Self-Intersection Numbers of Curves on the Punctured Torus

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CONTENTS

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On the punctured torus the number of essential self-intersections of a homotopy class of closed curves is bounded (sharply) by a quadratic function of its combinatorial length (the number of letters required for its minimal description in terms of the two generators of the fundamental group and their inverses). We show that if a homotopy class has combinatorial length L, then its number of essential self-intersections is bounded by $(L-2)^2/4$ if L is even, and $(L-1)(L-3)/4$ if L is odd. The classes attaining this bound can be explicitly described in terms of the generators; there are $(L-2)^2+4$ of them if L is even, and $2(L-1)(L-3)+8$ if L is odd. Similar descriptions and counts are given for classes with self-intersection number equal to one less than the bound. Proofs use both combinatorial calculations and topological operations on representative curves.

Computer-generated data are tabulated by counting for each nonnegative integer how many length-L classes have that self-intersection number, for each length L less than or equal to 13. Such experiments led to the results above. Experimental data are also presented for the pair-of-pants surface.

1. INTRODUCTION

The punctured torus has the homotopy type of a figure-eight. Its fundamental group is free on two generators: once these are chosen, say a, b, a free homotopy class of curves on the surface can be uniquely represented as a reduced cyclic word in the symbols a, b, A, B (where A stands for a^{-1} and B for b^{-1}). A *cyclic word* w is an equivalence class of words related by a cyclic permutation of their letters; we will write $w = \langle r_1 r_2 \dots r_n \rangle$, where the r_i are the letters of the word, and $\langle r_1 r_2 \dots r_n \rangle = \langle r_2 \dots r_n r_1 \rangle$, etc.

The term *reduced* means that the cyclic word contains no juxtapositions of a with A and none of b with B. Note here that we will call a free homotopy class (a reduced cyclic word) *primitive* if is not a proper power of another class (another word); and among the imprimitive classes are words we will call *pure powers*: those that are a proper power of a generator.

The *length* (with respect to the generating set (a, b)) of a free homotopy class of curves is the number of letters occurring in the corresponding reduced cyclic word.

This work studies the relation between length and the *self-intersection number* of a free homotopy class of curves: the smallest number of self-intersections among all general-position curves in the class. (General position in this context means as usual that there are no tangencies or multiple intersections.) The self-intersection number is a property of the free homotopy class and hence of the corresponding reduced cyclic word w; we denote it by $\mathrm{SI}(w)$.

Theorem 1.1. *The maximal self-intersection number for a primitive reduced cyclic word of length L on the punctured torus is*

$$\begin{cases} (L-2)^2/4 & \text{if } L \text{ is even,} \\ (L-1)(L-3)/4 & \text{if } L \text{ is odd.} \end{cases}$$

The words realizing the maximal self-intersection number are as follows (see Figure 1):

(1) *L even:*

- (i) $\langle r^{L/2} s^{L/2} \rangle$, $r \in \{a, A\}$, $s \in \{b, B\}$,
- (ii) $\langle r^i s^j r^{L/2-i} S^{L/2-j} \rangle$, $r \in \{a, A\}$, $s \in \{b, B\}$, $S = s^{-1}$, *and similar configurations interchanging r and s;*

(2) *L odd:*

- (i) $\langle r^{(L+1)/2} s^{(L-1)/2} \rangle$, $r \in \{a, A\}$, $s \in \{b, B\}$, *or vice versa,*
- (ii) $\langle r^i s^j r^{(L+1)/2-i} S^{(L-1)/2-j} \rangle$, $\langle r^i s^j r^{(L-1)/2-i} S^{(L+1)/2-j} \rangle$, $r \in \{a, A\}$, $s \in \{b, B\}$, $S = s^{-1}$, *and similar configurations interchanging r and s.*

Elementary counting with Theorem 1.1 yields the next result:

Theorem 1.2. *The number of distinct primitive free homotopy classes of length L realizing the maximal self-intersection number is*

$$\begin{cases} (L-2)^2 + 4 & \text{if } L \text{ is even,} \\ 2(L-1)(L-3) + 8 & \text{if } L \text{ is odd.} \end{cases}$$

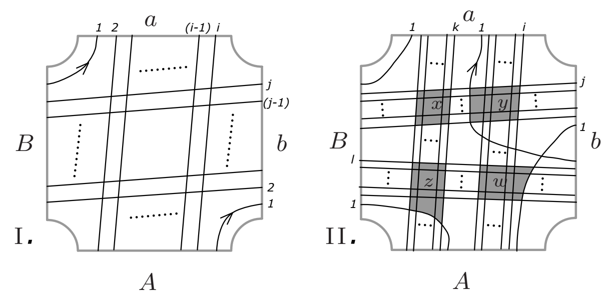

FIGURE 1. Curves of maximal self-intersection on the punctured torus. I. $w = \langle a^i b^j \rangle$ with $(i-1)(j-1)$ intersection points, a maximum when $i = j$ (even length) or $i = j \pm 1$ (odd length). II. $w = \langle a^i b^j a^k B^l \rangle$. Block x has $(k-1)(j-1)$ intersection points; block y has $i(j-1)$; block z has $l(k-1)$; block w has $i(l-1)$; and there are an additional i. The total is $(i+k-1)(j+l-1)$, a maximum when $i+k = j+l$ (even length) or $i+k = j+l\pm 1$ (odd length). Graphic conventions from Section 2.1; curve II drawn using the algorithm of [Blood 02]. Similar diagrams appear in [Chemotti and Rau 04].

Elementary computation with Theorem 1.2 allows the inequality to be reversed:

Theorem 1.3. *Let w be the reduced cyclic word corresponding to a primitive free homotopy class of curves on the punctured torus. Then if $\mathrm{SI}(w) \geq 1$, the length of w is greater than or equal to the smallest integer larger than $2\sqrt{\mathrm{SI}(w)} + 2$. Moreover, this bound is sharp.*

Remark 1.4. Pure-power words of length L between 2 and 6 do not fit the pattern of Theorems 1.1–1.3. Namely, $\mathrm{SI}(r^L) = L - 1 > (L-2)^2/4$ and $(L-1)(L-3)/4$ for integers in that range. But these theorems can be extended to all words of length seven or more, primitive or not.

Remark 1.5. The length of the word representing a free homotopy class depends on the choice of generating set (a, b) for the fundamental group, while its self-intersection number does not. Since the theorems above apply to *every* generating set, they can be rephrased in terms of the shortest such lengths.

Remark 1.6. The group of automorphisms of the fundamental group of the punctured torus acts on the set of cyclic words with a fixed self-intersection number n. Words with maximal self-intersection number minimize length in an orbit of this action. Igor Rivin asked us whether every orbit contains a word with maximal self-intersection number for its length. But $w = \langle ababAB \rangle$ is not in the orbit of such a word (this can be proved using [Lyndon and Schupp 01, Proposition 4.19]).

Theorems 4.9 and 4.11 treat curves on the punctured torus of self-intersection number one less than the maximum for their length; we do not have similar formulas for the distribution of other self-intersection numbers among curves of a given length. Here is some numerical evidence, computed using the algorithm given in [Cohen and Lustig 87]; see [Chas and Krongold 09] for a more detailed presentation.[1] This evidence was in fact the motivation for the research presented here.

Experimental Theorem 1.7. *The number of distinct primitive free homotopy classes with a given number of self-intersections corresponding to primitive reduced cyclic words of a given length appears, for length up to* 13*, in Table 1. (If one entry of a row of the table is* 0*, then all the entries to its right are also* 0*.)*

Experimental Theorem 1.8. *Let* $k \in \{1, 2, \ldots, 30\}$ *and let* K *be the set of all cyclic reduced words* v *corresponding to primitive free homotopy classes of curves on the punctured torus, with* $\mathrm{SI}(v) \geq k$*. If* w *is a word in* K *with minimal length, then the following statements hold:*

(1) *The length of* w *is equal to the smallest integer greater than or equal to* $2\sqrt{k}+2$*.*

(2) $\mathrm{SI}(w) = k$*.*

1.1 Related Results

For a reduced cyclic word w written in the symbols $\{a, A, b, B\}$, let $\alpha(w)$ and $\beta(w)$ denote the total number of occurrences of a, A and of b, B, respectively. In [Blood 02] is given a simple construction of a representative curve that has at most $(\alpha(w)-1)(\beta(w)-1)$ intersections; the author also finds some of the words whose representative curves require this number of self-intersections, namely those of the form $a^\alpha b^\beta$. Together, these two discoveries constitute a different proof of the first part of our Theorem 1.1 (compare Theorem 1.18). In addition, [Chemotti and Rau 04] gives elementary proofs of parts (2), (3), and (4) of our Proposition 3.7. This unpublished work came to our attention only during the final editing of this paper.

An algorithm is given in [Birman and Series 84] to decide whether a simple representative exists for a reduced cyclic word in the generators of the fundamental group of a surface with boundary. These ideas are extended in [Cohen and Lustig 87] (see also [Chas 04] and [Tan 96]), which gives an algorithm to compute the self-intersection of a reduced cyclic word. The program to compute Table 1.7 is based on these algorithms.

From the geometric point of view, the punctured torus can be studied as a manifold with boundary: the complement in $S^1 \times S^1$ of an open disk. This manifold admits a complete hyperbolic metric for which the boundary circle is a geodesic. Since every free homotopy class contains exactly one geodesic representative, and since a primitive geodesic cannot have excess intersections [Hass and Scott 85], the results in this section translate into results about counting geodesics on that Riemann surface.

Result 1.9. It follows from [Cohen and Lustig 87, Main Theorem] (see also [Chas 04, Proposition 2.9 and Remark 3.10]) that for any surface S with nonempty boundary and negative Euler characteristic, $\mathrm{SI}(w) \leq L(L-1)/2$ (using our notation) for w a primitive word of length L in the generators (and their inverses) of the fundamental group of S. For the torus with one boundary component, the special case examined here, our upper bound (Theorem 1.2) is lower.

On the punctured torus choose generators for the fundamental group and a metric for which the boundary is a geodesic. That metric, restricted to closed geodesics, is quasi-isometric to the word-length metric. (This is an elementary argument, based on the upper bound K for the length of geodesics representing the generators and on the lower bound k for the length of a transversal or a corner segment in the fundamental polygon; see Section 2.1 for this terminology; see also [Milnor 68, Bridson and Haefliger 91].)

Hence we can refer to word length as *combinatorial length*. (Note that the union of the four corner segments is the boundary, so our quasi-isometry evaporates as the length of the boundary goes to zero, i.e., as our surface approaches the torus minus a point. For a hyperbolic metric on that surface, the relation between length and self-intersection number can be expected to be quite different.)

Result 1.10. It is proved in [Lalley 96, Theorem 1] that on a compact hyperbolic closed surface, *most* closed geodesics of length approximately ℓ have about $C\ell^2$ self-intersections for some positive constant C depending on the surface. As a consequence of our Theorem 1.1, in the

[1] The Java program can be found online (http://www.math.sunysb.edu/~moira/CLB/CLB09/).

Length\ SI	0	1	2	3	4	5	6	7	8	9	10
1	**4**	0	0	0	0	0	0	0	0	0	0
2	**4**	0	0	0	0	0	0	0	0	0	0
3	**8**	0	0	0	0	0	0	0	0	0	0
4	*10*	**8**	0	0	0	0	0	0	0	0	0
5	16	*8*	**24**	0	0	0	0	0	0	0	0
6	8	16	32	*40*	**20**	0	0	0	0	0	0
7	24	16	32	48	112	*24*	**56**	0	0	0	0
8	16	24	52	76	116	156	136	104	*90*	**40**	0
9	24	32	64	120	144	240	384	208	376	136	304
10	16	32	72	168	272	332	492	628	644	700	700
11	40	48	80	160	272	584	664	1200	1280	1368	1608
12	16	40	104	208	372	660	1048	1408	2044	2696	3088
13	48	48	104	264	456	752	1216	2080	2496	4464	4752

Length\ SI	11	12	13	14	15	16	17	18	19	20
9	*48*	**104**	0	0	0	0	0	0	0	0
10	548	464	360	224	*160*	**68**	0	0	0	0
11	1368	2048	976	1704	528	1072	264	592	*80*	**168**
12	3580	3866	3792	3816	3612	3272	2820	2276	1808	1308
13	7048	6976	8968	8904	9328	10536	7984	10392	5760	8736

Length\ SI	21	22	23	24	25	26	27	28	29	30	31
11	0	0	0	0	0	0	0	0	0	0	0
12	960	680	392	*250*	**104**	0	0	0	0	0	0
13	3752	6616	2064	4016	976	2128	432	976	*120*	**248**	0

TABLE 1. The i, j entry in this table is the number of distinct reduced primitive cyclic words of length i with exactly j self-intersections, up to the maximum possible self-intersection number for each length. Boldface numbers and their location correspond to Theorems 1.1 and 1.2, italic numbers to Theorem 4.11.

case of the torus with one geodesic boundary component, for each hyperbolic metric there exists a positive constant C' such that the number of self-intersection points of *every* geodesic of length ℓ is less than $C'\ell^2$. (This fact also admits an elementary proof, as Lalley pointed out to us.) Lalley also studies the distribution on the surface of self-intersection points of a typical geodesic; [Lalley 96, Theorem 2] may be compared with the patterns in Figure 1.

Result 1.11. It is proved in [Basmajian 93, Corollary 1.2] that for any hyperbolic surface there exists an increasing sequence of constants $\{M_k\}$, $k \geq 1$, tending to infinity such that if ω is a closed geodesic with self-intersection number k, then the hyperbolic length of ω is greater than M_k. For the punctured torus and combinatorial length, our Theorem 1.3 gives explicit values for M_k, and our bounds are sharp.

In view of the quasi-isometry between combinatorial and hyperbolic length for the punctured torus (as manifold with boundary), the numbers in Experimental Theorem 1.7 are concordant with numbers or estimates from several other lines of research:

Result 1.12. It is known that for any hyperbolic surface the total number of primitive closed geodesics of length at most L is asymptotic to e^{hL}/L (h is the topological entropy of the geodesic flow; see [Buser 92] and references therein; similar results hold for the variable-curvature case [Lalley 89, Margulis 83, Parry and Pollicott 83]). On the punctured torus, the number of distinct primitive classes of combinatorial length L at most twelve, i.e., the sum of the numbers in row L of Table 1, appears to be very rapidly asymptotic to $3^L/L$.

Result 1.13. The numbers in the first column of Table 1, giving the number of simple classes for a given length, can be compared with the results of [McShane and Rivin 95] for the punctured torus and [Mirzakhani 08] for a general surface of negative Euler characteristic (see also [Rivin 01] for historical background). Mirzakhani, McShane, and Rivin prove that the number of simple closed geodesics of hyperbolic length at most L grows as a quadratic polynomial in L. (Contrast with Theorem 1.2, where the number of maximal curves of length *exactly* L grows quadratically with L.)

For the range of Table 1, we have data consistent with these: the number of simple curves of length exactly $2n+1$, $n \geq 1$, appears to grow more or less linearly with n; for $2n+1$ a prime, it is exactly $8n$.

Result 1.14. For L even, the numbers in the second column of Table 1 grow as $4(L-2)$. This is consistent with the determination from [Rivin 09] that the number of single-self-intersection geodesics of length at most L grows quadratically with L.

Result 1.15. For a closed surface S, [Basmajian 93, Proposition 1.3] states that there are constants N_k (depending on the genus of S) such that the shortest geodesic on S with at least k intersection points has length bounded above by N_k. Experimental Theorem 1.8 gives N_k an explicit value for curves of combinatorial length less than 13 on the punctured torus.

Result 1.16. It is proved in [Buser 92] that the shortest nonsimple closed geodesic on a hyperbolic surface has only one self-intersection; in our notation the shortest γ with $\mathrm{SI}(\gamma) \geq 1$ has $\mathrm{SI}(\gamma) = 1$. It is shown in [Basmajian 93, Corollary 1.4] that there exists an a priori constant, say K_k, depending on the genus of the surface, such that the shortest γ with $\mathrm{SI}(\gamma) \geq k$ has $\mathrm{SI}(\gamma) \leq K_k$. Our Table 1 shows that for $k \leq 30$, on the punctured torus and with respect to combinatorial length, $K_k = k$.

1.2 Sketch of Proof

The method of proof in this paper keeps track of three integer parameters of a reduced cyclic word w in the alphabet a, b, A, B: along with $\alpha(w)$ and $\beta(w)$ (see Section 1.1) there is $h(w)$, the total number of block-pairs in w; these are defined as follows:

Definition 1.17. Either a reduced cyclic word w is a pure power or there exist pairs of positive integers $j_1, k_1, \ldots, j_n, k_n$, $n \geq 1$, such that

$$w = \left\langle r_1^{j_1} s_1^{k_1} r_2^{j_2} s_2^{k_2} \ldots r_n^{j_n} s_n^{k_n} \right\rangle,$$

where $r \in \{a, A\}$ and $s \in \{b, B\}$. Each of the $r_i^{j_i} s_i^{k_i}$ occurring in this expression is a *block-pair*; the *number of block-pairs* of w is defined to be n in the second case, and zero in the first.

The main theorem in this paper is Theorem 1.18; it will be proved in Section 4.

Theorem 1.18. *For the punctured torus, let w be the reduced cyclic word corresponding to a free homotopy class of curves with a positive number h of block-pairs. If $h = 1$, then $\mathrm{SI}(w) = (\alpha(w)-1)(\beta(w)-1)$. If $h \geq 2$, then*

$$\mathrm{SI}(w) \leq (\alpha(w)-1)(\beta(w)-1) - h + 2.$$

The words w realizing the maximal self-intersection for non-pure-power words with given α and β (that is, $\mathrm{SI}(w) = (\alpha(w)-1)(\beta(w)-1)$) have one of the following forms:

(1) $\langle r^i s^j \rangle$, *$r \in \{a, A\}, s \in \{b, B\}$; here $\alpha(w) = i > 0, \beta(w) = j > 0$.*

(2) $\langle r^i s^j r^k S^l \rangle$, *with all $i, j, k, l > 0$, where $r \in \{a, A\}$ (and then $i + k = \alpha(w)$) and $s \in \{b, B\}$ (and then $j + l = \beta(w)$), or vice versa.*

This theorem has two immediate corollaries:

Corollary 1.19. *Let w be the reduced cyclic word corresponding to a primitive free homotopy class of curves on the punctured torus. Then*

$$\mathrm{SI}(w) \leq (\alpha(w)-1)(\beta(w)-1).$$

Corollary 1.20. *Among primitive words, those of maximal self-intersection number for their α and β values, i.e., with $\mathrm{SI}(w) = (\alpha(w)-1)(\beta(w)-1)$, have one of the following forms:*

(1) $\langle r \rangle$, *$r \in \{a, b, A, B\}$.*

(2) $\langle r^i s^j \rangle$, *$r \in \{a, A\}, s \in \{b, B\}$; here $\alpha(w) = i > 0$, $\beta(w) = j > 0$.*

(3) $\langle r^i s^j r^k S^l \rangle$, *all $i, j, k, l > 0$, where $r \in \{a, A\}$ (and then $i + k = \alpha(w)$) and $s \in \{b, B\}$ (and then $j + l = \beta(w)$), or vice versa.*

Remark 1.21. Since $\alpha(w) + \beta(w) = L$, where L is the length of w, an elementary calculation leads from Corollaries 1.19 and 1.20 to Theorem 1.1.

The next three sections carry the proof of Theorem 1.18. The strategy is to show that only words of the types listed in the statement of the theorem, i.e., $\langle r^i s^j \rangle$ and $\langle r^i s^j r^k S^l \rangle$, $r \in \{a, A\}$, $s \in \{b, B\}$, or vice versa, can have maximum self-intersection number for their length; this will be done by exhibiting, for every word that is not of these types, another word of the same length and with strictly larger self-intersection number. For most words w, "cross-corner surgery" (defined below) will produce a w' with the same α and β values (and so of the same length), with $\mathrm{SI}(w') > \mathrm{SI}(w)$ and with $h(w') < h(w)$.

For certain words with two, three, or four blocks, not candidates for surgery, the self-intersection number will be computed explicitly by counting "linked pairs" of subwords (definition below) and determining that it is indeed smaller than the self-intersection number of a word of the same length but of one of the two listed types (whose self-intersection numbers are also computed by counting linked pairs).

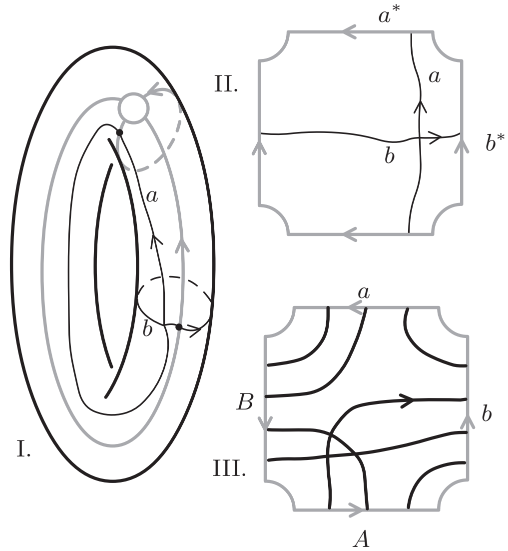

FIGURE 2. The punctured torus as a polygon with identifications. I, II. The generators a, b and their inverses A, B can be identified by their intersections with the dual cycles a^*, b^*, which appear among the edges of the fundamental polygon. III. When an oriented curve has been lifted to the fundamental polygon, the cyclic word corresponding to its free homotopy class can can be obtained by choosing a starting point and recording in sequence the edges it crosses, reading their names from *inside* the polygon. The lifted curve $\langle baBBAba \rangle$, with self-intersection number 3, is shown as an example.

2. CROSS-CORNER SURGERY

2.1 Preliminaries

Here, let M represent the punctured torus as a topological space. The choice of generators (a, b) for $\pi_1 M$ naturally implies a *fundamental polygon* from which M may be reconstructed by edge-identification. Namely, we can choose, as representative cycles for the homological duals $a^*, b^* \in H_1(M/\partial M)$, two disjoint connected arcs beginning and ending in ∂M; slicing M along these arcs gives a simply connected polygon that can serve as fundamental domain (for the action of $\pi_1(M)$ on the universal cover); for our purposes we will label a the edge keeping the orientation of a^*, and A its opposite edge with the opposite orientation (see Figure 2); similarly for b and B.

Lifting a curve in M to this fundamental polygon means representing the curve as a set of arcs with identifications; each of these *curve segments* leads from one of the edges a, b, A, B to another; the orientation of the curve defines a cyclic word in the four symbols: one records the positive intersections as they occur. By construction, this word represents the free homotopy class of the curve under consideration.

A curve segment is a *transversal* if it joins opposite edges of the fundamental domain, and a *corner* otherwise. Transversals correspond to consecutive aa, AA, bb, BB in the word; other combinations give corners. Two corners are *opposite* if they are diagonally opposed. Thus ab, ba and ab, AB correspond to diagonally opposed corners; ab and ba have the *same orientation*, whereas ab and AB have *reversed orientations*. In Figure 2 the curve $\langle baBBAba \rangle$ has two ba corner segments diagonally opposed to an ab (same orientation) and a BA (reversed orientation); one aB corner diagonally opposed to an ab (same orientation) and a BA (reversed orientation); one aB corner diagonally opposed to an Ab corner (reversed orientations); and one BB transversal.

A curve with only transversal self-intersections and with the smallest number of self-intersections for its homotopy class (multiple points count with multiplicity: a multiple intersection of n small arcs counts as $\binom{n}{2}$ intersections) is said to be *tight* (compare "taut" in [Thurston 10]).

Two-component multiwords $[w, w']$ enter into the surgery process. We define the *intersection number* $\mathrm{IN}(w, w')$ of two reduced cyclic words w, w' to be the minimum number of intersections between a general-position curve representing w and one representing w'. The self-intersection number of the multiword $[w, w']$ is then $\mathrm{SI}([w, w']) = \mathrm{SI}(w) + \mathrm{SI}(w') + \mathrm{IN}(w, w')$, and a pair of curves with that smallest number of self-intersections is also said to be *tight*. We also extend the α and β notation to multiwords: $\alpha(]w', w''])$ is the total number of occurrences of a or A in w and w'; $\beta([w', w''])$, the total number of occurrences of b or B.

2.2 The Surgery

Whenever a cyclic word w contains a pair of opposite corners, it may be cut in two places, once in the middle of each of the corners, to give two linear words. These two linear words may be reassembled (the corners themselves are reassembled into transversals) into either a new word w' or a new multiword $[w', w'']$ (according to the relative orientation of the corners); if a multiword $[v', v'']$ contains a pair of opposite corners, one in each component, the two corners may be cut and reassembled into two transversals, yielding a new single word v.

For a picture of the surgery on a curve, see Figure 3; in terms of the words, the cutting and reassembly take one of the following forms:

$$\langle \mathsf{x}r|\mathsf{s}\mathsf{y}s|r\mathsf{z}\rangle \to [\langle \mathsf{x}r|r\mathsf{z}\rangle, \langle s\mathsf{y}s|\rangle], \tag{2–1}$$

$$\langle \mathsf{x}r|s\mathsf{y}R|S\mathsf{z}\rangle \to \langle \mathsf{X}r|r\mathsf{Y}S|S\mathsf{z}\rangle, \tag{2–2}$$

$$[\langle \mathsf{x}r|s\mathsf{y}\rangle, \langle \mathsf{z}s|r\mathsf{w}\rangle] \to \langle \mathsf{x}r|r\mathsf{w}\mathsf{z}s|s\mathsf{y}\rangle, \tag{2–3}$$

$$\{\langle \mathsf{x}r|s\mathsf{y}\rangle, \langle \mathsf{z}R|S\mathsf{w}\rangle\} \to \langle \mathsf{x}r|r\mathsf{Z}\mathsf{X}s|s\mathsf{y}\rangle, \tag{2–4}$$

where $\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{w}$ are arbitrary (linear) subwords, and $R = r^{-1}$, $S = s^{-1}$, $\mathsf{X} = \mathsf{x}^{-1}$, etc.

Definition 2.1. This cutting and reassembly is called *cross-corner surgery* on the word w or the multiword $[v', v'']$.

It seems natural that transversals should contribute, more than corners, to the self-intersection number of a curve. Proposition 2.2 makes this quantitative by showing that cross-corner surgery, which eliminates two corners and adds two transversals, always increases the self-intersection number by at least one.

Proposition 2.2.

(1) *If a word w contains a pair of opposite corners with reversed orientation, then cross-corner surgery will produce a new word w', with*

$$\alpha(w') = \alpha(w), \quad \beta(w') = \beta(w),$$

with one fewer block-pair, and with

$$\mathrm{SI}(w') \geq \mathrm{SI}(w) + 1.$$

(2) *If a word w contains a pair of opposite corners with the same orientation, then cross-corner surgery will produce a multiword $[w', w'']$, with*

$$\alpha([w', w'']) = \alpha(w), \quad \beta([w', w'']) = \beta(w),$$

with one fewer block-pair, and with

$$\mathrm{SI}([w', w'']) \geq \mathrm{SI}(w) + 1.$$

(3) *If a multiword $[v', v'']$ contains a pair of opposite corners, one in each component, irrespective of orientation, then cross-corner surgery will produce a single word v with*

$$\alpha(v) = \alpha([v', v'']), \quad \beta(v) = \beta([v', v'']),$$

with one fewer block-pair, and with

$$\mathrm{SI}(v) \geq \mathrm{SI}([v', v'']) + 1.$$

This proposition is stated in terms of words, but its proof, given in the next subsection, works by examining curves representing the words before and after surgery; we first must fix a topological procedure for carrying out cross-corner surgery on a curve. Specifically, given a tight curve, or a tight pair of curves, representing the candidates w or $[v', v'']$ for cross-corner surgery, we need to establish a systematic way of generating curves representing the result w', $[w', w'']$ or v of the surgery. We do this as follows:

Definition 2.3. (Cross-corner surgery on curves.) Suppose $r|s$ and $s|r$ or $R|S$ are the loci (that is, two diagonally opposite corners) in the word w (or the multiword $[v', v''])$ chosen for surgery, and let K and L be the corresponding corners in a tight representative (see Figure 3).

- *Preparation for the surgery*: If the extension of any corner segment of the same type as K (i.e., corresponding to the same letter sequence rs or to the inverse sequence SR) intersects the extension of K in either direction before diverging, the curve is prepared for surgery by a homotopy sliding that (necessarily single) intersection onto the segment K itself. This deformation may be carried out by a sequence of Reidemeister type-III moves without changing the total number of intersections (see Figure 3, I and II). A similar operation is carried out on the corner L.

- *Cutting and sewing*: Corresponding to the word permutation, corners K and L are removed and replaced

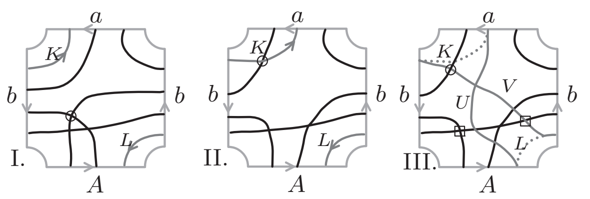

FIGURE 3. The cross-corner surgery $\langle baBB|Ab|a\rangle \to \langle baBB|Ba|a\rangle$ as carried out on a tight representative curve. I. The corner K corresponds to $b|a$; L corresponds to $B|A$. Note that the extension of K intersects that of one of its parallel corners (circled intersection). II. Before surgery, that intersection is "pushed," using a Reidemeister type-III move, into the center of the surgery. III. K and L are excised, U and V sewn in. The circled intersection migrates to an intersection with V. The intersection of V with the original BB spans a bigon with one of the original vertices (squared intersections); $\mathrm{SI}(\langle baBBBaa\rangle) = 6$.

by transversals U and V. More precisely, a line is drawn from a point on K to a point on L, in general position with respect to the rest of the curve, and cutting any segment no more than once; that line is expanded into an $\mathcal{X}$-junction: U routes the right edge of K to the left edge of L, and vice versa for V.

2.3 Proof of Proposition 2.2

We will obtain a lower bound on the increase in self-intersection number by counting the vertices added and those possibly annihilated by the surgery. Annihilation occurs through the creation of a *bigon*: an immersed planar polygon with two vertices and with two edges with disjoint preimages (a "singular 2-gon" in [Hass and Scott 85]); the bigon defines a homotopy of the curve leading to the disappearance of its two vertices. An intersection will be called *stable* if it is not the vertex of a bigon; a curve is tight if all its self-intersections are stable [Hass and Scott 85].

Lemma 2.4. *Cross-corner surgery does not create any bigons spanned by a pair of presurgery vertices.*

Proof. Since the initial curve is tight, the only way a pair x, y of curve portions starting from a presurgery ("old") vertex P can lead to a bigon with another old vertex is if one of those curve portions (say x) contains one of the new segments U and V, say U. Suppose the other one, i.e., y, enters inside the corner (L in Figure 4). Then (Figure 4, I) as y follows x across the frame, y must intersect L in an old vertex P', canceling P, contradicting

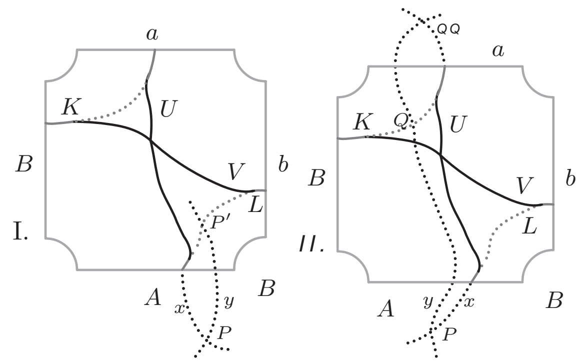

FIGURE 4. Cross-corner surgery cannot produce a bigon linking two presurgery vertices.

tightness of the original curve. So y must enter outside L; then running parallel to U across the frame, it must intersect the opposite corner K in an old vertex Q (Figure 4, II). Now if x and y meet in an old vertex Q' so as to form a bigon canceling P, then Q' and Q will span an old-vertex bigon. By tightness, this will require another use of the new segments. Since each of U and V can be used only once by each of x and y, after at most four passes through the frame all the possibilities will be exhausted; no such bigon can exist. □

of Proposition 2.2. The curve surgery described in Definition 2.3 yields one word if it is applied to a word that contains a pair of opposite corners with reversed orientation, a multiword if it is applied to a word that contains a pair of opposite corners with the same orientation, and a single word if it is applied to a multiword that contains a pair of opposite corners, one in each component, irrespective of orientation. Thus, to prove (1), (2), and (3) it is enough to prove that in a cross-corner surgery, the number of new vertices minus the number of vertices canceled by new bigons is greater than or equal to one. We start by classifying the new vertices introduced by the surgery and the possible bigons in which they may participate.

Vertices. The surgery creates three types of new vertices, shown as black, grey and white in Figure 5, as follows:

1. (black) Stable intersections between U and horizontal transversals (i.e., segments corresponding to *bb* or BB in the initial word w), between V and vertical transversals, and (bull's-eye) the stable intersection between U and V.

2. (gray) Intersections between U and other vertical transversals, and between V and other horizon-

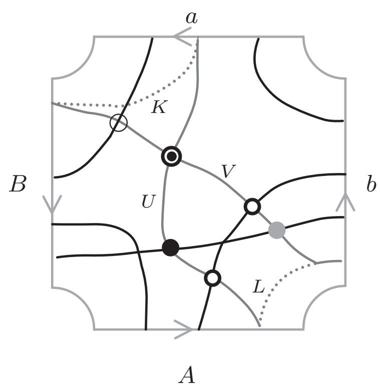

FIGURE 5. The new vertices created by a cross-corner surgery.

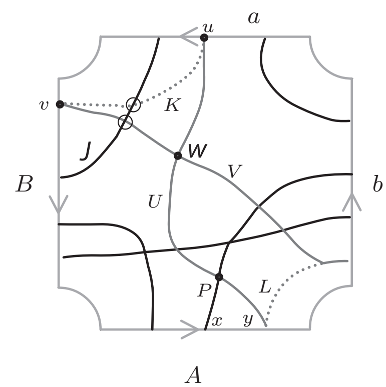

FIGURE 6. Vertices inherited by new curve from old; P is an example of a type-3 vertex.

tal transversals. These are potentially vertices of bigons.

3. (white) Intersections between U, V, and remaining corner segments. In Figure 5 only those of type ab or BA are shown; there is typically another family in the opposite corner corresponding to type ba or AB. These are also potentially vertices of bigons.

4. In addition, the circled vertices in Figure 5 are those inherited by the new curve from the old. These correspond to the intersections between K or L with other corners of the same type; such a corner is labeled J in Figure 6.

   Focusing on K, let us label with x and y the two ends of the segment K, and with w the intersection point of the new segments U and V. The segment K and the broken curve uwv are fixed-endpoint homotopic; it follows that for any original segment having exactly one endpoint between u and v (e.g., J), the intersection with K will migrate to an intersection with U or V during that homotopy (with V if the outside end of J is on the B side—as in Figure 6—and with U if it is on the a side).

Bigons. The only bigons that need to be examined are those for which one of the spanning vertices is an old vertex or a type-4 vertex; that is because if two new vertices form a bigon and cancel, that does not affect the inequality we need to prove. So, letting 1, 2, 3, and 4 represent vertices so labeled above, letting x, x' represent self-intersections of the original curve, and keeping in mind that type-1 vertices are stable and that bigons of type (x, x') cannot occur (Lemma 2.4), we need only examine bigons of type $(2, x)$, $(2, 4)$, $(3, x)$, $(3, 4)$, $(4, x)$, and $(4, 4)$:

(i) $(4, x)$ and $(4, 4)$. A vertex of type 4 can span a bigon in only one of its quadrants; but in that quadrant a bigon would imply a bigon with the old vertex from which the type-4 vertex was inherited; so a $(4, x)$ would imply an (x', x), and a $(4, 4)$ would imply a $(4, x)$; so neither $(4, 4)$ nor $(4, x)$ can occur.

(ii) $(2, x)$ and $(2, 4)$. A type-2 vertex y may span a bigon with an old vertex x; the type-2 vertex is either the intersection of U with another vertical transversal, or V with another horizontal. In the first case (the second case is similar), that vertical transversal must also intersect V, creating a new (type-1) stable intersection z. In total we will have added two vertices (y and z), and lost two vertices (y and x) to a bigon. The inequality is not affected.

   Since, arguing as in (i), a $(2, 4)$ bigon would imply a $(2, x)$ bigon, the loss of the 4 would be balanced by the gain of the corresponding new type-1 vertex, and again the inequality would not be affected.

(iii) $(3, x)$ and $(3, 4)$. Figure 6 shows a typical type-3 vertex P. It can span a bigon in only one quadrant; label with x and y the two segments issuing from P in that direction. Because of the way the curve is prepared for surgery, x and y cannot be continued with old segments to form a bigon canceling P. We need to discuss the possibility that after surgery their extensions could incorporate U or V or both and then form such a bigon. This P, x, and y exactly match the notation of Lemma 2.4; and the proof of that lemma applies here as well: no such bigon can exist.

> Since a $(3,4)$ bigon would imply a $(3,x)$ bigon, no type-$(3,4)$ bigons can exist either.

In summary, cross-corner surgery generates one special stable vertex (the intersection of U and V) plus other new vertices of types 1, 2, 3, and displaced vertices of type 4. Vertices of type 1 are stable. Some of the vertices of types 2 and 3 form bigons with each other and cancel out. Vertices of types 3 and 4 cannot form bigons with presurgery vertices, and any old or type-4 vertex canceled by a type-2 vertex can be replaced in the count by the corresponding type-1 vertex. It follows that cross-corner surgery increases the self-intersection number by at least one. □

3. LINKED PAIRS

Ultimately, the calculation of $\mathrm{SI}(w)$ or $\mathrm{IN}([w', w''])$ can be made directly from w or $[w', w'']$, by counting *linked pairs*. In this section we give a simplified definition appropriate for the punctured torus, we list two theorems from [Chas 04] giving the correspondence between linked pairs and intersection points, and we summarize explicit calculations of intersection and self-intersection numbers for certain families of words with a small number of block-pairs.

In earlier work, the similar concept of *linking pairs* was defined in [Cohen and Lustig 87], and the authors proved that the intersection and self-intersection numbers of primitive words can be calculated by counting linking pairs. Parts A and B of their main theorem are equivalent, respectively, to parts 1 and 3 of our Theorem 3.5. The linked pairs of [Chas 04], defined below, are somewhat better adapted to our purposes and will be used here. In particular, we need to be able to extend the calculation to certain imprimitive words.

Notation 3.1. From now on, we will use the symbols p, q, r, s, p_1, q_1, etc. to represent letters from the alphabet a, b, A, B, with $P = p^{-1}$, etc. The symbols v, w, v', w', etc., will represent cyclic words in that alphabet, e.g., $w = \langle abbaB\rangle = \langle aBabb\rangle$. Sans-serif symbols $\mathsf{u}, \mathsf{v}, \mathsf{y}$ will represent linear words in the alphabet $\{a, b, A, B\}$ with $\mathsf{r}, \mathsf{s}, \mathsf{R}, \mathsf{S}$ representing homogeneous blocks of letters $rr\ldots r, ss\ldots s, RR\ldots R, SS\ldots S$ respectively. As before, $\mathsf{V} = \mathsf{v}^{-1}, \mathsf{R} = \mathsf{r}^{-1}$, etc.

Remark 3.2. For the purpose of orientation, we identify the boundary of our fundamental domain with a clock face, with a, b, A, B at 3, 6, 9, and 12 o'clock. Given six letters p, q, r, p', q', r' from the alphabet a, b, A, B, we say that the triples p, q, r and p', q', r' are *similarly oriented* if the arcs pqr and $p'q'r'$ have the same orientation on the clock face. This implies that the three points in each triple are distinct.

Definition 3.3. Let u' and u'' be two *linear* words, both of the same length ≥ 2. The pair of words $\{\mathsf{u}', \mathsf{u}''\}$ is a *linked pair* if one of the following criteria is satisfied (see Figure 7):

I. $\{\mathsf{u}', \mathsf{u}''\}$ is one of the following pairs: $\{aa, bb\}, \{aa, BB\}, \{AA, bb\}, \{AA, BB\}$.

II. (i) (length 3) $\mathsf{u}' = p_1 r p_2$, $\mathsf{u}'' = q_1 r q_2$ (same r) with $P_1 Q_1 r$ and $p_2 q_2 R$ similarly oriented;
 (ii) (length n) $\mathsf{u}' = p_1 \mathsf{y} p_2$, $\mathsf{u}'' = q_1 \mathsf{y} q_2$, $\mathsf{y} = x_1 \mathsf{v} x_2$ (v possibly empty) with $P_1 Q_1 x_1$ and $p_2 q_2 X_2$ similarly oriented.

III. (i) (length 3) $\mathsf{u}' = p_1 r p_2$, $\mathsf{u}'' = q_1 R q_2$ ($R = r^{-1}$) with $P_1 q_2 r$ and $p_2 Q_2 R$ similarly oriented;
 (ii) (length n) $\mathsf{u}' = p_1 \mathsf{y} p_2$, $\mathsf{u}'' = q_1 \mathsf{Y} q_2$, $\mathsf{y} = x_1 \mathsf{v} x_2$ (v possibly empty) with $P_1 q_2 x_1$ and $p_2 Q_1 X_2$ similarly oriented.

Let w (respectively $[w', w'']$) be a reduced cyclic word (respectively a multiword with reduced cyclic components),

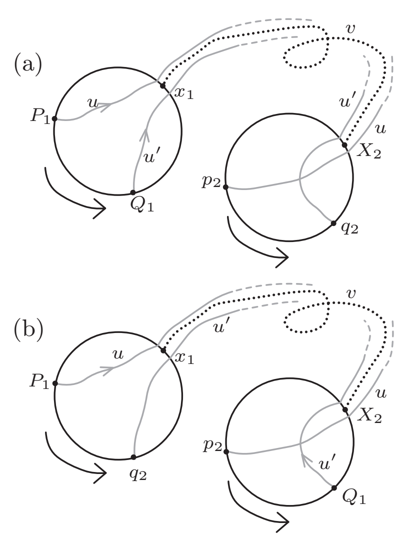

FIGURE 7. Linked pairs. Here $\mathsf{y} = x_1 \mathsf{v} x_2$. (a) The linked pair is $(p_1 \mathsf{y} p_2,\ q_1 \mathsf{y} q_2)$. Since the orientations of (P_1, Q_1, x_1) and (p_2, q_2, X_2) are the same, the curve segments must intersect. (b) The linked pair is $(p_1 \mathsf{y} p_2,\ q_2 \mathsf{Y} q_1)$. Since the orientations of (P_1, q_2, x_1) and (p_2, Q_1, X_2) are the same, the curve segments must intersect.

Words	$a^{i+1}b$	ba^{i+1}	$a^{i+1}B$	Ba^{i+1}	ba^ib	a^{i+2}	Ba^iB
$a^{i+1}b$	=	N	=	Y	=	=	Y
ba^{i+1}	N	=	Y	=	=	=	Y
$a^{i+1}B$	=	Y	=	N	Y	=	=
Ba^{i+1}	Y	=	N	=	Y	=	=
ba^ib	=	=	Y	Y	=	Y	Y
a^{i+2}	=	=	=	=	Y	=	Y
Ba^iB	Y	Y	=	=	Y	Y	=

TABLE 2. Linking of pairs of words with $\mathsf{Y} = a^i$. Notation is from Definition 3.3. The symbols =, Y, N are explained in the text.

Words	aa^ib^jb	ba^ib^ja	ba^ib^jb	aa^ib^ja
aa^ib^jb	=	N	=	=
ba^ib^ja	N	=	=	=
ba^ib^jb	=	=	=	Y
aa^ib^ja	=	=	Y	=

TABLE 3. Linking of pairs of words with $\mathsf{Y} = a^ib^j$ (notation as in Definition 3.3).

corresponding to a free homotopy class (respectively a pair of free homotopy classes) of curves on the punctured torus. We will say that $\{\mathsf{u}', \mathsf{u}''\}$ is a *linked pair of* w (respectively of $[w', w'']$) if $\mathsf{u}' \subset w$ and $\mathsf{u}'' \subset w$ (respectively $\mathsf{u}' \subset w'$ and $\mathsf{u}'' \subset w''$).

Remark 3.4. $\{\mathsf{u}, \mathsf{u}'\}$ is a linked pair of type (II) if and only if $\{\mathsf{u}, \mathsf{U}'\}$ is a linked pair of type (III).

Tables 2, 3, and 4 summarize for future reference the pairing between various subwords of a cyclic word w. In these tables "=" means that the row word and the column word have the same first or last letter (so they cannot form a linked pair); "N" means that there is no end matching but that the pair fails the orientation criterion; "Y" means that the row word and the column word form a linked pair.

The following theorem will be used to compute the self-intersection numbers of certain words and multiwords (see Proposition 3.7 and Section 6). This theorem is a direct consequence of [Chas 04, Theorems 3.9 and 3.10 and Remarks 3.10 and 3.11] and [Chas 04, Theorem 3.12 and Remark 3.13].

Theorem 3.5. *Let v and w be cyclic reduced words in the alphabet $\{a, b, A, B\}$. Suppose that $w = \langle u^k \rangle$ is the kth power ($k \geq 0$) of the primitive reduced cyclic word u.*

(1) *If $k = 1$, so that w is primitive,* $\mathrm{SI}(w)$ *is equal to the* number of linked pairs of *w, i.e., the cardinality of the set of unordered pairs $\{\mathsf{u}, \mathsf{u}'\}$, u and u' linear subwords of w, with u and u' linked as in Definition 3.3.*

(2) *In general,* $\mathrm{SI}(w)$ *is less than or equal to* $(k-1)$ *plus the number of linked pairs of w.*

(3) $\mathrm{IN}(\{v, w\})$ *equals the number of ordered pairs* $(\mathsf{u}, \mathsf{u}')$ *for which there exist positive integers j and k such that u is an occurrence of a subword of v^j, but not a subword of v^{j-1}; u' is an occurrence of a subword of w^k, but not a subword of w^{k-1}; and u, u' are linked as in Definition 3.3. (See Remark 3.6.)*

In this work, only the following simple instances of Theorem 3.5(3) will be necessary.

Remark 3.6.

(1) $\mathrm{IN}(\langle a^ib^j \rangle, \langle a^kB^l \rangle)$ equals the number of ordered pairs $(\mathsf{u}, \mathsf{u}')$ such that u is an occurrence of a subword of $\langle a^ib^j \rangle$, u' is an occurrence of a subword of $\langle a^kB^l \rangle$, and u, u' are linked as in Definition 3.3.

(2) $\mathrm{IN}(\langle a^ib^ja^kb^l \rangle, \langle a^mB^n \rangle)$ equals the number of ordered pairs $(\mathsf{u}, \mathsf{u}')$ such that u is an occurrence of a subword of $\langle a^ib^ja^kb^l \rangle$, u' is an occurrence of a subword of $\langle a^mB^n \rangle$, and u, u' are linked as in Definition 3.3.

This is because if $[v, w] = [\langle a^ib^j \rangle, \langle a^kB^l \rangle]$ or $(v, w) = [\langle a^ib^ja^kb^l \rangle, \langle a^mB^n \rangle]$, J and K are nonnegative integers, and u is a linear word that is an occurrence of a subword of v^J and w^K, then u is an occurrence of a subword of v and w.

In principle, the self-intersection number corresponding to any particular word can be ascertained combinatorially by a count of linked pairs. The number of steps in this calculation increases rapidly with the length of the word, but it can be carried out completely for words with a small number of block-pairs. The results of these calculations are given in Proposition 3.7, with the work itself presented in Section 6.

Words	$aa^ib^ja^kb$	$ba^ib^ja^ka$	$ba^ib^ja^kb$	$aa^ib^ja^ka$
$aa^ib^ja^kb$	=	N	=	=
$ba^ib^ja^ka$	N	=	=	=
$ba^ib^ja^kb$	=	=	=	Y
$aa^ib^ja^ka$	=	=	Y	=

TABLE 4. Linking of pairs of words with $\mathsf{Y} = a^ib^ja^k$ (notation as in Definition 3.3).

Proposition 3.7.

(1) $\mathrm{SI}(\langle a^i b^j \rangle) = (i-1)(j-1)$.

(2) $\mathrm{SI}(\langle a^i b^j a^k b^l \rangle)$

$$\begin{cases} \leq (i+k-2)(j+l-2)+1 \\ \quad \textit{if } k=i \textit{ and } l=j, \\ = (i+k-2)(j+l-2)+|i-k|+|j-l|-1 \\ \quad \textit{otherwise.} \end{cases}$$

(3) $\mathrm{SI}(\langle a^i b^j a^k B^l \rangle) = (i+k-1)(j+l-1)$.

(4) $\mathrm{SI}(\langle a^i b^j A^k B^l \rangle = (i+k-1)(j+l-1)-1$.

(5) $\mathrm{SI}(\langle a^i b^j a^k b^l a^m B^n \rangle) = (i+k+m-1)(j+l+n-1) - 2(k+\min(j,l)-1)$.

(6) $\mathrm{IN}(\langle a^i b^j \rangle, \langle a^k B^l \rangle) = il+kj$.

(7) $\mathrm{IN}(\langle a^i b^j a^k b^l \rangle, \langle a^m B^n \rangle) = (i+k)n + m(j+l)$.

Corollary 3.8.

$$\mathrm{SI}(\langle a^i b^j a^k b^l \rangle) \begin{cases} = 1 \\ \quad \textit{if } i=k \textit{ and } j=l \textit{ and } i=1 \textit{ or } j=1, \\ \leq (i+k-1)(j+l-1)-4 \\ \quad \textit{if } k=i \geq 2 \textit{ and } l=j \geq 2, \\ \leq (i+k-1)(j+l-1)-2 \\ \quad \textit{if } i \neq k \textit{ or } j \neq l. \end{cases}$$

Proof. It follows from Proposition 3.7 (2) that if $i=k$ and $j=l$ and either pair is 1, then $\mathrm{SI} \leq 1$; and if both are ≥ 2, then

$$\begin{aligned} &(i+k-2)(j+l-2)+1 \\ &\quad = (i+k-1)(j+l-1)-(i+k-1)-(j+l-1)+2 \\ &\quad \leq (i+k-1)(j+l-1)-4. \end{aligned}$$

If $i \neq k$ or $j \neq l$, then Proposition 3.7 (2) gives

$$\begin{aligned} &\mathrm{SI}(\langle a^i b^j a^k b^l \rangle) \\ &\quad = (i+k-2)(j+l-2)+|i-k|+|j-l|-1 \\ &\quad = (i+k-1)(j+l-1)-(i+k)+1-(j+l)+1+1 \\ &\qquad +|i-k|+|j-l|-1 \\ &\quad = (i+k-1)(j+l-1)-2\min(i,k)-2\min(j,l)+2 \\ &\quad \leq (i+k-1)(j+l-1)-2. \end{aligned}$$

□

The next remark is useful in the proof of Proposition 6.6.

Remark 3.9. In the punctured torus, it follows from Definition 3.3 that if $\mathsf{P} = rs\mathsf{u}sR$, where r and s are distinct letters and u is an arbitrary linear word, then $\{\mathsf{P}, \mathsf{Q}\}$ is not a linked pair for any Q.

4. PROOF OF THEOREM 1.18

4.1 Detailed Strategy of Proof

This subsection amplifies the sketch presented in Section 1.2, continuing with the notation from Definition 1.17 and Section 2.

Given an arbitrary reduced cyclic word, we prove that its self-intersection number must be less than or equal to that of a word of the same length with few enough block-pairs to be amenable to a linked-pair self-intersection-number calculation.

This "amalgamate and conquer" strategy is implemented by cross-corner surgery, which reduces the number of block-pairs in w while conserving $\alpha(w)$ and $\beta(w)$ and increasing $\mathrm{SI}(w)$.

The detailed procedure at each step in the reduction depends on the number of different letters occurring in the word (Figure 8). As we will see,

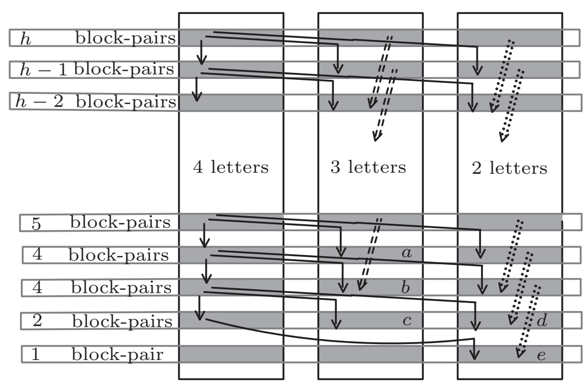

FIGURE 8. Flow chart of proof. The arrows correspond to possible cross-corner surgeries. Straight-line arrows: Lemma 4.1; dashed arrows: Lemma 4.2; dotted arrows: Lemma 4.4. Terminal cases: (a) = Lemma 4.3; (b) = Proposition 3.7 (5); (c) = Proposition 3.7 (3); (d) = Corollary 3.8; (e) = Proposition 3.7 (1). Note that the self-intersection number of a pure power (word with one letter) can be calculated directly ($\mathrm{SI}(r^k) = k-1$) and then compared with the maximum for general words of the same length; see Remark 1.4.

- a word that uses all four letters is always a candidate for cross-corner surgery using opposite corners with reversed orientation; the result will be a single word with one fewer block-pair (since this surgery reverses the orientation of part of the word, the number of different letters may change);

- if a word uses exactly three of the four letters and has at least five block-pairs, or if it uses only two of the four letters and has at least three block-pairs, then two cross-corner surgeries will reduce the number of block-pairs by two (the intermediate stage is a two-component multiword) and increase the self-intersection number by at least two; these surgeries permute the letters in the word, and so the new word still uses three letters or two letters if the old one did.

So the words remaining are

- words with three letters and
  - (a) four block-pairs ($\langle\mathsf{rsrsrsrS}\rangle$, $\langle\mathsf{rsrsrSrS}\rangle$, and $\langle\mathsf{rsrSrsrS}\rangle$),
  - (b) three block-pairs ($\langle\mathsf{rsrsrS}\rangle$), or
  - (c) two block-pairs ($\langle\mathsf{rsrS}\rangle$);
- words with two letters and
  - (d) two block-pairs ($\langle\mathsf{rsrs}\rangle$) or
  - (e) one block-pair ($\langle\mathsf{rs}\rangle$);
- pure powers.

4.2 Preparatory Lemmas

In these lemmas and their proofs, Notation 3.1 will be used.

Lemma 4.1. *If a reduced cyclic word w contains all four letters a, A, b, B, then there exists a word w' with the same α and β values, with one fewer block-pair, and with* $\mathrm{SI}(w') \geq \mathrm{SI}(w) + 1$.

Proof. We claim first that such a word must contain two corners with reverse orientation. In fact, let w be a reduced cyclic word that contains all four letters (such a word must have at least two block-pairs) and that does not contain two subwords of the form xy and XY, where $x \in \{a, A\}$ and $y \in \{b, B\}$ or vice versa. Now w must contain at least one of ab and aB; suppose w contains ab. Then w does not contain AB. So every B-block must be preceded by an a. Since there is at least one such block, w must contain aB, which implies that w does not contain Ab. Since w does not contain AB or Ab, there is no letter possible after an A-block. Since there is at least one such block, our hypothesis leads to a contradiction.

The lemma now follows from Proposition 2.2 (1). ☐

Lemma 4.2. *Suppose a cyclic word w uses exactly three distinct letters from the set $\{a, A, b, B\}$ and has five or more block-pairs. Then there exists a word w' with two fewer block-pairs and with the same α and β values such that* $\mathrm{SI}(w') \geq \mathrm{SI}(w) + 2$.

Proof. Suppose the three letters are a, b, and B. The block-pairs are either ab's or aB's. We may suppose there are at least three ab's. Hence w has the form $\langle\mathsf{ab\ au\ ab\ av\ ab\ ay}\rangle$, where $\mathsf{u, v, y}$ represent (possibly empty) blocks of letters.

We pick two *consecutive* ab block-pairs and apply Proposition 2.2 (cross-corner surgery) as follows: $\langle\mathsf{a|b\ au\ ab|av\ ab\ ay}\rangle \to [\langle\mathsf{b\ au\ ab|}\rangle, \langle\mathsf{a|av\ ab\ ay}\rangle] = [\langle\mathsf{baua}\rangle, \langle\mathsf{avabay}\rangle] = [v', v'']$.

We have lost one ab corner and one ba corner, so the number of block-pairs has gone down by one. On the other hand, Proposition 2.2 guarantees that $\mathrm{SI}([v', v'']) \geq \mathrm{SI}(w) + 1$.

Our consecutive corner condition guarantees that both v' and v'' contain both ab and ba, so the multiword $[v', v'']$ is a candidate for a second surgery, for example: $[\langle\mathsf{|baua}\rangle, \langle\mathsf{avab|ay}\rangle] \to \langle\mathsf{|baua|ayavab}\rangle = \langle\mathsf{bauayava}\rangle = w'$.

We have lost another pair of corners, so the number of block-pairs has gone down by one more; Proposition 2.2 guarantees that $\mathrm{SI}(w') \geq \mathrm{SI}([v', v'']) + 1$, and thus $\mathrm{SI}(w') \geq \mathrm{SI}(w) + 2$. The α and β values are clearly the same. ☐

Lemma 4.3. *If w has one of the forms $\langle\mathsf{abababaB}\rangle$, $\langle\mathsf{abaBabaB}\rangle$, $\langle\mathsf{ababaBaB}\rangle$, then there exists a word w' with two block-pairs and the same α and β as w such that* $\mathrm{SI}(w') \geq \mathrm{SI}(w) + 2$.

Proof. We give three steps:

Step 1. $\langle\mathsf{abababaB}\rangle$. We apply cross-corner surgery (Proposition 2.2) as follows: $\langle\mathsf{ab|aba|baB}\rangle \to [\langle\mathsf{ab|baB}\rangle, \langle\mathsf{|aba}] = [\langle\mathsf{abaB}\rangle, \langle\mathsf{ba}] = [v', v'']$. The multiword $[v', v'']$ has the same α and β values as w. Furthermore, $\mathrm{SI}([v', v'']) \geq \mathrm{SI}(w) + 1$. Another application of Proposition 2.2: $[\langle\mathsf{a|baB}\rangle, \langle\mathsf{b|a}\rangle] \to \langle\mathsf{a|ab|baB}\rangle = \langle\mathsf{abaB}\rangle = w'$ yields a word w' with two block-pairs, the same α and β values as w, and $\mathrm{SI}(w') \geq \mathrm{SI}(w) + 2$.

Step 2. $\langle \mathsf{abaBabaB} \rangle$. Cross-corner surgery $\langle \mathsf{abaB|aba|B} \rangle \to [\langle \mathsf{abaB|B} \rangle, \langle \mathsf{aba|} \rangle] = [\langle \mathsf{abaB} \rangle, \langle \mathsf{ba} \rangle]$ leads to the same half-way step as the previous case.

Step 3. $\langle \mathsf{ababaBaB} \rangle$. Apply Proposition 2.2:

$$\begin{aligned}\langle \mathsf{ababa|BaB|} \rangle &\to [\langle \mathsf{ababa|} \rangle, \langle \mathsf{BaB|} \rangle] = [\langle \mathsf{abab} \rangle, \langle \mathsf{aB} \rangle]\\ &= [\langle a^i b^j a^k b^l \rangle, \langle a^m B^n \rangle],\end{aligned}$$

say, (so $\alpha(w) = i + k + m,\ \beta(w) = j + l + n$), and

$$\mathrm{SI}(w) \le \mathrm{SI}([\langle a^i b^j a^k b^l \rangle, \langle a^m B^n \rangle]) - 1.$$

Now

$$\begin{aligned}&\mathrm{SI}([\langle a^i b^j a^k b^l \rangle, \langle a^m B^n \rangle])\\ &\quad = \mathrm{SI}(\langle a^i b^j a^k b^l \rangle) + \mathrm{SI}(\langle a^m B^n \rangle)\\ &\qquad + \mathrm{IN}(\langle a^i b^j a^k b^l \rangle, \langle a^m B^n \rangle).\end{aligned}$$

Because of the format of Corollary 3.8 we need to consider two cases:

Case (i): $i = k$ and $j = l = 1$ (by the construction, i and k cannot be 1). In that case $\mathrm{SI}(\langle a^i b^j a^k b^l \rangle) = 1$. By Proposition 3.7 (7),

$$\mathrm{IN}(\langle a^i b^j a^k b^l \rangle, \langle a^m B^n \rangle) = (i + k)n + m(j + l),$$

and by Proposition 3.7 (1),

$$\mathrm{SI}(\langle a^m B^n \rangle) = (m - 1)(n - 1).$$

This gives

$$\begin{aligned}\mathrm{SI}([\langle a^i b a^i b \rangle, \langle a^m B^n \rangle]) &= 1 + 2in + 2m + (m - 1)(n - 1)\\ &= (2i - 1)n + m(n + 1) + 2\end{aligned}$$

and

$$si(w) \le (2i - 1)n + m(n + 1) + 1.$$

On the other hand, the word $w' = \langle a^{2i} b^2 a^m B^n \rangle$ has the same α and β values as w and (Proposition 3.7 (3))

$$\mathrm{SI}(w') = (2i+m-1)(n+1) = (2i-1)n+m(n+1)+(2i-1).$$

Since as remarked above, $i \ge 2$, it follows that $\mathrm{SI}(w') \ge \mathrm{SI}(w) + 2$.

Case (ii): For all other $\langle a^i b^j a^k b^l \rangle$, Corollary 3.8 gives

$$\mathrm{SI}(\langle a^i b^j a^k b^l \rangle) \le (i + k - 1)(j + l - 1) - 2,$$

and

$$\begin{aligned}&\mathrm{SI}([\langle a^i b a^i b \rangle, \langle a^m B^n \rangle])\\ &\quad \le (i + k - 1)(j + l - 1) - 2 + (i + k)n + m(j + l)\\ &\qquad + (m - 1)(n - 1)\\ &\quad = (i + k + m - 1)(j + l + n - 1) - 1,\end{aligned}$$

so

$$\mathrm{SI}(w) \le (i + k + m - 1)(j + l + n - 1) - 2.$$

Comparing this estimate with

$$\mathrm{SI}(\langle a^{i+k} b^{j+l} a^m B^n \rangle) = (i + k + m - 1)(j + l + n - 1)$$

(Proposition 3.7 (3) again) completes the proof. □

Lemma 4.4. *If w uses exactly two letters and w has three or more block-pairs, then there exists a word w' with two fewer block-pairs and with the same α and β values such that $\mathrm{SI}(w') \ge \mathrm{SI}(w) + 2$.*

Proof. Suppose the two letters are a and b, so $w = \langle \mathsf{ababab} \dots \rangle$. Now proceed as in the proof of Lemma 4.2. □

4.3 End of the Proof

Proposition 4.5. *Let w be the reduced cyclic word corresponding to a free homotopy class of curves on the punctured torus, with $h(w) = h > 0$. Then there exists a word w' such that w' has one or two blocks, $\alpha(w') = \alpha(w)$ and $\beta(w') = \beta(w)$, and $\mathrm{SI}(w') \ge \mathrm{SI}(w) + h - 2$.*

Proof. If $h = 1$ or 2, then taking $w' = w$ satisfies the conclusions of the proposition.

For $h > 2$, we proceed by complete induction, and assume that the result holds for any word with a number of block-pairs smaller than h. Since h is positive, w contains two, three, or four distinct letters. We consider the cases separately.

Two letters. Suppose that w contains exactly two distinct letters. If w has more than two block-pairs, then by Lemma 4.4 there exists a word v with $h - 2$ block-pairs and with the same α and β values such that $\mathrm{SI}(v) \ge \mathrm{SI}(w) + 2$. By the induction hypothesis, there exists w' with one or two blocks and the same α and β as v such that

$$\mathrm{SI}(w') \ge \mathrm{SI}(v) + (h - 2) - 2 \ge \mathrm{SI}(w) + h - 2,$$

as desired.

Three letters. Suppose that w contains exactly three distinct letters. If $h > 4$, the result follows from combining Lemma 4.2 and the induction hypothesis. If $h = 4$ then w must have one of the following forms: $\langle \mathsf{abababaB} \rangle$, $\langle \mathsf{abaBabaB} \rangle$, or $\langle \mathsf{ababaBaB} \rangle$. Lemma 4.3 covers these three cases. In the case $h = 3$, the word can be supposed to be $w = \langle a^i b^j a^k b^l a^m B^n\rangle$. Taking $w' = \langle a^{i+k} b^{j+l} a^m B^n\rangle$ and applying Proposition 3.7 (3), (5) yields the desired result.

Four letters. Suppose that w contains all four letters a, b, A, B. By Lemma 4.1, there exists a word v with $h-1$ block-pairs, with $\alpha(v) = \alpha(w)$, $\beta(v) = \beta(w)$, and such that $\mathrm{SI}(v) \geq \mathrm{SI}(w)+1$. Now the result follows from our induction hypothesis. More explicitly, there exists a word w' with the same α and β as v and with one or two block-pairs such that $\mathrm{SI}(w') \geq \mathrm{SI}(v) + (h-1) - 2 \geq \mathrm{SI}(w) + h - 2$. □

of Theorem 1.18. If $h = 1$ or 2, the result follows from Proposition 3.7 (1)–(4). So suppose that $h > 2$. By Proposition 4.5, there exists a word w' with $\alpha(w') = \alpha(w)$ and $\beta(w') = \beta(w)$, $\mathrm{SI}(w) \leq \mathrm{SI}(w') - h + 2 < \mathrm{SI}(w')$, such that w' has one or two blocks. Referring to Proposition 3.7 (1)–(4), any such word satisfies $\mathrm{SI}(w') \leq (\alpha(w') - 1)(\beta(w') - 1)$. This proves part (1).

Part (2) of the theorem follows also, by inspection, from Proposition 3.7 (1)–(4). □

4.4 Words with Submaximal Intersection Number

Lemma 4.6. *If w is one of the words $\langle \mathsf{ababAB}\rangle$, $\langle \mathsf{abAbaB}\rangle$, $\langle \mathsf{abaBAB}\rangle$, and $\langle \mathsf{abABaB}\rangle$, then*

$$\mathrm{SI}(w) \leq (\alpha(w) - 1)(\beta(w) - 1) - 2.$$

Proof. Proposition 3.7 and Corollary 3.8 can be applied after one or two cross-corner surgeries (Proposition 2.2), each of which increases the self-intersection number by at least one:

$$\begin{aligned}
&\langle a^i b^j a^k b^l | A^m B^n|\rangle \\
&\quad \to \langle a^i b^j a^k b^l | b^n a^m |\rangle = \langle a^{i+m} b^j a^k b^{l+n}\rangle;\\
&\langle a^i b^j | A^k b^l a^m B^n|\rangle \\
&\quad \to \langle a^i b^j | b^n A^m B^l a^k|\rangle = \langle a^{i+k} b^{n+j} A^m B^l\rangle;\\
&\langle a^i b^j a^k | B^l A^m B^n|\rangle \\
&\quad \to [\langle a^i b^j a^k|\rangle, \langle B^l A^m B^n|\rangle]\\
&\qquad = [\langle a^{i+k} b^j\rangle, \langle A^m B^{n+l}\rangle][\langle a^{i+k} | b^j\rangle, \langle A^m | B^{n+l}\rangle]\\
&\quad \to \langle a^{i+k} | a^m b^{n+l} | b^j\rangle = \langle a^{i+k+m} b^{j+n+l}\rangle;\\
&\langle a^i | b^j A^k | B^l a^m B^n\rangle \\
&\quad \to \langle a^i | a^k B^j | B^l a^m B^n\rangle = \langle a^{i+k} B^{l+j} a^m B^n\rangle.
\end{aligned}$$

□

The following lemma will be used in the proof of Theorem 4.9. Note that the special case it covers admits a bound for the self-intersection number sharper than that of Theorem 1.19.

Lemma 4.7. *If w is a word with three block-pairs, then $\mathrm{SI}(w) \leq (\alpha(w) - 1)(\beta(w) - 1) - 2$. In particular, if we have length $L = \alpha(w) + \beta(w)$, then*

$$\mathrm{SI}(w) \leq \begin{cases} (L-2)^2/4 - 2 & \text{if } L \text{ is even,} \\ (L-1)(L-3)/4 - 2 & \text{if } L \text{ is odd.} \end{cases}$$

Proof. Without loss of generality, we may suppose that the number N of A and B blocks in w is at most three.

If $N = 0$, the result follows from Lemma 4.4 and Theorem 1.19.

If $N = 1$, we may suppose that $w = \langle \mathsf{abababB}\rangle$, which is covered by Proposition 3.7 (5).

If $N = 2$, we may suppose that w is one of $\langle \mathsf{abAbaB}\rangle$ and $\langle \mathsf{ababAB}\rangle$; if $N = 3$, we may suppose that w is one of $\langle \mathsf{abaBAB}\rangle$ and $\langle \mathsf{abABaB}\rangle$; for these cases, the result follows from Lemma 4.6. □

Lemma 4.8. *Let w be a word with two block-pairs and two letters, say a and b with length $L \geq 4$. Either $L = 4$ and $w = abab$ with $\mathrm{SI}(w) = 1$, or*

$$\mathrm{SI}(w) \leq \begin{cases} (L-2)^2/4 - 2 & \text{if } L \text{ is even,} \\ (L-1)(L-3)/4 - 2 & \text{if } L \text{ is odd.} \end{cases}$$

Proof. Refer to Corollary 3.8.

First note that $ab^j ab^j$ has length $L = 2j + 2$, an even number, and if $j \geq 2$, then

$$(L-2)^2/4 - 2 = j^2 - 2 \geq 1 = \mathrm{SI}(ab^j ab^j).$$

So the lemma holds for all words of the form $ab^j ab^j$ and $a^i b a^i b$.

For the rest of the words in question,

$$\mathrm{SI}(w) \leq (i + k - 1)(j + l - 1) - 2,$$

so the result follows as in Remark 1.21. □

Theorem 4.9. *Let w be a primitive reduced cyclic word of length $L > 3$ and self-intersection number*

$$\mathrm{SI}(w) = \begin{cases} (L-2)^2/4 - 1 & \text{if } L \text{ is even,} \\ (L-1)(L-3)/4 - 1 & \text{if } L \text{ is odd,} \end{cases}$$

i.e., one less than the maximum possible for its length. Then if L is odd, $w = r^i s^j R^k S^l$ with $i+k = \frac{L-1}{2}$ or $\frac{L+1}{2}$. And if L is even, w has one of the following forms:

(1) $\langle r^{L/2-1}s^{L/2+1}\rangle$;

(2) $\langle r^i s^j R^k S^l\rangle$, $i+k=\frac{L}{2}$;

(3) $\langle r^i s^j r^k S^l\rangle$, $i+k=\frac{L}{2}-1$, *or* $\frac{L}{2}+1$.

Here $r=a$ *or* A *and* $s=b$ *or* B, *or vice versa.*

Remark 4.10. The primitive reduced cyclic words of length $L \le 3$, namely those of the form a, ab, abb, all have self-intersection number zero, the maximum for those lengths (cf. Table 1).

of Theorem 4.9. By Proposition 4.5 and Lemma 4.7, $h(w)=1$ or 2 (the only primitive words with zero block-pairs are singletons, which do not satisfy the hypothesis).

We begin with the case $h(w)=2$. By Lemma 4.8, we can assume that $w=\langle \mathsf{abaB}\rangle$ or $\langle \mathsf{abAB}\rangle$.

First suppose $w=\langle \mathsf{abaB}\rangle$. By Proposition 3.7 (3), $\mathrm{SI}(w)=(\alpha(w)-1)(\beta(w)-1)$.

If L is odd, then $\mathrm{SI}(w)=(L-1)(L-3)/4-1$. Since $\beta(w)=L-\alpha(w)$, it follows that

$$(L-1)(L-3)/4-1=(\alpha(w)-1)(L-\alpha(w)-1).$$

This implies $\alpha(w)=(L\pm\sqrt{5})/2$, which is not an integer, a contradiction.

So L is even, and

$$(L/2-1)^2-1=(\alpha(w)-1)(L-\alpha(w)-1).$$

This implies $\alpha(w)=\frac{n}{2}-1$ or $\frac{n}{2}+1$, as desired.

Now suppose $w=\langle \mathsf{abAB}\rangle$. The result follows from Proposition 3.7 (4). This settles the case $h(w)=2$.

If $h(w)=1$, then by Proposition 3.7 (1),

$$\mathrm{SI}(w)=(\alpha(w)-1)(L-\alpha(w)-1).$$

The solutions of the equation

$$(\alpha(w)-1)(L-\alpha(w)-1)=(L-1)(L-3)/4-1$$

are $\alpha(w)=\frac{L-\sqrt{5}}{2}$ and $\alpha(w)=\frac{L+\sqrt{5}}{2}$. Hence there are no words of submaximal self-intersection with odd length L and one block-pair.

On the other hand, the solutions of the equation

$$(\alpha(w)-1)(L-\alpha(w)-1)=(L/2-1)^2-1$$

are $\alpha(w)=L/2-1$ and $L/2+1$; the result follows. □

Theorem 4.11. *If L is odd, there are $(L-1)(L-3)$ distinct reduced cyclic words with self-intersection number one less than the maximum for their length.*

If L is even, there are $5(L-2)^2/2$ distinct reduced cyclic words with self-intersection number one less than the maximum for their length.

Proof. Refer to Theorem 4.9. Suppose L is odd. If $i+k=\frac{L-1}{2}$, there are $1,\dots,\frac{L-3}{2}$ possibilities for i, and $1,\dots,\frac{L-1}{2}$ possibilities for j. The total is $\frac{L-3}{2}\frac{L-1}{2}$. Interchanging the roles of i and j, and those of a and b, we obtain $(L-1)(L-3)$.

Suppose L is even. then there are 8 words of the form $\langle r^{L/2-1}s^{L/2+1}\rangle$, together with $2(L/2-1)^2$ words of the form $\langle r^i s^j R^k S^l\rangle$ and $4L(L/2-2)$ words of the form $\langle r^i s^j r^k S^l\rangle$; the total is

$$\frac{5L^2}{2}-10L+10=\frac{5(L-2)^2}{2}.$$

□

Remark 4.12. The leading coefficient of the polynomial expression for the number of maximal words of odd length is twice that for even length, whereas for submaximal words the even leading coefficient is 2.5 times the odd leading coefficient. The discrepancies balance out to some extent, when one considers maximal and submaximal words together. For odd length L, this number is $3L^2-12L+17$, while for even length it is $7L^2/2-14L+18$.

5. EXPERIMENTAL RESULTS AND CONJECTURES FOR THE PAIR OF PANTS

The "pair of pants" is the usual name for the surface with boundary obtained by deleting three open disks from the sphere. The same computational methods that yielded Experimental Theorem 1.7 suggest that the dependence of maximum self-intersection number on length for the pair of pants is quadratic, just as it was for the punctured torus.

Experimental Theorem 5.1. *For $L \le 18$, the maximal self-intersection number for a primitive reduced cyclic word of length L on the pair of pants is*

$$\begin{cases} (L^2-1)/4 & \text{if } L \text{ is odd,}\\ L^2/4-1 & \text{if } L\equiv 0 \pmod 4,\\ L^2/4-2 & \text{if } L>2 \text{ and } L\equiv 2 \pmod 4,\\ 1 & \text{if } L=2.\end{cases}$$

Moreover, if L is odd, the (primitive) words realizing the above maximal self-intersection number are $r(rs)^{\frac{L-1}{2}}$, where $\{r,s\}=\{a,B\}$ or $\{r,s\}=\{A,b\}$ (primitive words *of even length follow a more complicated pattern, which cannot be easily reduced to a formula).*

Removing the restriction "primitive" leads to the following result:

Experimental Theorem 5.2. *For $L \leq 17$, the maximal self-intersection number for a reduced cyclic word of length L on the pair of pants is*

$$\begin{cases} (L^2-1)/4 & \text{if } L \text{ is odd,} \\ L^2/4+L/2-1 & \text{if } L \text{ is even.} \end{cases}$$

Moreover, if L is even, the words realizing the maximal self-intersection number are $(Ab)^{\frac{L-1}{2}}$ and $(aB)^{\frac{L-1}{2}}$; words of odd length L realizing the maximal self-intersection number can have the form $r(rs)^{\frac{L-1}{2}}$, where $\{r,s\} = \{a,B\}$ or $\{r,s\} = \{A,b\}$; but if L is not prime, this list of words is not always exhaustive.

The next two experimental theorems show radically different behavior from what we know for the punctured torus.

Experimental Theorem 5.3. *On the pair of pants, for $L \leq 12$ the number of distinct free homotopy classes of curves of length L realizing the maximal self-intersection number is*

$$\begin{cases} 6 & \text{if } L = 2, \\ 2 & \text{if } L \text{ is even and } L > 2, \\ 4 & \text{if } L \text{ is odd but not a multiple of } 3, \\ 8 & \text{if } L \text{ is an odd multiple of } 3. \end{cases}$$

Experimental Theorem 5.4. *On the pair of pants, for $L \leq 15$ the* minimal *self-intersection number for the free homotopy class representing a primitive reduced cyclic word of length L is 0 for $L = 1, 2$ and $[L/2]$ (the integer part of $L/2$) for $L \geq 3$.*

It seems reasonable to conjecture that all this behavior will persist for higher values of L.

Remark 5.5. Note that an analogue of Proposition 2.2 can be proved for any surface with boundary. So words with maximal self-intersection number cannot contain (the generalization of) diagonally opposed corners with reversed orientations.

6. APPENDIX: PROOF OF PROPOSITION 3.7

The seven parts of Proposition 3.7 correspond to Propositions 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7. In this appendix, the proofs of Propositions 6.1, 6.2, and 6.3 are given in detail; Propositions 6.4, 6.5, and 6.6 can be proved along the same lines as Proposition 6.4; Proposition 6.7 can be proved like Proposition 6.3; those proofs are omitted here but appear in full in the arXiv version [Chas and Phillips 09] of this work. The method of proof for each of these propositions is via Theorem 3.5: a counting of all occurrences of each of the three types of linked pairs given in Definition 3.3.

I. These pairs are easy to count. They have the form $\{rr, ss\}$, where $r \in \{a, A\}$ and $s \in \{b, B\}$.

II. These have the form $\{p_1\mathsf{y}p_2, q_1\mathsf{y}q_2\}$, with $p_1 \neq p_2$ and $q_1 \neq q_2$. One locates all subwords y with two occurrences and checks for each pair whether the corresponding $p_1\mathsf{y}p_2$ and $q_1\mathsf{y}q_2$ are linked.

III. Analogously, these pairs are found by locating subwords y that occur in our word or multiword along with their inverses Y. Such a pair will contribute to the count if the corresponding $p_1\mathsf{y}p_2$ and $\overline{q}_2\mathsf{y}\overline{q}_1$ are linked; see Remark 3.4.

Proposition 6.1. $\mathrm{SI}(\langle a^i b^j \rangle) = (i-1)(j-1)$.

Proof. There are $i-1$ occurrences of aa and $j-1$ occurrences of bb in $\langle a^i b^j \rangle$. Thus there are $(i-1)(j-1)$ pairs of type I. There are no pairs of the other two types. □

Proposition 6.2. $\mathrm{SI}(\langle a^i b^j a^k b^l \rangle) = (i+k-2)(j+l-2)+1$ *if $k = i$ and $j = l$; and $(i+k-2)(j+l-2)+|i-k|+|j-l|-1$, otherwise.*

Proof. I. There are $(i+k-2)(j+l-2)$ pairs of this kind.

II. In this case,

$$\mathsf{y} \in \left\{a^K, B^K, a^K b^J, b^K a^J, a^K b^J a^L, b^J a^K b^L\right\}$$

for some positive integers J, K, and L.

(i) $\mathsf{y} = a^K$. Analysis: Table 5, using Table 2. The total number is $|i-k|-1$ if $i \neq k$ and zero otherwise.

(ii) $\mathsf{y} = b^K$. With similar arguments as in case (i), it can be shown that the number of pairs here is $|j-l|-1$ if $j \neq l$ and zero otherwise.

configuration	with	if	add
$\{a^{k+2}, ba^k b\}$	a^{k+2} in a^i	$k+2 \le i$	$i-k-1$
$\{a^{i+2}, ba^i b\}$	a^{i+2} in a^k	$i+2 \le k$	$k-i-1$

TABLE 5. Linked pairs in $\langle a^i b^j a^k b^l\rangle$ of type II with $\mathsf{y} = a^K$.

configuration	with	if	add
$\{ba^i b^l b, aa^i b^l a\}$	$ba^i b^l b$ in $ba^i b^j$, $aa^i b^l a$ in $a^k b^l a$	$i < k$ and $j > l$	1
$\{ba^k b^j b, aa^k b^j a\}$	$ba^k b^j b$ in $ba^k b^l$, $aa^k b^j a$ in $a^i b^j a$	$k < i$ and $j < l$	1

TABLE 6. Linked pairs in $\langle a^i b^j a^k b^l\rangle$ of type II with $\mathsf{y} = a^K b^J$.

configuration	with	if	add
$\{bb^j a^i b, ab^j a^i a\}$	$bb^j a^i b$ in $b^l a^i b$, $ab^j a^i a$ in $ab^j a^k$	$i < k$ and $j < l$	1
$\{bb^l a^k b, ab^l a^k a\}$	$bb^l a^k b$ in $b^j a^k b$, $ab^l a^k a$ in $ab^l a^i$	$k < i$ and $j > l$	1

TABLE 7. Linked pairs in $\langle a^i b^j a^k b^l\rangle$ of type II with $\mathsf{y} = b^K a^J$.

	configuration	with	if	add
a	$\{a^K b, Ba^K\}$	$a^K b$ in $a^i b$, Ba^K in Ba^m	$K \in \{2 \dots \min(m,i)\}$	$\min(m,i)-1$
b	$\{a^i B, ba^i\}$	ba^K in ba^i, $a^K B$ in $a^m B$	$K \in \{2 \dots \min(m,i)\}$	$\min(m,i)-1$
c	$\{a^{m+2}, Ba^m B\}$	a^{m+2} in a^i	$m+2 \le i$	$i-m-1$
d	$\{a^{m+1}b, Ba^m B\}$	a^{m+1} in a^i	$m+1 \le i$	1
e	$\{ba^{m+1}, Ba^m B\}$	a^{m+1} in a^i	$m < i$	1
f	$\{a^{i+1}B, ba^i b\}$	a^{i+1} in a^m	$i < m$	1
g	$\{Ba^{i+1}, ba^i b\}$	a^{i+1} in a^m	$i < m$	1
h	$\{a^{i+2}, ba^i b\}$	a^{i+2} in a^m	$i+2 \le m$	$m-i-1$
i	$\{ba^i b, Ba^m B\}$		$i = m$	1

TABLE 8. Linked pairs of $\langle a^i b^j\rangle$ and $\langle a^m B^n\rangle$ of type II with $\mathsf{y} = a^K$.

(iii) $\mathsf{y} = a^K b^J$. By Table 3, the linked words of pairs with this y have the form $ba^K b^J b$ and $aa^K b^J a$. Analysis: Table 6. The three types of linked pairs can be added as follows:

(iv) $\mathsf{y} = b^K a^J$. By Table 3, the linked words have the form $ab^K a^J a$ and $bb^K a^J b$. Analysis: Table 7.

By (iii) and (iv) we add 1 if $k \neq i$ and $j \neq l$.

(v) $\mathsf{y} = a^K b^J a^L$. Since y has two occurrences, $j = l$. By Table 4, the linked pairs have the form $\{aa^K b^j a^L a, ba^K b^j a^L b\}$. There are two pairs of this kind, namely $\{aa^i b^j a^k a,\ ba^i b^j a^k b\}$ and $\{aa^k b^l a^i a,\ ba^k b^l a^i b\}$. Each of the possibilities implies that $i < k$ and $k < i$. Hence such pairs are impossible.

(vi) $\mathsf{y} = b^K a^J b^L$. As in case (v), there are no linked pairs of this form.

III. There are no pairs of type III because the word contains no occurrence of a letter and its inverse.

If $i = k$ and $j = l$ add 1, because the word has the form w^2, where w is a primitive word. Adding up all the contributions completes the proof. □

Proposition 6.3. $\mathrm{IN}(\langle a^i b^j\rangle, \langle a^m B^n\rangle) = in + mj$.

Proof. I. There are $(i-1)(n-1)+(j-1)(m-1)$ linked pairs of this type.

II. $\mathsf{y} = a^K$ for some positive integer K. Analysis: Table 8, using Table 2. The contributions of the different rows may be grouped in the following way: (a + c + d) $= i-1$, (b + f + h) $= m-1$, and (e + g + i) $= 1$.

III. $\mathsf{y} = b^K$. Combining Remark 3.4 with Table 2, we analyze these pairs in Table 9. Here (a + c + f) $= n-1$, (b + g + i) $= j-1$, and (d + e + h) $= 1$.

Adding the contributions from each of the three types yields the result. □

Proposition 6.4. $\mathrm{SI}(\langle a^i b^j a^k B^l\rangle) = (i+k-1)(j+l-1)$.

	configuration	with	if	add
a	$\{ab^K, aB^K\}$	ab^K in ab^j, aB^K in aB^n	$K \in \{2, \ldots, \min(j,n)\}$	$\min(j,n)-1$
b	$\{b^K a, B^K a\}$	ab^K in ab^j, aB^K in aB^n	$K \in \{2, \ldots, \min(j,n)\}$	$\min(j,n)-1$
c	$\{ab^j a, B^{j+1}a\}$	$B^{j+1}a$ in $B^n a$	$j < n$	1
d	$\{ab^j a, aB^{j+1}\}$	aB^{j+1} in aB^n	$j < n$	1
e	$\{ab^j a, aB^n a\}$		$j = n$	1
f	$\{ab^j a, B^{j+2}\}$	B^{j+2} in B^n	$j+2 \leq n$	$n-j-1$
g	$\{b^{n+1}a, aB^n a\}$	$b^{n+1}a$ in $b^j a$	$n < j$	1
h	$\{ab^{n+1}, aB^n a\}$	ab^{n+1} in ab^j	$n < j$	1
i	$\{b^{n+2}, aB^n a\}$	b^{n+2} in b^j	$n+2 \leq j$	$j-n-1$

TABLE 9. Linked pairs of $\langle a^i b^j \rangle$ and $\langle a^m B^n \rangle$ of type III with $\mathsf{y} = b^K$.

Proposition 6.5. $\mathrm{SI}(\langle a^i b^j A^k B^l \rangle = (i+k-1)(j+l-1)-1$.

Proposition 6.6.

$$\mathrm{SI}(\langle a^i b^j a^k b^l a^m B^n \rangle) = (i+k+m-1)(j+l+n-1) \\ - 2(k+\min(j,l)-1).$$

Proposition 6.7. $\mathrm{IN}(\langle a^i b^j a^k b^l \rangle, \langle a^m B^n \rangle) = (i+k)n + m(j+l)$.

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REFERENCES

[Basmajian 93] A. Basmajian. "The Stable Neighborhood Theorem and Lengths of Closed Geodesics." *Proc. Amer. Math. Soc.* 119:1 (1993), 217–224.

[Birman and Series 84] J. Birman and C. Series. "An Algorithm for Simple Curves on Surfaces." *J. London Math. Soc. (2)* 29 (1984), 331–342.

[Blood 02] A. Blood. "The Maximal Number of Transverse Self-Intersections on the Punctured Torus." In *Proceedings of the REU Program in Mathematics, Corvallis OR, August 2002*. Available online (http://www.math.oregonstate.edu/~math_reu/REU_Proceedings/Proceedings2002/Abnew.pdf).

[Bridson and Haefliger 91] M. Bridson and A. Haefliger. *Metric Spaces of Non-positive Curvature*. New York: Springer, 1991.

[Buser 92] P. Buser. *Geometry and Spectra of Compact Riemann Surfaces*. Boston: Birkhäuser, 1992.

[Chas 04] M. Chas. "Combinatorial Lie Bialgebras of Curves on Surfaces." *Topology* 43 (2004), 543–568.

[Chas and Krongold 09] M. Chas and F. Krongold. "An Algebraic Characterization of Simple Closed Curves on Surfaces with Boundary." arXiv:0801.3944 [math.GT], 2009.

[Chas and Phillips 09] M. Chas and A. Phillips. "Self-Intersection Numbers of Curves on the Punctured Torus." arXiv:0901.2974 [math.GT], 2009.

[Chemotti and Rau 04] F. Chemotti and A. Rau. "Intersection Numbers of Closed Curves on the Punctured Torus. In *Proceedings of the REU Program in Mathematics, Corvallis OR, August 2004*. Available online (http://www.math.oregonstate.edu/~math_reu/REU_Proceedings/Proceedings2004/chemotti_rau04.pdf).

[Cohen and Lustig 87] M. Cohen and M. Lustig. "Paths of Geodesics and Geometric Intersection Numbers I." In *Combinatorial Group Theory and Topology, Alta, Utah, 1984*, Ann. of Math. Studies 111, pp. 479–500. Princeton: Princeton Univ. Press, 1987.

[Hass and Scott 85] J. Hass and P. Scott. "Intersections of Curves on Surfaces." *Israel J. Math.* 51 (1985), 90–120.

[Lalley 89] S. Lalley. "Renewal Theorems in Symbolic Dynamics, with Applications to Geodesic Flows, Noneuclidean Tessellations, and Their Fractal Limits." *Acta. Math.* 163 (1989), 1–55.

[Lalley 96] S. Lalley. "Self-Intersections of Closed Geodesics on a Negatively Curved Surface: Statistical Regularities." In *Convergence in Ergodic Theory and Probability (Columbus, OH, 1993)*, Ohio State Univ. Math. Res. Inst. Publ. 5, pp. 263–272. Berlin: De Gruyter, 1996.

[Lyndon and Schupp 01] R. Lyndon and R. Schupp. *Combinatorial Group Theory*. New York: Springer, 2001.

[Margulis 83] G. Margulis. "Applications of Ergodic Theory to the Investigation of Manifolds of Negative Curvature." *Funct. Anal. and Appl.* 3 (1983), 573–591.

[McShane and Rivin 95] G. McShane and I. Rivin. "A Norm on Homology of Surfaces and Counting Simple Geodesics." *Internat. Math. Res. Notices*, February 1995.

[Milnor 68] J. Milnor. "A Note on Curvature and the Fundamental Group." *J. Differential Geometry* 2 (1968), 1–7.

[Mirzakhani 08] M. Mirzakhani. "Growth of the Number of Simple Closed Geodesics on a Hyperbolic Surface." *Ann. of Math.* 168 (2008), 97–125.

[Parry and Pollicott 83] W. Parry and M. Pollicott. "An Analogue of the Prime-Number Theorem for Closed Orbits of Axiom A Flows." *Ann. of Math.* 118 (1983), 573–591.

[Rivin 01] I. Rivin. "Simple Curves on Surfaces." *Geom. Dedicata* 87 (2001), 345–360.

[Rivin 09] I. Rivin. "Geodesics with One Self-Intersection and Other Stories." arXiv: 0901.2543[math.GT], 2009.

[Tan 96] S. Tan. "Self-Intersections of Curves on Surfaces." *Geom. Dedicata* 62:2 (1996), 209–225.

[Thurston 10] D. Thurston. "On Geometric Intersection of Curves in Surfaces." Preprint, available online (http://www.math.columbia.edu/~dpt/writing.html), 2010.

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