can state with an extremely high level of confidence that it is correct. Motivated by this result, we hope to find a direct argument to prove Conjecture 3.8.

4.4. Numerical versus Symbolic Computation

Finally, one might wonder why we chose to use numerical methods to test this conjecture rather than symbolic methods that will provide certainty. The main reasons are simple: time and space. Regarding time, we expect that without additional ideas to reduce the difficulty of computation, a related calculation using symbolic methods should take at least eight times as long as the calculation in Bertini, because Gröbner-basis algorithms are not completely parallelizable (but for an example of recent progress on this front see [Kredel 09]). In fact, based on the timings from an ongoing benchmarking project between the Bertini and Singular [Decker et al. 10] development teams, we suspect that any symbolic computation will actually take far more than eight times as long.

Regarding the issue of space, we must consider data storage at intermediate stages. While the initial input and final result may be relatively small, Gröbner-basis algorithms typically must store large intermediate results for subsequent calculations. On the other hand, homotopy continuation algorithms require a trivial amount of extra data in intermediate stages. Indeed, the amount of memory used grows linearly with the number of paths tracked (simply because the final point on each path must be stored). Bertini is thus much less likely to fail due to memory constraints.

Finally, one could also hope for a (symbolic) certificate of the validity of results obtained by numerical methods. At the EACA School in Tenerife, Spain, Wolfram Decker told us that the development of such certificates is among the current goals of the Singular team, and we hope to be able to use this feature in future work.

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