Hypergeometric Forms for Ising-Class Integrals

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CONTENTS

We apply experimental-mathematical principles to analyze the integrals

$$C_{n,k} := \frac{1}{n!}\int_0^\infty \cdots \int_0^\infty \frac{dx_1\, dx_2 \cdots dx_n}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}}.$$

These are generalizations of a previous integral $C_n := C_{n,1}$ relevant to the Ising theory of solid-state physics [Bailey et al. 06]. We find representations of the $C_{n,k}$ in terms of Meijer G-functions and nested Barnes integrals. Our investigations began by computing 500-digit numerical values of $C_{n,k}$ for all integers n, k, where $n \in [2, 12]$ and $k \in [0, 25]$. We found that some $C_{n,k}$ enjoy exact evaluations involving Dirichlet L-functions or the Riemann zeta function. In the process of analyzing hypergeometric representations, we found—experimentally and strikingly—that the $C_{n,k}$ almost certainly satisfy certain interindicial relations including discrete k-recurrences. Using generating functions, differential theory, complex analysis, and Wilf–Zeilberger algorithms we are able to prove some central cases of these relations.

1. BACKGROUND AND NOMENCLATURE

The primary entities on which the present work will focus are the n-dimensional integrals

$$C_{n,k} := \frac{1}{n!}\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{dx_1\, dx_2 \cdots dx_n}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}}. \tag{1–1}$$

These integrals are well defined—in fact absolutely convergent—for any positive integer n and any complex $k \in \mathcal{K}$, where we speak of the open half-plane

$$\mathcal{K} := (z \in \mathcal{C} : \Re(z) > -1).$$

The integrals $C_{n,k}$ can be traced back to the Ising theory of solid-state physics. As summarized in a previous work [Bailey et al. 06], there is interest in giving closed

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forms and growth bounds for n-dimensional *Ising susceptibility integrals*

$$D_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2}{\left(\sum_{j=1}^n (u_j+1/u_j)\right)^2}\frac{du_1}{u_1}\cdots\frac{du_n}{u_n}. \tag{1–2}$$

These D_n appear—with various normalizations—in the standard Ising literature [Orrick et al. 01, Palmer and Tracy 81, Wu et al. 76, Zenine et al. 05a, Zenine et al. 06, Zenine et al. 05b]. The quest for closed forms for Ising susceptibility integrals thus led to a definition in [Bailey et al. 06] of a class of structurally similar integrals, among which is the structure (1–2) but without the permutation product in the integrand, namely

$$C_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j+1/u_j)\right)^2}\frac{du_1}{u_1}\cdots\frac{du_n}{u_n},$$

which, as can be seen via a transformation $u_k \to e^{x_k}$, is the case $C_{n,1}$ of the key definition (1–1).

A brief digression here is worthwhile. There is an even more general class of integrals that likewise admit of analytical promise. We may define, for integer n, complex k, and an n-vector $\vec{r} := (r_1, \ldots, r_n)$ of complex numbers, the entities

$$C_{n,k,\vec{r}} := \frac{1}{n!}\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{\prod_{j=1}^n \cosh(r_j x_j)}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}} dx_1 \cdots dx_n.$$

Absolute convergence of the integral is ensured on the condition that k lie in the translated half-plane $\mathcal{K} + \Re\left(\sum r_j\right)$. Thus we can restrict indices to obtain integrals of our primary interest, e.g.,

$$C_{n,k} := C_{n,k,\vec{0}},$$
$$C_n := C_{n,1} := C_{n,1,\vec{0}}.$$

One reason to contemplate these generalized $C_{n,k,\vec{r}}$ is that they enjoy certain combinatorial relations when cast in so-called Bessel-kernel form, as we shall see later, in Section 7. In principle, one could also allow continuous n, and so a prefactor $1/\Gamma(n+1)$, with a fractional-dimensional integral defined in Bessel-kernel terms; so there could be yet more useful generalization. We will sometimes write $n!$ for the analytic quantity $\Gamma(n+1)$.

An outline of the paper is as follows: In Section 2 we examine hypergeometric and related expressions for our integrals. Then in Section 3 we describe closed forms and series for individual $C_{n,k}$. In Sections 4 and 5 we explore recurrence relations. In Section 6, related continued fractions are given, while in Section 7 we explore further analytic properties of the $C_{n,k}$. Finally, in Section 8 we discuss our extreme-precision numerics before concluding with some open problems.

2. HYPERGEOMETRIC CONNECTIONS

It turns out that the Ising-class integrals $C_{n,k}$ enjoy certain connections with hypergeometric functions and their powerful generalization, the *Meijer G-functions*. Such analysis gives rise to fascinating series representations, new closed forms, and rational relations between certain pairs of integrals. (We refer the reader also to our separate work on the quest for closed Ising forms [Bailey et al. 06].) Not surprisingly, the collection $(C_{n,k} : n \in \mathbb{Z}^+, k \in \mathcal{K})$ provides fertile ground for experimental-mathematical discovery, not to mention clues as to what symbolic behavior might be expected of Ising integrals in general. In addition, we derive some evidently new exact evaluations of Meijer G-functions themselves.

A Bessel-kernel representation we developed in [Bailey et al. 06] likewise generalizes to

$$C_{n,k} = \frac{2^n}{n!}\frac{1}{\Gamma(k+1)}c_{n,k}, \tag{2–1}$$

where we use $\Gamma(k+1) = k!$ to emphasize that k need not be an integer, and where the (lowercase) c definition is

$$c_{n,k} := \int_0^\infty t^k K_0(t)^n dt \tag{2–2}$$

(here K_0 is the modified Bessel function). This representation, as in [Bailey et al. 06], permits us to calculate explicit values to very high precision (our 500-digit values are available online [Bailey et al. 07]). Note that in regard to k-dependence, $c_{n,k}$ differs from $C_{n,k}$ by a prefactor of $\Gamma(k+1)$; this scaling will be convenient later, when we analyze recurrence relations.

It is clear from the definition (1–1) that for fixed integer n, $C_{n,k}$ is monotonic decreasing in real k. The arguments of theorems in [Bailey et al. 06] regarding the original C_n can be augmented to show first that for fixed real $k \geq 1$, the set $(C_{n,k})$ is monotonic decreasing in n, and that for any fixed k we have the large-n asymptote

$$C_{n,k} \sim \frac{1}{\Gamma(k+1)}\frac{2^{k+1+n}}{(k+1)^{n+1}}e^{-(k+1)\gamma},$$

for which our original, canonical case in [Bailey et al. 06] reads $C_n = C_{n,1} \sim_n 2e^{-2\gamma} \approx 0.63047\ldots$. This asymptotic behavior is revealed by extreme-precision numerical

n	C_n
4	0.70119986017642999981651392754834582794624200386529...
16	0.63050394617323726350529565756068741948431621720810...
64	0.63047350337438679648836208816533862535998880860015...
256	0.63047350337438679612204019271087890435458707871273...
1024	0.63047350337438679612204019271087890435458707871273...

TABLE 1. Extreme-precision numerical values for C_n.

values for C_n. Table 1 presents an example of the data downloadable at [Bailey et al. 07], where the asymptote $2e^{-2\gamma}$ is evident:

Another observation on the generalization $C_{n,k,\vec{r}}$ is in order. Some idea of the power of Bessel representation such as (2–1) can be gleaned by the observation that for vector $\vec{r} := (p, p, \ldots, p) = (\bar{p})$ we have again a one-dimensional integral

$$C_{n,k,(\bar{p})} := \frac{2^n}{n!} \frac{1}{\Gamma(k+1)} \int_0^\infty t^k K_p(t)^n dt.$$

It is interesting that for p half an odd integer, the Bessel function is *elementary* and we routinely obtain closed forms. For example, for general complex k we infer

$$C_{4,k,(3/2,3/2,3/2,3/2)} = \frac{2^{1-2k}\pi^2\Gamma(k-5)}{3\Gamma(k+1)} \left(k^4 + 2k^3 - 25k^2 - 10k + 56\right),$$

of which an instance is

$$C_{4,6,(\overline{3/2})} = \frac{103\pi^2}{552960}.$$

Though such cases do not shed much light on our main theme—the $C_{n,k}$ themselves—these tractable cases *do* suggest such notions as analytic continuation (in k, beyond the relevant half-plane) as well as the appearance of polynomials in k.

We shall be analyzing series representations and closed forms for various $C_{n,k}$. To this end, we state some exact integrals based on the *Adamchik algorithm* described in [Adamchik 95]:

$$c_{1,k} = \int_0^\infty t^k K_0(t) dt = 2^{k-1}\Gamma\left(\frac{k+1}{2}\right)^2, \tag{2–3}$$

$$c_{2,k} = \int_0^\infty t^k K_0^2(t) dt = \frac{\sqrt{\pi}\,\Gamma\left(\frac{k+1}{2}\right)^3}{4\Gamma\left(\frac{k}{2}+1\right)}, \tag{2–4}$$

$$c_{3,k} = \int_0^\infty t^k K_0^3(t) dt \tag{2–5}$$

$$= 2^{k-2}\sqrt{\pi}\,G_{3,3}^{3,2}\left(4 \,\middle|\, \begin{matrix} \frac{1-k}{2}, \frac{1-k}{2}, \frac{1}{2} \\ 0, 0, 0 \end{matrix}\right),$$

where the relevant Meijer G-function here is

$$G := \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma^2((k+1)/2 - s)\Gamma^3(s)}{\Gamma(s+1/2)} 4^{-s}\, ds.$$

Finally, we have

$$c_{4,k} = \int_0^\infty t^k K_0^4(t) dt = \frac{1}{8}\pi G_{4,4}^{3,3}\left(1 \,\middle|\, \begin{matrix} 1, 1, 1, \frac{k+2}{2} \\ \frac{k+1}{2}, \frac{k+1}{2}, \frac{k+1}{2}, \frac{1}{2} \end{matrix}\right), \tag{2–6}$$

where in this case the relevant Meijer G-function is

$$G := \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma^3(-s)\Gamma^3((k+1)/2 + s)}{\Gamma(1 + k/2 + s)\Gamma(1/2 - s)}\, ds.$$

In the above cases $n = 3, 4$, the contour $\mathcal{C}$ encompasses all poles of the first Γ form in the numerator, but no other poles, as is consistent with formal definitions of the Meijer G's as given in [Adamchik 95, Roach 97]. In our study, said contour can always be taken as a vertical run, upward, and intersecting the real s-axis at an appropriate place, say $s = -\frac{1}{2}$. It is unknown how to generalize such Meijer formulas beyond the fourth power of the Bessel-K: Once again, as happened in the work [Bailey et al. 06], we encounter a kind of theoretical blockade for $n \geq 5$.

In spite of the blockade for $n \geq 5$ in regard to Meijer-G representations, we shall still be able to represent, in our Section 7, arbitrary $C_{n,k}$ via yet more complicated structures.

3. CLOSED FORMS AND SERIES FOR INDIVIDUAL $C_{n,k}$

3.1 Evaluations of $C_{1,k}$

Immediately from relations (2–1), (2–3) we have

$$C_{1,k} = \frac{2^k\Gamma\left(\frac{k+1}{2}\right)^2}{\Gamma(k+1)}. \tag{3–1}$$

The first few exact evaluations are

$$(C_{1,0}, C_{1,1}, C_{1,2}, C_{1,3}, \ldots) = \left(\pi, 2, \frac{\pi}{2}, \frac{4}{3}, \ldots\right).$$

It is evident that for any $k \geq 1$,

$$C_{1,k} = p_{1,k} + q_{1,k}\pi,$$

where the p, q coefficients are always rational, with q vanishing for odd k and p vanishing for even k. This observation about the character of the p, q is trivial, but as we shall eventually see, such a "$p + qx$" pattern for larger n becomes radically more profound.

3.2 Evaluations of $C_{2,k}$

Next, from relations (2–1), (2–4) we obtain

$$C_{2,k} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{k+1}{2}\right)^3}{2\Gamma\left(\frac{k}{2}+1\right)\Gamma(k+1)}, \tag{3–2}$$

with the first few being

$$(C_{2,0}, C_{2,1}, C_{2,2}, C_{2,3}, \dots) = \left(\frac{\pi^2}{2}, 1, \frac{\pi^2}{32}, \frac{1}{9}, \dots\right).$$

In this $n = 2$ case we have

$$C_{2,k} = p_{2,k} + q_{2,k}\pi^2,$$

with the same vanishing rule on the rational p, q multipliers as for $n = 1$.

3.3 Evaluations of $C_{3,k}$

After all $C_{n,k}$ for $n = 1, 2$ have been resolved as above, the case $n = 3$ on $C_{n,k}$ suddenly becomes nontrivial, yet there are various approaches that yield new insight: at the very least, new closed-form evaluations of the appropriate Meijer-G. Choosing a contour and performing residue calculus (we leave out the intricate details) on the Meijer-G for identity (2–5), one may obtain quite efficient series developments. To summarize, define $\mu := \lfloor (k-2)/2 \rfloor$, a polynomial

$$P_\mu(x) := \prod_{a=0}^{\mu} (x-a)^2,$$

and an alternating harmonic number

$$H_c^{(-1)} := 1 - \frac{1}{2} + \frac{1}{3} - \cdots \pm \frac{1}{c}$$

with $H_0^{(-1)} := 0$. Then, for odd k, the residue calculus yields a linearly convergent series

$$C_{3,k} = \frac{2^k\sqrt{\pi}}{3!k!} \sum_{h=\mu+1}^{\infty} \frac{P_\mu(h)}{4^h} \frac{\Gamma(h+1)}{\Gamma(h+3/2)} \times \left(H_{2h+1}^{(-1)} - \frac{1}{2}\frac{P'_\mu(h)}{P_\mu(h)}\right). \tag{3–3}$$

Similarly, for even k, one obtains

$$C_{3,k} = \frac{2^{k+1}\sqrt{\pi}}{3!k!} \sum_{h=\mu+1}^{\infty} \frac{P_\mu(h)}{4^h} \frac{\Gamma^3(h+1/2)}{\Gamma^3(h+1)} \times \left(4\log 2 - 3H_{2h}^{(-1)} - \frac{1}{2}\frac{P'_\mu(h)}{P_\mu(h)}\right). \tag{3–4}$$

3.3.1 The $C_{3,\text{even}}$ integrals.

Yet another surprise in the world of Ising-class integrals is that the $C_{3,\text{even}}$ seem to be more mysterious than the $C_{3,\text{odd}}$. One way to think of this dichotomy is to observe the way that gamma functions appear in the respective series (3–3), (3–4). One may employ special hypergeometric identities, which we found in Mathematica and reconfirmed in Maple, such as

$$\sum_{h=0}^{\infty} \frac{\Gamma(h+1)}{\Gamma(h+3/2)} \sin^{2h}\theta = \frac{4}{\sqrt{\pi}} \frac{\theta}{\sin(2\theta)}$$

and

$$\sigma_0(\theta) := \sum_{h=0}^{\infty} \frac{\Gamma^3(h+1/2)}{\Gamma^3(h+1)} \sin^{2h}\theta = \frac{4}{\sqrt{\pi}} \mathrm{K}^2\left(\sin\frac{\theta}{2}\right),$$

where in the second identity $\mathrm{K}(k)$ is the (complete) *elliptic integral of the first kind* with modulus k.[1] We may also employ an integral identity

$$4\log 2 - 3H_{2h}^{(-1)} = \int_0^1 \frac{1+3t^{2h}}{1+t}\,dt = \log 2 + 3\int_0^1 \frac{t^{2h}}{1+t}\,dt.$$

Putting this all together for the special case

$$C_{3,0} = \frac{\sqrt{\pi}}{3} \sum_{h=0}^{\infty} \frac{1}{4^h} \frac{\Gamma^3(h+1/2)}{\Gamma^3(h+1)} \left(4\log 2 - 3H_{2h}^{(-1)}\right),$$

we arrive at the peculiar elliptic representation

$$C_{3,0} = \frac{4}{3}\mathrm{K}^2\left(\sin\frac{\pi}{12}\right)\log 2 + 8\int_0^{\pi/6} \frac{\mathrm{K}^2\left(\sin\frac{\theta}{2}\right)\cos\theta}{1+2\sin\theta}\,d\theta. \tag{3–5}$$

Moreover,

$$\mathrm{K}^2\left(\sin\frac{\pi}{12}\right) = \frac{2}{27}\frac{\sqrt{3}\sqrt[3]{2}\pi^4}{\Gamma^6(2/3)} = \frac{\sqrt[3]{2}\sqrt{3}}{24}\beta^2\left(\frac{1}{3},\frac{1}{3}\right)$$

is the integral at the third *singular value*, k_3 [Borwein et al. 04]. Correspondingly, the *Clausen product identity* [Borwein and Bailey 03, p. 50] shows that

$$8\int_0^{\pi/6} \frac{\mathrm{K}^2\left(\sin\frac{\theta}{2}\right)\cos\theta}{1+2\sin\theta}\,d\theta = \pi^2\int_0^1 {}_3F_2\left(\begin{matrix}1/2, 1/2, 1/2\\ 1, 1\end{matrix}; \frac{x^2}{4}\right)\frac{dx}{x+1}.$$

[1] Here we use the convention $\mathrm{K}(k) = \int_0^{\pi/2}(1-k^2\sin^2 s)^{-1/2}\,ds$. See [Borwein and Bailey 03, pp. 199–200]. One should beware: Some symbolic systems use $m := k^2$ as the argument; for example, in Mathematica one has `EllipticK[m]` $:= \mathrm{K}(\sqrt{m})$.

This elliptic-cum-hypergeometric form is a rather erudite result for the relatively innocent-looking integral

$$C_{3,0} := \frac{1}{6}\int_{R^3} \frac{dx\,dy\,dz}{\cosh x + \cosh y + \cosh z}.$$

There are other attractive representations equivalent to the elliptic form (3–5) such as

$$C_{3,0} = \pi \iint_0^\infty \frac{1}{\sqrt{x^2+1}\sqrt{y^2+1}\sqrt{(x+y)^2+1}}\,dx\,dy.$$

We next observe that $C_{3,2}$ possesses a corresponding closed form that also involves the *elliptic integral of the second kind* $\mathrm{E}(k_3)$, [Borwein and Borwein 87]. This may be similarly derived from (3–4) as follows.

Since $P_0(x) = x^2$, the building blocks for $C_{3,2}$ are

$$\begin{aligned}\sigma_1(\theta) &:= \sum_{h=0}^\infty \frac{h\Gamma^3(h+1/2)}{\Gamma^3(h+1)}\sin^{2h}\theta \\ &= \frac{4}{\sqrt{\pi}\cos\theta} \\ &\quad \times \left\{(\mathrm{E\,K})\left(\sin\frac{\theta}{2}\right) - \left(\cos^2\frac{\theta}{2}\right)\mathrm{K}^2\left(\sin\frac{\theta}{2}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}\sqrt{\pi}\sigma_2(\theta) &:= \sqrt{\pi}\sum_{h=0}^\infty \frac{h^2\Gamma^3(h+1/2)}{\Gamma^3(h+1)}\sin^{2h}\theta \\ &= \frac{(\cos\theta+1)\left(\cos^2\theta+\cos\theta-1\right)}{\cos^3\theta}\mathrm{K}^2\left(\sin\frac{\theta}{2}\right) \\ &\quad - 2\frac{(\cos\theta+1)(2\cos\theta-1)}{\cos^3\theta}(\mathrm{EK})\left(\sin\frac{\theta}{2}\right) \\ &\quad + \frac{2}{\cos^2\theta}\mathrm{E}^2\left(\sin\frac{\theta}{2}\right).\end{aligned}$$

Thus, we may use (3–4) to write

$$\begin{aligned}C_{3,2} &= \frac{2\log 2}{3}\sqrt{\pi}\sigma_2\left(\frac{\pi}{6}\right) - \frac{2}{3}\sqrt{\pi}\sigma_1\left(\frac{\pi}{6}\right) \\ &\quad + 4\int_0^{\pi/6}\sqrt{\pi}\sigma_2(\theta)\,\frac{\cos\theta}{1+2\sin\theta}\,d\theta.\end{aligned} \tag{3–6}$$

Also, for $\theta = \pi/6$, we have

$$\mathrm{E\,K} = \left(\pi + (2+2\sqrt{3})\mathrm{K}^2\right)\sqrt{3};$$

see [Borwein and Borwein 87]. Thus, using (3–6) we will get two more-complicated terms like the ones in $C_{3,0}$ but now involving both E and K. Note that $\cos\pi/12 = (\sqrt{3}+1)/\sqrt{8}$ and $\sin\pi/12 = (\sqrt{3}-1)/\sqrt{8}$ are reciprocals. Thus,

$$\sqrt{\pi}\sigma_1\left(\frac{\pi}{6}\right) = -\frac{2}{3}\mathrm{K}^2\left(\sin\frac{\pi}{12}\right) + \frac{2}{3}\pi$$

and

$$\sqrt{\pi}\sigma_2\left(\frac{\pi}{6}\right) = \frac{1}{9}\mathrm{K}^2\left(\sin\frac{\pi}{12}\right) + \frac{\pi^2}{18}\mathrm{K}^{-2}\left(\sin\frac{\pi}{12}\right).$$

In consequence of Theorem 5.4 below, all $C_{3,\mathrm{even}}$ are superpositions of $C_{3,0}$ and $C_{3,2}$ with polynomial (in k) weights; thus, the $C_{3,\mathrm{even}}$ can involve only algebraic combinations of the numbers above, such as $\log 2$, π, and the elliptic evaluations/integrals. PSLQ suggests that no relations exist between the seven monomials implicit in (3–6).

3.3.2 The $C_{3,\mathrm{odd}}$ integrals. A first observation in the cases $C_{k,\mathrm{odd}}$ is as follows. We recall the exact L-function evaluation given in [Bailey et al. 06]:

$$C_3 := C_{3,1} = \mathrm{L}_{-3}(2) := \sum_{m\geq 0}\left(\frac{1}{(3m+1)^2} - \frac{1}{(3m+2)^2}\right).$$

This knowledge about $C_{3,1}$ leads, via (3–3), to the remarkable L-function identity

$$\begin{aligned}\mathrm{L}_{-3}(2) &= \frac{2}{3}\sum_{h=0}^\infty \frac{1}{h+1}\frac{1}{\binom{2h+1}{h}} \\ &\quad \times\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2h+1}\right).\end{aligned}$$

Observe that via relation (2–5), this resolves the relevant Meijer-G in terms of an L-function; we believe this Meijer-G identity to be new.

Now, the $C_{3,\mathrm{odd}}$ seem to be pairwise rationally related, in the following sense. We discovered via numerical experiments the conjectures[2]

$$\begin{aligned}C_{3,3} &\stackrel{?}{=} -\frac{4}{27} + \frac{2}{9}\mathrm{L}_{-3}(2), \\ C_{3,5} &\stackrel{?}{=} -\frac{92}{1215} + \frac{8}{81}\mathrm{L}_{-3}(2),\end{aligned}$$

and several more, suggesting rational relations $aC_{3,k} + bC_{3,k'} = c$ for any distinct odd pair (k,k'), with a,b,c rational, $a \neq b$. We will prove these ($n=3$, odd k) conjectures below. We should mention that we found no such rational relations whatever between pairs of $C_{3,\mathrm{even}}$ (see Conjecture 4.3).

One might conceivably use the residue expansion (3–3) to prove our experimentally detected relations. However, there is another route, one that leads to an efficient algorithm for resolving the closed form of any $C_{3,\mathrm{odd}}$. We

[2] The notation $\stackrel{?}{=}$ means we experimentally suspect a given equality in absence of rigorous proof. Of course, we shall prove these $C_{3,\mathrm{odd}}$ closed forms, but we prefer to use $\stackrel{?}{=}$ when reporting on initial numerical discovery.

hark back to the dimensional-reduction methods in [Bailey et al. 06] and reduce to a two-dimensional integral

$$C_{3,k} = \frac{\sqrt{\pi}}{3!}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)} \times \iint_0^\infty \frac{dx\,dy}{xy}\left\{(1+x+y)\left(1+\frac{1}{x}+\frac{1}{y}\right)\right\}^{-(k+1)/2}.$$

Now for odd k we may assign $m := (k-1)/2$ and write

$$\iint_0^\infty \frac{dx\,dy}{xy}\left\{(1+x+y)\left(1+\frac{1}{x}+\frac{1}{y}\right)\right\}^{-(k+1)/2} = \frac{1}{m!^2}\left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\right)^m \times \iint_0^\infty \frac{dx\,dy}{xy}\left[(\alpha+x+y)\left(\beta+\frac{1}{x}+\frac{1}{y}\right)\right]^{-1}\Bigg|_{\alpha,\beta=1}.$$

The integral over x, say, may then be done, after which we put $y = z/\beta$ to reveal that, remarkably, the α,β-dependent integral is really a function only of the product $c := \alpha\beta$. In fact,

$$\iint_0^\infty \frac{dx\,dy}{xy}\frac{1}{(\alpha+x+y)(\beta+1/x+1/y)} = \int_0^\infty \frac{\log(1+1/z)+\log(c+z)}{z^2+cz+c}dz = \int_0^1 \frac{\log c - 2\log t}{t^2-ct+c}dt,$$

the final integral being obtained by making the substitutions $1+1/z = 1/t$ and $c+z = c/t$ respectively in the two parts of the preceding integral. Thus $C_{3,k}$ reduces to

$$C_{3,k} = \frac{2^{k+1}}{3!k!}\left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\right)^m \Upsilon(\alpha\beta)\Big|_{\alpha,\beta=1}, \qquad (3\text{–}7)$$

where

$$\Upsilon(c) := \int_0^1 \frac{\log\sqrt{c}-\log t}{t^2-ct+c}dt = \frac{1}{r_+-r_-}\Big(-\frac{1}{2}\log(r_+r_-)\log\frac{1-1/r_-}{1-1/r_+} + \mathrm{Li}_2(1/r_-) - \mathrm{Li}_2(1/r_+)\Big),$$

with

$$r_\pm := \frac{c\pm\sqrt{c^2-4c}}{2}.$$

Sure enough, for $k=1$, and so $m=0$ and no differentiation in (3–7), we obtain our original case $C_3 := C_{3,1} = (2/3)\Upsilon(1) = \mathrm{L}_{-3}(2)$.

More generally, our finite representation (3–7) leads to a proof of the evaluations above for $C_{3,3}$ and $C_{3,5}$ and indeed to a proof of our rational-relation conjecture. To this end, note that we can use the operator identity

$$\frac{\partial^2}{\partial\alpha\,\partial\beta} = \frac{\partial}{\partial c}\,c\,\frac{\partial}{\partial c},$$

valid on functions f, where $c = \alpha\beta$. In expanded form this means

$$\left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\right)^m f(c) = \sum_{k=0}^m \binom{m}{k}\frac{m!}{k!}c^k f^{(m+k)}(c).$$

From the above relations one may now derive, for nonnegative integers m,

$$C_{3,2m+1} = \frac{2^{2m+1}}{3(2m+1)\binom{2m}{m}} \times \sum_{k=0}^m \binom{m}{k}\binom{m+k}{k}(-1)^{m+k+1}I(m+k), \qquad (3\text{–}8)$$

where

$$I(\nu) := \int_0^1 \frac{t^\nu \log t}{(t^2-t+1)^{\nu+1}}\,dt. \qquad (3\text{–}9)$$

These observations lead us to the following theorem.

Theorem 3.1. *For odd $k \geq 1$, we have*

$$C_{3,k} = p_{3,k} + q_{3,k}\mathrm{L}_{-3}(2),$$

with the p, q coefficients always being rational, $q_{3,k}$ being given explicitly by (3–11) below.

Proof: In terms of the I function in (3–9), establishing the recurrence

$$\nu I(\nu-1) + (2\nu+1)I(\nu) - 3(\nu+1)I(\nu+1) + \frac{1}{\nu} = 0 \qquad (3\text{–}10)$$

is enough to prove the theorem, because

$$I(0) = -\frac{3}{2}\mathrm{L}_{-3}(2), \quad I(1) = -\frac{1}{2}\mathrm{L}_{-3}(2).$$

One may also derive

$$I(\nu) = a_\nu + b_\nu \mathrm{L}_{-3}(2)$$

with rational a_ν, b_ν satisfying the recurrences

$$\nu a_{\nu-1} + (2\nu+1)a_\nu - 3(\nu+1)a_{\nu+1} + \frac{1}{\nu} = 0,$$

with $a_0 = a_1 = 0$, and

$$\nu b_{\nu-1} + (2\nu+1)b_\nu - 3(\nu+1)b_{\nu+1} + \frac{1}{\nu} = 0,$$

with $b_0 = -\frac{3}{2},\ b_1 = -\frac{1}{2}$. So we now prove the recurrence (3–10). For $x \in (-1, 1)$ we have

$$y(x) := \sum_{\nu=0}^{\infty} I(\nu)x^{\nu} = \int_0^1 \frac{\log t}{t^2 - t(1+x) + 1}\,dt.$$

The recurrence (3–10) thus holds if and only if

$$(x+1)\sum_{\nu=0}^{\infty} I(\nu)x^{\nu} + \left(x + 2 - \frac{3}{x}\right)\sum_{\nu=0}^{\infty} \nu I(\nu)x^{\nu} \\ = I(0) - 3I(1) + \log(1-x),$$

which is equivalent to y satisfying the differential equation

$$(x+1)y + (x^2 + 2x - 3)y' = \log(1-x) - 3\mathrm{L}_{-3}(2),$$

subject to the initial condition

$$y(0) = -\frac{3}{2}\mathrm{L}_{-3}(2).$$

Maple verifies that $y(x)$ is indeed a solution. □

It turns out to be possible to give a finite expression for the $q_{3,k}$ rational in Theorem 3.1. What may be called the terminal term of the chain differentiation in (3–7), namely

$$\left\{\mathrm{Li}_2\left(\frac{1}{r_-}\right) - \mathrm{Li}_2\left(\frac{1}{r_+}\right)\right\} \cdot \left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\right)^m \frac{1}{r_+ - r_-}\Big|_{\alpha,\beta=1},$$

gives the rational coefficient of $\mathrm{L}_{-3}(2)$ as

$$q_{3,k} = \sqrt{3}\frac{2^{k-1}}{k!}\left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\right)^m \frac{1}{(\alpha\beta(4-\alpha\beta))^{1/2}}\Big|_{\alpha,\beta=1}.$$

In particular, a finite expression for the general q coefficient is, with $m := (k-1)/2$,

$$\begin{aligned} q_{3,k} &= \frac{2^{k-1}}{k!}\sum_{j=0}^{m}\binom{m}{j}(-1)^{m+j}\frac{m!}{j!} \\ &\quad\times \sum_{i=0}^{m+j}\binom{m+j}{i}\left(\frac{1}{2}\right)_i\left(\frac{1}{2}\right)_{m+j-i}\left(-\frac{1}{3}\right)^i \\ &= \sqrt{3}\frac{2^{2m-1}m!}{(2m-1)!}\sum_{j=0}^{m}\frac{(-1)^{m+j}\binom{m}{j}}{j!} \\ &\quad\times {}_2\mathrm{F}_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ \frac{1}{2} - m - j\end{matrix}; \frac{1}{4}\right)\prod_{i=0}^{j}\left(\frac{1}{2} + m + i\right). \end{aligned} \tag{3–11}$$

The above analysis provides closed forms for the relevant Meijer G-functions. The method also provides an algorithm for exact evaluation of any $C_{3,\mathrm{odd}}$ rather efficiently.[3] One may arrive quickly at such instances as

$$\begin{aligned} C_{3,15} &:= \frac{1}{3!}\iiint_{-\infty}^{\infty}\frac{dx\,dy\,dz}{(\cosh x + \cosh y + \cosh z)^{16}} \\ &= -\frac{11884272896}{837856594575} + \frac{4139008}{227988189}\mathrm{L}_{-3}(2). \end{aligned}$$

3.4 Evaluations of $C_{4,k}$

We begin with the first case of (2–6). Residue calculus—again we omit the intricacies—gives series such as

$$\begin{aligned} C_{4,0} &= \frac{1}{24}\sum_{h=0}\left(\frac{\Gamma^4(h+1/2)}{\Gamma^4(h+1)}\right)'' \\ &= \frac{1}{3}\sum_{h=0}^{\infty}\frac{\Gamma^4(h+1/2)}{\Gamma^4(h+1)} \\ &\quad\times\left(8\left(-\log 2 + H_{2h}^{(-1)}\right)^2 + \zeta(2) - 2H_{2h}^{(-2)}\right), \end{aligned} \tag{3–12}$$

where the double derivative $''$ is with respect to h, and the new sum is

$$H_{\mu}^{(-2)} := 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \pm \frac{1}{\mu^2}$$

with $H_0^{(-2)} := 0$. However, just as with the $C_{3,\mathrm{even}}$ cases of the previous section, we know not a single closed form for $C_{4,\mathrm{even}}$, and again, we found experimentally that $C_{4,\mathrm{odd}}$ are pairwise rationally related, meaning (see Table 2 for $C_4 := C_{4,1}$) that every $C_{4,\mathrm{odd}}$ would be $p + q\zeta(3)$ for rational p, q.

The finite-form evaluation of any $C_{4,\mathrm{odd}}$ is achieved as follows: Define integrals

$$\begin{aligned} U_h &:= \frac{i\pi}{2}\int_{-\infty}^{\infty}\frac{\sinh \pi t}{\cosh^3 \pi t}\left(-\frac{1}{2} + it\right)^h dt \\ &= (-1)^{h+1}h(h-1)\frac{\zeta(2-h)}{2\pi}. \end{aligned}$$

This latter identity actually holds for *any integer* h, with $U_1 := 1/(2\pi)$. Note that under the further constraint $h \geq 0$, the quantity πU_h *for* $h \geq 0$ *is rational*, as follows from the fact of known evaluations of $\zeta(2-h)$.

[3] One may explicitly differentiate and simplify in (3–7), but a faster algorithm is to use the finite expression for $q_{3,k}$ given after Theorem 3.1, an extreme-precision evaluation of series (3–3), then a function such as Mathematica's `Rationalize[ ]` to resolve $p_{3,k}$. This amounts to an interesting, systematic use of extreme precision *within* a general algorithm.

n	k	$C_{n,k}$
1	any	$\frac{2^k\Gamma\left(\frac{k+1}{2}\right)^2}{k!} = p_{1,k} + q_{1,k}\pi$
2	any	$\frac{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right)^3}{2\Gamma\left(\frac{k}{2}+1\right)\Gamma(k+1)} = p_{2,k} + q_{2,k}\pi^2$
3	0	Elliptic form (3–5)
3	1	$C_3 = \mathrm{L}_{-3}(2)$ (see [Bailey et al. 06])
3	2	Elliptic form (3–6)
3	3	$C_{3,3} = \frac{2}{9}\mathrm{L}_{-3}(2) - \frac{4}{27}$
3	any odd	$p_{3,k} + q_{3,k}\mathrm{L}_{-3}(2)$, Series (3–3)
3	any even	Order-2 recurrence (Theorem 5.4), Series (3–4)
3	any complex	Meijer integral (2–5)
4	0	Series (3–12)
4	1	$C_4 = \frac{7}{12}\zeta(3)$ (see [Bailey et al. 06])
4	3	$C_{4,3} = \frac{7}{288}\zeta(3) - \frac{1}{48}$
4	any odd	$p_{4,k} + q_{4,k}\zeta(3)$
4	any even	Order-2 recurrence (Theorem 5.4)
4	any complex	Meijer integral (2–6)
5	any complex	Nested-Barnes integral (7–2), Series (7–3)
large	fixed	$\sim \frac{1}{k!}\frac{2^{k+1+n}}{(k+1)^{n+1}}e^{-(k+1)\gamma}$

TABLE 2. Proven closed forms, series, and relations for the $C_{n,k}$. Every p or q coefficient above is proven rational, with the q having explicit finite forms. Our searches have uncovered no other closed forms, or pairwise rational relations not implicit above. Conjecture 4.1 gives a general recurrence relation for complex k

The relevance of the U_h is that a Meijer contour integral as in (2–6) can be developed as follows:

$$G := \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma^3(-s)\Gamma^3((k+1)/2+s)}{\Gamma(1+k/2+s)\Gamma(1/2-s)}\,ds$$
$$= \frac{i\pi}{2}\int_{-\infty}^{\infty}\frac{\sinh \pi t}{\cosh^3 \pi t}F\left(-\frac{1}{2}+it\right)dt,$$

where

$$F(s) := \frac{\Gamma^3((1+k)/2+s)\Gamma(1/2+s)}{\Gamma^3(1+s)\Gamma(1+k/2+s)}.$$

Now the key is that if we write

$$F(s) = f(s) + \phi(s),$$

where we express $F(s) = \sum_j f_j s^j$ as a polynomial and an error term $\phi(s) = o(s)$, then we can resolve the original Meijer-G by employing the U_h identity on the monomials $f_j s^j$, and using residue calculus for the ϕ term, to write

$$G = \sum_j f_j U_j + \frac{1}{2\pi}\sum_{h=0}^{\infty}\phi''(h). \tag{3–13}$$

This analysis now leads to a proof of the experimentally discovered conjecture on rational relations for any pair of $C_{4,\mathrm{odd}}$:

Theorem 3.2. *For odd $k \geq 1$, we have*

$$C_{4,k} = p_{4,k} + q_{4,k}\zeta(3),$$

with the p, q coefficients always being rational. In particular, a finite expression for the general q coefficient is, with $m := (k-1)/2$,

$$q_{4,k} = \frac{7}{12}\frac{(2m)!^3}{k!\cdot 64^m m!^4}\sum_{j=0}^{m}\frac{\binom{m}{j}^4}{\binom{2m}{2j}^3}.$$

Proof: For fixed odd k the function F is indeed polynomial plus a decay term, namely, set $m := (k-1)/2$ and write

$$F(s) = \frac{(1+s)^3(2+s)^3\cdots(m+s)^3}{(s+1/2)(s+3/2)\cdots(s+m+1/2)}$$
$$= \sum_{j=0}^{2m-1} f_j s^j + \sum_{j=0}^{m}\frac{A_j}{s+j+1/2}.$$

Here, the coefficients (f_j) and (A_j) are all rational, and can be calculated exactly, using polynomial remaindering and partial-fraction expansion, respectively. Thus the original Meijer G-function from (2–6) is given exactly by the result (3–13),

$$G = \sum_{j=0}^{2m-1} f_j U_j + \frac{1}{2\pi} \sum_{j=0}^{m} A_j \zeta\left(2, j + \frac{1}{2}\right),$$

where $\zeta(s,a) := \sum_{h \geq 0} 1/(h+a)^s$ is the *Hurwitz zeta function.*

Now, since each U_j here is (rational)/π, each $\zeta\left(2, j+\frac{1}{2}\right)$ is (rational) + (rational)$\zeta(3)$, and each $C_{4,\text{odd}}$ is (rational)πG, the theorem follows. The explicit evaluation of $q_{4,k}$ arises from the natural partial-fraction evaluation of the A_j terms and the accumulation of all normalizing factors. □

This result amounts to a closed-form resolution of the Meijer G-function in (2–6) for any odd k in terms of $\zeta(3)$, π, and rationals. Moreover,

$$\sum_{j=0}^{m} \frac{\binom{m}{j}^4}{\binom{2m}{2j}^3} = {}_4\mathrm{F}_3\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \\ \frac{-m+1}{2}, \frac{-m+1}{2}, \frac{-m+1}{2} \end{matrix} ; -1\right) - \frac{\binom{m}{m+1}^3}{\binom{2m}{2m+2}^3} {}_4\mathrm{F}_3\left(\begin{matrix} m+\frac{3}{2}, \frac{m+3}{2}, \frac{m+3}{2}, 1 \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{matrix} ; -1\right).$$

In this way, as for $n = 3$, polynomial-remaindering and rational-arithmetic algorithms quickly yield exact evaluations such as

$$C_{4,15} := \frac{1}{4!} \iiiint_{-\infty}^{\infty} \frac{dw\, dx\, dy\, dz}{(\cosh w + \cosh x + \cosh y + \cosh z)^{16}} = -\frac{1744313209}{578605547520000} + \frac{67697}{26990346240}\zeta(3).$$

In general, the odd Meijer-G form for $n = 4$ can be written explicitly as

$$C_{4,2k+1} = \frac{1}{(2k+1)!} \frac{\pi^2}{24} \times \int_{-\infty}^{\infty} \frac{\sinh(\pi t)}{t \cosh^3(\pi t)} \prod_{j=1}^{k} \frac{(t - i(j-1/2))^3}{t - ij} dt, \tag{3–14}$$

while the even form, as well as those for $n = 3$, offers less purchase. In particular, integration by parts in (3–14) yields

$$C_{4,1} = \frac{\pi^2}{12} \int_0^{\infty} \frac{\tanh(t)\,\mathrm{sech}^2(t)}{t} dt = \frac{\pi}{24} \int_0^{\infty} \frac{\tanh^2(\pi t)}{t^2}\, dt.$$

We next substitute the partial-fraction expansion

$$\frac{\tanh(\pi y)}{y} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{2y}{4y^2 + (2n+1)^2},$$

and expand, then interchange integration and summation to obtain from

$$\int_0^{\infty} \frac{4y^2}{(4y^2 + (2n+1)^2)(4y^2 + (2m+1)^2)}\, dy = \frac{\pi}{(2n+1)(2m+1)(2n+2m+2)}$$

that

$$C_{4,1} = \frac{2}{3} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2n+1)(2m+1)(2n+2m+2)}.$$

This double sum is a *Tornheim double sum* or a *Witten ζ-value*, see [Borwein 05], and equals

$$\int_0^1 \frac{\operatorname{arctanh}^2(x)}{x}\, dx = \int_0^1 \frac{\log^2 \sqrt{\frac{1-x}{1+x}}}{x}\, dx = \frac{1}{2} \int_0^1 \frac{\log^2 t}{1-t^2}\, dt = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} = \frac{7}{8}\zeta(3),$$

where the first integral and penultimate sum are obtained on integrating termwise. Thus,

$$C_{4,1} = \frac{7}{12}\zeta(3),$$

as before. Similar machinations lead to a corresponding evaluation of $C_{4,3}$.

4. RECURRENCE RELATIONS: EXPERIMENT

Based on extensive computational work we make the following conjecture:[4]

Conjecture 4.1. *For given $n \in \mathbb{Z}^+$ with $M := \lfloor (n+1)/2 \rfloor$, the integrals $(C_{n,k})$ enjoy an order-M recurrence involving $M+1$ terms with coefficients being integral polynomials $P_{n,j}$ each of degree n, that is,*

$$P_{n,0}(k)C_{n,k} + P_{n,1}(k)C_{n,k+2} + \cdots + P_{n,M}(k)C_{n,k+2M} = 0.$$

Moreover, this holds for all complex k in the sense of analytic continuation (the existence of poles in the k-plane is admitted).

[4]This conjecture has since been proved [Borwein and Salvy 07].

We shall eventually be able to prove certain instances of Conjecture 4.1, specifically, recurrence relations among the $C_{n,k}$ with fixed $n = 1, 2, 3, 4$. The first open cases of Conjecture 4.1 are $n = 5, 6$, specifically,

$$\begin{aligned} 0 \overset{?}{=} (k+1)^5 C_{5,k} &- (k+2)\big(35k^4 + 280k^3 + 882k^2 \\ &\qquad + 1288k + 731\big) C_{5,k+2} \\ &+ (k+2)(k+3)(k+4) \\ &\quad \times \left(259k^2 + 1554k + 2435\right) C_{5,k+4} \\ &- 225(k+2)(k+3)(k+4)(k+5)(k+6) C_{5,k+6} \end{aligned} \tag{4–1}$$

and

$$\begin{aligned} 0 \overset{?}{=} (k+1)^6 C_{6,k} &- 8(k+2)^2\big(7k^4 + 56k^3 + 182k^2 \\ &\qquad + 280k + 171\big) C_{6,k+2} \\ &+ 16(k+2)(k+3)^2(k+4) \\ &\quad \times \left(49k^2 + 294k + 500\right) C_{6,k+4} \\ &- 2304(k+2)(k+3)(k+4)^2(k+5)(k+6) C_{6,k+6}, \end{aligned} \tag{4–2}$$

where as before, the question mark is used to emphasize the fact that we have no formal proof.

Note that on this conjecture, our renormalized (lowercase-notated) $c_{n,k} = \Gamma(k+1) n! 2^{-n} C_{n,k}$ of equation (2–2) then satisfies a recurrence with a straightforward polynomial adjustment:

$$\sum_{i=0}^{M} (-1)^i p_{n,i}(k+i+1) c_{n,k+2i} = 0. \tag{4–3}$$

We write the "little-c" recurrence in this way for convenient connection with experimental results; for example, we have always encountered natural alternating signs, and some obvious factors of the polynomials p implicitly defined by (4–3). Note, for instance, that the experimental recurrences (4–1) and (4–2) can be recast compactly in the form of (4–3) by defining

$$\begin{aligned} p_{5,0}(x) &= x^6, & p_{6,0}(x) &= x^7, \\ p_{5,1}(x) &= 35x^4 + 42x^2 + 3, & p_{6,1}(x) &= x(56x^4 + 112x^2 + 24), \\ p_{5,2}(x) &= 259x^2 + 104, & p_{6,2}(x) &= x(784x^2 + 944), \\ p_{5,3}(x) &= 225, & p_{6,3}(x) &= 2304x. \end{aligned}$$

Table 3 has many other $p_{n,i}$ polynomials that we have found experimentally.

There is actually a substantial literature on such recurrences. Most authors abide by the nomenclature, as we do, that the order of the recurrence is M, meaning there are $M + 1$ different C terms (and $M + 1$ polynomial coefficients). Some researchers refer to any sequence such as C, satisfying such a recurrence, as *holonomic*, and observe that a generating function will satisfy a similar recurrence relation in its derivatives [van der Poorten and Shparlinski 05, Zudilin 97, Flajolet et al. 05].

We make two more conjectures that are experimentally motivated:

Conjecture 4.2. *Fix n and a complex rational k_0. Then for k lying in the arithmetic progression $\ldots, k_0 - 4, k_0 - 2, k_0, k_0 + 2, k_0 + 4, \ldots,$ the set $(C_{n,k} : k \in k_0 + 2\mathbb{Z})$ is rationally generated by any $M := \lfloor (n+1)/2 \rfloor$ distinct elements, but no fewer.*

Conjecture 4.3. *For a distinct complex pair (k, k'), the rational relation $pC_{n,k} + qC_{n,k'} = r$ with p, q, r complex rationals, $p \neq q$, is impossible for $n \geq 5$. For $n = 3, 4$ the rational relation is possible only for both k, k' odd integers.*

Since all of these conjectures have been experimentally motivated, we hereby start our recurrence discussion in the historical spirit, with experimental results first (and knowing that some of the tabulated recurrences in the present section are proven and some are not). We give our substantial evidence in Table 3, where $c_{n,k}$ (lowercase notation) is defined in (2–2), and in Table 4.

An example of our experimental forays runs as follows. The form of the nontrivial coefficients for a possible recurrence for the $C_{3,k}$ and $C_{4,k}$ was assisted by consulting *Sloane's Online Encyclopedia*,[5] which for $C_{4,k}$ connected the coefficients to the sequence A063495.[6] Having found these recurrences, it was then reasonable to assume that the coefficients were polynomials of the conjectured degree; and the tables were then built by numerical interpolation after the use of PSLQ. The predicted recurrences were then numerically checked to extreme precision at various values of k.

Table 3 shows recurrences for the renormalized $c_{n,k} := k! n! 2^{-n} C_{n,k}$, for $1 \leq n \leq 12$ and integer k. Using the recurrence form (4–3) we end up with simple (odd or even, positive) polynomials $p_{n,i}$. The explicit polynomials $p_{n,i}$ that we have found experimentally are shown in Tables 3

[5] See www.research.att.com/~njas/sequences/index.html.

[6] Consult A063495, which makes reference to equation (10) in [Martin 96].

n	$i = 1$
1	1
2	$4x$
3	$2 + 10x^2$
4	$x(12 + 20x^2)$
5	$3 + 42x^2 + 35x^4$
6	$x(24 + 112x^2 + 56x^4)$
7	$4 + 108x^2 + 252x^4 + 84x^6$
8	$x(40 + 360x^2 + 504x^4 + 120x^6)$
9	$5 + 220x^2 + 990x^4 + 924x^6 + 165x^8$
10	$x(60 + 880x^2 + 2376x^4 + 1584x^6 + 220x^8)$
11	$6 + 390x^2 + 2860x^4 + 5148x^6 + 2574x^8 + 286x^{10}$
12	$x(84 + 1820x^2 + 8008x^4 + 10296x^6 + 4004x^8 + 364x^{10})$
n	$i = 2$
3	9
4	$64x$
5	$104 + 259x^2$
6	$x(944 + 784x^2)$
7	$816 + 4752x^2 + 1974x^4$
8	$x(9024 + 17520x^2 + 4368x^4)$
9	$5376 + 54384x^2 + 52800x^4 + 8778x^6$
10	$x(70144 + 236544x^2 + 137808x^4 + 16368x^6)$
11	$32000 + 492544x^2 + 830544x^4 + 322608x^6 + 28743x^8$
12	$x(481280 + 2469376x^2 + 2498496x^4 + 693264x^6 + 48048x^8)$
n	$i = 3$
5	225
6	$2304x$
7	$7796 + 12916x^2$
8	$x(94976 + 52480x^2)$
9	$170298 + 625196x^2 + 172810x^4$
10	$x(2409216 + 2949056x^2 + 489280x^4)$
11	$2999076 + 18232188x^2 + 11161436x^4 + 1234948x^6$
12	$x(48354048 + 98000448x^2 + 36003968x^4 + 2846272x^6)$
n	$i = 4$
7	11025
8	$147456x$
9	$851976 + 1057221x^2$
10	$x(13036544 + 5395456x^2)$
11	$39605040 + 106102880x^2 + 21967231x^4$
12	$x(683253760 + 610355200x^2 + 75851776x^4)$
n	$i = 5$
9	893025
10	$14745600x$
11	$129879846 + 128816766x^2$
12	$x(2393358336 + 791691264x^2)$
n	$i = 6$
11	108056025
12	$2123366400x$

TABLE 3. Experimental polynomials $p_{n,i}$ for $1 \leq i \leq 6$ and $1 \leq n \leq 12$.

n	$i=0$	$i=1$	$i=2$	$i=3$
1	x^2	$\mathbf{1}$		
2	x^3	$\mathbf{4}x$		
3	x^4	$\overline{2}+\mathbf{10}x^2$	9	
4	x^5	$x(\overline{12}+\mathbf{20}x^2)$	$64x$	
5	x^6	$3+\overline{42}x^2+\mathbf{35}x^4$	$104+259x^2$	225
6	x^7	$x(24+\overline{112}x^2+\mathbf{56}x^4)$	$x(944+784x^2)$	$2304x$
7	x^8	$4+108x^2+\overline{252}x^4+\mathbf{84}x^6$	$816+4752x^2+1974x^4$	$7796+12916x^2$
8	x^9	$x(40+360x^2+\overline{504}x^4+\mathbf{120}x^6)$	$x(9024+17520x^2+4368x^4)$	$x(94976+52480x^2)$

TABLE 4. Polynomials $p_{n,i}$ for $0 \le i \le 3$ and $1 \le n \le 8$. Note that the coefficient of the rightmost polynomial is $(1\cdot 3 \cdots n)^2$ or $(2\cdot 4 \cdots n)^2$ respectively. Correspondingly, the bold numbers are of the form $\binom{n}{3}$, while the overlined numbers are of the form $2\binom{n}{5}$, etc. Generally, MacMahon's numbers, see Sloane's A008955, seem closely related: $T(n,k) = T(n,k-1) + k^2 T(n-1,k-1)$.

and 4. In particular, we *conjecture* from Table 4 that

$$
\begin{aligned}
p_{n,0}(x) &= x^{n+1},\\
p_{n,1}(x) &= \sum_{j=1}^{M} j\binom{n+2}{2j+1} x^{n+1-2j}\\
&= \frac{1}{4}\left(n+1+x\right)\left(x-1\right)^{n+1}\\
&\quad + \frac{1}{4}\left(x+1\right)^{n+1}\left(n+1-x\right),\\
p_{n,2}(x) &= \sum_{j=1}^{M-1} \frac{j4^{j-1}(2j+3)(n+2)+j+1}{j+2}\\
&\qquad \cdot \binom{n+2}{2j+3} x^{n+1-2j}\\
&= \frac{1}{32}\left((n+x+2)^2 - \frac{7n}{2} - \frac{11(x+2)}{4}\right)\\
&\qquad \cdot (x-2)^{n+1}\\
&\quad + \frac{1}{32}\left((n-x+2)^2 - \frac{7n}{2} + \frac{11(x-2)}{4}\right)\\
&\qquad \cdot (x+2)^{n+1}\\
&\quad - \frac{1}{16} x^{n+1}\left(x^2 - (n+2)^2\right) + \cdots,\\
p_{n,M}(x) &= \left\{\prod_{j=0}^{M} (n-2j)^2\right\} x^{n-2M}.
\end{aligned}
$$

Recall that $M := \lfloor (n+1)/2 \rfloor$ is the recurrence order, and we set $p_{n,i} = 0$ for $i \ge M$. If we consider the *graded* generating function, we equivalently *conjecture* that

$$
G_i(x,y) := \sum_{n=1}^{\infty} p_{n,i}(x) y^n,
$$

obtaining

$$
\begin{aligned}
G_0(x,y) &= \frac{x}{1-xy},\\
G_1(x,y) &= \frac{1}{(xy+y-1)^2(xy-y-1)^2},\\
G_2(x,y) &= \\
&\frac{y^3\left(-(1-xy)^3 + 10(1-xy)^2 + 4y^2(1-xy) - 8y^2\right)}{(xy+2y-1)^3(1-xy)^3(xy-2y-1)^3}.
\end{aligned}
$$

However, we have no idea what the general pattern should be.

5. RECURRENCE RELATIONS: THEORY

5.1 Direct Methods

An immediate but demonstrative result that does not require experimental mathematics is the following:

Theorem 5.1. *Conjecture 4.1 is true for $n = 1, 2$. In fact, for any complex k,*

$$
(k+1)C_{1,k} - (k+2)C_{1,k+2} = 0
$$

and

$$
(k+1)^2 C_{2,k} - 4(k+2)^2 C_{2,k+2} = 0.
$$

Proof: The desired recurrences follow immediately and analytically from (3–1) and (3–2) respectively. □

As intimated in Section 4, PSLQ in tandem with Sloane suggests that the $C_{3,k}$ satisfy a definite recurrence, at least for integers k. We can get a foothold on this, with a view to the general analytic Conjecture 4.1, with the following theorem.

Theorem 5.2. *Set $n = 3$, whence for positive odd integers k we have*

$$0 = (k+1)^3 C_{3,k} - 2(k+2)\left(5(k+2)^2+1\right) C_{3,k+2} + 9(k+2)(k+3)(k+4) C_{3,k+4}.$$

Remark 5.3. We shall eventually prove the recurrence for general complex k; however, the two "direct" methods of proof here for odd k are instructive and have indeed led us into the more general analytical forays to follow.

Proof (first method): For nonnegative integer m, we begin with the formulas for $C_{3,2m+1}$ and $I(\nu)$, namely (3–8), (3–9) respectively. We now make the crucial observation that

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{n+k}{k} t^{n+k}\log t}{(t-t^2-1)^{n+k+1}} = P_n\left(1 - \frac{2t}{-t+t^2+1}\right)\frac{t^n \log t}{(t-t^2-1)^{n+1}},$$

and so

$$C_{3,2m+1} = \frac{1}{3}\frac{2^{2m+1}}{(2m+1)\binom{2m}{m}} \times \int_0^1 P_m\left(1-\frac{2t}{t^2-t+1}\right)\frac{t^m \log t}{(t-t^2-1)^{m+1}}\,dt, \tag{5–1}$$

where P_n is the nth *Legendre polynomial* with ordinary generating function, see [Abramowitz and Stegun 70],

$$\sum_{n=0}^{\infty} P_n(x)\, y^n = \frac{1}{\sqrt{1-2xy+y^2}}. \tag{5–2}$$

Let J_m denote the integral on the right-hand side of (5–1). From (5–1) and (5–2), on justifying the exchange of sum and integral, we obtain that the generating function for J_m is

$$J(x) := \sum_{\mu\ge 0} J_\mu x^\mu = \int_0^1 \frac{\log t}{\sqrt{(-t+t^2+1+tx)^2 - 4t^2 x}}\,dt.$$

Now, our hypothesized recurrence, when written for J_m, is

$$m^2 J_{m-1} - \left(3+10m^2+10m\right) J_m + 9(m+1)^2 J_{m+1} = 0. \tag{5–3}$$

Thus it suffices to show that $J = v$ satisfies the ODE

$$(x-3)\,v + \left(3x^2-20x+9\right) v' + \left(x^3-10x^2+9x\right) v'' = 3. \tag{5–4}$$

This is indeed the case. Maple easily confirms that the value of the left-hand side of (5–4) is 3. □

Proof (second method): Alternatively we observe that (3–8) can be written as $C_{3,2m+1} = a_m J_m$, where $a(0) = \frac{2}{3}$ and

$$(-2m-2)\,a(m) + (2m+3)\,a(m+1) = 0,$$

while J_m satisfies (5–3), or via the proven recurrence,

$$(n+1)^2 u(n) + (n+1)(2n+3)\,u(n+1) - 3(n+2)(n+1)\,u(n+2) = -1,$$

for I. The INRIA-designed Maple package `gfun` provides an algorithm that will then produce a recurrence for $C_{3,2n+1}$ that simplifies to the vanishing of

$$4(m-1)^3 J_{m-2} - 2(2m-1)\left(3+10m^2-10m\right) J_{m-1} + 9(2m+1)(2m-1)\,m J_n,$$

which Maple easily confirms to be as claimed. This proof also can be obtained in Mathematica using Carsten Schneider's `Sigma` package available from Risc-Linz, [Schneider 06]. Both programs can certify the result, for example in Mathematica using `CreativeTelescoping`. □

The coefficients J_m are interesting in their own right. In fact,

$$J_m = q_m L_{-3}(2) - p_m \to_m 0,$$

where for $m \ge 1$,

$$q_m = \frac{1}{2} - \sum_{k=1}^{m-1} 9^{-k} {}_2F_2\left(\begin{matrix}\frac{1}{2}, -k, -k \\ 1, 2\end{matrix}; 4\right).$$

The first six values of p_m and q_m respectively are

$$(p_0,\ldots,p_5) = \frac{1}{3}, \frac{23}{108}, \frac{145}{972}, \frac{1331}{11664}, \frac{242353}{2624400}, \frac{5495507}{70858800},$$

and

$$(q_0,\ldots,q_5) = \frac{1}{2}, \frac{5}{18}, \frac{31}{162}, \frac{71}{486}, \frac{517}{4374}, \frac{11723}{118098}.$$

5.2 Analytic Method

Presumably there are direct methods, analogous to those used for Theorem 5.2, that would establish the experimentally motivated recurrence for the $C_{4,\text{odd}}$. However, it turns out that an analytic approach handles both $C_{3,k}$ and $C_{4,k}$ recurrences and moreover, does this for general complex k. Incidentally, by "general complex k" here and elsewhere, either we mean that $C_{n,k}$ is defined as its original integral (1–1) and all $k \in \mathcal{K}$ are being considered, or we are contemplating the analytic continuation $C_{n,k}$

over the entire complex k-plane (and at poles recurrences still make divergent sense).

The following method of proof, relying on a contour-integral application of the Zeilberger algorithm [Wilf and Zeilberger 92, Bećirović et al. 06, Zudilin 04], was suggested to us by W. Zudilin [Zudilin 06].

Theorem 5.4. *The recurrence in Theorem 5.2 for $C_{3,\mathrm{odd}\,k}$ extends to complex k; moreover, there is a recurrence of the same order ($M = 2$) for the $C_{4,k}$. Explicitly, both of the recurrences*

$$\begin{aligned}(k+1)^3C_{3,k} - 2(k+2)\left(5(k+2)^2+1\right)C_{3,k+2}\\ + 9(k+2)(k+3)(k+4)C_{3,k+4} = 0\end{aligned}$$

and

$$\begin{aligned}(k+1)^4C_{4,k} - 4(k+2)^2(5(k+2)^2+3)C_{4,k+2}\\ + 64(k+2)(k+3)^2(k+4)C_{4,k+4} = 0\end{aligned}$$

hold for general complex k.

Proof: (i) We focus on the $n = 4$ case—the $n = 3$ case follows the same logic—using a representation based on the Meijer form (2–6) and its associated contour integral. Contemplating t as a complex variable, we have

$$C_{4,2t-1} = -\frac{\pi^2}{24\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} F_4(t,s)\frac{\cos\pi s}{\sin^3\pi s}ds,$$

with the definition

$$F_4(t,s) := \frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma(s+t)^3}{\Gamma(2t)\Gamma(s+1)^3\Gamma\left(\frac{1}{2}+s+t\right)}.$$

If one then employs the Zeilberger algorithm,[7] one finds that the definition

$$\begin{aligned}G_4(t,s) := s^3\frac{1}{(t-1)(2s+2t-1)}\\ \times\big(12t^3+16t-2+26st^2-26t^2-37ts\\ +11s+18s^2t+4s^3-12s^2\big)F_4(t,s)\end{aligned}$$

leads to

$$\begin{aligned}16t^2(2t+1)(2t-1)F_4(t+1,s)\\ -(2t-1)^2(5t^2-5t+2)F_4(t,s)\\ +(t-1)^4F_4(t-1,s)\\ = G_4(t,s+1)-G_4(t,s).\end{aligned}$$

[7] Say, by calling in Maple `zeil(F4(t-1,s),s,t,N,2)`.

Inserting this F, G relation into the contour integral yields

$$\begin{aligned}16t^2(2t+1)(2t-1)C_{4,2t+1}\\ -(2t-1)^2(5t^2-5t+2)C_{4,2t-1}\\ +(t-1)^4C_{4,2t-3}\\ = \frac{\pi^2}{24\pi i}\int_{\mathcal{C}} G_4(t,s)\frac{\cos\pi s}{\sin^3\pi s}ds,\end{aligned}\tag{5–5}$$

where now the contour $\mathcal{C}$ is an infinitely tall, thin rectangle running vertically through $-\frac{1}{2}+0i$ and $\frac{1}{2}+0i$.

However, this rectangular integral is zero, since the only singularity is at $s = 0$, and as we saw in our previous Meijer analysis for $C_{4,k}$, the residue contribution is proportional to $\partial^2G_4(t,s)/\partial s^2|_{s=0}$, which is zero. Thus, the recurrence (5–5) holds in an analytic sense, and upon $t \to (k+3)/2$ becomes the order-2 recurrence desired.

(ii) For $n = 3$, the same procedure goes through; we first hark back to Meijer representation (2–5), then define

$$F_3(t,s) := \frac{\Gamma(s+1/2)\Gamma(s+t)^2}{\Gamma(2t)\Gamma(s+1)^3},$$

then run the Zeilberger algorithm to achieve

$$\begin{aligned}&G_3(t,s)\\ &:= s^3\frac{12t^3-17t^2+14st^2-10st+6t+4s^2t-s^2+2s-1}{2t(t-1)}\\ &\quad\times F_3(t,s)\end{aligned}$$

and

$$\begin{aligned}&(4t+1)(2t+1)(2t-1)(4t-1)F_3(t+1,s)\\ &\quad -t(2t-1)(10t^2-10t+3)F_3(t,s)+t(t-1)^3F_3(t-1,s)\\ &\quad = G_3(t,s+1)-G_3(t,s).\end{aligned}$$

Then, as with the $n = 4$ case above, we observe the vanishing of the relevant contour integral and arrive at the correct recurrence involving $C_{3,2t-1}$. □

6. CONTINUED FRACTIONS

It will have occurred to many readers that the order-$M = 2$ recurrences, namely for the $C_{3,k}$ and $C_{4,k}$, should give rise to continued fractions, since such fractions are also governed by order-2 recurrences. The classical *Pincherle theorem* [Lorentzen and Waadeland 92, Theorem 7, p. 202], [Bowman and McLaughlin, 02] runs thus:

Theorem 6.1. (Pincherle.) *Let $(a_N : N \in \mathbb{Z}^+), (b_N : N \in \mathbb{Z}^+), (G_N : N = -1, 0, 1, 2, \dots)$ be sequences of complex numbers related for all $N \in \mathbb{Z}^+$ by*

$$G_N = b_N G_{N-1} + a_N G_{N-2},$$

with each $a_N \neq 0$. Denote by P_N/Q_N the convergents to the continued fraction

$$x := \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots}}.$$

If $\lim_N G_N/Q_N = 0$, then the fraction converges and has the value

$$x = -\frac{G_0}{G_{-1}}.$$

Pincherle's theorem may be applied to recurrences of the form in Conjecture 4.1 when $n = 3$ or 4, as established in Theorem 5.4. For these n we have order-2 recurrences:

$$P_{n,0}(k)C_{n,k} + P_{n,1}(k)C_{n,k+2} + P_{n,2}(k)C_{n,k+4} = 0.$$

If we identify $G_N := C_{n,2N+2}$, Pincherle's theorem applies with

$$b_N := -\frac{P_{n,1}(2N-2)}{P_{n,2}(2N-2)}$$

and

$$a_N := -\frac{P_{n,0}(2N-2)}{P_{n,2}(2N-2)},$$

and we obtain a continued fraction with value $x = -C_{n,2}/C_{n,0}$. Similarly, setting $G_N := C_{n,2N+3}$ and suitably modifying the definitions of a_N, b_N gives us a fraction with value $-C_{n,3}/C_{n,1}$.

These machinations result in at least four attractive continued fractions having integer elements. Even though we do not know a single individual value of $C_{3,\mathrm{even}}$, we nevertheless have a fraction for the ratio $C_{3,2}/C_{3,0}$; specifically,

$$18\frac{C_{3,2}}{C_{3,0}} = \cfrac{9 \cdot 1^4}{d(1) - \cfrac{9 \cdot 3^4}{\ddots - \cfrac{9 \cdot (2N-1)^4}{d(N) - \cdots}}},$$

where $d(N) := 40N^2 + 2$. The very form of the fraction elements suggests that this ratio could well be a rational multiple of some brand of L-function, but we have not extensively searched for such.

For the L-function that appears in $C_{3,\mathrm{odd}}$ evaluations, we obtain

$$\frac{2}{\mathrm{L}_{-3}(2)} = 3 - \cfrac{9 \cdot 1^4}{f(1) - \cfrac{9 \cdot 2^4}{\ddots - \cfrac{9 \cdot N^4}{f(N) - \cdots}}}$$

with $f(N) := 10N^2 + 10N + 3$, and so $f(0) = 3$.

Along the same lines one derives a fraction

$$16\frac{C_{4,2}}{C_{4,0}} = \cfrac{1^6}{e(1) - \cfrac{3^6}{\ddots - \cfrac{(2N-1)^6}{e(N) - \cdots}}},$$

where $e(N) := N(20N^2 + 3)$.

Finally, for the $C_{4,\mathrm{odd}}$ we arrive at a fraction for $\zeta(3)$:

$$\frac{12}{7\zeta(3)} = 2 - \cfrac{16 \cdot 1^6}{g(1) - \cfrac{16 \cdot 2^6}{\ddots - \cfrac{16 \cdot N^6}{g(N) - \cdots}}},$$

where $g(N) := (2N+1)(5N^2 + 5N + 2)$, and so $g(0) = 2$. This fraction is structurally reminiscent of the Apéry continued fraction for $\zeta(3)$. (See [Borwein et al. 00] and the references therein.) However, the arguments presented in [Zudilin 03a]—where are derived Catalan-constant and $\zeta(4)$ fractions structurally similar to our L and $\zeta(3)$ fractions above—suggest that irrationality proofs using such fractions are rare. Typically, certain number-theoretic properties of a recurrence must be satisfied for an irrationality proof to be achievable.

Indeed, there are many literature connections involving recurrences, continued fractions, and irrationality [Apéry 79, McLaughlin and Wyshinski 04, Zudilin 02a, Prévost 96, van der Poorten 78, Zudilin 02b, Zudilin 04]. Our recurrence for $C_{4,k}$ in Theorem 5.4 (essentially a recurrence relevant to $\zeta(3)$) can be found in the literature [Almkvist and Zudilin 04, p. 23], and another one for the $C_{3,k}$, and so relevant to $L_{-3}(2)$, can be found also [Zudilin 03b]. We note that irrationality proofs of Apéry type do not appear to arise from the recurrences of the present paper. To our knowledge the number $L_{-3}(2)$ has never been proven irrational.

7. FURTHER ANALYTIC PROPERTIES OF THE $C_{n,k}$

We have investigated interindicial relations of k-variant form, i.e., recurrence relations, but now we turn to relations in which the first index, n, varies.

7.1 Analytic Convolution

Based on another idea of W. Zudilin [Zudilin 06], we sought relations on the *first* index, namely the n of $C_{n,k}$. One result is an analytic convolution theorem, where we recall the definition of the half-plane $\mathcal{K}$ from Section 1, and the renormalization $c_{n,k} := \Gamma(k+1)2^{-n}n!C_{n,k}$:

Theorem 7.1. *For complex $k \in \mathcal{K}$, positive integer n, and integer $q \in [1, n-1]$ we have*

$$c_{n,k} = \frac{1}{2\pi i}\int_{\mathcal{C}} c_{n-q,k+s}c_{q,-1-s}\,ds,$$

where the contour $\mathcal{C}$ runs vertically over $(\lambda - i\infty, \lambda + i\infty)$ with $\Re(\lambda) \in (-1, 0)$.

Remark 7.2. There are at least two remarkable features of this result. First, this is a kind of recurrence on the first index of the $c_{n,k}$ in contrast to the k-recurrences; and second, the convolution surprisingly takes the same form for any (legal) indicial offset q.

Proof: Write our original definition (1–1) in the form

$$\begin{aligned} C_{n,k} &:= \frac{1}{n!}\int dx_1 \cdots dx_n (A+B)^{-k-1} \\ &= \frac{1}{n!}\int dx_1 \cdots dx_n A^{-k-1}\left(1 + \frac{B}{A}\right)^{-k-1}, \end{aligned}$$

where A is the sum of the first $(n-q)$ cosh terms, and B is the sum of the remaining q cosh terms. We then invoke the hypergeometric form of the binomial theorem, namely

$$\begin{aligned} &\left(1+\frac{B}{A}\right)^{-k-1} \\ &\quad = \frac{1}{\Gamma(k+1)}\frac{1}{2\pi i}\int_{\mathcal{C}} \Gamma(1+k+s)\Gamma(-s)\left(\frac{B}{A}\right)^{s} ds. \end{aligned}$$

We can then contemplate integration of A terms over $dx_1 \cdots dx_{n-q}$, and B terms over $dx_{n-q+1} \cdots dx_n$, to obtain

$$\begin{aligned} C_{n,k} &= \frac{1}{\binom{n}{q}}\frac{1}{\Gamma(k+1)}\frac{1}{2\pi i} \\ &\quad \times \int_{\mathcal{C}} \Gamma(k+1+s)\Gamma(-s)C_{n-q,k+s}C_{q,-1-s}ds. \end{aligned}$$

But on renormalization to the little-c forms, this is the statement of the theorem. □

We have not explored all of the implications of this theorem. However, we *can* use it to extend the reach, if you will, of Meijer-G analysis. Though we encountered in Section 2 a certain blockade at $n = 5$—namely, we "ran out" of Meijer representations—we can nevertheless *cast $C_{n,k}$ as an order-$\lfloor (n-1)/2 \rfloor$ nested-Barnes integral.* Evidently, then, the Meijer representations (2–5), (2–6) can be considered in the larger scheme of things as the nested-Barnes cases for $n = 3, 4$.

The first nontrivial case of this "Meijer–Barnes extension" uses Theorem 7.1 with $n = 5, q = 2$ to yield

$$\begin{aligned} c_{5,k} &= \frac{1}{2\pi i}\int_{\mathcal{C}_s} c_{2,k+s}c_{3,-1-s}ds \\ &= -\frac{1}{4\pi^2}\int_{\mathcal{C}_s}\int_{\mathcal{C}_t} c_{2,k+s}c_{2,-1-s+t}c_{2,-1-t}ds, \end{aligned} \tag{7–1}$$

using the contours

$$\mathcal{C}_s := (\lambda - i\infty, \lambda + i\infty) \quad \text{and} \quad \mathcal{C}_t := (\rho - i\infty, \rho + i\infty),$$

where conditions simultaneously sufficient for these contours are

$$\begin{aligned} &\Re(k) + \lambda > -1, \quad -1 + \lambda + \rho \in (-1, 0), \\ &-1 + \rho \in (-1, 0). \end{aligned}$$

Using the explicit resolutions (3–1), (3–2) we arrive at the following twofold nested-Barnes integral (we also here have transformed $(s,t) \mapsto (2s, 2t)$ for notational convenience, and intentionally reverted back to "big-C" notation):

$$\begin{aligned} C_{5,k} &= -\frac{1}{240\pi}\int_{2\mathcal{C}_s}\int_{2\mathcal{C}_t} ds\,dt \\ &\quad \times \frac{\Gamma^3(s+(1+k)/2)\Gamma^3(t-s)}{\Gamma(s+1+k/2)\Gamma(t-s+1/2)}4^{-t}\Gamma^2(-t). \end{aligned} \tag{7–2}$$

It is of interest that another two-dimensional integral—but evidently of markedly different character—was derivable for $C_5 := C_{5,1}$ in the separate treatment [Bailey et al. 06].

7.2 Measure-Theoretic Representation

Again starting from the original definition (1–1) we denote the sum of cosh terms by U, and develop a measure-theoretic form,

$$C_{n,k} = \frac{1}{n!}\int_n^\infty \frac{dU}{U^{k+1}}\frac{\partial}{\partial U}\int_{\sum \cosh x_k \le U} dx_1 \cdots dx_n,$$

or, on integration by parts,

$$C_{n,k} = \frac{k+1}{n!}2^{n+2}\int_0^\infty \frac{rV_n(r)}{(2r^2+n)^{k+2}}\,dr,$$

where the volume V_n is that of a "hyperellipsoid" of "radius" r:

$$V_n(r) := \int_{\sum \sinh^2 y_k \le r^2} dy_1 \cdots dy_n.$$

A test case is $n = 1$, for which $V_1(r) = 2\,\mathrm{arcsinh}\, r$, and this measure-theoretic form agrees with (3–1).

This approach has not been taken further; however, note that we *always* have a one-dimensional integral for any n, an advantage shared by the Bessel-kernel representations. In the measure-theoretic case here, though, all involved functions are elementary. It is also interesting that if we had omniscience in regard to the properties of the hyperellipsoid, we would settle many questions about the $C_{n,k}$.

7.3 An n-Variant Recurrence and the Elusive C_5

Presumably the convolution Theorem 7.1 could be invoked, the resulting residue calculus giving us relations between the $c_{n,k}$ and entities $c_{p,j}$ with $p < n$. However, there is a much more direct way to establish an n-variant recurrence (i.e., now we have the *first* index n changing on $c_{n,k}$). The Bessel-kernel representation (2–1) together with the insertion of one copy of K_0 in the form of an ascending series

$$K_0^{(asc)}(t) = \sum_{k\geq 0} \frac{t^{2k}}{4^k k!^2} \left\{H_k - \gamma - \log(t/2)\right\},$$

see [Abramowitz and Stegun 70, Bailey et al. 06], immediately yields an n-variant recurrence (recall that $c_{n,k} := \Gamma(k+1)2^{-n}n!C_{n,k}$):

$$c_{n,k} = \sum_{m\geq 0} \frac{1}{4^m} \frac{1}{m!^2} \times \left\{(H_m^{(1)} - \gamma + \log 2)c_{n-1,k+2m} - c'_{n-1,k+2m}\right\}, \tag{7–3}$$

where the derivative is with respect to the second index, i.e., $c'_{n,q} := \partial c_{n,q}/\partial q$. Interestingly, for the problematic Ising integral $C_5 := C_{5,1} = c_{5,1}/450$, we actually *know* all of the $c_{4,2m+1}$ in principle, from Theorem 3.2 and the resulting algorithm. Unfortunately, we still do not have a convenient representation for $c'_{4,2m+1}$, but at least we have derived a computational series involving, say, numerical differentiation, for C_5.

7.4 Bessel-Moment Relation

Using an integration by parts, namely

$$\frac{1}{\Gamma(k+1)} \int_0^\infty t^k K_0^n(t)dt = \frac{1}{\Gamma(k+3)} \int_0^\infty t^{k+2} \left(K_0^n(t)\right)'' dt$$

in the original definition (1–1), we can iterate in view of the recurrence Conjecture 4.1 to write an equivalent conjecture as a Bessel-moment phenomenon, with $M := \lfloor (n+1)/2 \rfloor$ as in the conjecture

$$0 \stackrel{?}{=} \int_0^\infty t^{k+2M} \Big(P_M(k)K_0^n + P_{M-1}(k)(K_0^n)'' + \cdots + P_0(k)(K_0^n)^{(2M)}\Big)\, dt.$$

It is remarkable that polynomials $P_0, \ldots, P_M$ exist such that this moment integral appears to vanish for general complex $k \in \mathcal{K}$ (of course, the equivalent recurrence relations are likewise remarkable). Note that the suspected vanishing of the above moment integral has been proven for $n = 1, 2, 3, 4$ and appropriate respective polynomials.

We have not taken this moment relation any further than to make the following observation. Using the asymptotic series [Abramowitz and Stegun 70]

$$K_0^{(asy)}(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t} \sum_{m=0}^\infty \frac{(-1)^m((2m)!)^2}{m!^3(32t)^m}, \tag{7–4}$$

one may ask how the Bessel-moment integral above behaves when the asymptotic form is (naively, perhaps illegally) simply inserted into the integral. Surprisingly, if one truncates the sum (7–4) at a high enough m and solves for the polynomials that *minimize* the k-degree of the moment integral, one evidently finds the correct polynomials exactly.

For example, we took the summation index m up through 18 in (7–4). Then we solved symbolically for the coefficients of the higher powers of k that would make the moment integral's result vanish, and we found that we had detected this relation (4–2) previously, numerically. It was pleasing to find the same polynomials via this admittedly nonrigorous handling of the moment integral. The fascinating nuance here is that evidently, the recurrence polynomials depend in some profound sense on the coefficients of the asymptotic expansion in (7–4).

8. EXTREME-PRECISION NUMERICS

Using the Bessel-kernel representation (2–1), we have calculated to 500-digit accuracy values of $C_{n,k}$ for all integers n, k, where $n \in [2, 12]$ and $k \in [0, 25]$. This was done using the ARPREC arbitrary-precision software [Bailey et al. 02] and the tanh-sinh quadrature scheme [Bailey et al. 05]. We have placed a listing of these numerical values on a web site [Bailey et al. 07]. These were the raw data on which most of our discoveries were based.

9. CONCLUSION AND OPEN PROBLEMS

We wish to emphasize that the interaction of sophisticated numeric and symbolic computing has played an irreplaceable role in the work described herein. Indeed, we believe that these results would have been much more difficult, if not impossible, to deduce without reliance on heavy-duty computer power and sophisticated algorithms. Some of the techniques we employed include extreme-precision quadrature, PSLQ integer-relation-detection programs, generating function packages, high-accuracy least-squares polynomial fitting, and Wilf–Zeilberger theorem-proving software. We wish to thank those who have provided both the hardware and the software we have used.

We finish by recording some of the open problems we find the most compelling:

- While Conjectures 4.2 and 4.3 are probably out of current reach, what progress is possible on Conjecture 4.1? Specifically:
- How might one prove the conjectured recurrence for $n = 5$, from (4–1), using, say, the nested-Barnes representation (7–2)? This might amount to a higher-dimensional application of Wilf–Zeilberger methods [Wilf and Zeilberger 92].
- Is there a reasonable closed form for some or all of the following constants: $C_{5,1}, C_{4,0}, C_{3,2}/C_{3,0}, C_{4,2}/C_{4,0}$?

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