The Space of Clouds in Euclidean Space

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CONTENTS

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We study the space $\mathcal{N}_d^m$ of clouds in $\mathbb{R}^d$ (ordered sets of m points modulo the action of the group of affine isometries). We show that $\mathcal{N}_d^m$ is a smooth space, stratified over a certain hyperplane arrangement in $\mathbb{R}^m$. We give an algorithm to list all the chambers and other strata (this is independent of d). With the help of a computer, we obtain the list of all the chambers for $m \leq 9$ and all the strata when $m \leq 8$. As the strata are the product of a polygon space with a disk, this gives a classification of m-gon spaces for $m \leq 9$. When $d = 2,3$, $m = 5,6,7$, and modulo reordering, we show that the chambers (and so the different generic polygon spaces) are distinguished by the ring structure of their mod 2-cohomology.

1. INTRODUCTION

Let E be an oriented finite-dimensional Euclidean space. Let $\mathcal{N}_E^m$ be the space of ordered sets of m points in E, modulo the group of rigid motions of E; more precisely,

$$\mathcal{N}_E^m := G(E)\backslash E^m,$$

where the Lie group $G(E)$ is the semidirect product of the translation group of E by $SO(E)$, the group of linear orientation-preserving isometries of E, and the group $G(E)$ acts diagonally on E^m. We shall occasionally consider the space $\bar{\mathcal{N}}_E^m = \bar{G}(E)\backslash E^m$, where $\bar{G}(E)$ is the group of all affine isometries of E. Observe that $\bar{\mathcal{N}}_E^m$ is a subspace of $\mathcal{N}_{E'}^m$ when E is a proper subspace of E'. An element of $\mathcal{N}_E^m$ will be called a *cloud* of m points in E (the letter $\mathcal{N}$ stands for "nuage," meaning "cloud" in French). We abbreviate $\mathcal{N}_{\mathbb{R}^d}^m$ to $\mathcal{N}_d^m$. Observe that $\mathcal{N}_E^m$ is canonically homeomorphic to $\mathcal{N}_d^m$, when $d = \dim E$.

The space $\mathcal{N}_d^m$ plays a natural role in celestial mechanics, at least for $d = 2$ or 3 (see, for instance, [Albouy and Chenciner 98]). Moreover, its importance was recognized especially in statistical shape theory, a subject which has developed rapidly during the last two decades (see [Small 96] and [Kendall et al. 1999] for a history). There, the space $\mathcal{N}_d^m$ is called the *size-and-shape space* and is denoted by $S\Sigma_d^m$ [Kendall et al. 1999, Section 11.2].

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This terminology and notation emphasizes that $\mathcal{N}_d^m$ is the cone, with vertex $\mathcal{N}_0^m$, over the *shape space* Σ_d^m, defined as the quotient of $\mathcal{N}_d^m - \mathcal{N}_0^m$ by the homotheties. A great amount is known about the homotopy type of shape spaces. For instance, in [Kendall et al. 1999], Kendall, Barden, Carne, and Le show that Σ_d^m admits cellular decompositions leading to a complete computation of its homology groups.

In this paper, we present an alternative decomposition of the space $\mathcal{N}_d^m$. It is based on polygon spaces, a subject which has also encountered a rich development during the last decade, in connection with Hamiltonian geometry. This approach is completely different from that of statistical shape theory and this paper is essentially self-contained.

First of all, the point set topology of $\mathcal{N}_d^m$ is well behaved and $\mathcal{N}_d^m$ is endowed with a smooth structure. More precisely, the translations act freely and properly on E^m with quotient diffeomorphic to the vector subspace $\mathcal{K}(E^m) = \{(z_1, \dots, z_m) \in E^m \mid \sum z_i = 0\}$. Being therefore the quotient of $\mathcal{K}(E^m)$ by the action of the compact group $SO(E)$, the space of clouds $\mathcal{N}_E^m$ is locally compact (in particular Hausdorff). Classical invariant theory provides a proper topological embedding φ of $\mathcal{N}_d^m$ into a Euclidean space $\mathbb{R}^N$ (see Section 2.2). This embedding makes $\mathcal{N}_E^m$ a *smooth space*, i.e., a topological space together with an algebra $\mathcal{C}^\infty(\mathcal{N}_E^m)$ of *smooth functions* (with real values): those functions which are locally the composition of φ with a $\mathcal{C}^\infty$-function on $\mathbb{R}^N$. One can prove that $f \in \mathcal{C}^\infty(\mathcal{N}_E^m)$ if and only if $f \circ \pi$ is smooth on E^m, where $\pi : E^m \to \mathcal{N}_E^m$ is the natural projection (Proposition 2.1). Any subspace of a smooth space naturally inherits a smooth structure and, together with *smooth maps* (see Section 2.1), smooth spaces form a category whose equivalences are called *diffeomorphisms*. Finally, we mention that the space $\mathcal{N}_3^m$ has the special feature that the smooth maps admit a Poisson bracket (see Section 2.5).

Our main tool for stratifying the space $\mathcal{N}_E^m$ is the map $\ell : \mathcal{N}_E^m \to \mathbb{R}^m$ defined on $\rho = (\rho_1, \dots, \rho_m) \in E^m$ by

$$\ell(\rho) := \big(|\rho_1 - b(\rho)|, \dots, |\rho_m - b(\rho)|\big),$$

where $b(\rho) = \frac{1}{m}\sum \rho_i$ is the barycentre of ρ. The map ℓ is continuous and is smooth on $\ell^{-1}((\mathbb{R}_{>0})^m)$, the open subset of points $\rho \in \mathcal{N}_E^m$ such that no ρ_i is equal to $b(\rho)$.

We shall prove in Section 3. that the critical points of ℓ are the one-dimensional clouds $\bar{\mathcal{N}}_1^m \subset \mathcal{N}_E^m$. The space of critical values is then an arrangement of hyperplanes in $\mathbb{R}^m$ that we shall describe now. Let $\underline{m} := \{1, 2, \dots, m\}$ and denote by $\mathcal{P}(\underline{m})$ the family of subsets of $\underline{m}$. For $I \in \mathcal{P}(\underline{m})$, let $\mathcal{H}_I$ be the hyperplane of $\mathbb{R}^m$ defined by

$$\mathcal{H}_I := \Big\{(a_1, \dots, a_m) \in \mathbb{R}^m \;\Big|\; \sum_{i\in I} a_i = \sum_{i \notin I} a_i\Big\}.$$

We call these hyperplanes *walls*. They determine a stratification $\mathcal{H}(\mathbb{R}^m)$ of $\mathbb{R}^m$, i.e., a filtration

$$\{0\} = \mathcal{H}^{(0)}(\mathbb{R}^m) \subset \mathcal{H}^{(1)}(\mathbb{R}^m) \subset \cdots \subset \mathcal{H}^{(m)}(\mathbb{R}^m) = \mathbb{R}^m,$$

with $\mathcal{H}^{(k)}(\mathbb{R}^m)$ being the subset of those $a \in \mathbb{R}^m$ which belong to at least $m-k$ distinct walls $\mathcal{H}_I$. A *stratum* of dimension k is a connected component of $\mathcal{H}^{(k)}(\mathbb{R}^m) - \mathcal{H}^{(k-1)}(\mathbb{R}^m)$. Note that a stratum of dimension $k \geq 1$ is an open convex cone in a k-plane of $\mathbb{R}^m$. Strata of dimension m are called *chambers*. We denote by $\mathrm{Str}(a)$ the stratum of $a \in \mathbb{R}^m$. If $\mathrm{Str}(a)$ is a chamber, the m-tuple a is called *generic* and we will often denote $\mathrm{Str}(a)$ by $\mathrm{Ch}(a)$.

The stratification $\mathcal{H}(\mathbb{R}^m)$ induces a stratification of any subset U of $\mathbb{R}^m$, in particular of $U = (\mathbb{R}_{>0})^m$. Write $\mathrm{Str}(U)$ for the set of all the strata of U. $\mathcal{H}(\mathcal{N}_E^m)$ by defining $\mathcal{N}_E^m(a)$ the preimage $\ell^{-1}(\{a\})$ of $a \in \mathbb{R}^m$. Strengthening results of [Hausmann and Knutson 98], we shall prove the following theorem in Section 3.

Theorem 1.1. *Let $a \in (\mathbb{R}_{>0})^m$. Then there is a diffeomorphism from $\ell^{-1}(\mathrm{Str}(a))$ onto to $\mathcal{N}_E^m(a) \times \mathrm{Str}(a)$ intertwining the map ℓ with the projection to $\mathrm{Str}(a)$.*

In the proof of Theorem 1.1, we actually construct, when $\mathrm{Str}(a) = \mathrm{Str}(b)$, a diffeomorphism $\psi_{ba} : \mathcal{N}_E^m(a) \xrightarrow{\approx} \mathcal{N}_E^m(b)$. One has $\psi_{ab} = \psi_{ba}^{-1}$ and $\psi_{aa} = \mathrm{id}$. For $\alpha \in \mathrm{Str}((\mathbb{R}_{>0})^m)$, we will sometimes use the notation $\mathcal{N}_E^m(\alpha)$ for any of the spaces $\mathcal{N}_E^m(a)$ with $a \in \alpha$. This is in fact ambiguous because, in general, $\psi_{ca} \neq \psi_{cb} \circ \psi_{ba}$, so one cannot use the maps ψ_{ba} to define an equivalence relation on $\ell^{-1}(\mathrm{Str}(a))$ giving the points of $\mathcal{N}_E^m(\alpha)$. However, ψ_{ca} is isotopic to $\psi_{cb} \circ \psi_{ba}$, and so the homotopy invariants of $\mathcal{N}_E^m(\alpha)$, for instance, the elements of its cohomology ring, are well defined.

Theorem 1.1 may provide good local models for describing the evolution of a cloud. This is especially likely when $\dim E = 2, 3$, where, for generic a, the spaces $\mathcal{N}_E^m(a)$ and thus $\ell^{-1}(\mathrm{Str}(a))$ are smooth manifolds (see below).

Theorem 1.1 shows that $\mathcal{N}_E^m$ is obtained by gluing together pieces of the form $\mathcal{N}_E^m(\alpha) \times \alpha$ for various $\alpha \in \mathrm{Str}((\mathbb{R}_{>0})^m)$. From this point of view, the following questions are natural.

1. Describe the set of all strata of $(\mathbb{R}_{>0})^m$, in particular the set of chambers. This combinatorial problem does not depend on E.

2. Describe $\mathcal{N}_E^m(\alpha)$ for all $\alpha \in \mathrm{Str}((\mathbb{R}_{>0})^m)$.

3. Describe how a stratum of $\mathcal{H}(\mathcal{N}_E^m)$ is attached to its bordering strata of lower dimension.

The main issue of this paper is to answer Question 1 and partly Question 2 above. It is convenient to take advantage of the right action of the symmetric group Sym_m on $\mathcal{N}_E^m$ and on $\mathbb{R}^m$ by permutation of the coordinates (to deal directly with the smooth space $\mathcal{N}_E^m/\mathrm{Sym}_m$ and get a corresponding statement of Theorem 1.1, see Section 2.4). This action permutes the strata of $\mathcal{H}((\mathbb{R}_{>0})^m)$, and $\mathcal{N}_E^m(\alpha)$ is diffeomorphic to $\mathcal{N}_E^m(\alpha^\sigma)$ for $\sigma \in \mathrm{Sym}_m$. The map ℓ is equivariant and each $a \in (\mathbb{R}_{>0})^m$ has a unique representative in $\mathbb{R}^m_{\nearrow}$, where

$$\mathbb{R}^m_{\nearrow} := \{(a_1, \dots, a_m) \in \mathbb{R}^m \mid 0 < a_1 \leq \cdots \leq a_m\}.$$

Therefore, the set $\mathrm{Str}((\mathbb{R}_{>0})^m)/\mathrm{Sym_m}$ is in bijection with the set $\mathrm{Str}(\mathbb{R}^m_{\nearrow})$.

In Sections 4 and 5, we show how to obtain a complete list of the elements of $\mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ and $\mathrm{Str}(\mathbb{R}^m_{\nearrow})$. For this, we first show that the set of inequalities defining a chamber α of $\mathbb{R}^m_{\nearrow}$ can be recovered from some very concentrated information that we call the *genetic code* of α. Abstracting some properties of these genetic codes gives rise to the combinatorial notion of a *virtual genetic code*. We design an algorithm to find all virtual genetic codes, with the help of a computer (the program, written in C^{++}, is available at [Hausmann and Rodriguez 02]). Deciding which virtual genetic code is the genetic code of a chamber (realizability) is essentially done using the simplex algorithm of linear programming. We thus obtain the list of all the chambers of $\mathbb{R}^m_{\nearrow}$, with the restriction $m \leq 9$ due to the computer's limited capacities. The set $\mathrm{Str}(\mathbb{R}^{m-1}_{\nearrow})$ is determined using an injection of $\mathrm{Str}(\mathbb{R}^{m-1}_{\nearrow})$ into $\mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ (see Section 5). The number of elements of these sets is

m	3	4	5	6	7	8	9
$\lvert\mathrm{Ch}(\mathbb{R}^m_{\nearrow})\rvert$	2	3	7	21	135	2470	175428
$\lvert\mathrm{Str}(\mathbb{R}^m_{\nearrow})\rvert$	3	7	21	117	1506	62254	?

. (1–1)

It turns out that, for $m \leq 8$, all virtual genetic codes are realizable, but not for $m = 9$: only 175428 out of 319124 are realizable. The nonrealizable ones might well be of interest (see Problem 7.9).

Our algorithms produce, in each chamber α, a distinguished element $a_{\min}(\alpha) \in \mathbb{R}^m_{\nearrow}$ with integral coordinates and with $\sum a_i$ minimal. Several theoretical questions about these elements $a_{\min}(\alpha)$ remain open (see Section 4).

To describe the spaces $\mathcal{N}_E^m(a)$ for $a \in \alpha$ (Question 2 above), we note that

$$\mathcal{N}_E^m(a) = SO(E)\backslash\Big\{\rho \in E^m \;\Big|\; \sum_{i=1}^m \rho_i = 0 \text{ and } |\rho_i| = a_i\Big\}.$$

The condition $\sum_{i=1}^m \rho_i = 0$ suggests the picture of a closed m-step piecewise-linear path in E, whose ith step has length a_i. Therefore, the space $\mathcal{N}_E^m(a)$ is often called the m-gon space (in E) of type a (we could call it the space of clouds "calibrated at a"). These polygon spaces have been studied under different notations, especially for $\dim E = 2$ and 3 where, for generic a, they are manifolds: see, for instance, [Klyachko 94], [Kapovich and Millson 96], [Hausmann and Knutson 97], [Hausmann and Knutson 98]. For $\dim E > 3$ or for a nongeneric, see [Kamiyama 98] and [Kamiyama and Tezuka 99].

The classification of the polygon spaces $\mathcal{N}_E^m(a)$, for generic a, was previously known when $\dim E = 2, 3$ and $m \leq 5$ (see, for instance [Hausmann and Knutson 97, Section 6]). The genetic codes introduced in this paper extend this classification up to $m = 9$. In Section 6, we give handle-decomposition information about the 6-gon spaces $\bar{\mathcal{N}}_2^6$ for the 21 chambers of $\mathbb{R}^6_{\nearrow}$. This type of method could be applied to any space $\mathcal{N}_E^m(a)$ for generic a. In addition to these geometric descriptions, algorithms were previously found that compute cohomological invariants of the spaces $\mathcal{N}_3^m(a)$, for example, their Poincaré polynomial ([Klyachko 94, Theorem 2.2.4], [Hausmann and Knutson 98, Corollary 4.3]). This enables us, in Section 7., to compute the Betti numbers of the spaces $\mathcal{N}_3^m(\alpha)$ for $m \leq 9$. Moreover, presentations of the cohomology ring of $\mathcal{N}_3^m(\alpha)$ for any coefficients were given in [Hausmann and Knutson 98, Theorem 6.4]. This permits us to compute some invariants of the ring $H^*(\mathcal{N}_3^m(\alpha); \mathbb{F}_2)$ and prove in Section 7.1 the following result:

Proposition 1.2. *For $5 \leq m \leq 7$, the spaces $\mathcal{N}_3^m(\alpha)$ (or $\bar{\mathcal{N}}_2^m(\alpha)$), for distinct chambers α of $\mathbb{R}^m_{\nearrow}$, have nonisomorphic* mod 2*-cohomology rings.*

Here, the ring structure of $H^*(\mathcal{N}_3^m; \mathbb{Z}_2)$ is important: the Betti numbers alone do not distinguish the spaces. Interestingly enough, the virtual genetic codes which are not realizable also give rise to nontrivial graded rings.

We do not know if these rings are cohomology rings of a space or of a manifold (see Problem 7.9).

The paper is organized as follows. In Section 2, we set the background of the smooth structure on $\mathcal{N}_E^m$ which is used in Section 3 to prove Theorem 1.1. In Section 4, we introduce the genetic code of a chamber and show how to obtain the list of all chambers of $\mathbb{R}^m_{\nearrow}$ for $m \leq 9$. In Section 5, we study the injection $\mathrm{Str}(\mathbb{R}^{m-1}_{\nearrow})$ into $\mathrm{Str}(\mathbb{R}^m_{\nearrow})$ and show how to obtain the list of all strata of $\mathbb{R}^m_{\nearrow}$ for $m \leq 8$. Section 6 contains our information on the spaces $\mathcal{N}_3^m(a)$ and $\bar{\mathcal{N}}_2^m(a)$ for generic a. Section 7 is devoted to the cohomology invariants of the polygon spaces. Finally, the results of Sections 6 and 7 are applied in Section 8 to the case of hexagon spaces.

2. THE SMOOTH STRUCTURE ON $\mathcal{N}_E^m$

2.1 Smooth Spaces and Maps

For a topological space X, denote by $\mathcal{C}^0(X)$ the $\mathbb{R}$-algebra of continuous functions on X with real values. If $h : X \to Y$ is a continuous map, denote by $h^* : \mathcal{C}^0(Y) \to \mathcal{C}^0(X)$ the map $h^*(f) = f \circ h$.

Let X be a subspace of $\mathbb{R}^N$. A map $f : X \to \mathbb{R}$ is *smooth* if, for each $x \in X$, there exists an open set U of $\mathbb{R}^N$ containing x and a $\mathcal{C}^\infty$ map $F : U \to \mathbb{R}$ which coincides with f throughout $U \cap X$ (compare to [Milnor 65, Section 1]). The smooth maps on X constitute a subalgebra $\mathcal{C}^\infty(X)$ of $\mathcal{C}^0(X)$.

More generally, if $\varphi : X \to \mathbb{R}^N$ is a topological embedding of a space X into $\mathbb{R}^N$, one may consider the subalgebra $\mathcal{C}^\infty(X) = \varphi^*(\mathcal{C}^\infty(\varphi(X)))$. We call $\mathcal{C}^\infty(X)$ a *smooth structure* on X and X (or rather the pair $(X, \mathcal{C}^\infty(X))$) a *smooth space*.

A continuous map $h : X \to Y$ between smooth spaces is called *smooth* if $h^*(\mathcal{C}^\infty(Y)) \subset \mathcal{C}^\infty(X)$. The map h is a *diffeomorphism* if and only if it is a homeomorphism and h and h^{-1} are smooth. It is a *smooth embedding* if $h^*(\mathcal{C}^\infty(Y)) = \mathcal{C}^\infty(X)$. A smooth embedding is thus a diffeomorphism onto its image.

2.2 The Smooth Structure on $\mathcal{N}_E^m$

Let $\kappa : E^m \to E^m$ be the linear projection

$$\kappa(z_1, \dots, z_m) = (z_1 - b(z), \dots, z_m - b(z)),$$

where $b(z) = \frac{1}{m}\sum z_i$ is the barycentre of z. The image of κ is $\mathcal{K}(E^m)$ and its kernel is the diagonal Δ in E^m.

The normal subgroup E in $G(E)$ of translations acts freely and properly on E^m and the quotient space $E\backslash E^m$ is the same as the quotient vector space E^m/Δ. The projection κ descends to a linear isomorphism $\bar{\kappa} : E\backslash E^m \xrightarrow{\approx} \mathcal{K}(E^m)$. The space $\mathcal{N}_E^m$ is now the quotient of $\mathcal{K}(E^m)$ by the action of the compact group $SO(E)$. Therefore, $\mathcal{N}_E^m$ is a locally compact Hausdorff space.

Consider the m^2 polynomial functions on E^m given by $z \mapsto \langle \kappa(z_i), \kappa(z_j) \rangle$, where $\langle , \rangle$ denotes the scalar product on E. Choose an orientation on E. The determinants $|\kappa(z_{i_1}), \cdots, \kappa(z_{i_k})|$ with $i_1 < \cdots < i_k$ $(k = \dim E)$ are another family of $\binom{m}{k}$ polynomial functions on E^m. All these functions are $G(E)$-invariant and produce a continuous map $\varphi : \mathcal{N}_E^m \to \mathbb{R}^N$ with $N = m^2 + \binom{m}{k}$. It is an exercise to prove that φ is injective and proper. As $\mathcal{N}_E^m$ is locally compact, the map φ is a topological embedding of $\mathcal{N}_E^m$ into $\mathbb{R}^N$ and its image is closed (for a family of inequalities defining $\varphi(\mathcal{N}_E^m)$ as a semialgebraic set, see [Procesi and Schwarz 85]).

The embedding φ endows $\mathcal{N}_E^m$ with a smooth structure. The following proposition identifies $\mathcal{C}^\infty(\mathcal{N}_E^m)$ with the smooth functions on E^m which are $G(E)$-invariant.

Proposition 2.1. *Let $f \in \mathcal{C}^0(\mathcal{N}_E^m)$. The following are equivalent:*

(A) $f \in \mathcal{C}^\infty(\mathcal{N}_E^m)$.

(B) There is a global $\mathcal{C}^\infty$-function $F : \mathbb{R}^N \to \mathbb{R}$ such that $f = F \circ \varphi$.

(C) The map $f \circ \pi : E^m \to \mathbb{R}$ is $\mathcal{C}^\infty$, where $\pi : E^m \to \mathcal{N}_E^m$ denotes the natural projection.

Proof: It is clear that (B) implies (A). Conversely, let $f \in \mathcal{C}^\infty(\mathcal{N}_E^m)$. For every $\rho \in \mathcal{N}_E^m$, one has an open set U_ρ of $\mathbb{R}^N$ containing $\varphi(\rho)$ and a smooth function $F_\rho : U_\rho \to \mathbb{R}$ with $f_\rho \circ \varphi = f$ on $\varphi^{-1}(U_\rho)$. Call $U_\infty = \mathbb{R}^N - \varphi(\mathcal{N}_E^m)$ and $F_\infty : U_\infty \to \mathbb{R}$ the constant map to 0. As $\varphi(\mathcal{N}_E^m)$ is closed, the family $\mathcal{U} := \{U_\rho\}_{\rho \in \mathcal{N}_E^m \cup \{\infty\}}$ is an open covering of $\mathbb{R}^N$. Let $\mu_\rho : \mathbb{R}^N \to \mathbb{R}$ be a smooth partition of the unity subordinated to $\mathcal{U}$. Then $F(x) = \sum_{\rho \in \mathcal{N}_E^m \cup \{\infty\}} \mu_\rho(x) F_\rho(x)$ satisfies (B).

Statement (B) is obviously stronger than (C) (which, incidentally, implies that π is a smooth map). For the converse, one uses that the components of φ constitute a generating set for the algebra of $SO(E)$-invariant polynomial functions on $\mathcal{K}(E^m)$ [Weyl 39, Section II.9]. Then, any $SO(E)$-invariant smooth function on $\mathcal{K}(E^m)$ is of the form $F \circ \varphi$ by the Theorem of G. Schwarz [Schwarz 75]. □

2.3 Smooth Structure

The smooth structure on the space $\bar{\mathcal{N}}_E^m = G(E)\backslash E^m$ is obtained as in Section 2.2. The embedding $\varphi : \bar{\mathcal{N}}_E^m \to$

$\mathbb{R}^{m^2}$ is given by the polynomial functions $\rho \mapsto \langle \rho_i, \rho_j \rangle$. Proposition 2.1 holds true.

2.4 Clouds of Unordered Points

On $E^m = \{\rho : \underline{m} \to E\}$, the symmetric group Sym_m acts on the right, by precomposition (or by permuting the coordinates). This action descends to $\mathcal{N}_E^m$.

As in Section 2.2, the space $\mathcal{N}_E^m/\mathrm{Sym}_m = G(E)\backslash E^m/\mathrm{Sym}_m$ has a smooth structure, via a topological embedding $\varphi : E^m/\mathrm{Sym}_m \to \mathbb{R}^N$ given by a generating set of the algebra of polynomial functions on $\mathcal{K}(E^m)$ which are $(SO(E) \times \mathrm{Sym}_m)$-invariant. Proposition 2.1 holds true accordingly.

The space $\mathbb{R}^m/\mathrm{Sym}_m$ also has a smooth structure via the smooth embedding $\varphi : \mathbb{R}^m/\mathrm{Sym}_m \to \mathbb{R}^m$ given by the m elementary symmetric polynomials. The map ℓ descends to a continuous map $\bar{\ell} : \mathcal{N}_E^m/\mathrm{Sym}_m \to \mathbb{R}^m/\mathrm{Sym}_m$ which is smooth away from $\ell^{-1}(\{0\})$. The composition $\psi : \mathbb{R}^m_{\nearrow} \subset (\mathbb{R}_{>0})^m \to (\mathbb{R}_{>0})^m/\mathrm{Sym}_m$ is a smooth homeomorphism. The stratification $\mathcal{H}(\mathbb{R}^m_{\nearrow})$ can be transported via ψ to $(\mathbb{R}_{>0})^m/\mathrm{Sym}_m$, giving rise to a stratification $\mathcal{H}((\mathbb{R}_{>0})^m/\mathrm{Sym}_m)$. The map $\bar{\ell}$ is stratified and Theorem 1.1 holds true for $\bar{\ell}$. Indeed, the diffeomorphisms constructed in the proof of Theorem 1.1 given in Section 3 are natural with respect to the action of Sym_m.

We must be careful that the smooth homeomorphism $\psi : \mathbb{R}^m_{\nearrow} \to (\mathbb{R}_{>0})^m/\mathrm{Sym}_m$ is not a diffeomorphism: the projection onto the first coordinate is smooth on $\mathbb{R}^m_{\nearrow}$ but not on $(\mathbb{R}_{>0})^m/\mathrm{Sym}_m$.

2.5 Poisson Structures on $\mathcal{N}_3^m$

Recall that a Poisson structure on a smooth manifold X is a Lie bracket $\{,\}$ on $\mathcal{C}^\infty(X)$ satisfying the Leibnitz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$. See [Marsden and Ratiu 94] for properties of Poisson manifolds. The same definition makes sense on a smooth space.

The Euclidean space $E = \mathbb{R}^3$ has a standard Poisson structure by

$$\{f, g\}(x) := \langle \nabla f \times \nabla g, x \rangle.$$

We endow the product space E^m with the product Poisson structure. If $f, g : E^m \to \mathbb{R}$ are $SO(E)$-invariant, so is the bracket $\{f, g\}$. Thus, the quotient space $SO(E)\backslash E^m$ inherits a Poisson structure.

Using a canonical identification of $\mathbb{R}^3$ with $so(3)^*$, the above Poisson bracket on $\mathbb{R}^3$ corresponds, up to sign, to the classical Poisson structure on $so(3)^*$ [Marsden and Ratiu 94, page 287]. The map $\mu : z \mapsto \sum_{i=1}^m z_i$, from $\mathbb{R}^3 = so(3)^*$ to E is the moment map for the diagonal action of $SO(E)$. Let $\xi : \mathbb{R}^3 \to \mathbb{R}$ be a linear map. By the Theorem of Noether [Marsden and Ratiu 94, Theorem 11.4.1], if $f : E^m \to \mathbb{R}$ is a smooth $SO(E)$-invariant map, then $\{f, \xi \circ \mu\} = 0$. This proves that $\{f, g\} = 0$ for all $g \in \mathcal{C}^\infty(E^m)$ such that $g_{|\mathcal{K}(E^m)} = 0$. Thus, the space $\mathcal{N}_3^m$ inherits a Poisson structure so that the inclusion $\mathcal{N}_3^m \subset SO(E)\backslash E^m$ is a Poisson map.

When $a \in (\mathbb{R}_{>0})^m$ is generic, the spaces $\mathcal{N}_3^m(a)$ are manifolds and are the symplectic leaves of $\ell^{-1}(\mathrm{Str}(a))$. This accounts for the symplectic structures on the polygon spaces in $\mathbb{R}^3$ studied in [Klyachko 94], [Kapovich and Millson 96], and [HK1 and 2].

3. PROOF OF THEOREM 1.1

Throughout this section, the Euclidean space E and the number m of points are constant. Denote by $\dot{\mathcal{K}}$ the subset of m-tuples $\rho = (\rho_1, \ldots, \rho_m) \in E^m$ such that $\rho_i \neq 0$ and $\sum_{i=1}^m \rho_i = 0$. Define the map $\tilde{\ell} : \dot{\mathcal{K}} \to \mathbb{R}^m$ by $\tilde{\ell}(\rho) := (|\rho_1|, \ldots, |\rho_m|)$.

An element $\rho = (\rho_1, \ldots, \rho_m) \in E^m$ is called *one-dimensional* if the vector subspace of E spanned by $\rho_1, \ldots, \rho_m$ is of dimension 1 (therefore, ρ represents an element of $\bar{\mathcal{N}}_1^m \subset \mathcal{N}_E^m$). These are precisely the singularities of the map $\tilde{\ell} : \dot{\mathcal{K}} \to \mathbb{R}^m$. Indeed:

Lemma 3.1. *Suppose that $\rho \in \dot{\mathcal{K}}$ is not one-dimensional. Then $T_\rho \tilde{\ell}$ is surjective.*

Proof: Let $(a_1, \ldots, a_m) = \tilde{\ell}(\rho)$. As ρ is not one-dimensional, there are two vectors among $\rho_2, \ldots, \rho_m$ that are linearly independent. The orthogonal complements to these two vectors then span E. Thus, there are curves $\rho_i(t)$ for $i = 2, \ldots, m$ such that $|\rho_i(t)| = a_i$ and

$$\sum_{i=2}^m \rho_i(t) = -(1 + \frac{t}{a_1})\rho_1.$$

Therefore, the map

$$t \mapsto ((1 + \frac{t}{a_1})\rho_1, \rho_2(t), \ldots, \rho_m(t))$$

represents a tangent vector $v \in T_\rho \dot{\mathcal{K}}$ with $T_\rho \tilde{\ell}(v) = (1, 0, \ldots, 0)$. The same can be done for the other basis vectors of $\mathbb{R}^m$ proving that $T_\rho \tilde{\ell}$ is surjective. □

Let $\rho \in \dot{\mathcal{K}}$ be one-dimensional. One thus has $\rho_i = \lambda_i \rho_m$ with $\lambda_i \in \mathbb{R} - \{0\}$. Let $I(\rho) \in \mathcal{P}(\underline{m})$ defined by $i \in I(\rho)$ if and only if $\lambda_i < 0$. It is obvious that $\tilde{\ell}(\rho)$ belongs to the wall $\mathcal{H}_{I(\rho)}$.

Lemma 3.2. *Suppose that $\rho \in \dot{\mathcal{K}}$ is one-dimensional. Then the image of $T_\rho\tilde{\ell}$ is $\mathcal{H}_{I(\rho)}$.*

Proof: Let $I = I(\rho)$. One has $\sum_{i\in I} \rho_i = -\sum_{i\notin I} \rho_i$. The components $\rho_i(t)$ of a curve $\rho(t) \in E^m$ with $\rho(0) = \rho$ are of the form

$$\rho_i(t) = (1 + \frac{c_i(t)}{a_i})\rho_i + w_i(t),$$

with $c_i(0) = 0$ and $w_i(0) = 0$, where $c_i(t) \in \mathbb{R}$, and $w_i(t)$ is in the orthogonal complement of ρ_i. The curve $\rho(t)$ is in $\dot{\mathcal{K}}$ if and only if $\sum_{i=1}^m w_i(t) = 0$ and

$$\sum_{i\in I}(1 + \frac{c_i(t)}{a_i})\rho_i = -\sum_{i\notin I}(1 + \frac{c_i(t)}{a_i})\rho_i. \tag{3–1}$$

Let $c(t) = (c_1(t), \dots, c_m(t))$. The vector $\frac{\rho_i}{a_i}$ is constant when $i \in I$ and $\frac{\rho_i}{a_i} = -\frac{\rho_j}{a_j}$ if $i \in I$ and $j \notin I$. Therefore, Equation (3–1) is equivalent to $c(t) \in \mathcal{H}_I$. Finally, a direct computation shows that the tangent vector $v \in T_\rho\dot{\mathcal{K}}$ represented by $\rho(t)$ satisfies $T_\rho\tilde{\ell}(v) = \dot{c}(0)$. This proves the lemma. □

Proof of Theorem 1.1: Let $a, b \in (\mathbb{R}_{>0})^m$ be in the same stratum α. Let $X \subset \alpha$ be the segment joining a to b. For $\delta > 0$, write $U_\delta := \{x \in \mathbb{R}^m \mid d(x, X) < \delta\}$, where $d(x, X)$ is the distance from x to the segment X. We choose δ small enough so that the walls meeting U_δ, if any, are only those containing α. Let $\tilde{U}_\delta := \tilde{\ell}^{-1}(U_\delta) \subset \dot{\mathcal{K}}$.

Let V^b be a vector field on U_δ of the form $V^b_x = \lambda(x)(b - a)$, where $\lambda : U_\delta \to [0, 1]$ is a smooth function equal to 1 on $U_{\delta/3}$ and to 0 out of $U_{2\delta/3}$.

Put on $\dot{\mathcal{K}}$ and $\mathbb{R}^m$ the standard Riemannian metrics. For $\rho \in \dot{\mathcal{K}}$, define the vector subspace Δ_ρ of $T_\rho\dot{\mathcal{K}}$ by $\Delta_\rho := (T_\rho\tilde{\ell})^\sharp(T_{\ell(\rho)}(\mathbb{R}^m))$, where $(T_\rho\tilde{\ell})^\sharp$ is the adjoint of $T_\rho\tilde{\ell}$. The vector spaces Δ_ρ form a smooth distribution (of nonconstant rank) on $\dot{\mathcal{K}}$.

The tangent map $T_\rho\tilde{\ell}$ sends Δ_ρ isomorphically onto the image of $T_\rho\tilde{\ell}$. Since X lies in α, Lemmas 3.1 and 3.2 show that $V^b_{\tilde{\ell}(z)}$ is in the image of $T_z\tilde{\ell}$ for all $z \in \tilde{U}_\delta$. Therefore, there exists a unique vector field W^b on $\tilde{U}_\delta$ such that, for each $z \in \tilde{U}_\delta$, one has $W^b_z \in \Delta_z$ and $T_z\tilde{\ell}(W^b_z) = V^b_{\tilde{\ell}(z)}$. The map $\tilde{\ell}$ being proper, the vector field W^b has compact support, so its flow Φ_t is defined for all times t. Therefore, $z \mapsto \Phi_1(z)$ gives a diffeomorphism $\psi_{ba} : \tilde{\ell}^{-1}(b) \xrightarrow{\approx} \tilde{\ell}^{-1}(a)$.

As its notation suggests, the map ψ_{ba} depends only on b and not on the choices involved in the definition of V^b (δ and λ). One can thus define $\psi : \tilde{\ell}^{-1}(\alpha) \to \mathcal{N}^m_E(a) \times \alpha$ by $\psi(z) := (\tilde{\ell}(z), \psi_{\tilde{\ell}(z)a}(z))$. The vector fields V^b and W^b depending smoothly on $b \in \alpha$, the map is smooth as well as its inverse $(x, u) \mapsto \psi_{ax}(u)$. Therefore, ψ is a diffeomorphism. As the Riemannian metric on $\dot{\mathcal{K}}$ and the map $\tilde{\ell}$ are invariant with respect to the action of $SO(E) \times \mathrm{Sym}_m$, the map ψ descends to a diffeomorphism $\psi : \ell^{-1}(\alpha) \xrightarrow{\approx} \mathcal{N}^m_E(a) \times \alpha$, which proves Theorem 1.1. Actually, each diffeomorphism ψ_{ba} descends to a diffeomorphism $\psi_{ba} : \mathcal{N}^m_E(b) \xrightarrow{\approx} \mathcal{N}^m_E(a)$. □

Remark 3.3. Theorem 1.1 is also true for the spaces $\bar{\mathcal{N}}^m_E$.

4. THE GENETIC CODE OF A CHAMBER

Let $a \in (\mathbb{R}_{\geq 0})^m$. Following [Hausmann and Knutson 98, Section 2], we define $S(a) \subset \mathcal{P}(\underline{m})$ by

$$I \in S(a) \iff \sum_{i\in I} a_i \leq \sum_{i\notin I} a_i. \tag{4–1}$$

The very definition of the stratification $\mathcal{H}$ implies that $S(a) = S(a')$ if and only if $\mathrm{Str}(a) = \mathrm{Str}(a')$. Thus, for a stratum α of $(\mathbb{R}_{>0})^m$, we shall write $S(\alpha)$ for the common set $S(a)$ with $a \in \alpha$.

When α is a chamber, the inequalities in (4–1) are all strict. The elements of $S(\alpha)$ are then, as in [Hausmann and Knutson 98, Section 2], called *short subsets* of $\underline{m}$. Observe that $A \in \underline{m}$ is short if and only if its complement $\bar{A}$ is not short. Therefore, if α is a chamber, the set $S(\alpha)$ contains 2^{m-1} elements.

Define $S_m(\alpha) := S(\alpha) \cap \mathcal{P}_m(\underline{m})$, where $\mathcal{P}_m(\underline{m}) := \{X \in \mathcal{P}(\underline{m}) \mid m \in X\}$.

Lemma 4.1. *Let $\alpha \in \mathrm{Ch}((\mathbb{R}_{>0})^m)$. Then $S(\alpha)$ is determined by $S_m(\alpha)$.*

Proof: One has

$$I \in S(\alpha) \iff \begin{cases} m \in I \text{ and } I \in S_m(\alpha) \text{ or,} \\ m \notin I \text{ and } \bar{I} \notin S_m(\alpha). \end{cases} \tag{4–2}$$

□

Let us now restrict ourselves to chambers of $\mathbb{R}^m_{\nearrow}$. We shall determine them by a very concentrated information called their "genetic code." Define a partial order "$\hookrightarrow$" on $\mathcal{P}(\underline{m})$ by saying that $A \hookrightarrow B$ if and only if there exits a nondecreasing map $\varphi : A \to B$ such that $\varphi(x) \geq x$. For instance $X \hookrightarrow Y$ if $X \subset Y$ since one can take φ being the inclusion. The *genetic code* of α is the set of elements $A_1, \dots, A_k$ of $S_m(\alpha)$ which are maximal with respect to the order "$\hookrightarrow$." By Lemma 4.1, the chamber α is determined by its genetic code; we write $\alpha = \langle A_1, \dots, A_k\rangle$

and call the sets A_i the *genes* of α. Thanks to (4–2), the explicit reconstruction of $S(\alpha)$ out of its genetic code is given by the following recipe.

Lemma 4.2. *Let $\alpha = \langle A_1, \dots, A_k\rangle \in \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$. Let $I \in \mathcal{P}(\underline{m})$. Then*

$$I \in S(\alpha) \iff \begin{cases} m \in I \text{ and } \exists\, j \in \underline{k} \text{ with } I \hookrightarrow A_j \text{ or,} \\ m \notin I \text{ and } \bar{I} \not\hookrightarrow A_j \ \forall\, j \in \underline{k}. \end{cases}$$

Example 4.3. To unburden notations, a subset A of $\underline{m}$ is denoted by the number whose digits are the elements of A in decreasing order; example: $531 = \{5, 3, 1\}$. In $\mathbb{R}^3_{\nearrow}$, there are 2 chambers. One of them, say α_0, contains points such as $(1, 1, 3)$ that are not in the image of $\ell : \mathcal{N}^3_E \to \mathbb{R}^3$. One has $S_3(\alpha_0) = \emptyset$. Its genetic code is empty, and one has

$$\alpha_0 = \langle\rangle \quad ; \quad S(\alpha_0) = \{\emptyset, 1, 2, 21\}.$$

The other, α_1, contains $(1, 1, 1)$, and one has

$$\alpha_1 = \langle 3\rangle \quad ; \quad S(\alpha_1) = \{\emptyset, 1, 2, 3\}.$$

Let us now figure out which subset $\mathcal{A} \subset \mathcal{P}_m(\underline{m})$ is the genetic code of a chamber of $\mathbb{R}^m_{\nearrow}$. To reduce the number of trials, observe that if $\alpha = \langle A_1, \dots, A_k\rangle$, then

(a) $A_i \not\hookrightarrow A_j$ for all $i \neq j$ and

(b) $\bar{A}_i \not\hookrightarrow A_j$ for all i, j.

Indeed, one has Condition (a) since the sets A_i are maximal (and we do not write them twice). For Condition (b), if $\bar{A}_i \hookrightarrow A_j$, then A_i would be both short and not short and Inequalities (4–1) would have no solution. A finite set $\{A_1, \dots, A_k\}$, with $A_i \in \mathcal{P}_m(\underline{m})$, satisfying Conditions (a) and (b) is called a *virtual genetic code* (of type m), and we keep writing it as $\langle A_1, \dots, A_k\rangle$. Let $\mathcal{G}_m$ be the set of virtual genetic codes and $\mathcal{G}^{(k)}_m$ the subset of those virtual genetic codes containing k genes.

The determination of $\mathcal{G}_m$ is algorithmic:

1. $\mathcal{G}^{(0)}_m = \{\langle\rangle\}$.
2. Each $A \in \mathcal{P}_m(\underline{m})$ satisfying $\bar{A} \not\hookrightarrow A$ gives rise to a virtual genetic code $\langle A\rangle$. This gives the set $\mathcal{G}^{(1)}_m$.
3. Suppose, by induction, that we know the set $\mathcal{G}^{(k)}_m$ Then, each $(\langle A_1, \dots, A_k\rangle,\ \langle A_{k+1}\rangle)$ in $\mathcal{G}^{(k)}_m \times \mathcal{G}^{(1)}_m$, so that $\{A_1, \dots, A_{k+1}\}$ satisfies Conditions (a) and (b), gives rise to an element of $\mathcal{G}^{(k+1)}_m$.

When $\mathcal{G}^{(k+1)} = \emptyset$, the process stops and $\mathcal{G}_m = \bigcup_{k=0}^m \mathcal{G}^{(k)}_m$.

Examples 4.4. For $m = 3$, the family $\mathcal{P}_3(\underline{3})$ contains the sets 3, 31, 32 and 321 (with the notations introduced in Example 4.3). Only 3 satisfies $\bar{3} = 21 \not\hookrightarrow 3$. Thus $\mathcal{G}^{(1)}_3 = \{3\}$ while $\mathcal{G}^{(2)}$ is empty. We deduce that $\mathcal{G}_3 = \{\langle\rangle, \langle 3\rangle\}$. They correspond to the two chambers of $\mathbb{R}^3_{\nearrow}$ found in Example 4.3. In the same way, we easily find the following table:

m	Elements of $\mathcal{G}_m$
2	$\langle\rangle$.
3	$\langle\rangle$, $\langle 3\rangle$.
4	$\langle\rangle$, $\langle 4\rangle$, $\langle 41\rangle$.
5	$\langle\rangle$, $\langle 5\rangle$, $\langle 51\rangle$, $\langle 52\rangle$, $\langle 53\rangle$, $\langle 54\rangle$, $\langle 521\rangle$.

Having found the virtual genetic codes of type m, the next question is which of them are *realizable*, that is, which of them is the genetic code of a chamber of $\mathbb{R}^m_{\nearrow}$. We proceed as follows. Each virtual genetic codes $\langle A_1, \dots, A_k\rangle$ of type m determines, a subset $S_{\langle A_1,\dots,A_k\rangle}$ by the recipe of Lemma 4.2. Define the open polyhedral cone $P := P_{\langle A_1,\dots,A_k\rangle}$ by

$$P := \Big\{x \in \mathbb{R}^m_{\nearrow} \ \Big|\ \sum_{i\in I} x_i < \sum_{i\notin I} x_i \ \forall\, I \in S_{\langle A_1,\dots,A_k\rangle}\Big\}.$$

If there exits $\alpha \in \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ with $\alpha = \langle A_1, \dots, A_k\rangle$, then $\alpha = P_{\langle A_1,\dots,A_k\rangle}$. The realization problem is thus equivalent to P being nonempty. To find a point inside P, we "push" its walls and consider:

$$P_1 := \Big\{x \in \mathbb{R}^m_{\nearrow} \ \Big| \ \sum_{i\in I} x_i \leq \sum_{i\notin I} x_i - 1 \ \forall\, I \in S_{\langle A_1,\dots,A_k\rangle}\Big\} \subset P. \quad (4–3)$$

As P is an open cone in $\mathbb{R}^m$, then P is nonempty if and only if P_1 is nonempty. Indeed, if P is nonempty, then $\emptyset \neq P \cap \mathbb{Z}^m_{\nearrow} \subset P_1$. We then use the *simplex algorithm* of linear programming to minimize the ℓ_1-norm $\sum_{i=1}^m x_i$ on P_1. This algorithm either outputs an optimal solution, which is a vertex of P_1, or concludes that P_1 is empty [Chvátal 83].

A program in C^{++} was designed, following the above algorithms (comments on this program and the source code can be found in [Hausmann and Rodriguez 02]). A computer could thus list all the chambers of $\mathbb{R}^m_{\nearrow}$ for $m \leq 9$. Each chamber α is given by a distinguished element $a_{\min}(\alpha) \in \mathbb{Z}^m_{\nearrow}$ with minimal $\sum_{i=1}^m a_i$. The number of these chambers, $|\mathrm{Ch}(\mathbb{R}^m_{\nearrow})| = |\mathrm{Ch}((\mathbb{R}_{>0})^m)/\mathrm{Sym}_m|$, is the one given in the first line of Table (1–1) in the introduction.

Experimentally, it turned out that, for $m \leq 8$, all virtual genetic codes are realizable. This is not true for $m = 9$:

Lemma 4.5. *The virtual genetic code* $\langle 9642 \rangle \in \mathcal{G}_9$ *is not realizable.*

Proof: Let $S := S_{\langle 9642 \rangle}$. As $9531 \hookrightarrow 9642$, one has $9531 \in S$. On the other hand, $\overline{9642} = 87531 \notin S$. If $S = S(a)$ for some generic $a \in \mathbb{R}^9_{\nearrow}$, we would have $a_7 + a_8 > a_9$. Now $965 \notin S$ by Lemma 4.2, therefore $\overline{965} = 874321 \in S$. By the above inequality on the a_i's, this would imply that $94321 \in S$, which contradicts $94321 \not\hookrightarrow 9642$. □

Our algorithm found 319,124 elements in $\mathcal{G}_9$, out of which 175,428 are realizable.

The list of all the chambers α of $\mathbb{R}^m_{\nearrow}$ with their representative $a_{\min}(\alpha)$ can be found further in this paper for $m \leq 6$ (Sections 6. and 8.) and on the web page [Hausmann and Rodriguez 02] for $m = 7, 8, 9$.

Several theoretical questions about $a_{\min}(\alpha)$ remain open. For example, why does $a_{\min}(\alpha)$ have integral coordinates (with the ℓ_1-norm $|a|_1 = \sum a_i$ odd)? A priori, the vertices of P_1 should only be in $\mathbb{Q}^m_{\nearrow}$. Is $a_{\min}(\alpha)$ always unique? This suggests the following:

4.1 Conjectures

(a) Any stratum of $\alpha \in \mathcal{H}(\mathbb{R}^m_{\nearrow})$ contains a unique element $a_{\min}(\alpha) \in \mathbb{Z}^m_{\nearrow}$ with minimal ℓ_1-norm.

(b) α is a chamber if and only $|a_{\min}(\alpha)|_1$ is an odd integer.

(c) All vertices of $P_1(S)$ have integral coordinates.

Conjecture b) is supported by the following evidences. First, it is obvious that an element $a \in \mathbb{Z}^m_{\nearrow}$ with $|a|_1$ odd is generic. On the other hand, it is experimentally true for $m \leq 9$. Conjecture a) for nongeneric strata is experimentally true for $m \leq 8$ (see Section 5.). Conjecture c) has been checked for $m \leq 8$.

4.2 Cuts

One can prove that the set $\mathcal{G}_m$ of virtual genetic code of type m is in bijection with the set of "cuts" on $\underline{m}$ (the name is given in analogy with the Dedekind cuts of the rationals). A subset S of $\mathcal{P}(\underline{m})$ is a *cut* if, for all $I, J \subset \underline{m}$, the two following conditions are fulfilled:

(A) $I \in S \Leftrightarrow \bar{I} \notin S$.

(B) if $I \in S$ and $J \hookrightarrow I$, then $J \in S$.

The bijection sends a cut S of $\underline{m}$ to the set of maximal elements (with respect to the order "$\hookrightarrow$") of S_m. For details, see [Hausmann and Rodriguez 02].

5. NONGENERIC STRATA

If $a \in \mathbb{R}^m_{\nearrow}$ is not generic, some inequalities of (4–1) are equalities. Thus, an element $I \in S(a)$ is either a *short subset* of $\underline{m}$ (strict inequality) or an *almost short subset.* As in Lemma 4.1, $S(a)$ is determined by $S_m(a) = S(a) \cap \mathcal{P}_m(\underline{m})$ and the latter is determined by those elements which are maximal with respect to the order "$\hookrightarrow$" (the *genes* of $S(a)$). We denote the genes which are short subsets by $A_1, \ldots, A_k$ and those which are almost short by $B_1^=, \ldots, B_l^=$. For instance, when $m = 3$, one writes $S(1,1,1) = \langle 3 \rangle$ and $S(1,1,2) = \langle 3^= \rangle$. To be more precise on our conventions, let $I^=$ be an almost short gene of $S(a)$ and $J \hookrightarrow I$. If $|J| < |I|$, then J is automatically short (since $a \in \mathbb{R}^m_{\nearrow} \subset (\mathbb{R}_{>0})^m$). If $|J| = |I|$, then J is supposed to be almost short unless there is a short gene K with $J \hookrightarrow K$. For instance, $S(1,2,2,3,4) = \langle 51, 53^= \rangle$.

The set $\mathrm{Str}(\mathbb{R}^m_{\nearrow})$ of all the strata of $\mathcal{H}(\mathbb{R}^m_{\nearrow})$ will be studied via a map $\alpha \mapsto \alpha^+$ from $\mathrm{Str}(\mathbb{R}^{m-1}_{\nearrow})$ to $\mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ which we define now. Let $a \in \mathbb{R}^{m-1}_{\nearrow}$. If ε is small enough, the m-tuple $a^+ := (\delta, a_1, \ldots, a_{m-1})$ is a generic element of $\mathbb{R}^m_{\nearrow}$ for $\delta < \varepsilon$ and $\alpha^+ := \mathrm{Ch}(a^+)$ depends only on $\alpha = \mathrm{Str}(a)$.

If $\beta = \alpha^+$, we denote $\alpha = \beta^-$. This makes sense because of the following lemma.

Lemma 5.1. *The map* $\alpha \mapsto \alpha^+$ *is injective.*

Proof: Let $a, b \in \mathbb{R}^{m-1}_{\nearrow}$ such that $\mathrm{Str}(a) \neq \mathrm{Str}(b)$. The segment joining a to b will then cross a wall $\mathcal{H}_I$ that does not contain $\mathrm{Str}(a)$. But then, the segment joining a^+ and b^+ will also cross $\mathcal{H}_I$, showing that $\mathrm{Ch}(a^+) \neq \mathrm{Ch}(b^+)$. □

The correspondence $\alpha \mapsto \alpha^+$ can easily be described on the genetic codes. The genetic code of a^+ has the same number of genes as that of a. The correspondence goes as follows. If $\{p_1, \ldots, p_r\}$ is a gene of $S(a)$ which is short, then $\{p_1^+, \ldots, p_r^+, 1\}$ is a gene of $S(a)^+$, where $p_i^+ = p_i + 1$ (the genes of $S(a^+)$ are all short). If $\{p_1, \ldots, p_r\}^=$ is an almost short gene of $S(a)$, then $\{p_1^+, \ldots, p_r^+\}$ is a gene of $S(a^+)$.

The following convention will be useful.

Convention 5.2. Let $a = (a_1, \ldots, a_k)$ be a generic element of $\mathbb{Z}^k_{\nearrow}$. For $m \geq k$, the m-tuple $\hat{a} = (0, \ldots, 0, a_1, \ldots, a_k)$ determines a chamber $\hat{\alpha}$ represented by $(\delta_1, \ldots \delta_{m-k}, a_1, \ldots, a_k)$, where $\delta_i > 0$ and $\sum \delta_i < 1$. We say that $\hat{a}$ is a *conventional representative* of $\hat{\alpha}$. For instance, $\langle 521 \rangle$ having the conventional representative $(0, 0, 1, 1, 1)$ shows that $\langle 521 \rangle = \langle 3 \rangle^{++}$.

Lemma 5.3. *Let α be a chamber of $\mathbb{R}^m_{\nearrow}$. Then $\alpha = \beta^+$ if and only if one (at least) of the two following statement holds:*

1. *α has a conventional representative $(0, a_2, \ldots, a_m)$.*
2. *There exists $(a_1, \ldots, a_m) \in \alpha \cap \mathbb{Z}^m$ with $\sum a_i$ odd and $a_1 = 1$.*

Proof: It is clear that either Statement 1 or 2 implies $\alpha = \mathrm{Str}(a_2, \ldots, a_m)^+$. Also, if $\alpha = \beta^+$ for β generic, then α admits a conventional representative. It remains to show that, if $\alpha = \beta^+$ with β nongeneric, then Statement 2 holds true.

Observe that, since the walls $\mathcal{H}_I$ are defined by linear equations with integral coefficients, $\beta \cap \mathbb{Q}^{m-1}$ is dense in β. As β is a cone, it must contain a point in $b \in \mathbb{Z}^{m-1}_{\nearrow}$. As b is not generic, then $\sum b_i$ must be even and the m-tuples $(\delta, b_1, \ldots, b_{m-1})$ are all in α for $\delta < 2$. □

Tables 3–5 of Section 6 and Table 6 of Section 8 show that $a_{\min}(\alpha)$ satisfies the above conditions for all $\alpha \in \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ when $m \leq 6$. This proves the following:

Proposition 5.4. *The correspondence $\alpha \mapsto \alpha^+$ gives a bijection $\mathrm{Str}(\mathbb{R}^{m-1}_{\nearrow}) \to \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ for $m \leq 6$.*

Tables 1 and 2 make the bijection $\alpha \mapsto \alpha^-$ from $\mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ to $\mathrm{Str}(\mathbb{R}^{m-1}_{\nearrow})$ explicit (we put a conventional $a_{min}(\alpha)$ when there exists one).

In Table 2, Σ_g^{or} stands for the orientable surface of genus g, and the two graphs in the last column are the 2- and 3-fold covers of $S^1 \vee S^1$ without loops. The same work with the bijection $\mathrm{Ch}(\mathbb{R}^6_{\nearrow}) \xrightarrow{\approx} \mathrm{Str}(\mathbb{R}^5_{\nearrow})$ gives the classification of all the 21 pentagon spaces (not necessarily generic) obtained by A. Wenger [Wenger 88].

In the two tables, one sees that $\mathcal{N}_2^5(\alpha)$ is the boundary of a regular neighborhood (here in $\mathbb{R}^3$) of $\mathcal{N}_2^4(\alpha^-)$. This reflects the following fact: let $a_0 \in \alpha$ without zero coordinate. For any $a \in \alpha$, the Riemannian manifold $\mathcal{N}_E^m(a)$ is canonically diffeomorphic to $\mathcal{N}_E^m(a_0)$ by Theorem 1.1 and its proof. This produces a family of Riemannian metrics g_a on $\mathcal{N}_E^m(a_0)$, indexed by $a \in \alpha$. When a tends to a point $a^- \in \alpha^-$, the Riemannian manifold $(\mathcal{N}_E^m(a_0), g_a)$ converges, for the Gromov-Hausdorff metric, to the metric space $\mathcal{N}_E^{m-1}(a^-)$.

α	$a_{\min}(\alpha)$	$\mathcal{N}_2^4(\alpha)$	α^-	$a_{\min}(\alpha^-)$	$\mathcal{N}_2^3(\alpha^-)$
$\langle\rangle$	$(0,0,0,1)$	$\emptyset$	$\langle\rangle$	$(0,0,1)$	$\emptyset$
$\langle 4\rangle$	$(1,1,1,2)$	S^1	$\langle 3^=\rangle$	$(1,1,2)$	1 point
$\langle 41\rangle$	$(0,1,1,1)$	$S^1 \amalg S^1$	$\langle 3\rangle$	$(1,1,1)$	2 point

TABLE 1. The bijection $\mathrm{Ch}(\mathbb{R}^4_{\nearrow}) \xrightarrow{\approx} \mathrm{Str}(\mathbb{R}^3_{\nearrow})$.

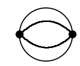
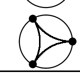
α	$a_{\min}(\alpha)$	$\mathcal{N}_2^5(\alpha)$	α^-	$a_{\min}(\alpha^-)$	$\mathcal{N}_2^4(\alpha^-)$
$\langle\rangle$	$(0,0,0,0,1)$	$\emptyset$	$\langle\rangle$	$(0,0,0,1)$	$\emptyset$
$\langle 5\rangle$	$(1,1,1,1,3)$	S^2	$\langle 4^=\rangle$	$(1,1,1,3)$	1 point
$\langle 51\rangle$	$(0,1,1,1,2)$	T^2	$\langle 4\rangle$	$(1,1,1,2)$	S^1
$\langle 52\rangle$	$(1,1,2,2,3)$	Σ_2^{or}	$\langle 41^=\rangle$	$(1,2,2,3)$	$S^1 \vee S^1$
$\langle 521\rangle$	$(0,0,1,1,1)$	$T^2 \amalg T^2$	$\langle 41\rangle$	$(0,1,1,1)$	$S^1 \amalg S^1$
$\langle 53\rangle$	$(1,1,1,2,2)$	Σ_3^{or}	$\langle 42^=\rangle$	$(1,1,2,2)$	
$\langle 54\rangle$	$(1,1,1,1,1)$	Σ_4^{or}	$\langle 43^=\rangle$	$(1,1,1,1)$	

TABLE 2. The bijection $\mathrm{Ch}(\mathbb{R}^5_{\nearrow}) \xrightarrow{\approx} \mathrm{Str}(\mathbb{R}^4_{\nearrow})$.

On the other hand, the map $\alpha \mapsto \alpha^+$ is not surjective when $m \geq 7$. For instance, $\langle 764 \rangle$, with $a_{\min} = (2, 2, 2, 2, 3, 3, 3)$, is not of the form $(\alpha^-)^+$. Then, α^- would be $\langle 653^= \rangle$. As $421 = \overline{653} \hookrightarrow 653$, this would imply that all a_i^- are equal and $\alpha^- = \langle 654^= \rangle$; but, $\langle 654^= \rangle^+ = \langle 765 \rangle \neq \langle 764 \rangle$.

The table of $\mathrm{Ch}(\mathbb{R}^7_{\nearrow})$ (giving the 135 7-gon spaces) shows 18 chambers with the first coordinate a_{min} not equal to 0 or 1 (see [Hausmann and Rodriguez 02]). One might ask whether there are other m-tuples a in these chambers with $a_1 = 0, 1$. But, by applying the simplex algorithm to minimize a_1 on the polytope P_1 of (4–3), we saw that this is not the case. Therefore, $|\mathrm{Str}(\mathbb{R}^6_{\nearrow})| = 118$. The same procedure succeeded for $m = 8$ and 9, giving the cardinality of $\mathrm{Str}((\mathbb{R}_{>0})^m)/\mathrm{Sym}_m = \mathrm{Str}(\mathbb{R}^m_{\nearrow})$ for $m \leq 8$ listed in the introduction.

6. GEOMETRIC DESCRIPTIONS OF $\mathcal{N}^m_{2,3}(\alpha)$

When $d = 2$ or 3 and a is generic, the spaces $\mathcal{N}_d^m(a)$ are smooth manifolds, since $SO(d)$ acts freely on the non-lined configurations. The space $\bar{\mathcal{N}}_2^m(a)$ is also a manifold, and the map $\mathcal{N}_2^m(a) \to \bar{\mathcal{N}}_2^m(a)$ is a 2-sheeted covering. The space $\bar{\mathcal{N}}_2^m(a)$ lies in $\mathcal{N}_3^m(a)$ as the fixed point set for the involution τ on $\mathcal{N}_3^m(a)$ obtained by reflection through a hyperplane. Observe that $\dim \mathcal{N}_3^m(a) = 2(m-3)$ while $\dim \bar{\mathcal{N}}_2^m(a) = m - 3$. The manifold $\bar{\mathcal{N}}_2^m(a)$ plays the role of a real locus of $\mathcal{N}_3^m(a)$, the latter being endowed with a natural Kaehler structure for which the involution

α	$a_{\min}(\alpha)$	$\mathcal{N}_3^3(\alpha)$	$\bar{\mathcal{N}}_2^3(\alpha)$	$\mathcal{N}_2^3(a)$
$\langle\rangle$	$(0,0,1)$	$\emptyset$	$\emptyset$	$\emptyset$
$\langle 3\rangle$	$(1,1,1)$	1 point	1 point	2 points

TABLE 3. The 3-gon spaces.

α	$a_{\min}(\alpha)$	$\mathcal{N}_3^4(\alpha)$	$\bar{\mathcal{N}}_2^4(\alpha)$	$\mathcal{N}_2^4(a)$
$\langle\rangle$	$(0,0,0,1)$	$\emptyset$	$\emptyset$	$\emptyset$
$\langle 4\rangle$	$(1,1,1,2)$	$\mathbb{C}P^1$	$\mathbb{R}P^1$	S^1
$\langle 41\rangle$	$(1,2,2,2)$	S^2	S^1	$S^1 \amalg S^1$

TABLE 4. The 4-gon spaces.

	α	$a_{\min}(\alpha)$	$\mathcal{N}_3^5(\alpha)$	$\bar{\mathcal{N}}_2^5(\alpha)$	$\mathcal{N}_2^5(\alpha)$
1	$\langle\rangle$	$(0,0,0,0,1)$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\langle 5\rangle$	$(1,1,1,1,3)$	$\mathbb{C}P^2$	$\mathbb{R}P^2$	S^2
3	$\langle 51\rangle$	$(0,1,1,1,2)$	$\mathbb{C}P^2\,\sharp\,\overline{\mathbb{C}P}^2$	Σ_1	T^2
4	$\langle 52\rangle$	$(1,1,2,2,3)$	$(S^2\times S^2)\,\sharp\,\overline{\mathbb{C}P}^2$	Σ_2	Σ_2^{or}
5	$\langle 521\rangle$	$(0,0,1,1,1)$	$S^2\times S^2$	T^2	$T^2 \amalg T^2$
6	$\langle 53\rangle$	$(1,1,1,2,2)$	$\mathbb{C}P^2\,\sharp\,3\overline{\mathbb{C}P}^2$	Σ_3	Σ_3^{or}
7	$\langle 54\rangle$	$(1,1,1,1,1)$	$\mathbb{C}P^2\,\sharp\,4\overline{\mathbb{C}P}^2$	Σ_4	Σ_4^{or}

TABLE 5. The 5-gon spaces.

τ is antiholomorphic (see [Hausmann and Knutson 98, Section 9]). It is shown in [Hausmann and Knutson 98, Theorem 9.1] that the cohomology rings $H^{2*}(\mathcal{N}_3^m(a);\mathbb{Z}_2)$ and $H^*(\bar{\mathcal{N}}_2^m(a);\mathbb{Z}_2)$ are isomorphic, by a graded ring isomorphism dividing the degrees by 2.

The above polygon spaces were previously known for $m \le 5$ (see, for instance, [Hausmann and Knutson 97, Section 6]). Our classification by genetic code produces the more systematic tables below. Conventional representatives $a_{\min}(\alpha)$ (see Convention 5.2) are used when available.

Our method produces a classification of the spaces $\mathcal{N}_d^m(\alpha)$ for $m \le 9$, α a chamber, and $d \ge 2$. Table 6 of Section 8 gives the list of hexagon spaces. The tables for generic m-gon spaces when $m = 7, 8, 9$ are too big to be included in this paper. They can be consulted on the web page [Hausmann and Rodriguez 02].

We shall now give procedures describing $\mathcal{N}_E^m(\beta^+)$ in terms of $\mathcal{N}_E^{m-1}(\beta)$ when β is generic and $\dim E = 2, 3$. A m-tuple $(\rho_1,\dots,\rho_m) \in \mathcal{K}(E^m)$ is called a *vertical configuration* if $\rho_m = (0,\dots,0,-|\rho_m|)$.

Proposition 6.1. *If $\beta \in \mathrm{Ch}(\mathbb{R}^{m-1}_{\nearrow})$, then $\mathcal{N}_2^m(\beta^+)$ is diffeomorphic to $\mathcal{N}_2^{m-1}(\beta)\times S^1$.*

Proof: Let $(b_2,\dots,b_m) \in \beta$ and let $\varepsilon > 0$ be small enough so that $a := (\varepsilon, b_2,\dots,b_m) \in \beta^+$. A class in $\mathcal{N}_2^m(a)$ has a unique representative $\rho = (\rho_1,\dots,\rho_m)$ which is a vertical configuration. As b is generic, if ε is small enough, then $(b_2,\dots,b'_m) \in \beta$ when $|b'_m - b_m| < \varepsilon$. The $(m-1)$-tuple $(\rho_2,\dots,\rho_m+\rho_1)$ thus represents an element $\rho' \in \mathcal{N}_2^{m-1}(\beta)$, and the correspondence $\rho \mapsto (\rho',\rho_1)$ produces a diffeomorphism from $\mathcal{N}_2^m(\beta^+)$ to $\mathcal{N}_2^{m-1}(\beta)\times S^1$. □

Let $\rho = (\rho_1,\dots,\rho_m) \in (\mathbb{R}^3)^m$. Let $\rho_m^\perp$ be the orthogonal complement of ρ_m, oriented by the vector ρ_m. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the composition of the orthogonal projection $\mathbb{R}^3 \to \rho_m^\perp$ with some chosen isometry $\rho_m^\perp \xrightarrow{\approx} \mathbb{R}^2$ preserving the orientation. If $a \in (\mathbb{R}_{>0})^m$ is generic, than $\pi(\rho_1)$, $\pi(\rho_1+\rho_2)$,..., $\pi(\rho_1+\cdots+\rho_{m-2})$ are not all zero. This defines a smooth map

$$r : \mathcal{N}_3^m(\alpha) \to \big((\mathbb{R}^2)^{m-2} - \{0\}\big)/SO(2), \qquad (6\text{–}1)$$

where $\alpha = \mathrm{Ch}(a)$. The right-hand member of Equation (6–1) is homotopy equivalent to $\mathbb{C}P^{m-3}$. The map r thus determines a cohomology class $R \in H^2(\mathcal{N}_3^m(\alpha);\mathbb{Z})$ which is the characteristic class of some principal circle bundle $\mathcal{E}(\alpha) \to \mathcal{N}_3^m(\alpha)$. The class R was introduced in [Hausmann and Knutson 98, Section 6 and 7] and will appear again in Section 6.

Lemma 6.2. (Compare [Hausmann and Knutson 98, Proposition 7.3]) *The total space $\mathcal{E}(\alpha)$ is S^1-equivariantly diffeomorphic to the space of representatives of $\mathcal{N}_3^m(a)$, ($\mathrm{Ch}(a) = \alpha$) which are vertical configurations.*

Proof: Let $\mathcal{E}'(\alpha) \subset (\mathbb{R}^3)^m$ be the space described in the statement. Any element of $\mathcal{N}_3^m(a)$ has at least one representative that is a vertical configuration, and any two of those are in the same orbit under the orthogonal action of $S^1 = SO(2)$ fixing the vertical axis. As a is generic, the quotient map $\mathcal{E}'(\alpha) \to \mathcal{N}_3^m(a)$ is then a principal circle bundle. If $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denotes the projection onto the first two coordinates, the correspondence $\rho \mapsto \big(\pi(\rho_1), \pi(\rho_1+\rho_2),\dots,\pi(\rho_1+\cdots+\rho_{m-2})\big)$ defines a smooth S^1-equivariant map $\tilde{r} : \mathcal{E}'(\alpha) \to (\mathbb{R}^2)^{m-2}$ that covers the map r. This proves that the characteristic class of $\mathcal{E}'(\alpha) \to \mathcal{N}_3^m(a)$ is R. □

Example 6.3. The chamber $\alpha = \langle m \rangle$ of $\mathbb{R}^m_\nearrow$ has its minimal representative $a = a_{\min}(\alpha) = (1, \ldots, 1, m-2)$. As, in a vertical configuration ρ of $[\rho] \in \mathcal{N}_3^m(a)$, one has $\sum_{i=1}^{m-1} \rho_i = (0, \ldots, 0, m-2)$, the sequence of the third coordinate of ρ_1, $\rho_1 + \rho_2$, $\ldots$, must be strictly increasing. This implies that the map $\tilde{r}$ of the proof of Lemma 6.2 is a smooth S^1-equivariant embedding. It induces a diffeomorphism from $\mathcal{N}_3^m(\langle m \rangle)$ onto $\mathbb{C}P^{m-3}$ and an identification of the bundle $\mathcal{E}(\langle m \rangle) \to \mathcal{N}_3^m(\langle m \rangle)$ with the Hopf bundle. (See also [Hausmann 89, Remark 4.2].)

Let $\mathcal{D}(\alpha)$ be the total space of the D^2-bundle associated to $\mathcal{E}(\alpha) \to \mathcal{N}_3^m(\alpha)$.

Proposition 6.4. *If $\beta \in \mathrm{Ch}(\mathbb{R}^{m-1}_\nearrow)$, then $\mathcal{N}_3^m(\beta^+)$ is diffeomorphic to the double of $\mathcal{D}(\beta)$.*

Proof: Let $(b_2, \ldots, b_m) \in \beta$ and let $\varepsilon > 0$ be small enough so that $a := (\varepsilon, b_2, \ldots, b_m) \in \beta^+$. A class in $\mathcal{N}_3^m(a)$ has a unique representative $\rho = (\rho_1, \ldots, \rho_m)$ that is a vertical configuration satisfying $\rho_1 = (\varepsilon \cos\theta, 0, \varepsilon \sin\theta)$ with $\theta \in [0, \pi]$. Let $\check{\mathcal{E}}(a)$ be the space of such representatives. The above argument produces a smooth map $f : \check{\mathcal{E}}(a) \to [0, \pi]$ by $f(\rho) := \theta$.

If $\rho \in \check{\mathcal{E}}(a)$, then $\rho_m + \rho_1$ is close to ρ_m. This defines a smooth map $P : \check{\mathcal{E}}(a) \to SO(3)$, sending ρ to P_ρ, characterized by $P_\rho(\rho_m + \rho_1) = (0, 0, -|\rho_m + \rho_1|)$ and $P_\rho(0, y, 0) = (0, y, 0)$. One has $P_\rho = \mathrm{id}$ if $f(\rho) = 0, \pi$. The smooth map $\check{F} : \check{\mathcal{E}}(a) \to [0, \pi] \times \mathcal{E}(\beta)$ given by

$$\check{F}(\rho) := \Big(f(\rho)\,,\, (P_\rho(\rho_2), \ldots, P_\rho(\rho_{m-1}), P_\rho(\rho_m + \rho_1)) \Big)$$

is a diffeomorphism. It induces a diffeomorphism

$$F : \mathcal{N}_3^m(\alpha) \cong \mathcal{N}_3^m(a) \longrightarrow [0, \pi] \times \mathcal{E}(\beta) \Big/ \sim, \qquad (6\text{–}2)$$

where $\sim$ is the equivalence relation given by $(0, \eta) \sim (0, g \cdot \eta)$ and $(\pi, \eta) \sim (\pi, g \cdot \eta)$ for all $g \in SO(2)$. The right member of (6–2) is diffeomorphic to the double of $\mathcal{D}(\beta)$ which proves the proposition. □

If, in Equation (6–1), one replaces $\mathbb{R}^3$ by $\mathbb{R}^2$, one gets a map

$$r : \bar{\mathcal{N}}_2^m(\alpha) \to \big(\mathbb{R}^{m-2} - \{0\}\big)/\{\pm 1\} \simeq \mathbb{R}P^{m-3}.$$

This produces a cohomology class $R \in H^1(\bar{\mathcal{N}}_2^m(\alpha); \mathbb{Z}_2)$, which is the Stiefel-Whitney class of the double covering $\mathcal{N}_2^m(\alpha) \to \bar{\mathcal{N}}_2^m(\alpha)$. Lemma 6.2 holds true and Example 6.3 becomes:

Example 6.5. For the chamber $\langle m \rangle$, realized by $a = (1, \ldots, 1, m-2)$, the map r is homotopic to a diffeomorphism from $\mathcal{N}_2^m(\langle m \rangle)$ onto $\mathbb{R}P^{m-3}$ and gives an identification of the double covering $\mathcal{N}_2^m(\langle m \rangle) \to \bar{\mathcal{N}}_2^m(\langle m \rangle)$ with $S^{m-3} \to \mathbb{R}P^{m-3}$.

If $\bar{\mathcal{D}}(\alpha)$ denotes the total space of the D^1-bundle associated to the double covering $\mathcal{N}_2^m(\alpha) \to \bar{\mathcal{N}}_2^m(\alpha)$, one proves, as in Proposition 6.4, the following:

Proposition 6.6. *If $\beta \in \mathrm{Ch}(\mathbb{R}^{m-1}_\nearrow)$, then $\bar{\mathcal{N}}_2^m(\beta^+)$ is diffeomorphic to the double of $\bar{\mathcal{D}}(\beta)$.*

Example 6.7. The chamber $\langle \{m, 1\} \rangle = \langle m-1 \rangle^+$ is represented by $a = (1/2, 1, \ldots, 1, m-3)$. As seen in Example 6.3, one has that $\mathcal{N}_3^m(\langle m-1 \rangle)$ is diffeomorphic to $\mathbb{C}P^{m-4}$ and $\mathcal{D}(\langle m-1 \rangle)$ is the disk bundle associated to the Hopf bundle. Therefore, $\mathcal{N}_3^m(\langle \{m, 1\} \rangle)$ is diffeomorphic to $\mathbb{C}P^{m-3} \sharp \overline{\mathbb{C}P}^{m-3}$. For planar polygons, one has $\bar{\mathcal{N}}_2^m(\langle m-1 \rangle)$ is diffeomorphic to $\mathbb{R}P^{m-4}$ and $\bar{\mathcal{E}}(\langle m \rangle) \to \bar{\mathcal{N}}_2^m(\langle m \rangle)$ is the double covering. Therefore, $\bar{\mathcal{N}}_2^m(\langle \{m, 1\} \rangle)$ is diffeomorphic to $\mathbb{R}P^{m-3} \sharp \overline{\mathbb{R}P}^{m-3}$ (of course, $\overline{\mathbb{R}P}^{m-3} = \mathbb{R}P^{m-3}$ when m is even).

6.1 Case Where β is Nongeneric

When $\alpha = \beta^+$ with β nongeneric, some partial information about $\mathcal{N}_3^m(\alpha)$ can still be gathered. We proceed as in the proof of Proposition 6.4, with the same notations. If ε is small enough, the $(m-1)$-tuple $b_\delta := (b_2, \ldots, b_m + \delta)$ is generic when $0 < |\delta| \leq \varepsilon$; set $\beta_{\pm\varepsilon} := \mathrm{Ch}(b_{\pm\varepsilon})$. The manifold $\check{\mathcal{E}}(a)$ is now a cobordism between $\mathcal{E}(\beta_\varepsilon)$ and $\mathcal{E}(\beta_{-\varepsilon})$. By [Hausmann 89, Theorem 3.2], the map $-\theta : \check{\mathcal{E}}(a) \to [-\pi, 0]$ is a Morse function. It has only one critical value, the angle for which the diagonal length $|\rho_m + \rho_1|$ is equal to b_m. The preimage of this critical value is diffeomorphic to $\mathcal{N}_3^{m-1}(\beta)$ and the (isolated) critical points are the lined configurations. There is one for each almost short subset $I^= \in S_{m-1}(\beta)$, and its index is equal to $2|I|$ (or $|I|$ for planar polygons). As in Equation (6–2), the space $\mathcal{N}_3^m(\alpha)$ is diffeomorphic to the quotient of the cobordism $\check{\mathcal{E}}(a)$ by the following identifications on its two ends: $\eta \sim g \cdot \eta$ for all $g \in SO(2)$, when $\theta(\eta) = 0, \pi$ (for planar polygons, $g \in O(1)$).

As an application of the results of this section, we will describe all the hexagon spaces in Section 8.

6.2 Use of Toric Manifolds

Recall that a symplectic manifold M^{2n} is called toric if it is endowed with a Hamiltonian action of a torus

T of dimension n (the maximal possible dimension for a Hamiltonian torus action). The moment map $\mu : M \to \mathrm{Lie}(T)^* \approx \mathbb{R}^n$ has for its image a convex polytope, the *moment polytope*, which determines M up to T-equivariant symplectomorphism (see [Guillemin 94]).

The spatial polygon space $\mathcal{N}_3^m(a)$ with its symplectic structure (see Section 2.5) may admit Hamiltonian torus actions by the so-called *bending flows* (see [Klyachko 94], [Kapovich and Millson 96], [Hausmann and Tolman 02]), which we recall now. For $I \in \underline{m}$, define $f_I : \mathcal{N}_3^m(a) \to \mathbb{R}$ by $f_I(\rho) := |\sum_{i\in I}\rho_i|$. If f_I does not vanish, it is a smooth map which generates a Hamiltonian circle action on $\mathcal{N}_3^m(a)$. This action rotates at constant speed the set of vectors $\{\rho_i \mid i \in I\}$ around the axis $\sum_{i\in I}\rho_i$ (see [Klyachko 94, Section 2.1], [Kapovich and Millson 96, Corollary 3.9]). The nonvanishing of f_I is equivalent to I being *lopsided*; that is, there exists $i \in I$ with $a_i > \sum_{j\in I-\{i\}} a_j$ (see [Hausmann and Tolman 02]).

Suppose that $\mathcal{I} \subset \mathcal{P}(\underline{m})$ is formed of lopsided subsets satisfying the following "absorption condition": *if $I, J \in \mathcal{I}$ with $I \neq J$, then either $I \cap J = \emptyset$ or one is contained in the other.* Then, the Hamiltonian flow of the f_I's for $I \in \mathcal{I}$ commute and generate a Hamiltonian action of a torus $T_{\mathcal{I}}$. (see [Klyachko 94, Section 2.1], [Hausmann and Tolman 02, Lemma 2.1]). Thus, when $\dim T_{\mathcal{I}} = m-3$, the manifold $\mathcal{N}_3^m(a)$ is a toric manifold that is determined by the moment polytope for the $T_{\mathcal{I}}$-action.

For example, when $m = 5$, each chamber $\alpha \in \mathrm{Ch}(\mathbb{R}^5_{\nearrow})$ has a representative $a \in \alpha$ with $a_1 \neq a_2$ and $a_3 \neq a_4$. Therefore, $\mathcal{N}_3^5(a)$ admits a Hamiltonian action of the two-dimensional torus $T_{\mathcal{I}}$ for $\mathcal{I} = \{\{1,2\},\{3,4\}\}$. This shows that the diffeomorphism type of $\mathcal{N}_3^5(a)$ is that of a toric manifold. The determination of all the two-dimensional moment polytopes was the principle of the classification of the 5-gon spaces given in [Hausmann and Knutson 97, Section 6].

The same holds for $m = 6$ since each chamber $\alpha \in \mathrm{Ch}(\mathbb{R}^6_{\nearrow})$ has a representative $a \in \alpha$ with $a_1 \neq a_2$, $a_3 \neq a_4$, and $a_5 \neq a_6$. Therefore, all $\mathcal{N}_3^6(\alpha)$ are diffeomorphic to toric manifolds. The three-dimensional moment polytopes can still be visualized but with more difficulties.

The above two cases generalize in the following:

Proposition 6.8. *Let $\alpha \in \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$. Suppose that there exists $a \in \alpha \cap \mathbb{Z}^m$ with $a_m \geq \sum_{i=1}^{m-5} a_i$. Then the diffeomorphism type of $\mathcal{N}_3^m(\alpha)$ is that of a toric manifold.*

Proof: One can find $a' \in \alpha$ arbitrarily close to a so that $a'_{m-4} \neq a'_{m-3}$, $a'_{m-2} \neq a'_{m-1}$, and $a'_m > \sum_{i=1}^{m-5} a'_i$. Therefore, the family of lopsided sets

$$\{m,1\}\,,\ \{m,2,1\}\,,\ \ldots\,,\ \{m, m-5, m-4, \ldots, 1\}\,,\ \{m-4, m-3\}\,,\ \{m-2, m-1\}$$

satisfies the absorption condition. Their bending flows generate a Hamiltonian action of a torus of dimension $m-3$, which proves the proposition. □

Examples 6.9. Consulting the table of the 135 7-gons (see [Hausmann and Rodriguez 02]), we see that there are only three $\alpha \in \mathrm{Ch}(\mathbb{R}^7_{\nearrow})$ for which $a = a_{\min}(\alpha)$ does not satisfy the hypothesis of Proposition 6.8, that is, here, $a_7 \geq a_1 + a_2$. These are

α	$a_{\min}(\alpha)$
$\langle 754, 762\rangle$	(3,3,3,4,4,5,5)
$\langle 764\rangle$	(2,2,2,2,3,3,3)
$\langle 765\rangle$	(1,1,1,1,1,1,1)

Thus, all the other 133 heptagon spaces are diffeomorphic to toric manifolds. We do not know whether the above three heptagon spaces are diffeomorphic to toric manifolds.

The same experiment with $m = 8$ or 9 gives the following results: 217 elements of $\mathrm{Ch}(\mathbb{R}^8_{\nearrow})$ (out of 2470) and 56550 elements of $\mathrm{Ch}(\mathbb{R}^9_{\nearrow})$ (out of 175428) do not satisfy the hypothesis of Proposition 6.8.

7. COHOMOLOGY INVARIANTS OF $\mathcal{N}_3^m(\alpha)$

Let α be a chamber of $(\mathbb{R}_{>0})^m$. In [Hausmann and Knutson 98], presentations of the cohomology rings $H^*(\mathcal{N}_3^m(\alpha);\mathbb{Z})$ and $H^*(\bar{\mathcal{N}}_2^m(\alpha);\mathbb{F}_2)$ were obtained in terms of α. Our algorithms allowed us, with the help of a computer, to find enough information about these rings to prove that, for $5 \leq m \leq 7$, $\alpha = \alpha'$ if and only if the mod 2 cohomology rings of $\mathcal{N}_3^m(\alpha)$ and of $\mathcal{N}_3^m(\alpha')$ are isomorphic.

We start by the Poincaré polynomial. Recall that $\mathcal{N}_3^m(\alpha)$ has a cellular decomposition with only even-dimensional cells [Hausmann and Knutson 98, Section 4], so its Poincaré polynomial,

$$P(t) = \sum_{i=0}^{2(m-3)} \dim_{\mathbb{F}} H^i(\mathcal{N}_3^m(\alpha);\mathbb{F})\, t^i\,,$$

is the same for any field $\mathbb{F}$ and has only terms of even degree. Moreover, the polynomial $P(\sqrt{t})$ is the Poincaré polynomial of $\bar{\mathcal{N}}_2^m(\alpha)$ for the coefficient field with two elements $\mathbb{F}_2$ [Hausmann and Knutson 98, Section 9]. The first formula for computing $P(t)$ in terms of α was found by A. Klyachko [Klyachko 94, Theorem 2.2.4]. We will use the more economical formula, using only elements of $S_m(\alpha)$, obtained in [Hausmann and Knutson 98, Corollary 4.3]. With our notation, this is the following:

Proposition 7.1. *Let $\alpha \in \mathrm{Ch}((\mathbb{R}_{>0})^m)$. The Poincaré polynomial of $\mathcal{N}_3^m(\alpha)$ is*

$$P(t) = \frac{1}{1-t^2} \sum_{J \in S_m(\alpha)} (t^{2(|J|-1)} - t^{2(m-1-|J|)}).$$

Remark 7.2. The difference between the formula in Proposition 7.1 and that of [Hausmann and Knutson 98, Corollary 4.3] comes from that, in the latter, the notation S_m is used for the set of $I \in \mathcal{P}(\underline{m-1})$ such that $J \cup \{m\} \in S$. Recall that, here, $S_m = S \cap \mathcal{P}_m(\underline{m})$. So, each occurrence of $|J|$ in [Hausmann and Knutson 98, Corollary 4.3] is replaced here by $|J|-1$.

For $\mathcal{I} \subset \mathcal{P}(\underline{m})$, denote by $\mathrm{NS}_i(\mathcal{I})$ the number of sets $I \in \mathcal{I}$ with $|I| = i+1$. The formula of Proposition 7.1 gives the Betti numbers $b_{2i} := b_{2i}(\alpha) := \dim_{\mathbb{F}} H^i(\mathcal{N}_3^m(\alpha); \mathbb{F})$ as the solution of the system of equations

$$b_{2i} - b_{2i-2} = \mathrm{NS}_i(S_m(\alpha)) - \mathrm{NS}_{m-2-i}(S_m(\alpha)), \quad (7\text{–}1)$$

starting with $b_{2i} = 0$ if $i < 0$. For instance, if $\alpha = \langle 54 \rangle$, realized by $(1,1,1,1,1)$, one has

$$S_5(\alpha) = \{5, 51, 52, 53, 54\};$$

thus, $\mathrm{NS}_0(S_5(\alpha)) = 1$, $\mathrm{NS}_1(S_5(\alpha)) = 4$, and the other $\mathrm{NS}_i(S_5(\alpha))$ vanish. This gives $b_0 = 1$, $b_2 = 5$, and $b_4 = 1$, which are indeed the Betti numbers of $\mathcal{N}_3^5(\langle 54 \rangle) = \mathbb{C}P^2 \sharp 4 \overline{\mathbb{C}P^2}$ [Hausmann and Knutson 98, Example 10.4].

As our computer algorithm had to list all the sets of $S_m(\alpha)$ (for instance, for the realization), the numbers $\mathrm{NS}_i(S_m(\alpha))$ are available and so are the b_{2i}'s.

We now recall the presentation of $H^*(\mathcal{N}_3^m(\alpha); \mathbb{Z})$ obtained in [Hausmann and Knutson 98, Theorem 6.4]. Taking care of Remark 7.2, this gives:

Proposition 7.3. *Let $\alpha \in \mathrm{Ch}((\mathbb{R}_{>0})^m)$. The cohomology ring of the polygon space $\mathcal{N}_3^m(\alpha)$ with coefficient in a ring Λ is*

$$H^*(\mathcal{N}_3^m(\alpha); \Lambda) = \Lambda[R, V_1, \dots, V_{m-1}]/\mathcal{I}(\alpha),$$

where R and V_i are of degree 2 and $\mathcal{I}(\alpha)$ is the ideal of $\Lambda[R, V_1, \dots, V_{m-1}]$ generated by the three families:

(R1) $V_i^2 + RV_i$ $\qquad i = 1, \dots, m-1,$

(R2) $\prod_{i \in L} V_i$ $\qquad$ *for all $L \in \mathcal{P}(\underline{m-1})$ with $L \cup \{m\} \notin S(\alpha)$,*

(R3) $\sum_{\substack{S \subset L \\ S \cup \{m\} \in \alpha}} (\prod_{i \in S} V_i) R^{|L-S|-1}$ $\qquad$ *for all $L \in \mathcal{P}(\underline{m-1})$ with $L \notin S(\alpha)$.*

We shall first use this presentation to compute a homotopy invariant $r_\cup(\alpha) \in \mathbb{N}$, defined as the rank of the linear map $x \mapsto x \cup x$ from $H^2(\mathcal{N}_3^m(\alpha); \mathbb{F}_2)$ to $H^4(\mathcal{N}_3^m(\alpha); \mathbb{F}_2)$ (recall that $x \mapsto x^2$ is a linear map in an algebra over $\mathbb{F}_2$). Observe that $r_\cup(\alpha)$ is also the rank of the same map from $H^1(\bar{\mathcal{N}}_2^m(\alpha); \mathbb{F}_2)$ to $H^2(\bar{\mathcal{N}}_2^m(\alpha); \mathbb{F}_2)$.

Proposition 7.4. *For a chamber α of $\mathbb{R}^m_{\nearrow}$, one has*

$$r_\cup(\alpha) = 1 + \mathrm{NS}_1(S_m(\alpha)) \\ - \mathrm{NS}_{m-3}(S_m(\alpha)) - \mathrm{NS}_{m-4}(S_m(\alpha)). \quad (7\text{–}2)$$

Proof: One has $1 + \mathrm{NS}_1(S_m(\alpha)) - \mathrm{NS}_{m-3}(S_m(\alpha)) = b_2$ by Equation (7–1). Let us first consider the case $\mathrm{NS}_{m-3}(S_m(\alpha)) \neq 0$. This means that $S_m(\alpha)$ contains a set with $m-2$ elements and thus contains the smallest of those for the order "$\hookrightarrow$", which is $I := \{m, m-3, m-4, \dots, 1\}$. Then, $\alpha = \langle I \rangle$. Indeed, if $\alpha \neq \langle I \rangle$, then α would contain $J := \{m, m-2\}$, which is impossible since $\bar{J} \hookrightarrow I$. Therefore,

$$\mathrm{NS}_1(S_m(\alpha)) = m-3, \\ \mathrm{NS}_{m-4}(S_m(\alpha)) = m-3, \\ \mathrm{NS}_{m-3}(S_m(\alpha)) = 1,$$

and the right hand member of Equation (7–2) is equal to zero. On the other hand, Relator (R3) of Proposition 7.3, with $L = \{m-2, m-1\}$ and $S = \emptyset$, gives the equality $R = 0$. By Relator (R1), all squares vanish and $r_\cup = 0$. Formula (7–2) is then proven in the case where $\mathrm{NS}_{m-3}(S_m(\alpha)) \neq 0$. Observe that $\mathcal{N}_3^m(\langle I \rangle)$ is diffeomorphic to a product of $m-3$ copies of the sphere S^2 [Hausmann and Knutson 98, Example 10.2].

Assume then that $\mathrm{NS}_{m-3}(S_m(\alpha)) = 0$. By Proposition 7.3, the vector space $H^2(\mathcal{N}_3^m(\alpha); \mathbb{F}_2) = H^2(\mathcal{N}_3^m(\alpha); \mathbb{Z}) \otimes \mathbb{F}_2$ has the basis $R, V_1, \dots, V_p$ for $p = \mathrm{NS}_1(S_m(\alpha))$. Indeed, $V_i = 0$ for $i > p$ by Relator (R2) with $L = \{i\}$. The image of $x \mapsto x^2$ is generated by $R^2, V_1^2, \dots, V_p^2$. The relations between these generators come from Relator (R3) of Proposition 7.3 with $|L| = 3$. For such an $L = \{i, j, k\}$ (denoted by ijk), the relation is

$$R^2 + RV_i + RV_j + RV_k + V_iV_j + V_iV_k + V_jV_k = 0. \quad (7\text{–}3)$$

The three last terms of the left-hand member vanish by Relator (R2). Indeed, since $ijk \notin \alpha$ and $ijk \hookrightarrow ijm$, then $ijm \notin \alpha$. By Relator (R1), Equation (7–3) becomes

$$R^2 - V_i^2 - V_j^2 - V_k^2 = 0. \tag{7–4}$$

To establish Proposition 7.4, it is enough to prove that, for distinct i, j, k, Equation (7–4) is independent. But, by Proposition 7.3, the vector space $H^4(\mathcal{N}_3^m(\alpha); \mathbb{F}_2)$ is generated by the b_1 elements $R^2, V_1^2, \ldots, V_p^2$, together with the $\mathrm{NS}_2(S_m(\alpha))$ nonvanishing products V_iV_j $(ijm \in \alpha)$, and these generators are just subject to Equation (7–4). This implies that $b_2 \geq b_1 + \mathrm{NS}_2(S_m(\alpha)) - \mathrm{NS}_{m-4}(S_m(\alpha))$. By Equation (7–1), this inequality is an equality, showing that Equation (7–4) is independent.

□

The idea of our last cohomology invariant was given to us by R. Bacher. Define $\mathcal{I}_k(\alpha)$ to be the ideal of $\mathbb{F}[R, V_1, \ldots, V_{m-1}]$ generated by the elements of $\mathcal{I}(\alpha)$, which are polynomials of degree $\leq k$ in the variables R and V_is (then giving elements of degree $\leq 2k$ in $H^*(\mathcal{N}_3^m(\alpha); \mathbb{F})$). Define $\mathrm{Sol}_k(\alpha; \mathbb{F}) \subset \mathbb{F}P^{m-1}$ to be the projective variety defined by the equations $W = 0$ for all $W \in \mathcal{I}_2(\alpha)$.

Proposition 7.5.

(a) Let α and α' be chambers of $\mathbb{R}^m_{\nearrow}$. Any graded ring isomorphism from $H^*(\mathcal{N}_3^m(\alpha); \mathbb{F})$ onto $H^*(\mathcal{N}_3^m(\alpha'); \mathbb{F})$ induces, for $k \geq 1$, a bijection from $\mathrm{Sol}_k(\alpha'; \mathbb{F})$ to $\mathrm{Sol}_k(\alpha; \mathbb{F})$.

(b) Suppose that $k \geq 2$. Then, any element $\zeta \in \mathrm{Sol}_k(\alpha; \mathbb{F})$ has a unique representative of the form $(-1, v_1, \ldots, v_{m-1}) \in \mathbb{F}^m$ with $v_i = 0$ or 1. In particular, the set $\mathrm{Sol}_k(\alpha; \mathbb{F})$ is finite.

(c) The finite set $\mathrm{Sol}_2(\alpha; \mathbb{F})$ does not depend on the field $\mathbb{F}$.

Proof:

Proof of a): Let $q : H^*(\mathcal{N}_3^m(\alpha); \mathbb{F}) \to H^*(\mathcal{N}_3^m(\alpha'); \mathbb{F})$ be a graded ring homomorphism. By Proposition 7.3, the homomorphism q is covered by a graded ring homomorphism $\tilde{q} : \mathbb{F}[R, V_1, \ldots, V_{m-1}] \to \mathbb{F}[R', V_1', \ldots, V_{m-1}']$, which sends $\mathcal{I}_k(\alpha)$ into $\mathcal{I}_k(\alpha')$ for all k. Such a lifting $\tilde{q}$ is well defined up to a homomorphism with image in $\mathcal{I}_1(\alpha')$; therefore, q functorially induces homomorphisms $\tilde{q} : \mathbb{F}[R, V_1, \ldots, V_{m-1}]/\mathcal{I}_k(\alpha) \to \mathbb{F}[R', V_1', \ldots, V_{m-1}']/\mathcal{I}_k(\alpha')$ for all $k \geq 1$. Observe that $\mathrm{Sol}_k(\alpha; \mathbb{F})$ can be identified with the projectivization of the vector space of ring homomorphisms from $\mathbb{F}[R, V_1, \ldots, V_{m-1}]/\mathcal{I}_k(\alpha)$ to $\mathbb{F}$. Therefore, the homomorphism q will functorially induce maps $\hat{q} : \mathrm{Sol}_k(\alpha'; \mathbb{F}) \to \mathrm{Sol}_k(\alpha; \mathbb{F})$ for all $k \geq 1$. By this functoriality, if q is an isomorphism, then $\hat{q}$ is a bijection, which proves a).

Proof of b): Let $z = (r, v_1, \ldots, v_{m-1}) \in \mathbb{F}^m - \{0\}$ represent an element $\zeta \in \mathrm{Sol}_k(\alpha; \mathbb{F})$. Then, $r \neq 0$, since otherwise Relator (R1) of Proposition 7.3 would give equations $v_i^2 = -rv_i$, implying that $z = 0$. Then, ζ has a unique representative with $r = -1$, and the equations $v_i^2 = -rv_i$ imply that $v_i \in \{0, 1\}$.

Proof of c): The equations defining $\mathrm{Sol}_2(\alpha; \mathbb{F})$, coming from Relators (R1)–(R3) of 7.3, are, for $i, j, k = 1, \ldots, m-1$:

(i) $v_i^2 = -rv_i$. Having normalized $r = -1$, these are equivalent to $v_i = 0, 1$.

(ii) $v_i = 0$, for $\{i, m\} \notin \alpha$ and $v_iv_j = 0$ for $\{i, j, m\} \notin \alpha$.

(iii) Equations (7–4) which, after (i), become $v_i^2 + v_j^2 + v_k^2 = 1$, for $\{i, j, k\} \notin \alpha$.

The solutions $v_i = 0, 1$ of Equations (ii) are clearly independent of the ground field $\mathbb{F}$. The solutions $v_i = 0, 1$ of an equation like $v_i^2 + v_j^2 + v_k^2 = 1$ seem, a priori, to depend on the characteristic of $\mathbb{F}$. But, as seen just before Equation (7–4), such an equation occurs only if $r_ir_j = r_jr_k = r_kr_i = 0$. Thus, Equation (iii) is equivalent to the fact that exactly one of the v_i, v_j, v_k is equal to one, a condition independent of $\mathbb{F}$. □

Definition 7.6. We set $s(\alpha)$ to be the number of elements of $\mathrm{Sol}_2(\alpha; \mathbb{F})$. This does not depend on the field $\mathbb{F}$ by Proposition 7.5.

It is not difficult to compute $s(\alpha)$ with or without the help of a computer. We select the elements of $W(R, V_1, \ldots, V_{m-1}) \in \mathcal{I}_2(\alpha)$ and count how many of them vanish when $R = -1$ and $V_i \in \{0, 1\}$.

7.1 Proof of Proposition 1.2

The following implies Proposition 1.2 of the introduction:

Proposition 7.7. *Let $5 \leq m \leq 7$ and let $\alpha, \alpha' \in \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$. Then $\alpha = \alpha'$ if and only if $\mathcal{N}_3^m(\alpha)$ and $\mathcal{N}_3^m(\alpha')$ have the same Betti numbers, $r_{\cup}(\alpha) = r_{\cup}(\alpha')$ and $s(\alpha) = s(\alpha)'$.*

Proof: For $m = 5$, by the list of Table 5, The only case where two 5-gon spaces have the same Betti numbers are $\mathcal{N}_3^5(\langle 52 \rangle) \approx \mathbb{C}P^2 \,\sharp\, \overline{\mathbb{C}P}^2$ and $\mathcal{N}_3^5(\langle 521 \rangle) \approx S^2 \times S^2$. But,

$r_\cup(\langle 52\rangle) = 1$ while $r_\cup(\langle 521\rangle) = 0$ (taking $\mathbb{F}_2$ coefficients is important here: for instance, these two spaces have isomorphic cohomology rings with real coefficients).

For $m = 6$, the list of Table 6 in Section 8 has been sorted by lexicographic order of the triple $b_2(\alpha), r_\cup(\alpha), s(\alpha)$. One thus can check that no such triples occur twice. The same holds for $m = 7$ with $b_2(\alpha), b_4(\alpha), r_\cup(\alpha), s(\alpha)$ (table in [Hausmann and Rodriguez 02]). □

Remark 7.8. By [Hausmann and Knutson 98, Section 9], the cohomology ring $H^*(\bar{\mathcal{N}}_2^m(\alpha);\mathbb{F}_2)$ admits the presentation of Proposition 7.3, with R and the V_is of degree 1. Therefore, the above invariants are mod 2 cohomology invariants of the spaces $\bar{\mathcal{N}}_2^m(\alpha)$ and Proposition 7.7 holds true for these spaces.

Problem 7.9. When a virtual genetic code $\gamma \in \mathcal{G}_m$ is not realizable (for instance, when $m = 9$), it gives rise as well to a nontrivial graded ring. Is this ring the cohomology ring of a space? Does it satisfy Poincaré duality? Is this ring the cohomology ring of a manifold?

8. THE HEXAGON SPACES

As for Tables 3–5 of Section 6, the first column of Table 6 in this section contains the list of the 21 genetic codes of type 6, all realized by a chamber α whose minimal realization $a_{\min}(\alpha)$, using conventional representatives (see Convention 5.2), is written in the second column. The next three columns give the cohomology invariants of $\mathcal{N}_3^6(\alpha)$ or $\bar{\mathcal{N}}_2^6(\alpha)$, which are defined in Section 7, with the abbreviations

$$b := \dim_{\mathbb{F}} H^2(\mathcal{N}_3^6(\alpha);\mathbb{F}) = \dim_{\mathbb{F}_2} H^1(\bar{\mathcal{N}}_2^6(\alpha);\mathbb{F}_2)\ ,\ r_\cup = r_\cup(\alpha)\ ,\ s = s(\alpha).$$

By Poincaré duality, the number b determines the Poincaré polynomial $P(t)$ of $\mathcal{N}_3^6(\alpha)$, which is

$$P(t) = 1 + bt^2 + bt^4 + t^6$$

(for $\bar{\mathcal{N}}_2^6(\alpha)$, this would be $P(t) = 1 + bt + bt^2 + t^3$). The 6-gon spaces have been listed by the lexicographic order of the triples $(b, r_\cup, s)$, showing that the homotopy type of the hexagon spaces in $\mathbb{R}^2$ or $\mathbb{R}^3$ is distinguished by these cohomology invariants.

The last two columns contain some geometric descriptions of $\bar{\mathcal{N}}_2^6(\alpha)$ and $\mathcal{N}_2^6(\alpha)$ obtained by the methods discussed in Section 6. In the last column, we see the hexagon spaces coming from the seven generic pentagons by adding a tiny vector; their descriptions uses Proposition 6.1. Lines 2 and 3 illustrate Examples 6.5 and 6.7. Line 4 is a special case of [Hausmann and Knutson 98, Example 10.2] (see also the proof of Proposition 7.4). The other descriptions come from the method of Section 6.1. Using the notations of Section 6.2, the 3-manifold $\check{\mathcal{E}}(a)$ is a cobordism between the orientable surfaces $\mathcal{E}(\beta_\varepsilon)$ and $\mathcal{E}(\beta_{-\varepsilon})$. When they are connected (all cases except Line 7), we denote their genus respectively by g_+ and g_-. The map $-\theta : \check{\mathcal{E}}(a) \to [-\pi, 0]$ is a Morse function with n_1 critical points of index 1 and n_2 critical points of index 2. This situation is indicated in Table 6 by the writing $[g_+; n_1, n_2; g_-]$ (observe that $g_- = g_+ + n_1 - n_2$). The orientable 3-manifold $\bar{\mathcal{N}}_2^6(\alpha)$ is diffeomorphic to the quotient of the cobordism $\check{\mathcal{E}}(a)$ by, on each end, a free involution reversing the orientation.

In other words, the notation $[g_+; n_1, n_2; g_-]$ tells us that the orientable 3-manifold $\bar{\mathcal{N}}_2^6$ is obtained in the following way. For a surface Σ, denote by $\bar{\mathcal{D}}(\Sigma)$ the mapping cylinder of the orientation covering of Σ. Let $W_+ = \bar{\mathcal{D}}(\Sigma_{g_+})$ with n_1 1-handles attached and $W_- = \bar{\mathcal{D}}(\Sigma_{g_-})$

	α	$a_{\min}(\alpha)$	b	$r_\cup$	s	$\bar{\mathcal{N}}_2^6(\alpha)$	$\mathcal{N}_2^6(\alpha)$
1	$\langle\rangle$	$(0,0,0,0,0,1)$	0	0	0	$\emptyset$	$\emptyset$
2	$\langle 6\rangle$	$(1,1,1,1,1,4)$	1	1	1	$\mathbb{R}P^3$	S^3
3	$\langle 61\rangle$	$(0,1,1,1,1,3)$	2	2	2	$\mathbb{R}P^3 \sharp \overline{\mathbb{R}P}^3$	$S^2 \times S^1$
4	$\langle 6321\rangle$	$(0,0,0,1,1,1)$	3	0	0	T^3	$T^3 \amalg T^3$
5	$\langle 621\rangle$	$(0,0,1,1,1,2)$	3	2	0	$[1;0,0;1]$	T^3
6	$\langle 62\rangle$	$(1,1,2,2,2,5)$	3	3	3	$[0;1,0;1]$	
7	$\langle 632\rangle$	$(1,1,1,3,3,4)$	4	1	1	see above	
8	$\langle 631\rangle$	$(0,1,1,2,2,3)$	4	2	0	$[2;0,0;2]$	$\Sigma_2^{or} \times S^1$
9	$\langle 621, 63\rangle$	$(1,1,2,3,3,5)$	4	3	1	$[1;1,0;2]$	
10	$\langle 63\rangle$	$(1,1,1,2,2,4)$	4	4	4	$[0;2,1;2]$	
11	$\langle 641\rangle$	$(0,1,1,1,2,2)$	5	2	0	$[3;0,0;3]$	$\Sigma_3^{or} \times S^1$
12	$\langle 632, 64\rangle$	$(1,1,1,2,3,3)$	5	2	2	$[2;1,1;2]$	
13	$\langle 631, 64\rangle$	$(1,2,2,3,4,5)$	5	3	1	$[2;1,0;3]$	
14	$\langle 621, 64\rangle$	$(1,1,2,2,3,4)$	5	4	2	$[1;2,0;3]$	
15	$\langle 64\rangle$	$(1,1,1,1,2,3)$	5	5	5	$[0;3,0;3]$	
16	$\langle 651\rangle$	$(0,1,1,1,1,1)$	6	2	0	$[4;0,0;4]$	$\Sigma_4^{or} \times S^1$
17	$\langle 641, 65\rangle$	$(1,2,2,2,3,3)$	6	3	1	$[3;1,0;4]$	
18	$\langle 632, 65\rangle$	$(1,1,1,2,2,2)$	6	3	3	$[2;2,1;3]$	
19	$\langle 631, 65\rangle$	$(1,2,2,3,3,4)$	6	4	2	$[2;2,0;4]$	
20	$\langle 621, 65\rangle$	$(1,1,2,2,2,3)$	6	5	3	$[1;3,0;4]$	
21	$\langle 65\rangle$	$(1,1,1,1,1,2)$	6	6	6	$[0;4,0;4]$	

TABLE 6. The 6-gon spaces.

with n_2 1-handles attached. If we require that $W_\pm$ are orientable, then they are well defined since there is only one way, up to diffeomorphism isotopic to the identity, to attach 1-handles to $\Sigma^{or}_{g_\pm} \times [0,1]$ in order to obtain an orientable manifold. Thus, $\bar{\mathcal{N}}^6_2$ is obtained by gluing W_+ to W_- by a diffeomorphism of their boundary. In the case where $n_1 = n_2 = 0$, one has $W_+ = W_-$ and Proposition 6.6 says that the gluing diffeomorphism is the identity. We were not able to identify this gluing diffeomorphism in the other cases, so, a priori, the numbers $[g_+; n_1, n_2; g_-]$ do not determine the homeomorphism type of $\bar{\mathcal{N}}^6_2$.

In the case $\alpha = \langle 632 \rangle$ (line 7 of the table), $\bar{\mathcal{N}}^5_2(\beta_\varepsilon) = T^2$ (the only case where it is orientable). Therefore, $W_+ = T^2 \times [-1,1]$ and $W_- = \bar{\mathcal{D}}(\Sigma_2)$.

Problem 8.1. Describe more precisely the three-dimensional manifolds $\bar{\mathcal{N}}^6_2$, for instance, in terms of the Kirby calculus.

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REFERENCES

[Albouy and Chenciner 98] A. Albouy and A. Chenciner. "Le problème des n corps et les distances mutuelles," *Invent. Math.* 131 (1998), 151–184.

[Chvátal 83] V. Chvátal. *Linear Programming.* New York: W. H. Freeman and Company, 1983.

[Guillemin 94] V. Guillemin. *Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces.* Boston: Birkhäuser, 1994.

[Hausmann 89] J. -C. Hausmann. "Sur la topologie des bras articulés." In *Algebraic Topology Poznań 1989*, Lecture Notes in Mathematics, 1474, pp. 146–159. New York: Springer, 1989.

[Hausmann and Knutson 97] J. -C. Hausmann and A. Knutson. "Polygon Spaces and Grassmannians." *Enseign. Math.* 43:2 (1997), 173–198.

[Hausmann and Knutson 98] J. -C. Hausmann and A. Knutson. "The Cohomology Rings of Polygon Spaces," *Ann. Inst. Fourier (Grenoble)* 48 (1998), 281–321.

[Hausmann and Rodriguez 02] J. -C. Hausmann and E. Rodriguez. "Additional Material to 'The Space of Clouds in Euclidean Space."' Available from World Wide Web (http://www.unige.ch/math/folks/hausmann/polygones), 2002.

[Hausmann and Tolman 02] J. -C. Hausmann and S. Tolman. "Maximal Hamiltonian Tori for Polygon Spaces." To appear in *Ann. Inst. Fourier (Grenoble).*

[Kamiyama 98] Y. Kamiyama. "The Homology of Singular Polygon Spaces." *Canad. J. Math.* 50 (1998), 581–594.

[Kamiyama and Tezuka 99] Y. Kamiyama and M. Tezuka. "Topology and Geometry of Equilateral Polygon Linkages in the Euclidean Plane." *Quart. J. Math. Oxford Ser.* 50:2 (1999), 463–470.

[Kapovich and Millson 96] M. Kapovich and J. Millson. "The Symplectic Geometry of Polygons in Euclidean Space." *J. Differential Geom.* 44 (1996), 479–513.

[Kendall et al. 1999] D. G. Kendall, D. Barden, T. K. Carne and H. Le. *Shape and Shape Theory.* Chichester: John Wiley & Sons Ltd., 1999.

[Klyachko 94] A. Klyachko. "Spatial Polygons and Stable Configurations of Points in the Projective Line." In *Algebraic Geometry and its Applications*, edited by Chandrajit L. Bujaj, pp. 67–84. New York: Springer-Verlag, 1994.

[Milnor 65] J. Milnor. *Topology from the Differential Viewpoint.* Charlottesville, VA: The University Press of Virginia, 1965.

[Marsden and Ratiu 94] J. Marsden and T. Ratiu. *Introduction to Mechanics and Symmetry.* New York: Springer-Verlag, 1994.

[Palais 61] R. Palais. "On the Existence of Slices for Actions of Non-Compact Lie Groups." *Ann. of Math.* 73:2 (1961), 295–323.

[Procesi and Schwarz 85] C. Procesi and G. W. Schwarz. "Inequalities Defining Orbit Spaces." *Invent. Math.* 81 (1985), 539–554.

[Small 96] C. Small. *The Statistical Theory of Shape.* New York: Springer-Verlag, 1996.

[Schwarz 75] G. W. Schwarz. "Smooth Functions Invariants Under the Action of a Compact Lie Group." *Topology* 14 (1975), 63–68.

[Wenger 88] A. Wenger. "Etudes des espaces de configurations de certains systèmes articulés." Diploma thesis, University of Geneva, 1988.

[Weyl 39] H. Weyl. *The Classical Groups: Their Invariants and Representations.* Princeton, NJ: Princeton University Press, 1939.

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