Subrings of the Asymptotic Hecke Algebra of Type H_4

Dean Alvis

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The structure of the subring $J^{\Gamma\cap\Gamma^{-1}}$ of the asymptotic Hecke algebra is described for Γ a left cell of the Coxeter group of type H_4. A small set of generators over $\mathbb{Z}$ is produced. The subalgebras spanned by a subset of the basis $\{t_x\}_{x\in\Gamma\cap\Gamma^{-1}}$ are determined.

1. INTRODUCTION

Let W be a finite Coxeter group with set of distinguished generators S, length function $\ell : w \mapsto \ell(w)$, and Bruhat order $\leq$. Let J be the *asymptotic Hecke algebra* of W, as defined in [Lusztig 87, Section 2] (see also [Lusztig 87, Section 18]). As an additive group, J is a free abelian group with basis $(t_w)_{w\in W}$ indexed by W. The multiplication operation of J is given by

$$t_x t_y = \sum_{z\in W} \gamma_{x,y,z^{-1}} t_z, \tag{1–1}$$

where the structure constants $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ are described in the next section. It is known that J is an associative ring with identity. Moreover, if Γ is a left cell of W, then

$$J^{\Gamma\cap\Gamma^{-1}} = \sum_{x\in\Gamma\cap\Gamma^{-1}} \mathbb{Z}\, t_x$$

is a $\mathbb{Z}$-subalgebra of J. We denote this ring by $J(\Gamma)$.

Fokko Du Cloux has computed $\gamma_{x,y,z^{-1}}$ for all $x, y, z \in W = W(H_4)$. In fact, Du Cloux has determined all of the coefficients, not just the leading coefficients, of the structure constants $h_{x,y,z}$ of the Hecke algebra; see [Du Cloux 06].

By Du Cloux's calculations, the coefficients of the constants $h_{x,y,z}$ are nonnegative integers. Since the same is known for the Kazhdan–Lusztig polynomials, results of [Lusztig 03, Chapter 15] show that all of the conjectures P1–P15 of [Lusztig 03, Chapter 14] hold in type H_4. In particular, each left cell Γ of $W(H_4)$ contains a unique element of $\mathcal{D}$, the set of distinguished involutions. Moreover, if $e \in \Gamma\cap\mathcal{D}$, then t_e is the identity element of $J(\Gamma)$.

The purpose of this investigation is to explicitly describe the algebras $J(\Gamma)$ when Γ is a left cell of $W =$

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$W(H_4)$. This is accomplished by determining the structure constants $\gamma_{x,y,z^{-1}}$ for $x, y, z \in \Gamma \cap \Gamma^{-1}$. These constants were calculated by computer using the algorithm described in Section 2, which differs from that used by Du Cloux.

There are 206 left cells altogether in type H_4 (see Section 3). For Γ a left cell of W, the associated W-graph gives rise to a corresponding $\mathbb{Q}W$-module as in [Kazhdan and Lusztig 79], which will be denoted by $M(\Gamma)$.

We say that a bijection $\pi : \Gamma_1 \cap \Gamma_1^{-1} \to \Gamma_2 \cap \Gamma_2^{-1}$ is a *permutation isomorphism* from $J(\Gamma_1)$ onto $J(\Gamma_2)$ if

$$\gamma_{x,y,z^{-1}} = \gamma_{\pi(x),\pi(y),\pi(z)^{-1}}$$

for all $x, y, z \in \Gamma_1 \cap \Gamma_1^{-1}$. A computer search of the matrices of structure constants reveals the following theorem. (The author knows of no a priori proof of this result.)

Theorem 1.1. *Suppose (W, S) is of type H_4 and Γ_1, Γ_2 are left cells of W such that the corresponding modules $M(\Gamma_1)$, $M(\Gamma_2)$ are isomorphic. Then there is a unique permutation isomorphism from $J(\Gamma_1)$ onto $J(\Gamma_2)$.*

Because of this result, it will be sufficient to describe the structure constants $\gamma_{x,y,z^{-1}}$ as $M(\Gamma)$ ranges over the isomorphism classes of left cell modules, greatly reducing the number of cases to be considered. The isomorphisms $\pi : \Gamma_1 \cap \Gamma_1^{-1} \to \Gamma_2 \cap \Gamma_2^{-1}$ are given in a data file available for download (see Section 9).

For a left cell Γ not contained in the largest two-sided cell of $W = W(H_4)$, we have $|\Gamma \cap \Gamma^{-1}| = 1$ or 2, and hence $J(\Gamma)$ is easily described (Section 4). Thus the interesting cases are those for which Γ is a subset of the largest two-sided cell.

There are three isomorphism classes of modules $M(\Gamma)$ for such Γ (see Section 3). Sections 5 through 7 describe the rings $J(\Gamma)$ for these three cases. A CAS (computer algebra system) program was used to find a set of generators over $\mathbb{Z}$, the characteristic polynomials for the left multiplication operators $(t_x)_L : J(\Gamma) \to J(\Gamma)$, and the subsets of $\{t_x\}_{x\in\Gamma\cap\Gamma^{-1}}$ that span subalgebras of $J(\Gamma)$.

2. THE COMPUTATION OF THE STRUCTURE CONSTANTS

We continue to assume that W is a finite Coxeter group with distinguished generators S. Let $\mathcal{H}$ be the corresponding Hecke algebra over $A = \mathbb{Z}[q^{1/2}, q^{-1/2}]$, q an indeterminate, with standard basis $(T_w)_{w\in W}$ satisfying

$$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w, \\ T_{sw} + \left(q^{1/2} - q^{-1/2}\right) T_w & \text{if } sw < w, \end{cases} \qquad (2\text{–}1)$$

for $s \in S$, $w \in W$. (This notation of [Lusztig 03] differs slightly from that in [Kazhdan and Lusztig 79].) The semilinear involution $a \mapsto \overline{a}$ of $\mathcal{H}$ is given by $\overline{q^{1/2}} = q^{-1/2}$, $\overline{T_w} = T_{w^{-1}}^{-1}$. The basis $(c_w)_{w\in W}$ for $\mathcal{H}$ (denoted by $(C'_w)_{w\in W}$ in [Kazhdan and Lusztig 79]) satisfies

$$c_w = \sum_{y\in W} p_{y,w} T_y,$$

where $p_{y,w} \in q^{-1/2}\mathbb{Z}[q^{-1/2}]$ when $y < w$, $p_{w,w} = 1$, $p_{y,w} = 0$ when $y \not\leq w$, and $\overline{c_w} = c_w$.

For $x, y, z \in W$, define $f'_{x,y,z} \in A$ by

$$T_x T_y = \sum_{z\in W} f'_{x,y,z} c_z.$$

Then $\gamma_{x,y,z^{-1}}$ is determined by

$$f'_{x,y,z} = \gamma_{x,y,z^{-1}} q^{a(z)/2} + \text{lower-degree terms} \qquad (2\text{–}2)$$

[Lusztig 03, 13.6(d)], where $a(z)$ is a nonnegative integer depending only on the two-sided cell containing z (see below).

Now, if $f_{x,y,z} \in A$ are given by

$$T_x T_y = \sum_{z\in W} f_{x,y,z} T_z,$$

then

$$f'_{x,y,z} = \sum_{w\in W} p'_{z,w} f_{x,y,w}$$

by [Lusztig 03, 13.1(b)], where $\left[p'_{z,w}\right]$ is the inverse matrix of $[p_{z,w}]$. Further, since W is finite, we have

$$p'_{z,w} = \varepsilon_z \varepsilon_w p_{w_0 w, w_0 z}$$

by [Lusztig 03, 11.4], where w_0 is the longest element of W and $\varepsilon_x = (-1)^{\ell(x)}$. Put $q_x^{1/2} = (q^{1/2})^{\ell(x)}$, and let

$$P_{x,y} = q_x^{-1/2} q_y^{1/2} p_{x,y},$$

so $P_{x,y}$ is the Kazhdan–Lusztig polynomial for x, y. Define

$$F_{x,y,z} = q_x^{1/2} q_y^{1/2} q_z^{-1/2} f_{x,y,z}.$$

Then

$$\begin{aligned} f'_{x,y,z} &= \sum_{w\in W} \left(\varepsilon_z \varepsilon_w q_w^{-1/2} q_z^{1/2} P_{w_0 w, w_0 z}\right) \\ &\qquad \times \left(q_x^{-1/2} q_y^{-1/2} q_w^{1/2} F_{x,y,w}\right) \\ &= q_x^{-1/2} q_y^{-1/2} q_z^{1/2} \sum_{w\in W} \varepsilon_z \varepsilon_w P_{w_0 w, w_0 z} F_{x,y,w}. \end{aligned}$$

Therefore formula (2–2) is equivalent to

$$\sum_{w\in W} \varepsilon_z\varepsilon_w P_{w_0w,w_0z}\, F_{x,y,w} = \gamma_{x,y,z^{-1}} q^{(a(z)-\ell(x)-\ell(y)+\ell(z))/2} + \text{lower-degree terms.} \tag{2–3}$$

To find the structure constants $\gamma_{x,y,z^{-1}}$ for $x, y, z \in \Gamma\cap\Gamma^{-1}$ in type H_4, the polynomials $F_{x,y,w}$ were evaluated by computer for a fixed $x \in \Gamma\cap\Gamma^{-1}$ and all $y \in \Gamma\cap\Gamma^{-1}$, $w \in W$, using a straightforward calculation based on (2–1). The leading term of the sum on the left side of (2–3) was then found for $y, z \in \Gamma\cap\Gamma^{-1}$, using the Kazhdan–Lusztig polynomials that were computed in the course of determining the left cells in [Alvis 87].

Varying x over $\Gamma\cap\Gamma^{-1}$ produced the value of the a-function on Γ: if $\delta(x, y, z)$ denotes the degree of the left side of (2–3) and

$$\mu = \max\{\, 2\delta(x, y, z) - (\ell(x) + \ell(y) - \ell(z)) \mid x, y, z \in \Gamma\cap\Gamma^{-1} \,\},$$

then $a(x) = \mu$ for $x \in \Gamma\cap\Gamma^{-1}$. Once the value $a(x)$ was found, the structure constants $\gamma_{x,y,z^{-1}}$ were then determined using (2–3). The results of these calculations are summarized in Sections 4 through 7.

3. THE LEFT CELLS IN TYPE H_4

For the remainder of this paper, (W, S) is of type H_4. In order to establish notation used in later sections, we briefly review some results on the left and two-sided cells of W.

Order the generators $S = \{a, b, c, d\}$ so that $(ab)^3 = (bc)^3 = (cd)^5 = (ac)^2 = (ad)^2 = (bd)^2 = 1$. For $w \in W$, put $R(w) = \{s \in S | ws < w\}$. For $I \subseteq S$, define $R_I = \{w \in W | R(w) = I\}$. Also, for $X \subseteq W$, put $X^* = w_0X$, where w_0 is the longest element of W.

The left cells of W will be denoted by A_i $(1 \le i \le 24)$, B_i, B_i^* $(1 \le i \le 36)$, C_i, C_i^* $(1 \le i \le 25)$, D_i, D_i^* $(1 \le i \le 16)$, E_i, E_i^* $(1 \le i \le 9)$, F_i, F_i^* $(1 \le i \le 4)$, $G_1 = \{1\}$, $G_1^* = \{w_0\}$. Each of these left cells has been explicitly described in terms of the subsets R_I, $I \subseteq S$. (See the discussion preceding [Alvis 87, Theorem 3.1].) For example,

$$\begin{aligned} A_1 &= R_{abc} \cap R_d dcbdcd, \\ A_9 &= (R_{acd} \cap R_c cdcb) \setminus R_{abc} bdc, \\ A_{19} &= R_{cd} \setminus (R_c cdcb \cup R_{abc} badc \cup R_{abd} acbd)\,. \end{aligned}$$

Expressions for the other left cells will not be repeated here. (One typographic error in [Alvis 87] should be noted: A_{12} is equal to $A_{11}d$, not $A_{10}d$.) The two-sided cells of W are $A = A^* = \cup A_i$, $B = \cup B_i$, B^*, $C = \cup C_i$, C^*, $D = \cup D_i$, D^*, $E = \cup E_i$, $F = \cup F_i$, F^*, $G = G_1$, and G^*.

The characters of the left cell modules $M(\Gamma)$ are also explicitly known. In particular, if Γ_1, Γ_2 are two left cells of W and $M(\Gamma_1)$ is isomorphic to $M(\Gamma_2)$, then Γ_1 and Γ_2 are contained in the same two-sided cell of W. The converse also holds unless $\Gamma_1, \Gamma_2 \subseteq A$, in which case there are three isomorphism classes of left cell modules represented by $M(A_1)$, $M(A_9)$, and $M(A_{19})$ (see the proof of Proposition 3.5 in [Alvis 87]).

4. SMALL LEFT CELLS

Suppose Γ is a left cell contained in one of the two-sided cells B, B^*, C, C^*, G, G^*. Then $|\Gamma\cap\Gamma^{-1}| = 1$. In this case, $J(\Gamma) = \mathbb{Z}t_e$, where $\Gamma\cap\Gamma^{-1} = \{e\}$ and $t_e^2 = t_e$.

Now suppose Γ is contained in one of the two-sided cells D, D^*, E, E^*, F, F^*. In this case, $|\Gamma\cap\Gamma^{-1}| = 2$ and $\Gamma\cap\Gamma^{-1} = \{e, s\}$, where e is the distinguished involution and s is the other involution in Γ.

Then

$$J(\Gamma) = \mathbb{Z}t_e \oplus \mathbb{Z}t_s,$$

with identity element t_e. Moreover, the calculations described in Section 2 show that

$$t_s^2 = \begin{cases} t_e & \text{if } \Gamma \subseteq D \cup D^*, \\ t_e + t_s & \text{if } \Gamma \subseteq E \cup E^* \cup F \cup F^*. \end{cases}$$

From these results and the structure of the modules $M(\Gamma)$ given in [Alvis 87], Theorem 1.1 holds for left cells Γ not contained in A.

5. THE CASE $\Gamma = A_1$

It remains only to consider the left cells Γ such that $\Gamma \subseteq A$. Suppose Γ is the left cell A_1, so $|\Gamma| = 326$ and $\Gamma\cap\Gamma^{-1} = 14$ [Alvis 87]. The elements $x_1, \dots, x_{14}$ of $\Gamma\cap\Gamma^{-1}$ are indexed according to the list of reduced expressions given in Table 1.

Put $m_{ij}^k = \gamma_{x_k,x_i,x_j^{-1}}$ for $1 \le i, j, k \le 14$, and define $M_k = \left[m_{ij}^k\right]_{1\le i,j\le 14}$. The structure constants $\gamma_{x,y,z^{-1}}$ are described below by giving the matrices $M_1, \dots, M_{14}$. To save space, only a set of generators is given explicitly, and the other matrices are then described in terms of those generators. (Data files containing all of the matrices M_k are available for download: see Section 9.)

k	x_k
1	*abcaba*
2	*abcdabcaba*
3	*abcdabcdabcaba*
4	*abcdabcdabcdabcaba*
5	*bcdabcdabcdbcdabcaba*
6	*abcdabcdabcdabcdabcaba*
7	*abcdbcdabcdabcdbcdabcaba*
8	*bcdabcdabcdbcdabcdabcaba*
9	*abcdabcdabcdabcdabcdabcaba*
10	*abcdbcdabcdabcdbcdabcdabcaba*
11	*abcdabcdabcdabcdabcdabcdabcaba*
12	*abcdabcdabcdabcdabcdabcdabcdabcaba*
13	*abcdabcdabcdabcdabcdabcdabcdabcdabcaba*
14	*abcdabcdabcdabcdabcdabcdabcdabcdabcdabcaba*

TABLE 1. The elements of $\Gamma \cap \Gamma^{-1}$, $\Gamma = A_1$.

Note that M_k is the transpose of the left multiplication operator $(t_{x_k})_L : J(\Gamma) \to J(\Gamma)$ by (1–1):

$$t_{x_k} t_{x_i} = \sum_j m_{ij}^k t_{x_j}.$$

Thus the mapping $t_{x_k} \mapsto M_k$ extends to an anti-isomorphism from $J(\Gamma)$ onto the subring of $\mathrm{gl}(14, \mathbb{Z})$ generated by $M_1, M_2, \ldots, M_{14}$.

If F denotes the free ring with identity generated by indeterminates $\tau_1, \ldots, \tau_{14}$, then $J(\Gamma) \approx F/I$, where I is the ideal generated by

$$\Big\{ \tau_k \tau_i - \sum_j m_{ij}^k \tau_j | 1 \le i, j, k \le 14 \Big\}.$$

(Similar observations hold for any left cell, not just A_1.)

The calculations described in Section 2 yield $M_1 = I$, the identity matrix, so x_1 is the distinguished involution of A_1. Also,

$$M_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},$$

$$M_4 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 3 & 4 & 2 & 4 & 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & 4 & 2 & 4 & 2 & 2 & 4 & 1 & 3 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 2 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 3 & 1 & 4 & 2 & 2 & 4 & 2 & 4 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 3 & 1 & 1 & 4 & 2 & 4 & 3 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 3 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},$$

$$M_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 3 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & 4 & 2 & 4 & 2 & 2 & 4 & 1 & 3 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 2 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 3 & 4 & 2 & 5 & 2 & 2 & 5 & 2 & 4 & 2 & 1 & 0 \\
0 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 0 \\
0 & 1 & 2 & 4 & 2 & 5 & 2 & 2 & 5 & 2 & 4 & 3 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 1 & 3 & 1 & 4 & 2 & 2 & 4 & 2 & 4 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 3 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Moreover,

$$\begin{aligned}
M_3 &= -I + M_2^2, \\
M_5 &= I - M_4 - M_6 - M_2^2 + M_2 M_4, \\
M_7 &= I - M_2 - 2M_4 - M_2^2 - M_2 M_4 - M_6 M_2 \\
&\quad + M_2^2 M_4, \\
M_8 &= I - M_2 - 2M_4 - M_2^2 - M_2 M_4 - M_2 M_6 \\
&\quad + M_2^2 M_4, \\
M_9 &= -I + M_2 + M_4 - M_6 + M_2^2 + M_2 M_4 + M_2 M_6 \\
&\quad + M_6 M_2 - M_2^2 M_4 \\
M_{10} &= I - 2M_2 - 3M_4 + 2M_6 - 2M_2^2 - 4M_2 M_4 \\
&\quad - 2M_2 M_6 + 2M_4^2 + M_4 M_6 - 2M_6 M_2 \\
&\quad + 3M_2^2 M_4 - M_2 M_4^2, \\
M_{11} &= 2M_4 - M_6 + M_2 M_4 - M_4^2 - M_4 M_6 - 2M_2^2 M_4 \\
&\quad + M_2 M_4^2, \\
M_{12} &= -I + M_2 - M_4 + 3M_6 + 2M_2^2 - M_2 M_4 + M_4^2 \\
&\quad + 3M_4 M_6 + 3M_2^2 M_4 - 2M_2 M_4^2, \\
M_{13} &= I + M_4 - 4M_6 - 2M_2^2 + 3M_2 M_4 + M_2 M_6 - 2M_4^2 \\
&\quad - 4M_4 M_6 + M_6 M_2 + M_6^2 - 3M_2^2 M_4 + 2M_2 M_4^2, \\
M_{14} &= 2I - M_2 + M_4 - 6M_6 - 2M_2^2 + 4M_2 M_4 + M_2 M_6 \\
&\quad - 2M_4^2 - 6M_4 M_6 + M_6 M_2 - 2M_6^2 - 5M_2^2 M_4 \\
&\quad + 3M_2 M_4^2 + M_2 M_6^2.
\end{aligned}$$

This completes the description of the $J(\Gamma)$ when $\Gamma = A_1$. The characteristic polynomials of the matrices M_k appear in Table 2. After similar calculations were carried out for the left cells $A_2, \ldots, A_8$, a computer search established the corresponding cases of Theorem 1.1. In addition, we have the following.

Theorem 5.1. *Suppose $\Gamma = A_1$ and X is a nonempty subset of $\Gamma \cap \Gamma^{-1}$. Then $\sum_{x \in X} \mathbb{Z} t_x$ is a subring (with 1) of $J(\Gamma)$ if and only if X is one of the sets $\{x_1\}$, $\{x_1, x_{14}\}$, $\Gamma \cap \Gamma^{-1}$.*

6. THE CASE $\Gamma = A_9$

In this section Γ is the left cell A_9 of W, so $|\Gamma| = 392$ and $\Gamma \cap \Gamma^{-1} = 18$. The notation of the previous section will be adapted to this case. The elements $x_1, \ldots, x_{18}$ of $\Gamma \cap \Gamma^{-1}$ are indexed as in Table 3.

k	$\det(uI - M_k)$
1	$(-1+u)^{14}$
2	$(-2+u)(-1+u)u(1+u)(-2+u^2)^2(1-3u+u^2)(-4-2u+u^2)(-1+u+u^2)$
3	$(-3+u)(-1+u)^4u^2(1+u)(5-10u+u^2)(-5-5u+u^2)(-1-u+u^2)$
4	$(-1+u)^3(1+u)^5(1-18u+u^2)(1-7u+u^2)(1+3u+u^2)$
5	$u^2(1+u)^2(-2+u^2)^2(-4-8u+u^2)(5-5u+u^2)(1-3u+u^2)$
6	$(-1+u)u(1+u)(2+u)(-2+u^2)^2(-4-22u+u^2)(1-3u+u^2)(-1-u+u^2)$
7	$u^4(1+u)^2(3+u^2)^2(5-10u+u^2)(-4+2u+u^2)$
8	$u^4(1+u)^2(3+u^2)^2(5-10u+u^2)(-4+2u+u^2)$
9	$(-1+u)^2u(2+u)(-2+u^2)^2(-4-22u+u^2)(-1-u+u^2)(1+3u+u^2)$
10	$(-1+u)u^2(1+u)(-2+u^2)^2(-4-8u+u^2)(5-5u+u^2)(1+3u+u^2)$
11	$(-1+u)^6(1+u)^2(1-18u+u^2)(1+3u+u^2)(1+7u+u^2)$
12	$(-3+u)u^2(1+u)^5(5-10u+u^2)(-1-u+u^2)(-5+5u+u^2)$
13	$(-2+u)(-1+u)^2u(-2+u^2)^2(-4-2u+u^2)(-1+u+u^2)(1+3u+u^2)$
14	$(-1+u)^7(1+u)^7$

TABLE 2. Characteristic polynomials for $\Gamma = A_1$.

k	x_k
1	*cdcdca*
2	*dcdabcdbcdca*
3	*cdcdabcdabcdca*
4	*dcdabcdbcdbcdbca*
5	*cdcdabcdabcdabcdca*
6	*cdcdbcdabcdabcdbcdca*
7	*dcdabcdbcdabcdabcdca*
8	*cdcdabcdabcdabcdabcdca*
9	*dcdabcdbcdabcdabcdabcdca*
10	*cdabcdbcdbcdabcdabcdbcdbca*
11	*cdcdabcdabcdabcdabcdabcdca*
12	*cdcdabcdabcdbcdabcdabcdbcdca*
13	*dcdabcdbcdabcdabcdabcdabcdca*
14	*cdcdabcdabcdabcdabcdabcdabcdca*
15	*cdcdabcdabcdabcdabcdabcdabcdabcdca*
16	*dcdabcdbcdabcdabcdbcdabcdabcdbcdca*
17	*dcdabcdbcdabcdabcdabcdabcdabcdabcdca*
18	*cdcdabcdabcdabcdabcdabcdabcdabcdabcdabcdca*

TABLE 3. The elements of $\Gamma \cap \Gamma^{-1}$, $\Gamma = A_9$.

Let $M_k = \left[m^k_{ij}\right]$ be the 18×18 matrix with (i,j) entry $m^k_{ij} = \gamma_{x_k, x_i, x_j^{-1}}$. Then $M_1 = I$,

$$M_2 = \begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&2&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&1&1&0&1&0&1&0&0&0&0&0\\
0&0&0&0&1&1&0&1&2&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&1&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&1&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&1&1&2&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix},$$

$$M_3 = \begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&2&1&1&1&1&0&1&0&0&0&0&0&0&0\\
0&1&0&0&1&1&0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&1&1&1&0&1&0&0&1&0&0&0&0\\
0&0&0&0&1&1&1&1&2&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&1&0&1&0&1&1&1&0&0&0\\
0&0&0&0&0&1&0&0&1&0&0&1&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&1&1&2&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0
\end{pmatrix},$$

and

$$M_4 = \begin{pmatrix}
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}$$

by the calculations described in Section 2.

Further,

$$\begin{aligned}
M_5 &= -I - M_2 - M_3 - M_4 + M_2^2,\\
M_6 &= I + M_4 - M_2^2 + M_2M_3,\\
M_7 &= -M_2 + M_2M_4,\\
M_8 &= M_4 - M_2^2 + M_3^2,\\
M_9 &= I + M_2 + M_3 - M_2^2 - 2M_2M_3 - 2M_2M_4 - M_3^2\\
&\quad + M_2^3,\\
M_{10} &= -I - M_4 + M_4^2,\\
M_{11} &= I + 2M_2 + M_2M_4 - M_3^2 - M_2^3 + M_2^2M_3,\\
M_{12} &= -2I - 2M_2 - M_4 + 2M_2^2 + M_2M_3 + 2M_2M_4\\
&\quad + M_3^2 - M_2^3 - M_2^2M_3 + M_2M_3^2,\\
M_{13} &= -2M_2M_4 + M_2M_4^2,
\end{aligned}$$

k	$\det(uI - M_k)$
1	$(-1+u)^{18}$
2	$(-1+u)^2u^2(-5-5u+u^2)(-1-4u+u^2)(-5-u+u^2)^2(-1-u+u^2)^2(-1+u+u^2)$
3	$(-1+u)(1+u)^3(1-7u+u^2)(-1-4u+u^2)(1-3u+u^2)^2(-1-u+u^2)^2(-1+u+u^2)$
4	$(1+u)^2(1-3u+u^2)^4(-1-u+u^2)^4$
5	$(-1+u)^7(1+u)^5(1-18u+u^2)(1-7u+u^2)(1+3u+u^2)$
6	$u^2(1+u)^2(1+u^2)^2(5-10u+u^2)(1-3u+u^2)(-1-u+u^2)(5+2u+u^2)^2$
7	$u^2(1+u)^2(1+u^2)^2(5-10u+u^2)(1-3u+u^2)(-1-u+u^2)(5+2u+u^2)^2$
8	$u^2(-5-15u+u^2)(1-3u+u^2)(-1-u+u^2)^4(-5+u+u^2)^2$
9	$(-1+u)u^8(1+u)(-4-22u+u^2)(-4+2u+u^2)^3$
10	$(-1+u)^2u^8(-4-2u+u^2)^4$
11	$u^2(-5-15u+u^2)(1-3u+u^2)(-5+u+u^2)^2(-1+u+u^2)^4$
12	$(-1+u)^2u^2(1+u^2)^2(5-10u+u^2)(-1-u+u^2)(5+2u+u^2)^2(1+3u+u^2)$
13	$(-1+u)^2u^2(1+u^2)^2(5-10u+u^2)(-1-u+u^2)(5+2u+u^2)^2(1+3u+u^2)$
14	$(-1+u)^9(1+u)^3(1-18u+u^2)(1+3u+u^2)(1+7u+u^2)$
15	$(-1+u)(1+u)^3(1-7u+u^2)(1-3u+u^2)^2(-1-u+u^2)(-1+u+u^2)^2(-1+4u+u^2)$
16	$(1+u)^2(1-3u+u^2)^4(-1+u+u^2)^4$
17	$(-1+u)^2u^2(-5-5u+u^2)(-5-u+u^2)^2(-1-u+u^2)(-1+u+u^2)^2(-1+4u+u^2)$
18	$(-1+u)^{10}(1+u)^8$

TABLE 4. Characteristic polynomials for $\Gamma = A_9$.

$$\begin{aligned}
M_{14} &= -2I - 2M_2 - 2M_3 - M_4 + 2M_2^2 + 2M_2M_3 + M_3^2 \\
&\quad - 2M_2^2M_3 + M_3^3, \\
M_{15} &= 3I + 3M_2 + 5M_3 + 3M_4 - M_2^2 + 4M_2M_3 + 6M_2M_4 \\
&\quad + M_3^2 - M_4^2 - M_2^3 + 2M_2^2M_3 - 3M_2M_3^2 - 2M_2M_4^2 \\
&\quad - 4M_3^3 + M_2^2M_3^2, \\
M_{16} &= I - M_4 - 2M_4^2 + M_4^3, \\
M_{17} &= 5I + 6M_2 + 2M_3 + 4M_4 - 4M_2^2 - 3M_2M_3 + 2M_2M_4 \\
&\quad - M_4^2 + 6M_2^2M_3 - 3M_2M_3^2 - M_2M_4^2 - 2M_3^3 - M_2^2M_3^2 \\
&\quad + M_2M_3^3, \\
M_{18} &= 3M_4 - 3M_4^3 + M_4^4.
\end{aligned}$$

This completes the description of the structure constants for the left cell $\Gamma = A_9$. After the structure constants for $A_{10}, \ldots, A_{18}$ were also computed, the relevant cases of Theorem 1.1 were verified by a computer search. We also have the following.

Theorem 6.1. *Suppose $\Gamma = A_9$ and X is a nonempty subset of $\Gamma \cap \Gamma^{-1}$. Then $\sum_{x\in X} \mathbb{Z}t_x$ is a subring (with 1) of $J(\Gamma)$ if and only if X is one of the sets*

$$\{x_1\}, \quad \{x_1, x_{18}\}, \quad \{x_1, x_4, x_{10}, x_{16}, x_{18}\}, \quad \Gamma \cap \Gamma^{-1}.$$

The characteristic polynomials of the matrices M_k appear in Table 4.

7. THE CASE $\Gamma = A_{19}$

Suppose Γ is the left cell A_{19}. Thus $|\Gamma| = 436$ and $\Gamma \cap \Gamma^{-1} = 24$. A notation similar to that in the previous two sections is used for the elements of $\Gamma \cap \Gamma^{-1}$ and the matrices of structure constants. Table 5 lists the elements $x_1, \ldots, x_{24}$ of $\Gamma \cap \Gamma^{-1}$.

k	x_k
1	*dcdbcdbcdc*
2	*dcdbcdbcdbcdbc*
3	*cdcdbcdabcdabcdc*
4	*cdbcdbcdbcdabcdabcdc*
5	*cdcdbcdabcdabcdabcdc*
6	*cdcdbcdabcdabcdbcdbc*
7	*dcdbcdbcdabcdabcdabcdc*
8	*cdbcdbcdbcdabcdabcdbcdbc*
9	*cdcdbcdabcdabcdabcdabcdc*
10	*cdcdbcdabcdbcdabcdabcdbcdc*
11	*dcdbcdabcdabcdbcdabcdabcdc*
12	*dcdbcdbcdabcdabcdabcdabcdc*
13	*cdcdbcdabcdabcdabcdabcdabcdc*
14	*dcdbcdbcdabcdabcdabcdabcdabcdc*
15	*dcdbcdabcdabcdbcdabcdabcdbcdbc*
16	*cdcdbcdabcdabcdabcdabcdabcdabcdc*
17	*dcdbcdbcdabcdabcdbcdabcdabcdbcdc*
18	*dcdbcdbcdabcdabcdabcdabcdabcdabcdc*
19	*cdcdbcdabcdabcdabcdabcdabcdabcdabcdc*
20	*cdcdbcdabcdbcdabcdabcdbcdabcdabcdabcdc*
21	*dcdbcdbcdabcdabcdabcdabcdabcdabcdabcdc*
22	*cdcdbcdabcdabcdabcdabcdabcdabcdabcdabcdc*
23	*cdcdbcdabcdbcdabcdabcdbcdabcdabcdabcdabcdc*
24	*cdcdbcdabcdabcdabcdabcdabcdabcdabcdabcdabcdc*

TABLE 5. The elements of $\Gamma \cap \Gamma^{-1}$, $\Gamma = A_{19}$.

Let $M_k = \left[m_{ij}^k\right]$ be the 24×24 matrix with (i,j) entry $m_{ij}^k = \gamma_{x_k, x_i, x_j^{-1}}$. Then $M_2 = I$,

$$M_4 = \left(\begin{smallmatrix}
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&1&1&1&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&1&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0
\end{smallmatrix}\right),$$

$$M_8 = \left(\begin{smallmatrix}
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0
\end{smallmatrix}\right),$$

and

$$M_{21} = \left(\begin{smallmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&1&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&1&0&1&1&1&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{smallmatrix}\right).$$

Also,

$$\begin{aligned}
M_1 &= M_4 + M_{21} + 2M_4M_8 - M_4M_8^2,\\
M_3 &= M_4^2 + M_4M_{21} + 2M_4^2M_8 - M_4^2M_8^2,\\
M_5 &= M_8M_4,\\
M_6 &= M_8M_4 + M_8M_{21} + 2M_4^3 - 2M_4^2M_8 - M_4^3M_8\\
&\quad + M_4^2M_8^2,\\
M_7 &= M_4^2,\\
M_9 &= -M_8 - M_4^2 + M_8M_4 + M_4M_8M_4 - M_8^2M_4,\\
M_{10} &= -M_4 - M_8M_4 + M_8^2M_4,\\
M_{11} &= M_4 + M_8M_4 - M_8^2 + M_4^3 - 2M_4^2M_8 - M_8^2M_4\\
&\quad + M_4^4 - M_4^3M_8 - M_4^2M_8M_4 + M_4^2M_8^2,\\
M_{12} &= -2M_8M_4 - M_8M_{21} - M_4^3 + M_4^2M_8 + M_4^3M_8\\
&\quad - M_4^2M_8^2,\\
M_{13} &= -2M_4^2 - M_4M_{21} - M_4^2M_8 + M_4^2M_8^2,\\
M_{14} &= -2M_4 - M_{21} - M_4M_8 + M_4M_8^2,\\
M_{15} &= M_8 + M_4^2 - M_4M_8 - M_8M_4 + M_8^2 - 2M_4^3\\
&\quad + 3M_4^2M_8 - M_4M_8M_4 + M_8^2M_4 - M_4^4 + M_4^3M_8\\
&\quad + 2M_4^2M_8M_4 - 2M_4^2M_8^2,\\
M_{16} &= M_4M_{21},\\
M_{17} &= -I - M_8 + M_8^2,\\
M_{18} &= M_8M_{21},\\
M_{19} &= 2M_4^2 + M_4^2M_8 + M_4M_{21}M_8 - M_4^2M_8^2,\\
M_{20} &= 2M_8M_4 + M_4^3 - M_4^2M_8 + M_4^2M_{21} - M_4M_{21}M_8\\
&\quad - M_4^3M_8 + M_4^2M_8^2,\\
M_{22} &= M_4 + M_8 - M_{21} + M_4^2 - M_4M_{21} - M_4M_8M_4\\
&\quad + M_4M_8M_{21},\\
M_{23} &= I - M_4 + M_{21} + M_4M_8 + M_8M_4 - M_8M_{21}\\
&\quad - M_{21}M_8 + M_4^3 + M_4^2M_8 - M_4^2M_{21} + M_4M_8M_4\\
&\quad - M_4M_8M_{21} - M_4M_{21}M_8 - M_4^4 + M_4^3M_{21},\\
M_{24} &= 2M_4 + M_4M_8 + M_{21}M_8 - M_4M_8^2.
\end{aligned}$$

As before, a computer search was used to verify the cases of Theorem 1.1 corresponding to left cells with $\mathbb{C}W$-modules isomorphic to $M(\Gamma) = M(A_{19})$. Moreover, the following holds.

Theorem 7.1. *Suppose* $\Gamma = A_{19}$ *and* X *is a nonempty subset of* $\Gamma \cap \Gamma^{-1}$*. Then* $\sum_{x \in X} \mathbb{Z}t_x$ *is a subring (with* 1*) of* $J(\Gamma)$ *if and only if* X *is one of sets*

$$\{x_2\}, \quad \{x_1, x_2\}, \quad \{x_2, x_{23}\}, \quad \{x_1, x_2, x_{23}, x_{24}\},$$
$$\{x_2, x_8, x_{17}, x_{22}, x_{23}\}, \quad \Gamma \cap \Gamma^{-1}.$$

Table 6 contains the characteristic polynomials of the matrices M_k.

k	$\det(uI - M_k)$
1	$(-1-u+u^2)^{12}$
2	$(-1+u)^{24}$
3	$(1+u)^4(1-7u+u^2)(-1-4u+u^2)(1-3u+u^2)^4(-1-u+u^2)^2(-1+u+u^2)(1+3u+u^2)$
4	$(-1+u)^2(1+u)^4(1+u^2)^2(-1-4u+u^2)(1-3u+u^2)(-1+u+u^2)(1+3u^2+u^4)^2$
5	$(-1-11u+u^2)(-1-4u+u^2)^3(-1-u+u^2)^3(-1+u+u^2)^5$
6	$(-1+u)^2(1+u)^4(1+u^2)^2(-1-4u+u^2)(1-3u+u^2)(-1+u+u^2)(1+3u^2+u^4)^2$
7	$(-1+u)^6(1+u)^4(1-18u+u^2)(1-7u+u^2)(1-3u+u^2)(1+3u+u^2)^4$
8	$(1+u)^6(1-3u+u^2)^5(-1-u+u^2)^4$
9	$(-2+u)^6u^8(1+u)^2(2+u)^2(-4-8u+u^2)(-1+4u+u^2)^2$
10	$u^8(4-14u+u^2)(-1+u+u^2)(4+2u+u^2)^2(-4+6u+7u^2-u^3+u^4)^2$
11	$(-2+u)^2u^8(2+u)^2(-4-8u+u^2)(4-6u+u^2)^2(1-3u+u^2)(-1+u+u^2)^2$
12	$u^8(4-14u+u^2)(-1+u+u^2)(4+2u+u^2)^2(-4+6u+7u^2-u^3+u^4)^2$
13	$u^8(-4-22u+u^2)(1-3u+u^2)^2(-4+2u+u^2)^4(1+3u+u^2)$
14	$u^8(4-6u+u^2)(-1-u+u^2)(4+2u+u^2)^2(-4-4u-3u^2-u^3+u^4)^2$
15	$u^8(4-6u+u^2)(-1-u+u^2)(4+2u+u^2)^2(-4-4u-3u^2-u^3+u^4)^2$
16	$(-1+u)^8(1+u)^2(1-18u+u^2)(1-3u+u^2)(1+3u+u^2)^4(1+7u+u^2)$
17	$(-1+u)^6u^8(-4-2u+u^2)^5$
18	$(-1-11u+u^2)(-1-4u+u^2)^2(-1-u+u^2)^2(-1+u+u^2)^6(-1+4u+u^2)$
19	$(1+u)^4(1-7u+u^2)(1-3u+u^2)^4(-1-u+u^2)(-1+u+u^2)^2(1+3u+u^2)(-1+4u+u^2)$
20	$(-1+u)^4(1+u)^2(1+u^2)^2(-1-4u+u^2)(-1+u+u^2)(1+3u+u^2)(1+3u^2+u^4)^2$
21	$(-1+u)^4(1+u)^2(1+u^2)^2(-1-4u+u^2)(-1+u+u^2)(1+3u+u^2)(1+3u^2+u^4)^2$
22	$(1+u)^6(1-3u+u^2)^5(-1+u+u^2)^4$
23	$(-1+u)^{16}(1+u)^8$
24	$(-1-u+u^2)^8(-1+u+u^2)^4$

TABLE 6. Characteristic polynomials for $\Gamma = A_{19}$.

8. CONCLUDING REMARKS

The Wedderburn structure of the rings $J(\Gamma)$ can be described in a uniform way if scalars are extended to a splitting field. Put $K = \mathbb{Q}[\sqrt{5}]$. Then K is a splitting field for W and $K(\sqrt{q})$ is a splitting field for $\mathcal{H}$ by [Alvis and Lusztig 82]. For Γ a left cell of W and F a field, put

$$J(\Gamma)_F = F \otimes_{\mathbb{Z}} J(\Gamma).$$

Since the coefficients of the structure constants $h_{x,y,z}$ are nonnegative for H_4 by the calculation of [Du Cloux 06], a result of [Lusztig 03, 21.9] shows that $J(\Gamma)_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} J(\Gamma)$ is semisimple. Thus $J(\Gamma)_K$ is semisimple.

A CAS program was used to compute the dimension of the derived algebra

$$[J(\Gamma)_{\mathbb{Q}}, J(\Gamma)_{\mathbb{Q}}] = \langle rs - sr \mid r, s \in J(\Gamma)_{\mathbb{Q}} \rangle .$$

This dimension is 0 unless $\Gamma \subset A$, and is 3, 6, and 12 if $\Gamma = A_1$, $\Gamma = A_9$, and $\Gamma = A_{19}$, respectively. Another CAS program has verified that the number of central idempotents in $J(\Gamma)_K$ is 11, 12, and 12 if $\Gamma = A_1, \Gamma = A_9$, and $\Gamma = A_{19}$, respectively. From these observations and the structure of the modules $K \otimes_{\mathbb{Q}} M(\Gamma)$ given in [Alvis 87], the following holds.

Theorem 8.1. *Let Γ be a left cell of $W = W(H_4)$. Then*

$$J(\Gamma)_K = K \otimes_{\mathbb{Z}} J(\Gamma)$$

is split semisimple over K, and is isomorphic to the endomorphism algebra of the KW-module $K \otimes_{\mathbb{Q}} M(\Gamma)$.

9. DATA FILES

The following files are available for download.

- http://mypage.iusb.edu/~dalvis/h4data/rtran.txt contains a multiplication table for $W = W(H_4)$, in terms of a fixed indexing $0, \ldots, 14399$ for the elements of W.
- http://mypage.iusb.edu/~dalvis/h4data/basic.txt contains basic information about the elements of W, including lengths, a reduced expression, the sets $L(w)$ and $R(w)$, and inverses.
- http://mypage.iusb.edu/~dalvis/h4data/pyw.txt contains the Kazhdan–Lusztig polynomials P_{yw} for the set of "reduced" pairs $y \leq w$.
- http://mypage.iusb.edu/~dalvis/h4data/lcells.txt contains, for each left cell Γ of W, a list of the elements of W and a description of the associated W-graph, which determines the module $M(\Gamma)$.

- http://mypage.iusb.edu/~dalvis/h4data/gammas.txt contains the matrices $M_k = [m_{ij}^k]$, where $m_{ij}^k = \gamma_{x_k, x_i, x_j^{-1}}$, for a specified ordering of the elements x_k of $\Gamma \cap \Gamma^{-1}$.
- http://mypage.iusb.edu/~dalvis/h4data/isos.txt contains a description of the permutation isomorphisms $\pi : \Gamma_1 \cap \Gamma_1^{-1} \to \Gamma_2 \cap \Gamma_2^{-1}$ of Theorem 1.1.
- http://mypage.iusb.edu/~dalvis/h4data/README.txt contains additional information on the content and format of the files above.

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Dean Alvis, Department of Mathematical Sciences, Indiana University South Bend, South Bend, IN, 46634 (dalvis@iusb.edu)

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