CW Complexes for Complex Algebraic Surfaces

Andrew Kresch

CONTENTS

We describe CW complexes for complex projective algebraic surfaces in the context of practical computation of topological invariants.

1. INTRODUCTION

Let X be a nonsingular projective algebraic surface over the complex numbers given by explicit defining equations. The practical computation of topological invariants of the underlying complex manifold $X(\mathbb{C})$ is a desirable goal in many settings in algebraic geometry. That topological invariants are effectively computable is a well-established fact, since $X(\mathbb{C})$ may be embedded in a Euclidean space, and a cell complex on $X(\mathbb{C})$ may be obtained from a suitable subdivision of the Euclidean space. This paper explores the practicality of the computation.

For the computation, one first uses the defining equations of X to obtain discrete data, and then computes the desired invariants from those data. The discrete data could be a structure of a CW complex on $X(\mathbb{C})$, presented in such a way that the boundary of a cell can be effectively represented in terms of cells of the next lower dimension. Then the computation of, say, homology or cohomology with integer coefficients reduces to computation of the Smith normal form of an integer matrix. The focus in this paper is on the first step; practicality is assessed by comparing the numbers of cells of various dimensions obtained in several examples against the practical capability of Smith normal form computation as presently known.

The link between the algebraic geometry of X and the algebraic topology of $X(\mathbb{C})$ underlies some of the anticipated applications. For instance, the verification that a collection of algebraic divisor classes on X (given by explicit equations) generates a saturated sublattice of $H_2(X(\mathbb{C}), \mathbb{Z})/\text{tors}$ could be approached directly, given

the ability to compute the topological intersection number of an algebraic divisor class and a topological homology class. (The need for this arises in arithmetic geometry, such as in [Várilly-Alvarado and Viray 10], where it is addressed by a purely algebraic technique of Stoll and Testa.)

2.   BACKGROUND

The computational topology of manifolds has been treated extensively. From Whitney we have not only his famous embedding theorem [Whitney 44], but also an effective algorithm for the computation of topological invariants of an (explicitly given) compact manifold in Euclidean space, by restricting a subdivision of the ambient Euclidean space [Whitney 57]. A general algorithm with complexity analysis is given in [Schwartz and Sharir 83].

2.1   Cell Decompositions of Manifolds

The practicality of the computation of a cell decomposition of a manifold is the subject of intensive study in dimensions one and two, with surfaces treated, for instance, in [Berberich et al. 10] and [Cheng et al. 05]. See also [Alberti et al. 09], and especially the introduction, with its thorough overview and references. As mentioned there, an analysis of cylindrical algebraic decomposition [Collins 75], an algorithm that has been both well studied and widely implemented, yields a bound of $O(d^{2^n+3^n})$ for a degree-d real algebraic hypersurface of dimension n. A recent preprint [Kerber and Sagraloff 10] focuses on the case of hypersurfaces and delivers triangulations with $O(d^{3 \cdot 2^{n-1}-1})$ cells.

2.2   Smith Normal Form Computation

Focusing on the case of (co)homology with integer coefficients, once the discrete data have been obtained as in Section 2.1, the computation is reduced to finding the Smith normal form of an integer matrix. This is known to be possible in polynomial time [Kannan and Bachem 79], and algorithms that do not exploit any particular sparseness structure can in practice handle matrices with up to 1000 rows and columns [Jäger 03].

Exploiting sparseness, which is a characteristic of the matrices representing the boundary maps that arise, probabilistic analysis yields an algorithm quadratic in the number of cells [Donald and Chang 91]. This raises the practicality limit to the tens of thousands, with improvements for simplicial or cubical cell complexes permitting computations with hundreds of thousands of cells

[Mrozek and Batko 09, Mrozek et al. 08]. Regular CW complexes have been treated more recently [Dłotko et al. 10], but since the CW complexes that we produce below are not regular, we accept tens of thousands of cells as a practical limit.

2.3   Case of a Complex Surface

Given a nonsingular projective surface X, a first approach to the computation of topological invariants of $X(\mathbb{C})$ would be to view it as a real manifold, embed it in a Euclidean space, and apply existing techniques. If we seek a triangulation, then we face the bounds attached to known algorithms described in Section 2.1. These are theoretical bounds (we have no other guide due to the lack of treatment of practicality issues in dimensions greater than two in the literature), applicable to the case of a hypersurface (which $X(\mathbb{C})$ will not generally be; for example, in the simplest case $X = \mathbb{P}^2$ it is known that a 7-dimensional Euclidean space is required [Whitney 44]), but they suggest that for the present technology it is infeasible to determine a cell decomposition of $X(\mathbb{C})$ in this way.

Another method, also for real manifolds, is Morse theory. This has been implemented for surfaces, for example in [Fortuna et al. 03], but in ways that rely on the structure of the nonsingular fibers of a Morse function (unions of circles). In higher dimensions we would face practicality issues such as the determination of boundary relations between the critical points (corresponding to gradient flow lines). The literature offers computational approaches such as [Gyulassy et al. 08], but these require a cell decomposition of the manifold as part of the input data. Beyond the practicality issues, a further issue would be to maintain the link between algebraic geometry and topology mentioned in the introduction.

A third approach, using that X is a complex projective variety, is to consider a Lefschetz pencil on X (an algebraic analogue of a Morse function that yields topological information about X). While computations abound (see, e.g., [Gompf and Stipsicz 99]), the general use of Lefschetz pencils brings up the same kinds of issues as with Morse theory.

3.   CELL COMPLEXES ON COMPLEX CURVES AND SURFACES

Following [Moishezon 81], we start with X presented as a branched cover of $\mathbb{P}^2$ with some branch curve $B \subset \mathbb{P}^2$, and by projecting, B as a branched cover of $\mathbb{P}^1$. We will

construct a cell complex on $\mathbb{P}^2$ that extends one on B; this then lifts to X.

3.1   Cell Complexes on Curves

Given a nonsingular complex projective algebraic curve, a general projection to $\mathbb{P}^1$ of suitable degree will represent the curve as a branched cover of $\mathbb{P}^1$ such that over each branch point there is just one ramification point, which is a double point. The curve may have nodes and cusps as singularities, in which case for a general projection the branch points will include as well the images of the nodes and cusps. A cell complex on $\mathbb{P}^1$ with the branch points as 0-cells will lift to a cell complex on the algebraic curve.

We define a finite cell complex on $\mathbb{P}^1$ to be *standard* when the union of 1-cells is homeomorphic to a circle. The complement of the union of 1-cells then has two connected components, which must be the 2-cells.

3.2   Cell Complexes on Surfaces

Now let X be a nonsingular complex projective algebraic surface. A morphism $f : X \to \mathbb{P}^2$ will be called a *generic covering* if f is a finite covering whose only singularities are double points (analytically equivalent to the projection to the (x, y)-plane from the surface $x = z^2$) and cuspidal-type singular points (analytically equivalent to $y = z^3 + xz$), the branch curve $B \subset \mathbb{P}^2$ has only nodes and cusps as singularities, and the restriction of f to the ramification locus of f is a birational isomorphism onto B. Given an embedding of X in a projective space, it is known that a general linear projection $X \to \mathbb{P}^2$ is a generic covering [Ciliberto and Flamini 10].

We suppose that a generic covering $f : X \to \mathbb{P}^2$ has been fixed, and we choose coordinates $(x : y : z)$ on $\mathbb{P}^2$ such that (i) the point $(0 : 0 : 1)$ does not lie on the branch curve B of f, and the projection $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $g(x : y : z) = (x : y)$ has the property that (ii) the restriction of g to B is unramified over $\infty = (0 : 1)$, and the preimage in B of every branch point contains just one ramification point, which is a double point (possibly a singular point of B).

We let d denote the degree of the curve B. Our construction is the following.

Algorithm 3.1.

Step 1. Choose an ordering $p_1, \ldots, p_N$ of the points in the branch locus of $B \to \mathbb{P}^1$ and a standard cell

complex

$$
\begin{aligned}
\{p_1, \ldots, p_N\} \quad &\text{(0-cells),} \\
\{L_1, \ldots, L_N\} \quad &\text{(1-cells),} \\
\{H_+, H_-\} \quad &\text{(2-cells),}
\end{aligned}
$$

on $\mathbb{P}^1$ such that the 1-cell L_1 joining p_N with p_1 passes through ∞.

Step 2. Choose a continuously varying family of standard cell complexes on $g^{-1}(t) \cong \mathbb{P}^1$ for $t \in L_1 \cup \cdots \cup L_N$:

$$
\begin{aligned}
\{(0 : 0 : 1)\} \cup (g^{-1}(t) \cap B) \quad &\text{(0-cells)} \\
\{M_1(t), \ldots, M_e(t)\} \quad &\text{(1-cells)}
\end{aligned}
$$

(where $e = d$ if $t \in \{p_1, \ldots, p_N\}$, and $e = d + 1$ otherwise),

$$
\{H_+(t), H_-(t)\} \quad \text{(2-cells).}
$$

Step 3. Describe a continuous extension of the cell complexes of Step 2 to cell complexes on $g^{-1}(t)$ for $t \in H_+$, as well as to $g^{-1}(t)$ for $t \in H_-$, such that the cells of $g^{-1}(p_j)$ for $j = 1, \ldots, N$, the unions over $t \in L_j$ of the respective cells of $g^{-1}(t)$ for $j = 1, \ldots, N$, and the unions over $t \in H_\pm$ of the respective cells of $g^{-1}(t)$ together form a cell complex on $\mathbb{P}^2$.

Step 4. Lift the cell complex on $\mathbb{P}^2$ to obtain one on X.

In Step 2, a natural definition of the $M_k(t)$ for t in the interior of some L_j suggests itself, provided that the points of $g^{-1}(t)$ that approach each other as t approaches an endpoint of L_j are adjacent for the ordering in the standard cell complexes. In the first example (Section 4.1), this holds. Generally, we carry out Step 2 using a scheme described in Section 4.2 based on adding extra 0-cells to $\{p_1, \ldots, p_N\}$. We will also need to add extra 0-cells in order to accomplish Step 3, as explained in Section 4.1. With these modifications, the standard cell complexes of Step 2 will be supplemented with 1- and 2-cells; for instance, for some t, we may have $M_{1+}(t)$ and $M_{1-}(t)$ instead of just $M_1(t)$. The construction is then determined by fixing the standard cell complexes used: in Section 4.1 they are given explicitly, while in Sections 4.2 and 4.3 the 0-cells are connected in the order of their real parts, using line segments in $\mathbb{C}$ as 1-cells. The latter choice is one that could be applied in any example, as long as the points of $g^{-1}(p_j)$ have distinct real parts for each j (which after a complex rescaling of coordinates is always the case).

4. EXAMPLES

4.1 Quadric

For the first example we take X to be the double cover of

$$B : x^2 + y^2 + z^2 = 0$$

in $\mathbb{P}^2$, i.e., X is a quadric surface. The projection $B \to \mathbb{P}^1$ is branched over the points $p_1 = i$ and $p_2 = -i$, where $i = \sqrt{-1}$ and we identify $t \in \mathbb{C}$ with $(1 : t) \in \mathbb{P}^1 \setminus \{\infty\}$.

In Step 1, we take the 1-cells to be

$$L_1 = \{i + t \mid t \in \mathbb{R}_{\leq 0}\} \cup \{\infty\} \cup \{-i + t \mid t \in \mathbb{R}_{\geq 0}\},$$
$$L_2 = \{ti \mid t \in \mathbb{R}, \, |t| \leq 1\}.$$

For Step 2, when $t = \infty$ we have $g^{-1}(\infty) \cong \mathbb{P}^1$ by $(0 : y : z) \mapsto (y : z)$, and there is the standard cell complex with 1-cells

$$M_1(\infty) = i + \mathbb{R}_{\leq 0} \cup \{\infty\},$$
$$M_2(\infty) = i[-1, 1],$$
$$M_3(\infty) = -i + \mathbb{R}_{\geq 0}.$$

For $t \in \mathbb{P}^1 \setminus \{\infty\}$ we identify $g^{-1}(t)$ with $\mathbb{P}^1$ by $(x : tx : z) \mapsto (x : z)$. We have the following cell complexes on $g^{-1}(t)$, $t \in L_1 \setminus \{\infty, i, -i\}$, compatible with the one at ∞:

$$M_{2-\mathrm{sgn}(\mathrm{Re}(t))}(t) = i\sqrt{1 + t^2} + \mathbb{R}_{\leq 0},$$
$$M_2(t) = i\sqrt{1 + t^2}[-1, 1],$$
$$M_{2+\mathrm{sgn}(\mathrm{Re}(t))}(t) = -i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0},$$

where the analytic function $\sqrt{1 + z^2}$ is extended to $\mathbb{C} \setminus \{ti \mid t \in \mathbb{R}, \, |t| \geq 1\}$, by convention taking the value 1 at $z = 0$. The cell complex at $t = \pm i$ is taken to have 1-cells $\mathbb{R}_{\leq 0}$ and $\mathbb{R}_{\geq 0}$. For $t \in L_2 \setminus \{i, -i\}$ we take

$$M_1(t) = -i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0},$$
$$M_2(t) = i\sqrt{1 + t^2}[-1, 1],$$
$$M_3(t) = i\sqrt{1 + t^2} + \mathbb{R}_{\leq 0}.$$

With only these definitions it is impossible to complete Step 3. This is because when the 1-cells $M_1(\infty)$ and $M_3(\infty)$ vary continuously to t in a neighborhood of ∞, there must be 1-cells with paths to ∞ along line segments of varying slope, not only line segments parallel to $\mathbb{R} \subset \mathbb{C}$. Here $g^{-1}(t)$ for $t \neq \infty$ is identified with $\mathbb{P}^1$ and $\mathbb{C}$ with $\mathbb{P}^1 \setminus \{\infty\}$ as indicated above.

A remedy is to choose

$$0 = t_2 < t_3 < t_4 < t_5 < t_6 < t_7 < t_8 < t_9 \in \mathbb{R},$$

$$
\begin{aligned}
j = 3: \quad & M_1(t) = -i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}, \\
& M_2(t) = i\sqrt{1 + t^2}[-1, 1], \\
& M_3(t) = i\sqrt{1 + t^2} + \mathbb{R}_{\leq 0}, \\[4pt]
j = 4: \quad & M_{1+} = -i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}, \\
& M_{1-} = -i\sqrt{1 + t^2} + e^{-\pi i \frac{t - t_3}{t_4 - t_3}} \mathbb{R}_{\geq 0}, \\
& M_2 = i\sqrt{1 + t^2}[-1, 1], \\
& M_{3+} = i\sqrt{1 + t^2} + \mathbb{R}_{\leq 0}, \\
& M_{3-} = i\sqrt{1 + t^2} + e^{-\pi i \frac{t - t_3}{t_4 - t_3}} \mathbb{R}_{\leq 0}, \\[4pt]
j = 5: \quad & M_{1\pm} = -i\sqrt{1 + t^2} \pm \mathbb{R}_{\geq 0}, \\
& M_{2+} \cup M_{2-} = i\sqrt{1 + t^2}[-1, 1], \\
& M_{3+} = i\frac{t_4 + t_5 - 2t}{t_5 - t_4}\sqrt{1 + t^2} + \mathbb{R}_{\leq 0}, \\
& M_{3-} = i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}, \\[4pt]
j = 6: \quad & M_1 = i\frac{t_5 + t_6 - 2t}{t_5 - t_6}\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}, \\
& M_{2+} \cup M_{2-} = i\sqrt{1 + t^2}[-1, 1], \\
& M_{3+} = -i\sqrt{1 + t^2} + \mathbb{R}_{\leq 0}, \\
& M_{3-} = i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}, \\[4pt]
j = 7: \quad & M_{1+} = i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}, \\
& M_{1-} = i\sqrt{1 + t^2} + e^{\pi i \frac{t - t_6}{t_7 - t_6}} \mathbb{R}_{\geq 0}, \\
& M_2 = i\sqrt{1 + t^2}[-1, 1], \\
& M_{3+} = -i\sqrt{1 + t^2} + \mathbb{R}_{\leq 0}, \\
& M_{3-} = -i\sqrt{1 + t^2} + e^{\pi i \frac{t - t_6}{t_7 - t_6}} \mathbb{R}_{\leq 0}, \\[4pt]
j = 8: \quad & M_{1\pm} = i\sqrt{1 + t^2} \pm \mathbb{R}_{\geq 0}, \\
& M_{2+} \cup M_{2-} = i\sqrt{1 + t^2}[-1, 1], \\
& M_{3+} = i\frac{t_7 + t_8 - 2t}{t_7 - t_8}\sqrt{1 + t^2} + \mathbb{R}_{\leq 0}, \\
& M_{3-} = -i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}, \\[4pt]
j = 9: \quad & M_1 = i\frac{t_8 + t_9 - 2t}{t_9 - t_8}\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}, \\
& M_{2+} \cup M_{2-} = i\sqrt{1 + t^2}[-1, 1], \\
& M_{3+} = i\sqrt{1 + t^2} + \mathbb{R}_{\leq 0}, \\
& M_{3-} = -i\sqrt{1 + t^2} + \mathbb{R}_{\geq 0}.
\end{aligned}
$$

TABLE 1. Definition of the $M_k(t)$ on the segments where $t + i$ is real and strictly between t_{j-1} and t_j, for $j = 3, \ldots, 9$.

set $p_j = -i + t_j$ for $3 \leq j \leq 9$, redefine

$$L_1 = \{i + t \mid t \in \mathbb{R}_{\leq 0}\}$$
$$\cup \{\infty\} \cup \{-i + t_9 + t \mid t \in \mathbb{R}_{\geq 0}\}$$

(this is a subset of the original L_1), and introduce

$$L_j = \{-i + t \mid t_{j-1} \leq t \leq t_j\}$$

for $3 \leq j \leq 9$ with the $M_k(t)$ appearing in Table 1. (The specification of $M_k(t)$ for t in the interiors of the L_j, given in the table, determines $M_k(p_j)$ for $3 \leq j \leq 9$.)

Then there exist families of cell complexes over $t \in H_+$ and $t \in H_-$, built out of line segments in $\mathbb{C}$, extending the $M_k(t)$ for $t \in L_1 \cup \cdots \cup L_9$. For instance, the line segments that approach ∞ may be taken parallel to $\mathbb{R} \subset \mathbb{C}$ when

Images of basis elements of homological chain complex under ∂_2 (with $w^\pm = L_3^\pm + \cdots + L_9^\pm$)

2-cell	boundary	2-cell	boundary	2-cell	boundary
H_+^+	$L_1^+ + L_2^+ + w^+$	L_{53-}	$L_5^+ + p_{53-} - p_{43-}$	p_1^-	$p_{11} + p_{12}$
H_+^-	$L_1^- + L_2^- + w^-$	L_{61}	$p_{61} - L_6^0 - p_{51}$	p_2^+	$-p_{21} - p_{22}$
H_-^+	$-L_1^+ - L_2^- - w^+$	L_{62+}	$L_6^0 - L_6^+ - p_{52}$	p_2^-	$p_{21} + p_{22}$
H_-^-	$-L_1^- - L_2^+ - w^-$	L_{62-}	$L_6^- + p_{62} - L_6^0$	p_3^+	$-p_{31} - p_{32} - p_{33}$
L_{11}	$p_{11} - L_1^+ + p_{93}$	L_{63+}	$L_6^- + p_{63} - p_{53+}$	p_3^-	$p_{31} + p_{32} + p_{33}$
L_{12}	$L_1^- - L_1^+ - p_{92}$	L_{63-}	$L_6^+ - p_{61} - p_{53-}$	p_4^+	$-p_{43+} + p_{43-}$
L_{13}	$L_1^- + p_{12} + p_{91}$	L_{71+}	$p_{71+} - L_7^+ - p_{61}$	$p_4^<$	$p_{41-} + p_{42} + p_{43+}$
L_{21}	$p_{21} - L_2^- - p_{11}$	L_{71-}	$p_{71-} - L_7^+ - p_{61}$	$p_4^>$	$-p_{41+} - p_{42} - p_{43-}$
L_{22}	$L_2^- - L_2^+$	L_{72}	$L_7^- + p_{72} - L_7^+ - p_{62}$	p_4^-	$p_{41+} - p_{41-}$
L_{23}	$L_2^+ + p_{22} - p_{12}$	L_{73+}	$L_7^- + p_{73+} - p_{63}$	p_5^+	$p_{52} - p_{53+} + p_{53-}$
L_{31}	$p_{31} - L_3^- - p_{21}$	L_{73-}	$L_7^- + p_{73-} - p_{63}$	$p_5^>$	$-p_{51} - p_{52} - p_{53-}$
L_{32}	$L_3^- + p_{32} - L_3^+$	L_{81+}	$p_{81} - L_8^+ - p_{71+}$	p_5^-	$p_{51} + p_{53+}$
L_{33}	$L_3^+ + p_{33} - p_{22}$	L_{81-}	$-p_{83+} - L_8^+ - p_{71-}$	p_6^+	$-p_{61} + p_{62} - p_{63}$
L_{41+}	$p_{41+} - L_4^- - p_{31}$	L_{82+}	$L_8^0 - L_8^+ - p_{72}$	p_6^-	$p_{61} - p_{62} + p_{63}$
L_{41-}	$p_{41-} - L_4^- - p_{31}$	L_{82-}	$L_8^- + p_{82} - L_8^0$	p_7^+	$-p_{71+} + p_{71-}$
L_{42}	$L_4^- + p_{42} - L_4^+ - p_{32}$	L_{83+}	$L_8^0 + p_{83+} - p_{73+}$	$p_7^<$	$-p_{71-} + p_{72} - p_{73+}$
L_{43+}	$L_4^+ + p_{43+} - p_{33}$	L_{83-}	$L_8^- + p_{83-} - p_{73-}$	$p_7^>$	$p_{71+} - p_{72} + p_{73-}$
L_{43-}	$L_4^+ + p_{43-} - p_{33}$	L_{91}	$p_{91} - L_9^0 - p_{81}$	p_7^-	$p_{73+} - p_{73-}$
L_{51+}	$p_{51} - L_5^- - p_{41+}$	L_{92+}	$L_9^0 + p_{92} - L_9^+$	p_8^+	$-p_{81} - p_{83+}$
L_{51-}	$-p_{53+} - L_5^- - p_{41-}$	L_{92-}	$L_9^- - L_9^0 - p_{82}$	$p_8^>$	$p_{81} - p_{82} + p_{83-}$
L_{52+}	$L_5^0 + p_{52} - L_5^+$	L_{93+}	$L_9^+ + p_{93} - p_{83+}$	p_8^-	$p_{82} + p_{83+} - p_{83-}$
L_{52-}	$L_5^- - L_5^0 - p_{42}$	L_{93-}	$L_9^- - p_{91} - p_{83-}$	p_9^+	$-p_{91} - p_{92} - p_{93}$
L_{53+}	$L_5^0 + p_{53+} - p_{43+}$	p_1^+	$-p_{11} - p_{12}$	p_9^-	$p_{91} + p_{92} + p_{93}$

and ∂_3 (with $u = L_{32} + \cdots + L_{92\pm}$ and $v_{j\pm} = (2-j)H_\pm^{2-j} \pm (L_{1j} + L_{2j} + L_{3j})$)

$$v_{1+} + u + L_{22} + L_{41-} + L_{51-} - L_{63+} - L_{73+} - L_{82-} - L_{83+} - L_{92\pm} - L_{93+},$$
$$v_{3+} + u + L_{22} + L_{43-} + L_{53-} + L_{63-} - L_{71+} - L_{81+} - L_{92+} - L_{91},$$
$$v_{1-} - L_{32} - L_{42} - L_{41+} - L_{51+} - L_{52\pm} - L_{61} - L_{62+} - L_{71-} - L_{81-} + L_{93+},$$
$$v_{3-} - L_{32} - L_{42} - L_{43+} - L_{52-} - L_{53+} - L_{63+} - L_{73-} - L_{83-} - L_{93-},$$
$$H_+^+ - H_+^- + u + L_{12} + L_{22}, \ H_-^+ - H_-^- - u - L_{12} + L_{22},$$
$$p_1^\pm - p_9^\mp \pm (L_{11} - L_{12} + L_{13}), \ p_j^\pm - p_{j-1}^\pm \pm (L_{j1} + L_{j2} + L_{j3}) \ (j = 2,3),$$
$$p_4^> - p_3^+ + L_{41+} + L_{42} + L_{43-}, \ p_4^+ + L_{43+} - L_{43-}, \ p_4^- - L_{41+} + L_{41-},$$
$$p_4^< - p_3^- - L_{41-} - L_{42} - L_{43+}, \ p_5^+ - p_4^+ - L_{52+} + L_{53+} - L_{53-},$$
$$-p_4^< - L_{51-} - L_{52-} - L_{53+}, \ p_5^> - p_4^> + L_{51+} + L_{52\pm} + L_{53-},$$
$$p_5^- - p_4^- - L_{51+} + L_{51-}, \ p_6^+ - p_5^+ - L_{62\pm} + L_{63+} - L_{63-}, \ -p_5^> + L_{61} + L_{62+} + L_{63-},$$
$$p_6^- - p_5^- - L_{61} + L_{62-} - L_{63+}, \ p_7^+ - p_6^+ + L_{71-} - L_{72} + L_{73+}, \ p_7^- - L_{73+} + L_{73-},$$
$$p_7^+ + L_{71+} - L_{71-}, \ p_7^> - p_6^- - L_{71+} + L_{72} - L_{73-}, \ p_8^+ - p_7^+ + L_{81+} - L_{81-},$$
$$-p_7^< + L_{81-} - L_{82+} + L_{83+}, \ p_8^> - p_7^> - L_{81+} + L_{82\pm} - L_{83-},$$
$$p_8^- - p_7^- - L_{82-} - L_{83+} + L_{83-}, \ p_9^+ - p_8^+ + L_{91} + L_{92+} + L_{93+},$$
$$-p_8^> - L_{91} + L_{92-} - L_{93-}, \ p_9^- - p_8^- - L_{92\pm} - L_{93+} + L_{93-}$$

TABLE 2. Data for computation in Section 4.1.

$|\operatorname{Re}(t)| \le t_9 + 1$, $|\operatorname{Im}(t)| \le 2$, and

$$\begin{cases} \operatorname{Re}(t) \le t_3 \text{ or } \operatorname{Re}(t) \ge t_4, & \text{when } t \in H_+, \\ \operatorname{Re}(t) \le t_6 \text{ or } \operatorname{Re}(t) \ge t_7, & \text{when } t \in H_-. \end{cases}$$

Suitable cell complexes on $g^{-1}(t)$ for $t \in H_-$ close to L_k for $k = 5$, 6, and 7 are depicted in Figure 1.
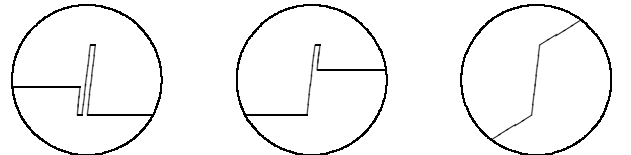

FIGURE 1. Neighborhood of $0 \in \mathbb{C}$ of cell complex on $g^{-1}(t)$ for $t \in H_-$ close to L_5 (left), to L_6 (middle), and to L_7 (right).

The cell complex of Step 3 has 17 0-cells, 53 1-cells, 69 2-cells, 34 3-cells, and 4 4-cells. If we orient the cells so that

- L_j is a path from t_{j-1} (or t_9, when $j = 1$) to t_j,

- $M_1(t)$ (or $M_{1\pm}(t)$) is a path from $(0 : 0 : 1)$ to a point of $g^{-1}(t) \cap B$, $M_e(t)$ (or $M_{e\pm}(t)$) is a path from a point of $g^{-1}(t) \cap B$ to $(0 : 0 : 1)$, and $M_2(t)$ for $t \neq \pm i$ with nonnegative real part is a path from the point with negative imaginary part to the point with positive imaginary part,

- open subsets of $\mathbb{C}$ are given the canonical orientation,

then we have, in a straightforward manner, a homological chain complex for $\mathbb{P}^2$. For instance, we may compute $H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z}) = \ker(\partial_2)/\operatorname{im}(\partial_3) \cong \mathbb{Z}$, where $\partial_2 : \mathbb{Z}^{69} \to \mathbb{Z}^{53}$ and $\operatorname{im}(\partial_3)$ are given in Table 2.

In the table, 1-cells $L_j^{\pm}$ denote the point of $g^{-1}(t) \cap B$ ($t \in L_j$) with indicated sign of the imaginary part, L_j^0 the point with varying sign of the imaginary part, $p_{jk(\pm)}$ (respectively $L_{jk(\pm)}$) the 1- (respectively 2-) cells in $g^{-1}(t)$ (respectively over L_j), and $p_j^>$, $p_j^<$, p_j^+, p_j^- the 2-cells in $g^{-1}(t)$ (with $>$ and $<$ indicating the respective signs of the real parts, and $+$ and $-$ the signs of the imaginary parts).

Step 4 produces a cell complex with 18 0-cells, 88 1-cells, 134 2-cells, 68 3-cells, and 8 4-cells.

4.2 Cubic

It is known classically that the branch curve of a general cubic surface is a sextic curve with six cusps lying on a conic [Zariski 29]. For this example we take

$$X : x^2z + x^2t + y^3 + z^2t + t^3 = 0.$$

This is a triple cover of $\mathbb{P}^2$ branched over

$$B : 4x^6 + 39x^4z^2 + 54x^2y^3z + 12x^2z^4 + 27y^6 + 4z^6 = 0.$$

Under projection $B \to \mathbb{P}^1$ we find 12 ordinary branch points and 6 images of cusps, for a total of 18 branch points.

As mentioned in Section 3.2, for the $M_k(t)$ we join the points of $g^{-1}(t) \cap B$ in the order of their real parts (under the usual identification of $\mathbb{C}$ with the complement of ∞ in $g^{-1}(t) \cong \mathbb{P}^1$) using line segments in $\mathbb{C}$. This means that on some L_j the order in which the points are joined will change. This may be accomplished by suitably enlarging the set $\{p_1, \ldots, p_N\}$ just as in the first example, e.g., with L_5 and L_6. To accomplish m adjacent point swaps along

L_j we replace L_j by $2(m + 1)$ 1-cells (by adding $2m + 1$ additional points on L_j to $\{p_1, \ldots, p_N\}$).

Choosing $\{L_1, \ldots, L_N\}$ (Step 1) and making modifications as in the previous paragraph for Step 2, we find (for Step 3 adding additional cells in analogy with the specifications of Table 1) the following total numbers of cells in Step 3: 2143 0-cells, 5012 1-cells, 3636 2-cells, 768 3-cells, and 4 4-cells. Taking account of the cells that make up B (2142 0-cells, 2160 1-cells, and 12 2-cells, with cusps accounting for 6 0-cells), we obtain 4281 0-cells, 12876 1-cells, 10896 2-cells, 2304 3-cells, and 12 4-cells in Step 4.

4.3 K3 surface

A K3 surface of degree 2 is a double cover of the plane branched along a nonsingular sextic curve [Mayer 72]. The construction may be carried out just as in Section 4.2; we report the total numbers of cells (Step 4) obtained in a particular example ($B : x^6 + y^5z = z^6$): 1676 0-cells, 6120 1-cells, 5692 2-cells, 1232 3-cells, and 8 4-cells.

5. CONCLUSION

We have described a construction of CW complexes for complex algebraic surfaces that in examples yields sizes amenable to computation. The examples encompass simple classes of rational surfaces as well as a first instance of K3 surfaces, which are actively studied with the interactions of the algebraic and topological points of view playing an important role due to the presence of transcendental (nonalgebraic) homology classes.

REFERENCES

[Alberti et al. 09] L. Alberti, B. Mourrain, and J.-P. Técourt. "Isotopic Triangulation of a Real Algebraic Surface." *J. Symbolic Comput.* 44:9 (2009), 1291–1310.

[Berberich et al. 10] E. Berberich, M. Kerber, and M. Sagraloff. "An Efficient Algorithm for the Stratification and Triangulation of an Algebraic Surface." *Comput. Geom.* 43:3 (2010), 257–278.

[Cheng et al. 05] J.-S. Cheng, X.-S. Gao, and M. Li. "Determining the Topology of Real Algebraic Surfaces." In *Mathematics of Surfaces XI (Loughborough, 2005)*, Lect. Notes in Comput. Sci. 3604, pp. 121–146. Berlin: Springer 2005.

[Ciliberto and Flamini 10] C. Ciliberto and F. Flamini. "On the Branch Curve of a General Projection of a Surface to a Plane." To appear in *Trans. Amer. Math. Soc.*, 2010.

[Collins 75] G. E. Collins. "Quantifier Elimination for Real Closed Fields by Cylindrical Algebraic Decomposition." In *Automata Theory and Formal Languages, 2nd GI Conference (Kaiserslautern, 1975)*, Lect. Notes in Comput. Sci. 33, pp. 134–183. Berlin: Springer, 1975.

[Dłotko et al. 10] P. Dłotko, T. Kaczynski, M. Mrozek, and T. Wanner. "Coreduction Homology Algorithm for Regular CW-Complexes." To appear in *Discrete Comput. Geom*, 2010.

[Donald and Chang 91] B. R. Donald and D. R. Chang. "On the Complexity of Computing the Homology Type of a Triangulation." In *Proceedings of the 32nd Annual IEEE Symposium on Foundations of Computer Science (San Juan, PR, 1991)*, pp. 650–661. Los Alamitos: IEEE Comput. Soc. Press, 1991.

[Fortuna et al. 03] E. Fortuna, P. Gianni, P. Parenti, and C. Traverso. "Algorithms to Compute the Topology of Orientable Real Algebraic Surfaces." *J. Symbolic Comput.* 36:3–4 (2003), 343–364.

[Gompf and Stipsicz 99] R. E. Gompf and A. I. Stipsicz. *4-Manifolds and Kirby Calculus.* Providence: Amer. Math. Soc., 1999.

[Gyulassy et al. 08] A. Gyulassy, P.-T. Bremer, B. Hamann, and V. Pascucci. "A Practical Approach to Morse–Smale Complex Computation: Scalability and Generality." *IEEE Trans. Visual. Comput. Graphics* 14:6 (2008), 1619–1626.

[Jäger 03] G. Jäger. "Parallel Algorithms for Computing the Smith Normal Form of Large Matrices." In *Recent Advances in Parallel Virtual Machine and Message Passing Interface (Venice, 2003)*, Lect. Notes in Comput. Sci. 2840, pp. 170–179. Berlin: Springer, 2003.

[Kannan and Bachem 79] R. Kannan and A. Bachem. "Polynomial Algorithms for Computing the Smith and Hermite Normal Forms of an Integer Matrix." *SIAM J. Comput.* 8:4 (1979), 499–507.

[Kerber and Sagraloff 10] M. Kerber and M. Sagraloff. "A Note on the Complexity of Real Algebraic Hypersurfaces." Preprint, 2010.

[Mayer 72] A. L. Mayer. "Families of K-3 Surfaces." *Nagoya Math. J.* 48 (1972), 1–17.

[Moishezon 81] B. G. Moishezon. "Stable Branch Curves and Braid Monodromies." In *Algebraic Geometry (Chicago, Ill., 1980)*, Lect. Notes in Math. 862, pp. 107–192. Berlin: Springer, 1981.

[Mrozek and Batko 09] M. Mrozek and B. Batko. "Coreduction Homology Algorithm." *Discrete Comput. Geom.* 41:1 (2009), 96–118.

[Mrozek et al. 08] M. Mrozek, P. Pilarczyk, and N. Żelazna. "Homology Algorithm Based on Acyclic Subspace." *Comput. Math. Appl.* 55:11 (2008), 2395–2412.

[Schwartz and Sharir 83] J. T. Schwartz and M. Sharir. "On the 'Piano Movers' Problem, II: General Techniques for Computing Topological Properties of Real Algebraic Manifolds." *Adv. in Appl. Math.* 4:3 (1983), 298–351.

[Várilly-Alvarado and Viray 10] A. Várilly-Alvarado and B. Viray. "Failure of the Hasse Principle for Enriques Surfaces." Preprint, 2010.

[Whitney 44] H. Whitney. "The Self-Intersections of a Smooth n-Manifold in $2n$-Space." *Ann. of Math. (2)* 45:2 (1944), 220–246.

[Whitney 57] H. Whitney. *Geometric Integration Theory.* Princeton: Princeton Univ. Press, 1957.

[Zariski 29] O. Zariski. "On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve." *Amer. J. Math.* 51:2 (1929), 305–328.

Andrew Kresch, Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zurich, Switzerland (andrew.kresch@math.uzh.ch)