The Cubic Chan–Chua Conjecture

Shaun Cooper

CONTENTS

A conjecture that expresses the nth power of the cubic theta function $a(q)=\sum_j\sum_k q^{j^2+jk+k^2}$ in terms of Eisenstein series is formulated. It is an analogue of four conjectures of H. H. Chan and K. S. Chua for powers of $\varphi^2(q)=\sum_j\sum_k q^{j^2+k^2}$. With the help of a computer, the conjecture is shown to be true for $6\le n\le 100$. It is conjectured that the result continues to hold for $n>100$.

1. INTRODUCTION

Let n be a positive integer and let q be a complex number that satisfies $|q|<1$. The Bernoulli numbers B_n are defined by

$$\sum_{n=0}^{\infty} B_n \frac{u^n}{n!} = \frac{u}{e^u-1}.$$

Let

$$E_{2n}(q) = 1 - \frac{4n}{B_{2n}} \sum_{j=1}^{\infty} \frac{j^{2n-1}q^j}{1-q^j},$$

and put

$$S_{2n} = S_{2n}(q) = \frac{2^{2n}E_{2n}(q^2)-E_{2n}(q)}{2^{2n}-1}.$$

Let

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}.$$

One of the results in the epic work of S. C. Milne [Milne 02, (1.25)] is

$$\varphi^{24}(-q) = \frac{1}{9}\left(17S_4S_8 - 8S_6^2\right).$$

Inspired by this, H. H. Chan and K. S. Chua [Chan and Chua 03] discovered and proved the result

$$\varphi^{32}(-q) = \frac{1}{4725}\left(11056S_4S_{12} - 12400S_6S_{10} + 6069S_8^2\right)$$

and made the following general conjecture.

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Conjecture 1.1. (Chan and Chua.) *For $n \geq 2$ there exist rational constants $c_{n,j}$, depending only on n and j, such that*

$$\varphi^{8n}(-q) = \sum_{j=2}^{n} c_{n,j} S_{2j} S_{4n-2j}.$$

Chan and Chua showed (see Lemma 2.2 in [Chan and Chua 03] and the subsequent comments) that $\varphi^4(-q)$ and $S_{2n}(q)$ may both be expressed as polynomials in two variables. Hence, given enough computing power, the truth of Conjecture 1.1 can be established for any particular value of n by verifying a polynomial identity. As a consequence, Conjecture 1.1 is known to be true for a large number of values of n. For example, H. Y. Lam [Lam 06, p. 174] reports that Conjecture 1.1 is true for $2 \leq n \leq 86$.

Chan and Chua made conjectures of a similar nature for $\varphi^{8n+2}(-q)$, $\varphi^{8n+4}(-q)$, and $\varphi^{8n+6}(-q)$, and some additional conjectures have since been given by Lam [Lam 06]. Apart from the computational evidence and some results of Ö. Imamoğlu and W. Kohnen [Imamoğlu and Kohnen 05], almost no progress has been made in proving Conjecture 1.1 or any of the other conjectures.

The purpose of this short article is to formulate an analogue of Conjecture 1.1 for the cubic theta function $a(q)$. That is, we consider powers of $\sum_j \sum_k q^{j^2+jk+k^2}$ instead of powers of $\sum_j \sum_k q^{j^2+k^2}$. The main result of this work is Theorem 3.1, which is shown to be true for $6 \leq n \leq 100$. Conjecture 3.2 predicts that Theorem 3.1 continues to hold for $n > 100$.

2. NOTATION

Let n be a nonnegative integer and let $p \equiv 3 \pmod 4$ be prime. The generalized Bernoulli numbers $B_{n,p}$ are defined by

$$\sum_{n=0}^{\infty} B_{n,p} \frac{u^n}{n!} = \frac{u}{e^{pu}-1} \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) e^{ju}.$$

The generalized Eisenstein series F_n is defined by

$$F_n = F_{n,p}(q)$$
$$= \begin{cases} 1 - \dfrac{2}{B_{1,p}} \displaystyle\sum_{j=1}^{\infty} \left(\frac{j}{p}\right) \frac{q^j}{1-q^j}, & \text{if } n = 1, \\ \dfrac{E_n(q) + (-p)^{n/2} E_n(q^p)}{1+(-p)^{n/2}}, & \text{if } n \text{ is even}, \\ 1 - \dfrac{2n}{B_{n,p}} \displaystyle\sum_{j=1}^{\infty} \left(\frac{j}{p}\right) \frac{j^{n-1} q^j}{1-q^j} & \\ \quad -(-p)^{(n-1)/2} \dfrac{2n}{B_{n,p}} \displaystyle\sum_{j=1}^{\infty} \frac{j^{n-1}}{1-q^{pj}} \sum_{\ell=1}^{p-1} \left(\frac{\ell}{p}\right) q^{j\ell}, & \\ \quad \text{if } n \geq 3 \text{ is odd}. & \end{cases}$$

The cubic theta function is defined by

$$a = a(q) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+jk+k^2}.$$

For the rest of this work we take $p = 3$.

3. MAIN RESULTS

The main results of this work are Theorem 3.1 and Conjecture 3.2.

Theorem 3.1. *With a and F_n as defined in Section 2, we have*

$$a = F_1, \quad a^2 = F_2, \quad a^3 = F_3, \quad a^4 = F_4, \quad a^5 = F_5,$$
$$a^6 = F_2F_4, \quad a^7 = F_2F_5, \quad a^8 = F_4^2, \quad a^9 = F_4F_5,$$
$$a^{10} = -\frac{41}{63} F_2F_8 + \frac{104}{63} F_4F_6,$$
$$a^{11} = -\frac{809}{535} F_2F_9 + \frac{1344}{535} F_4F_7,$$
$$a^{12} = -\frac{88}{35} F_2F_{10} + \frac{123}{35} F_4F_8,$$
$$a^{13} = -\frac{118208}{33075} F_2F_{11} + \frac{151283}{33075} F_4F_9.$$

Furthermore, if $14 \leq n \leq 100$, there exist rational constants $c_{n,j}$ depending only on n and j such that

$$a^n = \sum_{1 \leq j \leq \frac{n}{6}+1} c_{n,j} F_{2j} F_{n-2j}. \tag{3–1}$$

Proof: The results for a^n for $1 \leq n \leq 5$ are classical. For example, S. Ramanujan knew the results for $1 \leq n \leq 4$ [Andrews and Berndt 05, pp. 402–403], and the results for $3 \leq n \leq 5$ were given by H. Petersson [Petersson 82, p. 90]. For a more recent proof of the results for $1 \leq n \leq 5$ and generalizations, see [Chan and Cooper 08]. The results for $6 \leq n \leq 9$ are trivial consequences of the results for $n = 2$, 4, and 5.

We shall give a detailed proof for the case $n = 16$. The proofs for other values of n in the range $10 \leq n \leq 100$ are similar. The identity we seek to establish is

$$a^{16} = c_{16,1} F_2F_{14} + c_{16,2} F_4F_{12} + c_{16,3} F_6F_{10}. \tag{3–2}$$

If we expand both sides in powers of q and equate coefficients of q^i for $0 \leq i \leq 2$, we obtain a system of linear equations whose solution is

$$c_{16,1} = -\frac{7471748}{1142505}, \quad c_{16,2} = \frac{1261075}{228501}, \quad c_{16,3} = \frac{6578}{3255}. \tag{3–3}$$

Next, let

$$c(q) = 3q^{1/3} \prod_{j=1}^{\infty} \frac{(1-q^{3j})^3}{(1-q^j)}$$

and define

$$x = x(q) = \frac{c^3(q)}{a^3(q)}.$$

It is known (for example, see [Cooper 06, Lemma 13.8 and Theorem 13.11]) that

$$\frac{F_{m,3}(q)}{a^m(q)} = p_m(x), \tag{3–4}$$

where $p_m(x)$ is a polynomial in x of degree at most $m/3$. (It is also true that $p_m(1-x) = p_m(x)$, but this is not crucial to the proof.) The polynomials $p_m(x)$ may be computed explicitly by setting

$$p_m(x) = \sum_{0 \le j \le m/3} b_{m,j} x^j,$$

and equating the first $1 + \lfloor m/3 \rfloor$ terms in the q-expansion of (3–4) to determine the numbers $b_{m,j}$. For $1 \le m \le 15$, the results are

$$\begin{aligned}
&p_1(x) = p_2(x) = p_3(x) = p_4(x) = p_5(x) = 1,\\
&p_6(x) = 1 - \frac{8}{13}y, \quad p_7(x) = 1 - \frac{8}{7}y,\\
&p_8(x) = 1 - \frac{64}{41}y, \quad p_9(x) = 1 - \frac{1536}{809}y,\\
&p_{10}(x) = 1 - \frac{24}{11}, \quad p_{11}(x) = 1 - \frac{4488}{1847}y,\\
&p_{12}(x) = 1 - \frac{134272}{50443}y + \frac{3200}{50443}y^2,\\
&p_{13}(x) = 1 - \frac{160544}{55601}y + \frac{12160}{55601}y^2,\\
&p_{14}(x) = 1 - \frac{3400}{1093}y + \frac{512}{1093}y^2,\\
&p_{15}(x) = 1 - \frac{23070808}{6921461}y + \frac{5568256}{6921461}y^2,
\end{aligned}$$

where $y = x(1-x)$.

Observe that for the values of $c_{16,j}$ in (3–3) we have

$$c_{16,1}p_2(x)p_{14}(x) + c_{16,2}p_4(x)p_{12}(x) + c_{16,3}p_6(x)p_{10}(x) \equiv 1. \tag{3–5}$$

Multiplying (3–5) by a^{16} and using (3–4), we obtain (3–2). This completes the proof of Theorem 3.1 in the case $n = 16$. Proofs for the other values $10 \le n \le 100$ may be given similarly, and the calculations have been carried out using a computer. □

Conjecture 3.2. *If $n > 100$, there exist rational constants $c_{n,j}$ depending only on n and j such that the identity (3–1) holds.*

Remark 3.3.

The identity (3–1) provides a formula for the number of representations of a positive integer by the form $\sum_{j=1}^{n} x_j^2 + x_j y_j + y_j^2$ in terms of convolutions of divisor sums.

4. DISCUSSION

The procedure described in the proof of Theorem 3.1 has been automated to carry out similar calculations for (in principle) any value of n. The resulting computer program was executed for $14 \le n \le 100$. For each value of n, the coefficients $c_{n,j}$ were calculated for $1 \le j \le n/6 + 1$, and the polynomial

$$\sum_{1 \le j \le \frac{n}{6}+1} c_{n,j} p_{2j}(x) p_{n-2j}(x)$$

was computed. In each case, the polynomial simplified identically to 1. The numbers $c_{n,j}$ are enormous. For example, $c_{100,1}$ is a rational number whose numerator and denominator contain 650 and 639 digits, respectively. Furthermore, the unsimplified polynomial

$$\sum_{j=1}^{17} c_{100,j} p_{2j}(x) p_{100-2j}(x)$$

requires 63 screens of output to display, yet (according to the computer algebra program) simplifies identically to 1.

In [Chan and Cooper 08] it was shown that $a^n - F_n$ is a sum of $\lfloor n/6 \rfloor$ linearly independent cusp forms. Moreover, for $14 \le n \le 100$, computer calculations show that the products $F_{2j}F_{2n-2j}$, $1 \le j \le \frac{n}{6} + 1$, are linearly independent and span the same space as the space spanned by F_n and the $\lfloor n/6 \rfloor$ cusp forms. This is why there are $\lfloor \frac{n}{6} \rfloor + 1$ terms in the sum in (3–1). All of a^n, F_n, $F_{2j}F_{n-2j}$ and the cusp forms are modular forms of weight n for the modular group $\Gamma_0(3)$; the precise details are given in [Chan and Cooper 08, Section 2].

Undoubtedly, an approach that is different from the method in Section 3 will be required to prove Conjecture 3.2. Until then, we believe that we have presented significant computational evidence in support of the conjecture.

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Shaun Cooper, Institute of Information and Mathematical Sciences, Massey University – Albany, Private Bag 102904, North Shore Mail Centre, Auckland, New Zealand (s.cooper@massey.ac.nz)

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