

# Heuristics for the Whitehead Minimization Problem

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In this paper we discuss several heuristic strategies which allow one to solve the Whitehead minimization problem much faster (on most inputs) than the classical Whitehead algorithm. The mere fact that these strategies work in practice leads to several interesting mathematical conjectures. In particular, we conjecture that the length of most nonminimal elements in a free group can be reduced by a Nielsen automorphism which can be identified by inspecting the structure of the corresponding Whitehead graph.

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## 1. INTRODUCTION TO THE WHITEHEAD METHOD

Let  $X = \{x_1, \dots, x_m\}$  be a finite alphabet and  $X^{-1} = \{x^{-1} \mid x \in X\}$  be the set of formal inverses of letters from  $X$  and  $X^{\pm 1} = X \cup X^{-1}$ . A word  $w = y_1 \dots y_n$  in the alphabet  $X^{\pm 1}$  is called *reduced* if  $y_i \neq y_{i+1}^{-1}$  for  $i = 1, \dots, n-1$  (here we assume that  $(x^{-1})^{-1} = x$ ). Applying reduction rules  $xx^{-1} \rightarrow \varepsilon$ ,  $x^{-1}x \rightarrow \varepsilon$  (where  $\varepsilon$  is the empty word), one can reduce each word  $w$  in the alphabet  $X^{\pm 1}$  to a reduced word  $\bar{w}$ . The word  $\bar{w}$  is uniquely defined and does not depend on a particular sequence of reductions. Denote by  $F = F(X)$  the set of reduced words over  $X^{\pm 1}$ . The set  $F$  forms a group with respect to the multiplication  $u \cdot v = \overline{uv}$ , which is called a *free group* with *basis*  $X$ . The cardinality  $|X|$  is called the *rank* of  $F(X)$ . Sometimes we write  $F_n$  instead of  $F$  to indicate that the rank of  $F$  is equal to  $n$ .

A bijection  $\phi : F \rightarrow F$  is called an *automorphism* of  $F$  if  $\phi(uv) = \phi(u)\phi(v)$  for every  $u, v \in F$ . The set  $\text{Aut}(F)$  of all automorphisms of  $F$  forms a group with respect to composition of maps. Every automorphism  $\phi \in \text{Aut}(F)$  is completely determined by the images  $\phi(x)$  of elements  $x \in X$ . The following two subsets of  $\text{Aut}(F)$  play an important part in group theory and topology.

An automorphism  $t \in \text{Aut}(F)$  is called a *Nielsen automorphism* if for some  $x \in X$ ,  $t$  fixes all elements  $y \in X$ ,  $y \neq x$  and maps  $x$  to one of the elements  $x^{-1}$ ,  $y^{\pm 1}x$ ,  $xy^{\pm 1}$ . By  $N(X)$  we denote the set of all Nielsen automorphisms of  $F$ .

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An automorphism  $t \in \text{Aut}(F)$  is called a *Whitehead automorphism* if either  $t$  permutes elements of  $X^{\pm 1}$  or  $t$  fixes a given element  $a \in X^{\pm 1}$  and maps each element  $x \in X^{\pm 1}$ ,  $x \neq a^{\pm 1}$  to one of the elements  $x$ ,  $xa$ ,  $a^{-1}x$ , or  $a^{-1}xa$ . Obviously, every Nielsen automorphism is also a Whitehead automorphism. By  $W(X)$  we denote the set of nontrivial Whitehead automorphisms of the second type.

Observe that

$$|N(X)| = 4n(n-1), \quad |W(X)| = 2n4^{(n-1)} - 2n$$

where  $n = |X|$  is the rank of  $F$ .

It is known [Lyndon and Schupp 77] that every automorphism from  $\text{Aut}(F)$  is a product of finitely many Nielsen (hence Whitehead) automorphisms.

The automorphic orbit  $\text{Orb}(w)$  of a word  $w \in F$  is the set of all automorphic images of  $w$  in  $F$ :

$$\text{Orb}(w) = \{v \in F \mid \exists \varphi \in \text{Aut}(F) \text{ such that } w^\varphi = v\}.$$

A word  $w \in F$  is called *minimal* (or *automorphically minimal*) if  $|w| \leq |w^\varphi|$  for any  $\varphi \in \text{Aut}(F)$ . By  $w_{\min}$  we denote a word of minimal length in  $\text{Orb}(w)$ . Notice that  $w_{\min}$  is not unique.

**Problem 1.1. (Minimization Problem (MP).)** For a word  $u \in F$  find an automorphism  $\varphi \in \text{Aut}(F)$  such that  $u^\varphi = u_{\min}$ .

In 1936 J. H. C. Whitehead proved the following result which gives a solution to the minimization problem [Whitehead 36].

**Theorem 1.2. (Whitehead.)** *Let  $u, v \in F_n(X)$  and  $v \in \text{Orb}(u)$ . If  $|u| > |v|$ , then there exists  $t \in W(X)$  such that*

$$|u| > |ut|.$$

An automorphism  $\phi \in \text{Aut}(F)$  is called a length-reducing automorphism for a given word  $u \in F$  if  $|u\phi| < |u|$ . The theorem above claims that the finite set  $W(X)$  contains a length-reducing automorphism for every non-minimal word  $u \in F$ . This allows one to design a simple search algorithm for MP.

Let  $u \in F$ . For each  $t \in W(X)$  compute the length of the tuple  $ut$  until  $|u| > |ut|$ , then set  $t_1 = t$ ,  $u_1 = ut_1$ . Otherwise stop and set  $u_{\min} = u$ . The procedure above is called the *Whitehead Length Reduction* routine (WLR). Now the *Whitehead Reduction Algorithm* (WRA) proceeds as follows. Repeat WLR on  $u$ , and then on  $u_1$ , and

so on, until on some step  $k$  WLR gives an output  $u_{\min}$ . Then  $ut_1 \dots t_{k-1} = u_{\min}$ , so  $\phi = t_1 \dots t_{k-1}$  is a required automorphism.

Notice that the iteration procedure WRA simulates the classical greedy descent method ( $t_1$  is a successful direction from  $u$ ,  $t_2$  is a successful direction from  $u_1$ , etc.). Theorem 1.2 guarantees that the greedy approach will always converge to the global minimum.

Clearly, there could be at most  $|u|$  repetitions of WLR on an input  $u \in F$

$$|u| > |ut_1| > \dots > |ut_1 \dots t_l| = u_{\min}, \quad l \leq |u|.$$

Hence the worst case, complexity of the algorithm WRA is bounded from above by

$$cA_n|u|^2,$$

where  $A_n = 2n4^{(n-1)} - 2n$  is the number of Whitehead automorphisms in  $W(X)$ . Therefore, in the worst case scenario, the algorithm seems to be impractical for free groups with large ranks. One can try to improve on the number of steps it takes to find a length-reducing automorphism for a given nonminimal element from  $F$ . In this context the main question of interest is the complexity of the following:

**Problem 1.3. (Length Reduction Problem.)** For a given nonminimal element  $u \in F$  find a length-reducing automorphism.

We refer to [Miasnikov and Myasnikov 04] for a general discussion on this problem.

In the next section, we give some empirical evidence that using smart strategies in selecting Whitehead automorphisms  $t \in W(X)$  one can dramatically improve the average complexity of WRA in terms of the rank of a group.

## 2. HEURISTICS FOR THE LENGTH REDUCTION PROBLEM

### 2.1 Nielsen First

The first heuristic comes from a very naive approach: replace  $W(X)$  by  $N(X)$  in WLR and denote the resulting routine by NLR. Since the size of  $N(X)$  is quadratic and the size of  $W(X)$  is exponential in the rank of  $F$ , the algorithm NRA may really speedup computations. However, it is known (see [Lyndon and Schupp 77]) that the Whitehead theorem, Theorem 1.2, above does not hold after replacement of  $W(X)$  by  $N(X)$ . Therefore, the algorithm NRA will not give the correct answer for at least

Dataset	Group	Dataset Size	Min. length	Avg. length	Max. length
$D_3$	$F_3$	10143	3	558.2	1306
$D_4$	$F_4$	10176	4	570.9	1366
$D_5$	$F_5$	10165	5	581.3	1388

**TABLE 1.** Statistics of sets of nonminimal elements.

Dataset	$D_3$	$D_4$	$D_5$
Fraction	0.998	0.997	0.998
95% Conf. Interval	[0.9970,0.9988]	[0.9957,0.9979]	[0.9970,0.9988]

**TABLE 2.** Fraction of elements in the sets  $D_3$ ,  $D_4$ , and  $D_5$  with length-reducing Nielsen automorphisms.

some inputs. But this is not the end of the story. Now the question is: *how often does the length reduction routine NLR give the correct answer?*

To get some insights, we perform a simple experiment. For free groups  $F_3$ ,  $F_4$ , and  $F_5$  we generate test sets of nonminimal elements of Whitehead complexity 1 (see definitions in [Miasnikov and Myasnikov 04]), described in Table 1. For a detailed description of the data generation procedures we refer to [Haralick et al. 04].

For each set  $D_n$  we compute the fraction of elements from  $D_n$  which have length-reducing Nielsen automorphisms. The results of the computations together with the corresponding 95% confidence intervals are given in Table 2. We can see that most of the words have been reduced by Nielsen automorphisms. We would like to mention here, that it can be shown statistically that increasing the length of elements in the datasets does not significantly change the results of experiments.

Based on these experiments one can speculate that, with very high probability, Nielsen automorphisms reduce the length of a given nonminimal element in  $F$ . More precisely, we state the following:

**Conjecture 2.1.** *Let  $U_n$  be the set of all nonminimal elements in  $F$  of length  $n$  and  $NU_n \subset U_n$  the subset of elements which have Nielsen length-reducing automorphisms. Then*

$$\lim_{n \rightarrow \infty} \frac{|NU_n|}{|U_n|} = 1.$$

Our first heuristic is based on this conjecture and simply suggests to try Nielsen automorphisms first in the routine WLR, i.e., in this case we assume that in the fixed listing of automorphisms of  $W(X)$  the automorphism from  $N(X)$  come first. We refer to this heuristic as *Nielsen First* (NF) and denote the corresponding length reduction routine and the Whitehead reduction

algorithm (with respect to this ordering of  $W(X)$ ) by  $WLR_{NF}$  and  $WRA_{NF}$ .

The expected number of steps for the routine  $WLR_{NF}$  to find a length-reducing automorphism on an input  $u \in F$  of length  $n$  is equal to

$$P_n |N(X)| + (1 - P_n)(|W(X)| - |N(X)|),$$

where  $P_n = NU_n/U_n$ .

Given that Conjecture 2.1 is true, we expect  $WRA_{NF}$  to perform much better on average. In the next section, we describe experimental results supporting this strategy.

## 2.2 Cluster Analysis

According to the heuristic NF, one has to apply Nielsen automorphisms to a given input  $w$  in some fixed order, which is independent of the word  $w$ . Intuitively, we expect some automorphisms to be more likely to reduce the length of a given word than the others. It suggests that the conditional probabilities

$$Prob(|wt| < |w| \mid w), \quad t \in N(X)$$

may not be equal for different nonminimal words  $w \in F$ , so the order in which Nielsen automorphisms are applied to an input  $w$  should depend on the word  $w$  itself.

The question we would like to address next is whether it is possible to find a dependence between a nonminimal word  $w$  and its length-reducing Nielsen automorphisms. For this purpose we employ methods from statistical pattern recognition.

Briefly, pattern recognition aims to classify a variety of given objects into categories based on the existing statistical information. The objects are typically presented by collections of measurements or observations (called *features*) that are real numbers. In this event the tuple of features that corresponds to a given object is called a *feature vector*; it can be viewed as a point in the appropriate multidimensional vector space  $\mathbb{R}^d$ . Most of the approaches in statistical pattern recognition are based

on statistical characterizations of features, assuming that objects are generated by a probabilistic system. The detailed description of pattern recognition methods is beyond the scope of this report. We refer interested readers to [Duda et al. 00, Fukunaga 90, Theodoridis and Koutroumbas 99] for a general introduction to the subject, and [Haralick et al. 04] for applications of pattern recognition methods in groups.

Unsupervised learning or *clustering* methods of pattern recognition are used when no a priori information about the objects is available. In this case, there are general algorithms to group the feature vectors of objects into some “natural classes” (called *clusters*) relative to the specified similarity assumptions. Intuitively, the objects whose feature vectors belong to the same cluster are more similar to each other than to the objects with feature vectors in different clusters.

The most simple and widely used clustering scheme is called *K-means*. It is an iterative method. Let  $D = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  be a set of given objects, represented by the corresponding feature vectors  $\mathbf{x} \in \mathbb{R}^d$ . *K-means* begins with a set of  $K$  randomly chosen cluster centers  $\mu_1^0, \dots, \mu_K^0 \in \mathbb{R}^d$ . At iteration  $i$  each feature vector is assigned to the nearest cluster center (in some metric  $\|\cdot\|$  on  $\mathbb{R}^d$ ). This forms the cluster sets  $C_1^i, \dots, C_K^i$ , where

$$C_j^i = \{\mathbf{x} \mid \|\mathbf{x} - \mu_j^i\| \leq \|\mathbf{x} - \mu_m^i\|, \mathbf{x} \in D, m = 1, \dots, K\}.$$

Then each cluster center is redefined as the mean of the feature vectors assigned to the cluster:

$$\mu_k^{i+1} = \frac{1}{|C_k^i|} \sum_{\mathbf{x} \in C_k^i} \mathbf{x}.$$

Each iteration reduces the criterion function  $J^i$ , defined as

$$J^i = \sum_{k=1}^K \sum_{\mathbf{x} \in C_k^i} \|\mathbf{x} - \mu_k^i\|.$$

As this criterion function is bounded below by zero, the iterations must converge. This method works well when clusters are mutually exclusive and compact around their center means.

Here we claim that the *K-means* algorithm allows one to discover some natural classes of nonminimal words. We show below that the analysis of the corresponding cluster structures sheds some light on the relation between nonminimal words and their length-reducing automorphisms.

We define features of elements  $w \in F(X)$  as follows. Recall that the Labelled Whitehead Graph  $WG(w) =$

$(V, E)$  of an element  $w \in F(X)$  is a weighted nonoriented graph, where the set of vertices  $V$  is equal to the set  $X^{\pm 1}$ , and for  $x_i, x_j \in X^{\pm 1}$  there is an edge  $(x_i, x_j) \in E$  if the subword  $x_i x_j^{-1}$  (or  $x_j x_i^{-1}$ ) occurs in the word  $w$  viewed as a cyclic word. Every edge  $(x_i, x_j)$  is assigned a weight  $l_{ij}$  which is the number of times the subwords  $x_i x_j^{-1}$  and  $x_j x_i^{-1}$  occur in  $w$ .

Let  $l(w)$  be a vector of edge weights in the Whitehead graph  $WG(w)$  with respect to a fixed order. We define a feature vector  $f(w)$  by

$$f(w) = \frac{1}{|w|} l(w).$$

To execute the *K-means* algorithm one has to define in advance the expected number of clusters  $K$ . Since we would like these clusters to be related to the set of Nielsen automorphisms  $N(X)$  we put  $K = |N(X)|$ .

To evaluate the usefulness of the clustering we use the goodness measure  $R_{max}$  defined below. Let  $\mathcal{C} \subset D$  be a cluster of the data set  $D \subset F_n = F(X)$ . For  $t \in N(X)$  define

$$R(t, \mathcal{C}) = \frac{|\{w \in \mathcal{C} \mid |wt| < |w|\}|}{|\mathcal{C}|}.$$

The number  $R(t, \mathcal{C})$  shows how many elements in  $\mathcal{C}$  are reducible by  $t$ . Now set

$$R_{max}(\mathcal{C}) = \max\{R(t, \mathcal{C}) \mid t \in N(X)\}$$

and denote by  $t_{\mathcal{C}}$  a Nielsen automorphism  $t \in N(X)$  such that  $R(t_{\mathcal{C}}, \mathcal{C}) = R_{max}(\mathcal{C})$ . The number  $R_{max}(\mathcal{C})$  shows how many elements in  $\mathcal{C}$  can be reduced by a single automorphism, in this case by  $t_{\mathcal{C}}$ . We also define the average value of the goodness measure

$$avg(R_{max}) = \frac{1}{K} \sum_{i=1}^K R_{max}(C_i),$$

where  $K$  is the number of clusters.

The results of the *K-mean* cluster analysis of sets of randomly generated nonminimal elements in free groups  $F_3, F_4, F_5$  are given in Table 3. It shows that more than 70 percent of the elements in every cluster can be reduced by the same Nielsen automorphism. In the free group  $F_3$ , where the number of clusters is significantly smaller, the corresponding number of elements is over 98 percent. Moreover, our experiments show that  $t_{C_i} \neq t_{C_j}$  for  $i \neq j$ . In other words, there are no two distinct clusters such that the same Nielsen automorphism reduces most of the elements in both clusters.

The discovered cluster structure gives rise to the following strategy in solving the length reduction problem

Free group	$F_3$	$F_4$	$F_5$
number of clusters, $K$	24	48	80
$avg(R_{max})$ , $K$ -means	0.985	0.879	0.731

**TABLE 3.** Average values of the goodness measure  $R_{max}$  for  $K$ -means clustering.

for a given word  $w$ . Let  $\mu_1, \dots, \mu_K$  be the centers of clusters  $\mathcal{C}_1, \dots, \mathcal{C}_K$  computed by the  $K$ -means procedure. We compute the distance  $\|f(w) - \mu_i\|$  for each  $i = 1, \dots, K$ . Now we list the Nielsen automorphisms in  $N(X)$  in the order  $t_{i_1}, t_{i_2}, \dots, t_{i_K}$  with respect to the distances

$$\|f(w) - \mu_{i_1}\| \leq \|f(w) - \mu_{i_2}\| \leq \dots \leq \|f(w) - \mu_{i_K}\|.$$

To find a length-reducing automorphism for a given word  $w$  we consecutively apply automorphisms from  $N(X)$  in the prescribed order until we find an automorphism  $t_i \in N(X)$  that reduces the length of  $w$ . If such an automorphism does not exist we proceed with the remaining automorphisms from  $W(X) - N(X)$  as in the NF heuristic.

From the description of the  $K$ -means method we know that clusters are characterized by the center means of the feature vectors of elements in the same cluster. The observations above lead us to the following vaguely stated conjecture, which gives a model to describe behavior of nonminimal elements from  $F$  in terms of their feature vectors.

**Conjecture 2.2.** *The feature vectors of weights of the Whitehead graphs of elements from  $F$  are separated into bounded regions in the corresponding space. Each such region can be bounded by a hypersurface and corresponds to a particular Nielsen automorphism in the sense that all elements in the corresponding class can be reduced by that automorphism.*

### 2.3 Improvement on the Clustering

Experiments with the  $K$ -means clustering algorithm show that clustering is a useful tool in solving the length reduction problem. Now, the goal is to make clustering more effective. Further analysis of the clusters suggests that to some extent they correspond to partitions of elements in  $F$  that can be reduced by one and only one Nielsen automorphism. To verify this conjecture we perform the following experiment.

Let  $S \subset F_n = F(X)$  be a set of randomly generated nonminimal elements and  $D$  the set used for cluster analysis in the previous section. Note that  $S$  is generated independently from the set  $D$ . For each automorphism

Free group	$F_3$	$F_4$	$F_5$
number of clusters, $K$	24	48	80
$avg(R_{max})$ , distance to $\lambda_t$	0.998	0.993	0.991

**TABLE 4.** Average values of the goodness measure  $R_{max}$  for the clustering based on the distance to the estimated centers  $\lambda_t$ .

$t \in N(X)$  set

$$O_t = \{w \in S \mid \forall r \in N(X) (|wr| < |w| \iff r = t)\}$$

and define new cluster centers by

$$\lambda_t = \frac{1}{|O_t|} \sum_{w \in O_t} f(w) \quad (2-1)$$

as the mean feature vector of the elements from  $S$  that can be reduced only by  $t$  and no other automorphisms.

We cluster elements from  $D$  based on the distance between the corresponding feature vector and centers  $\lambda_t$ :

$$\mathcal{C}_t = \{w \in D \mid \forall r \in N(X) (\|f(w) - \lambda_t\| \leq \|f(w) - \lambda_r\|)\}.$$

The results of evaluation of the clusters  $\mathcal{C}_t$  are given in Table 4. One can see that the goodness measure is improved and is close to 1 in every case.

Similar to the strategy based on the centers of the  $K$ -means clusters, we define a new search procedure  $WRA_C$  which employs a heuristic based on the distances to centers  $\lambda_t$ . Let  $w$  be a word and  $\langle \lambda_{t_1}, \dots, \lambda_{t_K} \rangle$  be the centers corresponding to each of the Nielsen automorphisms  $t_i \in N(X)$ . Set  $d(i) = \|f(w) - \lambda_{t_i}\|$  and construct a vector

$$\langle d(m_1), d(m_2), \dots, d(m_K) \rangle,$$

where

$$d(m_1) \leq d(m_2) \leq \dots \leq d(m_K).$$

To find a length reducing automorphism for a given word  $w$ , the algorithm  $WRA_C$  applies Whitehead automorphisms to  $w$  in the following order. First, Nielsen automorphisms  $t_{m_1}, \dots, t_{m_K}$  are applied consecutively. If none of the Nielsen automorphisms reduces the length of  $w$ , the algorithm  $WRA_C$  proceeds with the remaining automorphisms  $W(X) - N(X)$  in some fixed order.

Based on the results of the cluster analysis from Table 4, we expect the algorithm  $WRA_C$  to reduce a non-minimal word  $w$  using, on average, very few elementary automorphisms.

### 2.4 Maximal Weight Edges

Now we would like to take a closer look at the edges' weight distributions at the cluster centers. First, observe that every edge in the Whitehead graph  $WG$ , except for the ones which correspond to subwords of type

$x^2$ ,  $x \in X^{\pm 1}$ , will correspond to subwords reducible by two particular Nielsen transformations. For example, edge connecting nodes  $a$  and  $b$  correspond to subwords  $(ab^{-1})^{\pm 1}$  both of which are reduced by automorphisms

$$\begin{aligned} (a \rightarrow ab, b \rightarrow b), \\ (a \rightarrow a, b \rightarrow ba). \end{aligned}$$

In fact there is no other Nielsen transformation that will reduce the length of words  $(ab^{-1})^{\pm 1}$ .

To generalize, let  $WG(w)$  be a Whitehead graph of a word  $w$  with the vertex set  $V$  and the set of edges  $E$ . Let  $e = (x, y^{-1})$ , where  $x, y^{-1} \in V$ , be an edge in  $E$ . By construction  $e$  corresponds to subwords  $s_e = (xy)^{\pm 1}$  of the word  $w$ . The only Nielsen automorphisms that reduce the length of the subwords  $s_e$  are

$$\psi_e^x : x \rightarrow xy^{-1}, z \rightarrow z, \forall z \neq x, z \in X$$

and

$$\psi_e^y : y \rightarrow x^{-1}y, z \rightarrow z, \forall z \neq y, z \in X.$$

We will call automorphisms  $\psi_e^x, \psi_e^y$  the length-reducing Nielsen automorphisms with respect to the edge  $e = (x, y^{-1})$  and denote  $\psi_e = \{\psi_e^x, \psi_e^y\}$ .

The following phenomenon has been observed for all clusters in free groups  $F_3, F_4$ , and  $F_5$ . Let  $C_t$  be a cluster of a test set  $D_n$ ,  $n = 3, 4, 5$ , then for all  $t \in N(X)$ ,

$$t \in \psi_{e_{max}},$$

where  $e_{max}$  is the edge having the maximal weight in the cluster center  $\lambda_t$ . This suggests that at least in the case of free groups  $F_3, F_4$ , and  $F_5$ , one can try to estimate a length-reducing automorphism for a given word  $w$  by taking the length-reducing Nielsen automorphisms of the highest weight edge in the Whitehead graph  $WG(w)$ .

To evaluate the goodness of the heuristic based on the maximal edge weight in the Whitehead graph we compute the fraction of elements in the sets  $D_3, D_4$ , and  $D_5$  reducible by the Nielsen automorphisms corresponding to the maximal weight edge. The corresponding goodness measure, evaluated on a set  $D$ , is given by

$$\mathcal{G}_{MAX} = \frac{1}{|D|} |\{w \in D \mid \exists t \in \psi_{e_{max}(w)}, \text{ s.t. } |wt| < |w|\}|.$$

Values of the goodness measure  $\mathcal{G}_{MAX}$  for test sets in free groups  $F_3, F_4$ , and  $F_5$  are given in Table 5. It shows that this heuristic is surprisingly effective. Nevertheless, the centroid-based method still yields better results. Note that  $\mathcal{G}_{MAX}$  measures the success of applying two automorphisms corresponding to the maximal weight

Dataset	$D_3$	$D_4$	$D_5$
$\mathcal{G}_{MAX}$	0.991	0.986	0.986

TABLE 5. Values of the goodness measure  $\mathcal{G}_{MAX}$  for sets of nonminimal elements in free groups  $F_3, F_4$ , and  $F_5$ .

edge, whereas the centroid-based method was evaluated by the success rate of only one automorphism which corresponds to the closest center.

The observation provides a new search procedure which we denote by  $WRA_{MAX}$ . Let  $w$  be a word and  $WG(w) = (V, E)$  be the corresponding Whitehead graph. Denote by  $E'$  the set of edges which do not correspond to the subwords of type  $x^{\pm 2}$ ,  $x \in X$ ,

$$E' = \{e \in E \mid e \neq (v, v^{-1}), v \in V\}.$$

It has been shown above, that for each edge  $e$  from  $E'$  there exists two unique length-reducing automorphisms. Note that  $2|E'| = |N(X)|$ , where  $N(X)$  is the set of Nielsen automorphisms for free group  $F(X)$ .

We can order Nielsen automorphisms  $\psi_{e_i} \subset N(X)$ :

$$\langle \psi_{e_1}, \psi_{e_2}, \dots, \psi_{e_{|E'|}} \rangle \tag{2-2}$$

such that edges  $e_1, \dots, e_{|E'|}$  are chosen according to the decreasing order of the values of the corresponding weights

$$\omega_{e_1} \geq \omega_{e_2} \geq \dots \geq \omega_{e_{|E'|}}.$$

Note that  $\psi_e$  is not a single automorphism, but a pair of Nielsen length-reducing automorphisms with respect to the edge  $e$ . Here we do not give any preference in ordering automorphisms in  $\psi_e$ .

To find a length-reducing automorphism for  $w$ , procedure  $WRA_{MAX}$  first applies Nielsen automorphisms in the order given by (2-2). If none of the Nielsen automorphisms reduces the length of  $w$ ,  $WRA_{MAX}$  proceeds with the remaining automorphisms from  $W(X) - N(X)$ .

### 3. COMPARISON OF THE STRATEGIES

In this section we describe experiments designed to compare the performance of WRA implemented with different search strategies. We compare four variations of the algorithm.  $WRA_R$  is the variation of WRA, where a random order of the elements from  $W_n$  is used when searching for length-reducing automorphisms.  $WRA_{NF}$  and  $WRA_C$  correspond to the implementations with Nielsen first and centroid-based heuristics respectively. The algorithm  $WRA_{MAX}$  employs the strategy which applies automorphisms corresponding to the largest edge weights

Dataset	Group	Dataset Size	Min. length	Avg. length	Max. length
$S_3$	$F_3$	5645	3	1422.1	143020
$S_4$	$F_4$	5241	4	2513.1	168353
$S_5$	$F_5$	3821	5	2430.5	160794

**TABLE 6.** Statistics of the test sets of primitive elements.

Strategy	$N_{total}$	$N_{red}$	$N_{LRP}$
$WRA_C$	19.9	18.4	1.1
$WRA_{MAX}$	47.1	23.9	1.9
$WRA_{NF}$	207.8	28.2	7.37
$WRA_R$	374.8	29.8	12.6

(a)  $F_3$ ;

Strategy	$N_{total}$	$N_{red}$	$N_{LRP}$
$WRA_C$	58.8	34.1	1.4
$WRA_{MAX}$	152.8	42.5	3.0
$WRA_{NF}$	1052.6	56.2	18.7
$WRA_R$	2610.4	58.8	44.4

(b)  $F_4$ ;

Strategy	$N_{total}$	$N_{red}$	$N_{LRP}$
$WRA_C$	162.0	50.9	2.4
$WRA_{MAX}$	342.2	58.8	4.5
$WRA_{NF}$	2307.6	75.4	30.6
$WRA_R$	15939.6	78.8	202.0

(c)  $F_5$ .

**TABLE 7.** Results of experiments with sets of primitive elements in free groups  $F_3$ ,  $F_4$ , and  $F_5$ . Counts are averaged over all inputs.

of the Whitehead graph. The algorithms were compared on randomly generated sets of primitive elements  $S_3$ ,  $S_4$ ,  $S_5$  in free groups  $F_3$ ,  $F_4$ , and  $F_5$ , respectively. Some descriptive statistics of the test sets  $S_n$  are given in Table 6.

Let  $\mathcal{A}$  be one of the variations  $WRA_R$ ,  $WRA_{NF}$ ,  $WRA_C$ , or  $WRA_{MAX}$  of the Whitehead reduction algorithm. By an elementary step of the algorithm  $\mathcal{A}$  we mean one application of a Whitehead automorphism to a given word. Below we evaluate the performance of  $\mathcal{A}$  with respect to the number of elementary steps that are required by  $\mathcal{A}$  to execute a particular routine.

Let  $N_{total} = N_{total}(\mathcal{A}, S_n)$  be the average of the total number of elementary steps required by  $\mathcal{A}$  to reduce a given primitive element  $w \in S_n$  to a generator.

By  $N_{red} = N_{red}(\mathcal{A}, S_n)$  we denote the average number of elementary length-reducing steps required by  $\mathcal{A}$  to reduce a given primitive element  $w \in S_n$  to a generator, so  $N_{red}$  is the average number of “productive” steps performed by  $\mathcal{A}$ . It follows that if  $t_1, \dots, t_l$  are all the length-reducing automorphisms found by  $\mathcal{A}$  when execut-

ing its routine on an input  $w \in S_n$ , then  $|wt_1 \dots t_l| = 1$  and the average value of  $l$  is equal to  $N_{red}$ .

Finally, denote by  $N_{LRP} = N_{LRP}(\mathcal{A}, S_n)$  the average number of elementary steps required by  $\mathcal{A}$  to find a length-reducing automorphism for a given nonminimal input  $w$ .

In Table 7 we present results of our experiments on the performance of the algorithms  $WRA_R$ ,  $WRA_{NF}$ ,  $WRA_C$ , and  $WRA_{MAX}$  on the test sets  $S_n$ ,  $n = 3, 4, 5$ . The algorithms compare as expected. The algorithms  $WRA_C$  and  $WRA_{MAX}$  perform very efficiently with the numbers  $N_{LRP}$  and  $N_{red}$  being small. Algorithm  $WRA_C$ , based on the centroid approach, shows best overall performance. The growth of the numbers  $N_{LRP}$  with the rank could be explained by the occasional occurrence of nonminimal words that cannot be reduced by Nielsen automorphisms. In this event the algorithm tries Whitehead automorphisms from  $W(X) - N(X)$  the number of which grow exponentially with the rank. Notice that in every one of our experiments the number  $N_{red}$  of length reductions performed by  $WRA_C$  is less than the corresponding number in the other approaches.

In Table 8 we give the correlation coefficients showing dependence of the number of elementary steps required by a particular algorithm to find a length-reducing automorphism with respect to the length of the input words. The coefficients are negative in all cases which indicates that the values of  $N_{LRP}$  do not increase when the words’ lengths increase.

Strategy	$F_3$	$F_4$	$F_5$
$WRA_C$	-0.008	-0.001	-0.006
$WRA_{MAX}$	-0.024	-0.023	-0.038
$WRA_{NF}$	-0.009	-0.022	-0.022
$WRA_R$	-0.038	-0.014	-0.035

**TABLE 8.** Correlation coefficients between word lengths and values of  $N_{LRP}$ . Negative coefficients indicate that  $N_{LRP}$  does not increase when length increases.

#### 4. CONCLUSIONS

The experimental results presented in this paper show that using appropriate heuristics in the algorithm  $WRA$ ,

one can significantly reduce the complexity of the Whitehead minimization problem on most inputs with respect to the group rank. Suggested heuristics reduce the average number of Whitehead automorphisms required to find a length-reducing automorphism for a given word. The performance of heuristic algorithms tested on the sets of randomly generated primitive elements shows robust behavior and does not deteriorate when the lengths of the input words increase.

One of the interesting contributions of this paper is the empirically discovered properties of nonminimal elements of free groups formulated in Conjectures 2.1 and 2.2. These conjectures suggest that the length of “generic” nonminimal elements in a free group can be reduced by a Nielsen automorphism. Moreover, the feature vectors of the weights of the Whitehead graphs of nonminimal elements are divided into “compact” regions in the corresponding vector space. Each such region is related to a particular Nielsen automorphism, that reduces the length of all elements in the region. We believe this is one of those few cases when a meaningful, rigorous, but not intuitively clear conjecture, in group theory was obtained by using experimental simulations and statistical analysis of the problem.

It remains to be seen why the algorithm  $WRA_C$  is, on average, able to find minimal elements using a smaller number of length reductions. We are going to address this issue in a subsequent paper.

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