Borwein and Bradley's Apéry-Like Formulae for $\zeta(4n+3)$

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Granville is a Presidential Faculty Fellow and is supported, in part, by the National Science Foundation.

We prove a formula for $\zeta(4n+3)$ discovered by Borwein and Bradley (*Experimental Mathematics* **6**:3 (1997), 181–194).

1. INTRODUCTION

The Riemann zeta function is defined by

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}, \quad \text{for } \operatorname{Re}(s) > 1.$$

For every even positive integer $2m$, it is known that

$$\zeta(2m) = (-1)^{m-1}(2\pi)^{2m}\frac{B_{2m}}{2(2m)!},$$

where B_{2m}, the $(2m)$-th Bernoulli number, is rational. The numbers $\zeta(3), \zeta(5), \zeta(7), \dots$ remain rather more mysterious; just about the only useful arithmetic fact known is Apéry's result that $\zeta(3)$ is irrational (see [Apéry 1981] or [van der Poorten 1979]). His proof is based on finding a series for $\zeta(3)$ that converges exponentially fast, and so he uses

$$\zeta(3) = \frac{5}{2}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Analogously, it is known that

$$\zeta(2) = 3\sum_{n\geq 1}\frac{1}{n^2\binom{2n}{n}} \quad \text{and} \quad \zeta(4) = \frac{36}{17}\sum_{n\geq 1}\frac{1}{n^4\binom{2n}{n}}.$$

It seems unlikely that there are any such simple formulae for either $\zeta(5)$ or $\zeta(7)$, though Gosper [van der Poorten 1980, footnote 10] noted that one can obtain a slightly more complicated formula for $\zeta(5)$:

$$\zeta(5) = \frac{5}{2}\sum_{n\geq 1}\frac{(-1)^n}{n^3\binom{2n}{n}}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots\frac{1}{(n-1)^2}\right) - 2\sum_{n\geq 1}\frac{(-1)^n}{n^5\binom{2n}{n}}.$$

1058-6458/1999 $0.50 per page

Presumably Gosper's identity is just the tip of the iceberg, and there is a whole slew of such identities just waiting to be discovered. A big problem in trying to uncover these new identities is the difficulty in determining new ones without a general method of proof. Borwein and Bradley [1997] came up with an extraordinary new approach: If such identities do exist then one can find them by computing the values of all such relevant series to many decimal places and then one can look for a linear combination that equals zero; or, in reality, equals zero to many decimal places. In fact finding all such linear combinations with small coefficients is easy using standard lattice reduction algorithms. One then conjectures, and tries to prove, that these identities, discovered by computation, really are identities.

Borwein and Bradley found many such "identities", and then naturally proceeded to look for some general patterns. They came up with the following incredible identity, which would imply the existence of fast converging series (of Gosper-type) for all $\zeta(4n+3)$:

Conjecture 1 [Borwein and Bradley 1997]. *For any complex number z, with $|z|<1$, we have*

$$\sum_{k\geq 0}\zeta(4k+3)z^{4k} = \sum_{n=1}^{\infty}\frac{1}{n^3(1-z^4/n^4)} = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3\binom{2n}{n}}\frac{1}{1-z^4/n^4}\prod_{m=1}^{n-1}\frac{1+4z^4/m^4}{1-z^4/m^4}. \tag{1}$$

We will prove this conjecture here. In their paper, Borwein and Bradley gave several fascinating reformulations of (1). We will actually prove one of these reformulations, due to Wenchang Chu; this is shown to be equivalent to our Conjecture 1 in [Borwein and Bradley 1997, Lemma 5.2]:

Conjecture 2 (Wenchang Chu). *For all positive integers n we have*

$$\sum_{k=1}^{n}\frac{2n^2}{k^2}\frac{\prod_{i=1}^{n-1}(i^4+4k^4)}{\prod_{\substack{i=1\\ i\neq k}}^{n}(k^4-i^4)} = \binom{2n}{n}. \tag{2}$$

Our main result is the following, seemingly unrelated, identity:

Theorem 1. *For all t and for all integers $n\geq 1$ we have*

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\prod_{\substack{0\leq j<n-k\\ \text{or}\\ n<j<n+k}}(k^2t+j^2) = \left(\frac{(2n-1)!}{n!}\right)^2. \tag{3}$$

We deduce:

Corollary 1. *Conjecture 2 is true. Thus Conjecture 1 is also true.*

Borwein and Bradley [1997] give several amusing consequences of this result (see their Corollaries 2.3 and 2.4 and equation (6–1) for strange hypergeometric series evaluations, Lemma 4.1 for another strange sequence of "finite identities", and Corollary 5.1 for a marvellous integral to compute $\binom{2n}{n}$). In attempting to prove Theorem 1 we came across the following result (which follows from, and implies, Theorem 1), as well as several others, noted in Section 4.

Corollary 2. *For all integers $r>n\geq 1$ we have*

$$\sum_{1\leq k\leq n}(-1)^{k-1}rk\binom{2n}{n-k}\frac{\binom{(r+1)k+n-1}{2n-1}\binom{(r-1)k+n-1}{2n-1}}{\binom{rk+n}{2n+1}} = (2n+1)\binom{2n}{n}. \tag{4}$$

As we will see in the proof of Corollary 1, our objective, we only actually need to prove Theorem 1 in the case $t=1$. We were unable to prove this directly, and so searched for a generalization that might be easier to prove, by trying various numeric experiments in Maple, in ad hoc manner. It was somewhat of a shock when (3) was turned up by such an unlikely procedure!

2. PROVING THE MAIN THEOREM

For the sake of completeness we prove a well-known identity:

Lemma 1. *For all integers $n>r\geq 1$ we have*

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}k^{2r} = 0.$$

Proof. This follows immediately by combining

$$e^{-nx}(e^x-1)^{2n} = \left(1-nx+\tfrac{1}{2}n^2x^2+\cdots\right)\left(x+\tfrac{1}{2}x^2+\cdots\right)^{2n} = x^{2n}+\tfrac{1}{12}nx^{2n+2}+\cdots.$$

with

$$\begin{aligned}
e^{-nx}(e^x-1)^{2n} &= \sum_{k=-n}^{n}\binom{2n}{n+k}(-1)^{n+k}e^{kx}\\
&= \binom{2n}{n}(-1)^n+\sum_{k=1}^{n}\binom{2n}{n-k}(-1)^{n-k}(e^{kx}+e^{-kx})\\
&= 2\sum_{r\ge 1}\left(\sum_{k=1}^{n}\binom{2n}{n-k}(-1)^{n-k}k^{2r}\right)\frac{x^{2r}}{(2r)!}. \qquad \square
\end{aligned}$$

Our next result may be of some independent interest. It shows us a way to deal with the peculiar limits in the product in (3).

In the proof we use the fact that for any nonnegative integers i and m, one has

$$\sum_{j=0}^{m-1} j^i = \frac{1}{i+1}(B_{i+1}(m) - B_{i+1}),$$

where $B_k(x) := \sum_{l=0}^{k}\binom{k}{l}B_{k-l}x^l$, and the B_js are the Bernoulli numbers.

Proposition 1. *Let $g(x)$ be a given polynomial of degree d. Fix a positive integer n. There exist polynomials $c_r(x)$ for $0 \le r \le n-1$, of degree at most $r\lfloor \frac{d+1}{2}\rfloor$, such that, for any k in the range $1 \le k \le n$,*

$$f_k(x) := \prod_{\substack{0\le j<n-k\\ \text{or}\\ n<j<n+k}} (x - g(j)) = \sum_{r=0}^{n-1} c_r(k^2)x^{n-1-r}.$$

Remark. There are exactly $n-1$ elements in any set $\{0 \le j < n-k\} \cup \{n < j < n+k\}$. Also, the polynomials $c_r(x)$ are defined independent of the choice of k.

Proof. Throughout the proof we think of $g(x)$ and n as being fixed. Write $g(x)^r = \sum_{i=0}^{dr} g_{r,i}x^i$. The sum of the r-th powers of the roots of f_k is

$$\begin{aligned}
\sigma_r &= \sum_{j=0}^{n-k-1} g(j)^r + \sum_{j=n+1}^{n+k-1} g(j)^r\\
&= \sum_{i=0}^{dr} g_{r,i}\left(\sum_{j=0}^{n-k-1} j^i + \sum_{j=n+1}^{n+k-1} j^i\right)\\
&= \sum_{i=0}^{dr}\frac{g_{r,i}}{i+1}\\
&\quad\times\big(B_{i+1}(n-k)+B_{i+1}(n+k)-B_{i+1}(n)-B_{i+1}\big),
\end{aligned}$$

which evidently is a polynomial of degree at most $dr+1$ in k, and is an even function of k. Thus we can write $\sigma_r = \sigma_r(k^2)$ a polynomial of degree at most $\lfloor\frac{dr+1}{2}\rfloor \le r\lfloor\frac{d+1}{2}\rfloor$.

Note that $c_0 = 1$. Sir Isaac Newton showed that for any r with $0 \le r \le n-1$ one has the recurrence relation

$$rc_r = -\sum_{i=0}^{r-1} c_i\sigma_{r-i}.$$

It then follows from this formula, via an induction hypothesis on r, that we can write $c_r = c_r(k^2)$, a polynomial of degree at most $r\lfloor\frac{d+1}{2}\rfloor$. $\square$

Combining Lemma 1 and Proposition 1 we can easily prove the following generalization of Theorem 1:

Theorem 1′. *Let $g(x)$ be any polynomial of degree ≤ 2. For all t and for all integers $n \ge 1$ we have*

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\left(\prod_{\substack{0\le j<n-k\\ \text{or}\\ n<j<n+k}}(k^2t+g(j)) - \prod_{\substack{0\le j<n-k\\ \text{or}\\ n<j<n+k}} g(j)\right) = 0. \quad (5)$$

Remark. Theorem 1 is the special case $g(x) = x^2$, as we will verify after the proof.

Proof. Write each $c_r(x)$ in the form $\sum_{i=0}^{D_r} c_{r,i}x^i$ in Proposition 1, where $D_r \le r$ (since $\lfloor\frac{d+1}{2}\rfloor \le 1$); then

$$f_k(x) - f_k(0) = \sum_{r=0}^{n-2}\sum_{i=0}^{D_r} c_{r,i}k^{2i}x^{n-1-r}.$$

Therefore the left side of (5) is

$$\begin{aligned}
&(-1)^{n-1}\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}(f_k(-k^2t) - f_k(0))\\
&\quad= (-1)^{n-1}\sum_{r=0}^{n-2}\sum_{i=0}^{D_r} c_{r,i}(-t)^{n-1-r}\\
&\qquad\times\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}k^{2(i+(n-1-r))} = 0
\end{aligned}$$

by Lemma 1, since

$$1 \le i + (n-1-r) \le r + (n-1-r) = (n-1)$$

in the range of our sums. $\square$

Deduction of Theorem 1 from Theorem 1′. Taking $g(x) = x^2$ in Theorem 1′, we see that the left side of (3) is

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\prod_{\substack{0\le j<n-k\\ \text{or}\\ n<j<n+k}} j^2.$$

However, except when $k=n$, the product vanishes, since it contains the factor $j=0$. Thus this sum becomes

$$(-1)^0\binom{2n}{0}\prod_{n<j<2n} j^2 = \left(\frac{(2n-1)!}{n!}\right)^2. \qquad \square$$

3. DEDUCING THE COROLLARIES

Deduction of Corollary 1 from Theorem 1. Take $t=1$ in (3) and multiply through by $(2n)^2/(2n)!$, to get

$$\begin{aligned}\binom{2n}{n} &= \sum_{k=1}^{n}(-1)^{n-k}\frac{4n^2}{(n-k)!\,(n+k)!}\prod_{0\le j\le n-k-1}(k^2+j^2)\prod_{n+1\le j\le n+k-1}(k^2+j^2)\\ &= \sum_{k=1}^{n}(-1)^{n-k}\frac{4n^2}{(n-k)!\,(n+k)!}\prod_{k\le i\le n-1}(k^2+(i-k)^2)\prod_{n+1-k\le i\le n-1}(k^2+(i+k)^2),\end{aligned}$$

via the simple changes of variables $j\to i-k$ and $j\to i+k$ respectively. Multiply top and bottom of the k-th term in this sum through by

$$\prod_{1\le i\le k-1}(k^2+(i-k)^2)\prod_{1\le i\le n-k}(k^2+(i+k)^2) = \prod_{\substack{1\le j\le n\\ j\ne k}}(k^2+j^2),$$

respectively, to get

$$(-1)^{n-k}\frac{4n^2}{(n-k)!\,(n+k)!}\;\frac{\prod_{1\le i\le n-1}(i^4+4k^4)}{\prod_{\substack{1\le j\le n\\ j\ne k}}(k^2+j^2)}$$

since $(k^2+(i-k)^2)(k^2+(i+k)^2) = i^4+4k^4$. Next multiply bottom and top through by

$$\prod_{\substack{1\le j\le n\\ j\ne k}}(k^2-j^2) = \prod_{\substack{1\le j\le n\\ j\ne k}}(k-j)\prod_{\substack{1\le j\le n\\ j\ne k}}(k+j) = \big((k-1)!\,(-1)^{n-k}(n-k)!\big)\left(\frac{(n+k)!}{k!\,(2k)}\right)$$

respectively, which gives the k-th term in the sum in (2), and thus (2) is proved. $\square$

Corollary 2: Equivalence of (3) and (4). The left side of (3) is a polynomial of degree $\le n-1$ in t. Thus, in order to establish the identity in (3) it suffices to show that (3) holds for at least n different values of t. We will show that (4) is essentially the same as (3) with $t=-r^2$, and so proving Corollary 2 establishes Theorem 1; the converse is clear.

If we take $t=-r^2$ then $j^2-(kr)^2 = -(kr-j)(kr+j)$, which is nonzero since $k\ge 1>(n-1)/(r-1)$ so that $kr+j\ge kr-j\ge kr-(n+k-1)>0$. Therefore we obtain, from this substitution,

$$\begin{aligned}(-1)^{n-k}\prod_{\substack{0\le j<n-k\\ \text{or}\\ n<j<n+k}}(k^2(-r^2)+j^2) &= (-1)^{(n-k)+(n-1)}\prod_{0\le j\le n-k-1}(kr-j)(kr+j)\prod_{n+1\le j\le n+k-1}(kr-j)(kr+j)\\ &= (-1)^{k-1}\frac{(kr)!}{(kr+k-n)!}\,\frac{(kr+n-k-1)!}{(kr-1)!}\,\frac{(kr-n-1)!}{(kr-k-n)!}\,\frac{(kr+n+k-1)!}{(kr+n)!}\\ &= \frac{(2n-1)!^2}{(2n+1)!}\,(-1)^{k-1}rk\,\frac{\binom{(r+1)k+n-1}{2n-1}\binom{(r-1)k+n-1}{2n-1}}{\binom{rk+n}{2n+1}}.\end{aligned}$$

Equation (4) follows immediately from substituting this into (3). $\square$

4. FURTHER CONSEQUENCES

Many identities arise as consequences of Theorem 1′; perhaps one of those is already known and might itself imply our results (in the same way that we showed (3) and (4) to be equivalent). For example, using the Γ-function we can rederive Corollary 2 (replacing r by x) as

$$\sum_{1\le k\le n}(-1)^{n-k}\binom{2n}{n-k}\frac{\Gamma(n+kx+k)\Gamma(n-kx+k)\Gamma(n+kx-k)\Gamma(n-kx-k)}{\Gamma(kx)\Gamma(-kx)\Gamma(n+kx+1)\Gamma(n-kx+1)}=\left(\frac{(2n-1)!}{n!}\right)^2. \tag{6}$$

Below we list a few identities for binomial coefficients that are easy deductions from our results. It is convenient to use, at times, a variant on the "falling factorial" notation:

$$\begin{aligned} x^{(n)} &= x(x-1)\dots(x-n+1) && \text{if } n\ge 1,\\ x^{(-n)} &= 1/(x+1)(x+2)\dots(x+n) && \text{if } n\ge 1,\\ x^{(0)} &= 1. \end{aligned}$$

More generally than in Corollary 2, we now take $t=r^2$ and $g(j)=-(j+a-n)^2$ in Theorem 1′ to obtain:

For any given integers a and n the value of

$$\sum_{1\le k\le n}(-1)^{n-k}\binom{2n}{n-k}\times\frac{((r+1)k+a-1)^{(2a-1)}((r-1)k+a-1)^{(2a-1)}}{(rk+a)^{(2a+1)}(rk+a-n-1)^{(2a-2n-1)}} \tag{7}$$

is independent of r. (Corollary 2 is the case $a=n$.)

Next we take $t=r$ and $g(j)=j+a-n$ in Theorem 1′, and then $t=r$ and $g(j)=-j+a+n$ in Theorem 1′, to obtain:

For any given integers a and n the values of

$$\sum_{1\le k\le n}(-1)^{n-k}\frac{\binom{n+1}{k+1}\binom{k^2r+a+k-1}{n+k}}{\binom{k^2r+a}{k+1}} \tag{8}$$

and of

$$\sum_{1\le k\le n}(-1)^{n-k}\frac{\binom{n+1}{k+1}\binom{k^2r+a+n}{n+k}}{\binom{k^2r+a+k}{k+1}} \tag{9}$$

are independent of r.

In results such as this, where we write that the value is independent of the variable r, we can obtain a prettier identity if we can just evaluate the sum for one particular value of r. For example, a nice special case of (8), when we take $a=n+1$, gives

$$\sum_{1\le k\le n}(-1)^{n-k}\frac{\binom{n+1}{k+1}\binom{k^2r+n+k}{n+k}}{\binom{k^2r+n+1}{k+1}}=\begin{cases}1 & \text{if } n \text{ is odd},\\ 0 & \text{if } n \text{ is even}.\end{cases} \tag{10}$$

Next we take $t=2r$ and $g(j)=2(j+a-n)+1$ in Theorem 1′, and then, in the same theorem, $t=2r$ and $g(j)=2(-j+a+n)-1$, to obtain:

For any given integers a and n the values of

$$\sum_{1\le k\le n}(-1)^{n-k}\binom{2n}{n-k}\times\frac{(2(rk^2+a-k))^{(2(n-k))}}{(rk^2+a-k)^{(n-k)}}\frac{(2(rk^2+a+k))^{(2k-1)}}{(rk^2+a+k)^{(k)}} \tag{11}$$

and of

$$\sum_{1\le k\le n}(-1)^{n-k}\binom{2n}{n-k}\times\frac{(2(rk^2+a+n))^{(2(n-k))}}{(rk^2+a+n)^{(n-k)}}\frac{(2(rk^2+a-1))^{(2k-1)}}{(rk^2+a-1)^{(k)}} \tag{12}$$

are independent of r.

There are similar, though more complicated, identities to be obtained from Theorem 1′ by taking $t=2r+1$ and $g(j)$ equal to each of the values

$$2(j+a-n)+1,\quad 2(-j+a+n)-1,\quad 2(j+a-n),\quad 2(-j+a+n).$$

We now move on to another class of identities, obtained by studying the coefficients of t^{m+1} in Theorem 1, for various values of $n-2\ge m\ge 0$. Let S_k be the set of integers in $(0,\,n-k)\cup(n,\,n+k)$. Equating the coefficient of t^{m+1} in the $k=n$ term with the others gives, in general,

$$n^{2m}\sum_{\substack{n<j_1<j_2<\dots\\ \dots<j_{m+1}<2n}}\frac{1}{(j_1j_2\dots j_{m+1})^2}=$$
$$4\sum_{k=1}^{n-1}(-1)^{n-1-k}\frac{\left(\frac{k^{m+1}}{n^2-k^2}\right)^2}{\binom{2n}{n-k}}\sum_{\substack{j_1<j_2<\dots<j_m\\ j_1,\dots,j_m\in S_k}}\frac{1}{(j_1\dots j_m)^2}. \tag{13}$$

Taking $m=0$ here gives us

$$\sum_{n<j<2n}\frac{1}{j^2}=4\sum_{k=1}^{n-1}(-1)^{n-1-k}\frac{\left(\frac{k}{n^2-k^2}\right)^2}{\binom{2n}{n-k}}, \tag{14}$$

which is a fast converging series to approximate this sum. Taking $m=1$ gives

$$\frac{8}{n^2}\sum_{k=1}^{n-1}(-1)^{n-1-k}\frac{\left(\frac{k^2}{n^2-k^2}\right)^2}{\binom{2n}{n-k}}\sum_{j\in S_k}\frac{1}{j^2}$$
$$=\left(\sum_{n<j<2n}\frac{1}{j^2}\right)^2-\sum_{n<j<2n}\frac{1}{j^4}.$$

Similar results follow if we work with Theorem 1′ instead of Theorem 1; though it is a matter of taste as to what constitutes a nice identity and what an eyesore. We hope the reader will play with Theorem 1′ to discover further pretty identities.

5. MORE FORMULAS

It seems worth recording here several formulas of Apéry type that we found in the literature [van der Poorten 1979; 1980]. Define $\varphi=(\sqrt{5}-1)/2$ and $\tau=\log(1/\varphi)$. Then

$$\sum_{n\ge1}\frac{1}{\binom{2n}{n}}=\frac{2\pi\sqrt{3}+9}{27},\qquad \sum_{n\ge1}\frac{(-1)^{n-1}}{\binom{2n}{n}}=\frac{4\tau}{5\sqrt{5}}+\frac{1}{5};$$
$$\sum_{n\ge1}\frac{1}{n\binom{2n}{n}}=\frac{\pi\sqrt{3}}{9},\qquad \sum_{n\ge1}\frac{(-1)^{n-1}}{n\binom{2n}{n}}=\frac{2\tau}{\sqrt{5}};$$
$$\sum_{n\ge1}\frac{1}{n^2\binom{2n}{n}}=\frac{\pi^2}{18},\qquad \sum_{n\ge1}\frac{(-1)^{n-1}}{n^2\binom{2n}{n}}=2\tau^2;$$
$$\sum_{n\ge1}\frac{1}{n^4\binom{2n}{n}}=\frac{17\pi^4}{3240};\qquad \sum_{n\ge1}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}=\frac{2\zeta(3)}{5}.$$

For similar formulas, though with the numerator a polynomial in n, see [Lehmer 1985]. Zucker [1985] showed how such sums are related to values of Dirichlet L-functions and the polylogarithm function $\mathrm{Li}_k\,x:=\sum_{n\ge1}x^n/n^k$. For example:

$$\sum_{n\ge1}\frac{1}{n^3\binom{2n}{n}}=\frac{\sqrt{3}\pi}{2}L\left(2,\left(\frac{-3}{\cdot}\right)\right)-\frac{4\zeta(3)}{3},$$
$$\sum_{n\ge1}\frac{1}{n^5\binom{2n}{n}}=\frac{9\sqrt{3}\pi}{8}L\left(4,\left(\frac{-3}{\cdot}\right)\right)+\frac{\pi^2\zeta(3)}{9}-\frac{19\zeta(5)}{3},$$
$$\sum_{n\ge1}\frac{(-1)^{n-1}}{n^4\binom{2n}{n}}=\tfrac{3}{2}\tau^4-7\zeta(4)-\tau^2\,\mathrm{Li}_2\,\varphi^2-\tau\,\mathrm{Li}_3\,\varphi^2$$
$$-\tfrac{1}{2}\,\mathrm{Li}_4\,\varphi^2+4\tau^2\,\mathrm{Li}_2\,\varphi+8\tau\,\mathrm{Li}_3\,\varphi+8\,\mathrm{Li}_4\,\varphi,$$
$$\sum_{n\ge1}\frac{(-1)^{n-1}}{n^5\binom{2n}{n}}=\tfrac{4}{3}\tau^5-2\zeta(5)+\tfrac{10}{3}\tau^3\,\mathrm{Li}_2\,\varphi^2$$
$$+5\tau^2\,\mathrm{Li}_3\,\varphi^2+5\tau\,\mathrm{Li}_4\,\varphi^2+\tfrac{5}{2}\,\mathrm{Li}_5\,\varphi^2.$$

Adamchik recently informed us that he has found similar expressions for these same sums. However, it doesn't seem that such expressions are known for any higher exponents.

There are also other generalizations of Gosper's formula [Koecher 1980; Leshchiner 1981]:

$$\sum_{k\ge0}\zeta(2k+3)z^{2k}$$
$$=\sum_{n\ge1}\frac{1}{n^3(1-z^2/n^2)}$$
$$=\sum_{n\ge1}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}\left(\frac{1}{2}+\frac{2}{(1-z^2/n^2)}\right)\prod_{m=1}^{n-1}\left(1-\frac{z^2}{m^2}\right)$$

and

$$\sum_{k\ge0}\left(1-\frac{1}{2^k}\right)\zeta(2k+2)z^{2k}$$
$$=\sum_{n\ge1}\frac{1}{n^2}\left(\frac{1}{1-z^2/n^2}-\frac{1}{1-z^2/2n^2}\right)$$
$$=\sum_{n\ge1}\frac{1}{n^2\binom{2n}{n}}\left(-\frac{1}{2}+\frac{2}{(1-z^2/n^2)}\right)\prod_{m=1}^{n-1}\left(1-\frac{z^2}{m^2}\right).$$

These formulas have a slightly different flavour from the generalization given in Conjecture 1.

Using the revolutionary method of Wilf and Zeilberger [1990], Amdeberhan and Zeilberger [1997] found the striking, and fast converging, formula

$$\zeta(3) = \frac{1}{2}\sum_{n\geq 1} \frac{(-1)^{n-1}(205n^2 - 160n + 32)}{n^5\binom{2n}{n}^5},$$

amongst several others.

ACKNOWLEDGEMENTS

Thanks are due to David Bradley for pointing us to several of the formulas cited in this last section.

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Received February 6, 1998; accepted July 6, 1998