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GLOBAL EXISTENCE AND FINITE DIMENSIONAL GLOBAL ATTRACTOR FOR A 3D DOUBLE VISCOUS MHD-α MODEL*

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Abstract. We consider a magnetohydrodynamic-α model with kinematic viscosity and magnetic diffusivity for an incompressible fluid in a three-dimensional periodic box (torus). Similar models are useful to study the turbulent behavior of fluids in presence of a magnetic field because of the current impossibility to handle non-regularized systems neither analytically nor via numerical simulations.

We prove the existence of a global solution and a global attractor. Moreover, we provide an upper bound for the Hausdorff and the fractal dimension of the attractor. This bound can be interpreted in terms of degrees of freedom of the system. In some sense, this result provides an intermediate bound between the number of degrees of freedom for the simplified Bardina model and the Navier–Stokes-α equation.

Key words. Magnetohydrodynamics, MHD-α model, Bardina model, regularizing MHD, turbulence models, incompressible fluid, global attractor.

AMS subject classifications. 35Q35, 76D03.

1. Introduction

The basic system of equations that one can consider in magnetohydrodynamics is obtained by combining Maxwell's equations, which rule the magnetic field, with the Navier–Stokes equation, which governs the fluid motion; this system has form

$$\boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}+\nabla\left(p+\frac{1}{2}|\boldsymbol{B}|^2\right)=\nu\Delta\boldsymbol{v}, \tag{1.1a}$$

$$\boldsymbol{B}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}=\mu\Delta\boldsymbol{B}, \tag{1.1b}$$

$$\nabla\cdot\boldsymbol{v}=\nabla\cdot\boldsymbol{B}=0, \tag{1.1c}$$

$$(\boldsymbol{v},\boldsymbol{B})|_{t=0}=(\boldsymbol{v}_0,\boldsymbol{B}_0),\qquad \boldsymbol{x}\in\mathbb{R}^n,\qquad n=2,3, \tag{1.1d}$$

where the fluid velocity field $\boldsymbol{v}(\boldsymbol{x},t)$, the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$, and the pressure $p(\boldsymbol{x},t)$ are the unknowns, while $\nu\geqslant 0$ is the constant kinematic viscosity and $\mu\geqslant 0$ is the constant magnetic diffusivity. In this case, an incompressible fluid is considered.

This problem has been deeply studied. If $\nu>0$ and $\mu>0$, then there exists a unique global solution in time when $n=2$, while for $n=3$ the problem is still open, as discussed in [14].

When $n=2$, $\nu=0$, and $\mu=1$, local existence and small data global existence results have been established by Kozono [11] for bounded domains and by Casella–Secchi–Trebeschi [4] for unbounded domains.

When $n=2$, $\nu=1$, and $\mu=0$, there is a regularity criterion for the solution $\boldsymbol{B}$ provided by Jiu–Niu [10], but the problem in its generality is still open.

As pointed out in [13] (see also the suggested bibliography), at the moment there is no possibility to compute the turbulent behavior of fluids neither analytically nor via direct numerical simulation (this task is prohibitively expensive and disputable as

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well due to sensitivity of perturbation errors in the initial data). Hence, one can try to focus only on certain statistical features of the physical phenomenon through the employment of suitable models. This is sufficient in many practical applications.

Because of the success of Navier–Stokes-α models in producing solutions in excellent agreement with empirical data for a wide range of large Reynolds numbers and flow in infinite channels or pipes, it is natural to consider such a kind of regularization for magnetohydrodynamic models as well.

In α models, a function (or several functions) is substituted in one or more of its occurrences with a regularized function; more precisely, the function $\boldsymbol{v}$ is substituted with $\boldsymbol{u}$, where

$$\boldsymbol{v}=(1-\alpha^2\Delta)\boldsymbol{u}, \qquad \alpha>0.$$

This substitution is performed in nonlinear terms to make the nonlinearity milder, so that the solution becomes smoother.

Linshiz–Titi [13] have suggested several models. For instance, filtering only the velocity field, one can consider the following model, referred to as Navier–Stokes-α-MHD (NSαMHD):

$$\boldsymbol{v}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}+\sum_{j=1}^{n} v_j\nabla u_j-(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}+\nabla\left(p+\frac{1}{2}|\boldsymbol{B}|^2\right)=\nu\Delta\boldsymbol{v}, \tag{1.2a}$$

$$\boldsymbol{B}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}=\mu\Delta\boldsymbol{B}, \tag{1.2b}$$

$$\boldsymbol{v}=(1-\alpha^2\Delta)\boldsymbol{u}, \qquad \alpha>0, \tag{1.2c}$$

$$\nabla\cdot\boldsymbol{v}=\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{B}=0, \tag{1.2d}$$

$$(\boldsymbol{v},\boldsymbol{B})|_{t=0}=(\boldsymbol{v}_0,\boldsymbol{B}_0). \tag{1.2e}$$

In this case, Linshiz–Titi [13] have shown a global existence result in a three-dimensional periodic box when $\nu>0$ and $\mu>0$, while Fan–Ozawa [8] have achieved the same result in the whole space $\mathbb{R}^2$ for both $(\nu=1,\ \mu=0)$ and $(\nu=0,\ \mu=1)$.

Let us note that, in the ideal case, i.e., when $\nu=\mu=0$, the NSαMHD model possesses three quadratic invariants: the energy $E^\alpha=\frac{1}{2}\int_\Omega\left(\boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})+|\boldsymbol{B}(\boldsymbol{x})|^2\right)\mathrm{d}\boldsymbol{x}$, the cross helicity $H_{\mathrm{C}}^\alpha=\frac{1}{2}\int_\Omega\boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{B}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, and the magnetic helicity $H_{\mathrm{M}}^\alpha=\frac{1}{2}\int_\Omega\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{B}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, where $\boldsymbol{A}$ is the vector potential, so that $\boldsymbol{B}=\nabla\times\boldsymbol{A}$. Moreover, as $\alpha\to 0$, these quantities reduce to the ideal invariants of the MHD equations.

Another model is the so-called simplified Bardina model, which is studied by Cao–Lunasin–Titi in [3].

In [5], the following magnetohydrodynamic-α model, derived from Bardina model for incompressible fluids, is considered:

$$\boldsymbol{v}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}+\nabla\left(p+\frac{1}{2}|\boldsymbol{B}|^2\right)=\Delta\boldsymbol{v} \qquad \text{in } [0,T]\times\mathbb{R}^2, \tag{1.3a}$$

$$\boldsymbol{B}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}=\boldsymbol{0} \qquad \text{in } [0,T]\times\mathbb{R}^2, \tag{1.3b}$$

$$\boldsymbol{v}=(1-\alpha^2\Delta)\boldsymbol{u}, \qquad \alpha>0 \qquad \text{in } [0,T]\times\mathbb{R}^2, \tag{1.3c}$$

$$\nabla\cdot\boldsymbol{v}=\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{B}=0 \qquad \text{in } [0,T]\times\mathbb{R}^2, \tag{1.3d}$$

$$(\boldsymbol{v},\boldsymbol{B})|_{t=0}=(\boldsymbol{v}_0,\boldsymbol{B}_0) \qquad \boldsymbol{x}\in\mathbb{R}^2. \tag{1.3e}$$

Once again, a global existence result is obtained.

In this paper, we consider the following model, referred to as Simplified Bardina MHD (SBMHD):

$$\begin{aligned}
&\boldsymbol{v}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}+\nabla p=\nu\Delta\boldsymbol{v}+\boldsymbol{f} && \text{in } \Omega\times[0,T], && (1.4a)\\
&\boldsymbol{B}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}=\mu\Delta\boldsymbol{B} && \text{in } \Omega\times[0,T], && (1.4b)\\
&\boldsymbol{v}=(1-\alpha^2\Delta)\boldsymbol{u}, \qquad \alpha>0 && \text{in } \Omega\times[0,T], && (1.4c)\\
&\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{B}=0 && \text{in } \Omega\times[0,T], && (1.4d)\\
&(\boldsymbol{v},\boldsymbol{B})|_{t=0}=(\boldsymbol{v}_0,\boldsymbol{B}_0) && \boldsymbol{x}\in\Omega, && (1.4e)
\end{aligned}$$

where $\alpha,\nu,\mu>0$ and $\Omega=[0,2\pi L]^3$, $L>0$, with periodic boundary conditions and hence periodic solutions. Moreover, we assume that the forcing term $\boldsymbol{f}$ does not depend on time and has zero mean: $\int_\Omega \boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0$. We assume the same hypothesis of zero mean for the initial data, so that also the solutions $\boldsymbol{v}$ (and $\boldsymbol{u}$ as well) and $\boldsymbol{B}$ have zero mean.

The ideal version of system SBMHD conserves the energy and the magnetic helicity, but at the moment we are unable to find an invariant quantity corresponding to cross helicity.

We will prove the following results.

THEOREM 1.1 (Global Existence). *Assume that the initial data satisfy*

$$\boldsymbol{v}_0\in \mathrm{L}^2(\Omega), \qquad \boldsymbol{B}_0\in \mathrm{H}^1(\Omega),$$

$\nabla\cdot\boldsymbol{v}_0=\nabla\cdot\boldsymbol{B}_0=0$.

Then, problem (1.4) *has a unique global solution* $(\boldsymbol{v},\boldsymbol{B})$ *such that, for each time* $T>0$, *one has*

$$\begin{aligned}
\boldsymbol{v}&\in \mathrm{L}^\infty(0,\infty;\mathrm{L}^2(\Omega))\cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)),\\
\boldsymbol{B}&\in \mathrm{L}^\infty(0,\infty;\mathrm{H}^1(\Omega))\cap \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega)).
\end{aligned}$$

Note that local existence, uniqueness and continuous dependence on the initial data can be achieved through Galerkin method following [3, 13] and using the a priori estimates that we provide in section 3.

From now on, to simplify notation we set $\|\cdot\|=\|\cdot\|_{\mathrm{L}^2(\Omega)}$ and denote by the subscript σ a space of divergence-free and zero mean functions.

THEOREM 1.2 (Finite Dimensional Global Attractor). *There is a (unique) compact global attractor* $\mathscr{A}\subset \mathrm{H}^1_\sigma(\Omega)\times \mathrm{L}^2_\sigma(\Omega)$ *in terms of the solution* $(\boldsymbol{u},\boldsymbol{B})$ *to* (1.4). *Moreover, we have an upper bound for the Hausdorff dimension* $d_{\mathrm{H}}(\mathscr{A})$ *and the fractal dimension* $d_{\mathrm{F}}(\mathscr{A})$ *of the attractor* $\mathscr{A}$*; in particular, there is a positive constant* C *such that*

$$d_{\mathrm{H}}(\mathscr{A})\leqslant d_{\mathrm{F}}(\mathscr{A})\leqslant CG^{6/5}\left(\frac{L}{\alpha}\right)^3\left[\left(\frac{L}{\alpha}\right)^{\frac{3}{5}}+G^{6/5}\left(\frac{L}{\alpha}\right)^{\frac{9}{5}}+G^{3/10}\right],$$

where, setting $\eta=\min\{\nu,\mu\}$,

$$G=\frac{L^{3/2}\|\boldsymbol{f}\|}{\eta^2}$$

is the modified Grashoff number.

We can interpret the estimate for the attractor dimension in terms of the mean rate of energy dissipation, defined by

$$\bar{\varepsilon} = \frac{1}{L^3} \sup_{(\boldsymbol{u}_0,\boldsymbol{B}_0)\in\mathscr{A}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left(\nu\|\nabla\boldsymbol{u}(t)\|^2 + \nu\alpha^2\|\Delta\boldsymbol{u}(t)\|^2 + \mu\|\nabla\boldsymbol{B}(t)\|^2\right)\mathrm{d}t.$$

Moreover, in analogy with Kolmogorov dissipation length in the classical theory of turbulence, we define the dissipation length as

$$\ell_d = \left(\frac{\eta^3}{\bar{\varepsilon}}\right)^{1/4},$$

so that

$$\begin{aligned} &\sup_{(\boldsymbol{u}_0,\boldsymbol{B}_0)\in\mathscr{A}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left(\nu\|\nabla\boldsymbol{u}(t)\|^2 + \nu\alpha^2\|\Delta\boldsymbol{u}(t)\|^2 + \mu\|\nabla\boldsymbol{B}(t)\|^2\right)\mathrm{d}t \\ &= \frac{L^3\eta^3}{\ell_d^4}. \end{aligned} \tag{1.5}$$

We have the following result.

THEOREM 1.3. *The unique compact global attractor* $\mathscr{A}\subset \mathrm{H}^1_\sigma(\Omega)\times \mathrm{L}^2_\sigma(\Omega)$ *in terms of the solution* $(\boldsymbol{u},\boldsymbol{B})$ *to* (1.4) *has Hausdorff dimension* $d_{\mathrm{H}}(\mathscr{A})$ *and fractal dimension* $d_{\mathrm{F}}(\mathscr{A})$ *bounded by*

$$D \doteq C\max\left\{\left(\frac{L}{\alpha}\right)^{12/5}\left(\frac{L}{\ell_d}\right)^{12/5}, \left(\frac{L}{\alpha}\right)^{3/2}\left(\frac{L}{\ell_d}\right)^{3}\right\},$$

where C *is a positive constant.*

Identifying the dimension of the global attractor with the number of degrees of freedom of the long-time dynamics of the solution, this means that the number of degrees of freedom of problem (1.4) is bounded from above by a quantity which scales like D. This information is useful to establish the validity of the model as a large-eddy simulation model of turbulence.

Let us observe that, in space dimension $n=3$, the number of degrees of freedom of the simplified Bardina model (with no magnetic field) is bounded from above by $C(L/\alpha)^{12/5}(L/\ell_d)^{12/5}$, while for the Navier–Stokes-α model this upper bound is $C(L/\alpha)^{3/2}(L/\ell_d)^3$ (see [3] for the first result and further references). Hence, in some sense, our result provides an intermediate bound.

As a final remark, let us note that the nonlinearity in the SBMHD model considered in this paper is milder than the one in the NSαMHD model studied by Linshiz–Titi. This means that the SBMHD is easier to handle than NSαMHD from the point of view of global existence, and one expects the same behavior for the estimates of the global attractor dimension, that is to say, new difficulties might arise in the proof of the dimension bounds for the NSαMHD case. Nevertheless, this problem has been addressed in [6], where using an approach analogous to the one presented in this paper, bounds for the global attractor dimension of the NSαMHD and the Modified Leray-α-MHD models are provided.

Outline of the paper. In section 2 we provide some preliminary results that we will use in the following sections. Section 3 is devoted to the prove of some a priori estimates which imply, in particular, the global existence of the solution. In section 4 we prove the existence of the unique global attractor, whose dimension is estimated in section 5.

2. Preliminary results

We consider functions with zero mean over Ω. This assumption is taken for all functions in all the remainder of the paper.

If $\boldsymbol{f}$ is a divergence free function, then

$$\int (\boldsymbol{f}\cdot\nabla)\boldsymbol{g}\cdot\boldsymbol{h}\,\mathrm{d}\boldsymbol{x} = -\int (\boldsymbol{f}\cdot\nabla)\boldsymbol{h}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x} \tag{2.1}$$

and

$$\int (\boldsymbol{f}\cdot\nabla)\boldsymbol{g}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x} = 0, \tag{2.2}$$

where the integrals are intended over Ω.

Let us denote λ_1 to be the minimal eigenvalue of $-\Delta$; we have $\lambda_1 = L^{-2}$ and the Poincaré inequality becomes

$$\lambda_1 \|\boldsymbol{g}\|^2 \leqslant \|\nabla\boldsymbol{g}\|^2. \tag{2.3}$$

We consider norms in nondimensional form

$$\|\boldsymbol{g}\|_{\mathrm{H}^1(\Omega)} = \|\boldsymbol{g}\| + L\|\nabla\boldsymbol{g}\|,$$
$$\|\boldsymbol{g}\|_{\mathrm{H}^2(\Omega)} = \|\boldsymbol{g}\| + L\|\nabla\boldsymbol{g}\| + L^2\|\Delta\boldsymbol{g}\|,$$

so that the Agmon inequality

$$\|\boldsymbol{g}\|_{\mathrm{L}^\infty(\Omega)} \leqslant C\|\boldsymbol{g}\|^{1/2}_{\mathrm{H}^1(\Omega)}\|\boldsymbol{g}\|^{1/2}_{\mathrm{H}^2(\Omega)} \tag{2.4}$$

can be recast in homogeneous form

$$\|\boldsymbol{g}\|_{\mathrm{L}^\infty(\Omega)} \leqslant C\|\nabla\boldsymbol{g}\|^{1/2}\|\Delta\boldsymbol{g}\|^{1/2}. \tag{2.5}$$

We recall the Gagliardo–Nirenberg inequality: let us set

$$|\boldsymbol{f}|_{k,p} = \left(\sum_{|l|=k} |\partial^l \boldsymbol{f}|^p\right)^{1/p};$$

then

$$|\boldsymbol{f}|_{j,s} \leqslant C|\boldsymbol{f}|^a_{m,r}\|\boldsymbol{f}\|^{1-a}_{\mathrm{L}^q} \tag{2.6}$$

provided

$$\frac{1}{s} - \frac{j}{n} = a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, \qquad \frac{j}{m} \leqslant a \leqslant 1,$$

where n is the space dimension (there are some exceptional cases that we will not consider).

In particular, we will use the following estimates:

$$\|\boldsymbol{g}\|_{\mathrm{L}^{5/2}} \leqslant C\|\boldsymbol{g}\|^{7/10}\|\boldsymbol{g}\|^{3/10}_{\mathrm{L}^6}, \tag{2.7}$$
$$\|\nabla\boldsymbol{g}\|_{\mathrm{L}^4} \leqslant C\|\Delta\boldsymbol{g}\|^{7/8}\|\boldsymbol{g}\|^{1/8}. \tag{2.8}$$

Moreover, the following estimate holds:

$$\|\boldsymbol{g}\|_{\mathrm{L}^6} \leqslant C\|\nabla\boldsymbol{g}\|. \tag{2.9}$$

3. Global existence

The global existence of the solution follows from the local existence (see the introduction) and the following estimates. Let us note that such estimates imply uniform in time estimates, hence the global existence result.

3.1. Estimate for $\boldsymbol{u}\in \mathrm{H}^1$, $\boldsymbol{B}\in \mathrm{L}^2$.

PROPOSITION 3.1 (Some A Priori Estimates). *Assume that a solution* $(\boldsymbol{v},\boldsymbol{B})$ *of problem* (1.4) *is defined in the time interval* $[0,T]$. *Let us set*

$$k_0=\|\boldsymbol{u}(0)\|^2+\alpha^2\|\nabla\boldsymbol{u}(0)\|^2+\|\boldsymbol{B}(0)\|^2,$$
$$K_1=\min\left\{\frac{\|A^{-1}\boldsymbol{f}\|^2}{\nu\alpha^2},\frac{\|A^{-1/2}\boldsymbol{f}\|^2}{\nu}\right\},$$

$\eta=\min\{\nu,\mu\}$, *and* $\lambda_1=L^{-2}$ *(minimum eigenvalue of* $-\Delta$*).*

Then, the following estimates hold:

$$\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla\boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2\leqslant k_0\,\mathrm{e}^{-\eta\lambda_1 t}+\frac{K_1}{\eta\lambda_1}(1-\mathrm{e}^{-\eta\lambda_1 t}), \tag{3.1}$$

$$\int_t^{t+r}(\nu\|\nabla\boldsymbol{u}(\tau)\|^2+\nu\alpha^2\|\Delta\boldsymbol{u}(\tau)\|^2+\mu\|\nabla\boldsymbol{B}(\tau)\|^2)\,\mathrm{d}\tau\leqslant rK_1+k_1 \tag{3.2}$$

provided $r>0$ *and* $t+r\leqslant T$,

$$\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\left(\nu\|\nabla\boldsymbol{u}(t)\|^2+\nu\alpha^2\|\Delta\boldsymbol{u}(t)\|^2+\mu\|\nabla\boldsymbol{B}(t)\|^2\right)\mathrm{d}t\leqslant\frac{4K_1}{\eta\lambda_1}+k_0. \tag{3.3}$$

Proof. We test the first equation by $\boldsymbol{u}$ (i.e. we take the scalar product with $\boldsymbol{u}$ and integrate over Ω) and the second one by $\boldsymbol{B}$; summing up, using (2.1) and (2.2), and integrating by parts when needed (in particular, the term with the pressure p disappears, since $\nabla\cdot\boldsymbol{u}=0$), we obtain the energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}\|^2+\alpha^2\|\nabla\boldsymbol{u}\|^2+\|\boldsymbol{B}\|^2\right)+\nu\|\nabla\boldsymbol{u}\|^2+\nu\alpha^2\|\Delta\boldsymbol{u}\|^2+\mu\|\nabla\boldsymbol{B}\|^2=(\boldsymbol{f},\boldsymbol{u}),$$

where $(\cdot,\cdot)$ denotes the standard scalar product in $\mathrm{L}^2(\Omega)$, i.e.,

$$(\boldsymbol{f},\boldsymbol{u})=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}.$$

Now, setting $A=-\Delta$, where Δ is the Laplace operator with domain $\mathrm{D}(A)=(\mathrm{H}^2_\sigma(\Omega))^3$, we have that A is a positive self-adjoint operator and

$$|(\boldsymbol{f},\boldsymbol{u})|\leqslant\begin{cases}\|A^{-1}\boldsymbol{f}\|\,\|\Delta\boldsymbol{u}\| \\ \|A^{-1/2}\boldsymbol{f}\|\,\|\nabla\boldsymbol{u}\|\end{cases}\leqslant\begin{cases}\frac{\|A^{-1}\boldsymbol{f}\|^2}{2\nu\alpha^2}+\frac{\nu\alpha^2}{2}\|\Delta\boldsymbol{u}\|^2 \\ \frac{\|A^{-1/2}\boldsymbol{f}\|^2}{2\nu}+\frac{\nu}{2}\|\nabla\boldsymbol{u}\|^2\end{cases} \tag{3.4}$$

by Young's inequality.

If we set

$$K_1=\min\left\{\frac{\|A^{-1}\boldsymbol{f}\|^2}{\nu\alpha^2},\frac{\|A^{-1/2}\boldsymbol{f}\|^2}{\nu}\right\},$$

we have

$$|(\boldsymbol{f},\boldsymbol{u})| \leqslant (K_1+\nu\alpha^2\|\Delta\boldsymbol{u}\|^2+\nu\|\nabla\boldsymbol{u}\|^2)/2,$$

hence we deduce the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\boldsymbol{u}\|^2+\alpha^2\|\nabla\boldsymbol{u}\|^2+\|\boldsymbol{B}\|^2\big)+\nu\|\nabla\boldsymbol{u}\|^2+\nu\alpha^2\|\Delta\boldsymbol{u}\|^2+\mu\|\nabla\boldsymbol{B}\|^2 \leqslant K_1. \tag{3.5}$$

We set $\eta=\min\{\nu,\mu\}>0$ and use Poincaré inequality (2.3) to recast the previous estimate in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\boldsymbol{u}\|^2+\alpha^2\|\nabla\boldsymbol{u}\|^2+\|\boldsymbol{B}\|^2\big)+\eta\lambda_1(\|\boldsymbol{u}\|^2+\alpha^2\|\nabla\boldsymbol{u}\|^2+\|\boldsymbol{B}\|^2)\leqslant K_1$$

and obtain, by the Gronwall lemma,

$$\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla\boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2\leqslant k_0\,\mathrm{e}^{-\eta\lambda_1 t}+\frac{K_1}{\eta\lambda_1}(1-\mathrm{e}^{-\eta\lambda_1 t}),$$

where

$$k_0=\|\boldsymbol{u}(0)\|^2+\alpha^2\|\nabla\boldsymbol{u}(0)\|^2+\|\boldsymbol{B}(0)\|^2.$$

This gives (3.1).

Setting $k_1=k_0+\frac{K_1}{\eta\lambda_1}$, we have from (3.1)

$$\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla\boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2\leqslant k_1. \tag{3.6}$$

Moreover, integrating (3.5) for $\tau\in[t,t+r]$, where $r>0$, yields

$$\int_t^{t+r}(\nu\|\nabla\boldsymbol{u}(\tau)\|^2+\nu\alpha^2\|\Delta\boldsymbol{u}(\tau)\|^2+\mu\|\nabla\boldsymbol{B}(\tau)\|^2)\,\mathrm{d}\tau\leqslant rK_1+k_1,$$

that is (3.2).

In the following, we will need an exponential variant of the previous estimate. Let us multiply (3.5) by $\mathrm{e}^{-\eta\lambda_1 t/4}>0$ and integrate in time over the interval $[0,T]$:

$$\begin{aligned}\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla\boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2\big)\,\mathrm{d}t&\\ +\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\big(\nu\|\nabla\boldsymbol{u}(t)\|^2+\nu\alpha^2\|\Delta\boldsymbol{u}(t)\|^2+\mu\|\nabla\boldsymbol{B}(t)\|^2\big)\,\mathrm{d}t&\\ \leqslant\int_0^T K_1\,\mathrm{e}^{-\eta\lambda_1 t/4}\,\mathrm{d}t=\frac{4K_1}{\eta\lambda_1}(1-\mathrm{e}^{-\eta\lambda_1 T/4})\leqslant\frac{4K_1}{\eta\lambda_1}.&\end{aligned}$$

An integration by parts of the first integral yields

$$\begin{aligned}&\big[\mathrm{e}^{-\eta\lambda_1 t/4}(\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla\boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2)\big]_0^T\\ &\quad+\frac{\eta\lambda_1}{4}\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\big(\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla\boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2\big)\,\mathrm{d}t,\end{aligned}$$

hence we have

$$
\begin{aligned}
&\mathrm{e}^{-\eta\lambda_1 T/4}(\|\boldsymbol{u}(T)\|^2+\alpha^2\|\nabla\boldsymbol{u}(T)\|^2+\|\boldsymbol{B}(T)\|^2)\\
&\quad+\frac{\eta\lambda_1}{4}\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\left(\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla\boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2\right)\mathrm{d}t\\
&\quad+\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\left(\nu\|\nabla\boldsymbol{u}(t)\|^2+\nu\alpha^2\|\Delta\boldsymbol{u}(t)\|^2+\mu\|\nabla\boldsymbol{B}(t)\|^2\right)\mathrm{d}t\\
\leqslant&\frac{4K_1}{\eta\lambda_1}+k_0;
\end{aligned}
$$

neglecting the first two terms, which are positive quantities, we conclude

$$
\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\left(\nu\|\nabla\boldsymbol{u}(t)\|^2+\nu\alpha^2\|\Delta\boldsymbol{u}(t)\|^2+\mu\|\nabla\boldsymbol{B}(t)\|^2\right)\mathrm{d}t\leqslant\frac{4K_1}{\eta\lambda_1}+k_0,
$$

that is (3.3). □

3.2. Estimate for $\boldsymbol{u}\in \mathrm{H}^2$ and $\boldsymbol{B}\in \mathrm{H}^1$.

PROPOSITION 3.2 (Further A Priori Estimate). *Let us assume that a solution $(\boldsymbol{v},\boldsymbol{B})$ of problem (1.4) is defined in the time interval $[0,T]$. Let us set*

$$
k_0'=\|\boldsymbol{v}(0)\|^2+\|\boldsymbol{B}(0)\|^2+\alpha^2\|\nabla\boldsymbol{B}(0)\|^2,
$$

$\eta=\min\{\nu,\mu\}$ *and* $\lambda_1=L^{-2}$ *(minimum eigenvalue of* $-\Delta$*).*

Then, the following estimate holds for a suitable $k_2'=k_2'(\boldsymbol{f},k_0,\alpha,\eta,\lambda_1)>0$*:*

$$
\|\boldsymbol{v}(t)\|^2+\|\boldsymbol{B}(t)\|^2+\alpha^2\|\nabla\boldsymbol{B}(t)\|^2\leqslant k_0'\mathrm{e}^{-\eta\lambda_1 t/4}+\frac{4k_2'}{\eta\lambda_1}(1-\mathrm{e}^{-\eta\lambda_1 t/4}). \tag{3.7}
$$

Proof. We test Equ. (1.4a) by $\boldsymbol{v}$ and (1.4b) by $(1-\alpha^2\Delta)\boldsymbol{B}$; proceeding similarly as before, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{v}\|^2+\|\boldsymbol{B}\|^2+\alpha^2\|\nabla\boldsymbol{B}\|^2\right)+\nu\|\nabla\boldsymbol{v}\|^2+\mu\|\nabla\boldsymbol{B}\|^2+\mu\alpha^2\|\Delta\boldsymbol{B}\|^2\\
=&-\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}+\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\cdot\boldsymbol{v}+\alpha^2\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\cdot\Delta\boldsymbol{B}\\
&\quad+\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\cdot(\boldsymbol{B}-\alpha^2\Delta\boldsymbol{B})+\int\boldsymbol{f}\cdot\boldsymbol{v}\\
=&\alpha^2\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\Delta\boldsymbol{u}-\alpha^2\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\cdot\Delta\boldsymbol{u}+\alpha^2\int(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\cdot\Delta\boldsymbol{B}\\
&\quad-\alpha^2\int(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\cdot\Delta\boldsymbol{B}+\int\boldsymbol{f}\cdot\boldsymbol{v}\\
=&-\alpha^2\int\partial_k u_i\partial_i u_j\partial_k u_j+\alpha^2\int\partial_k B_i\partial_i B_j\partial_k u_j-\alpha^2\int\partial_k u_i\partial_i B_j\partial_k B_j\\
&\quad+\alpha^2\int\partial_k B_i\partial_i u_j\partial_k B_j+\int\boldsymbol{f}\cdot\boldsymbol{v},
\end{aligned}
$$

where the sum over $i,j,k=1,2,3$ is assumed.

We denote ∂ to be a generic first order spatial derivative. First, we have to estimate nonlinear terms of form

$$\begin{aligned}\alpha^2\int|\partial\boldsymbol{u}||\partial\boldsymbol{u}|^2 &\leqslant C\alpha^2\|\nabla\boldsymbol{u}\|_{\mathrm{L}^\infty}\|\nabla\boldsymbol{u}\|^2\\ &\leqslant C\alpha^2\|\Delta\boldsymbol{u}\|^{1/2}\|\nabla\Delta\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{u}\|^2\\ &\leqslant C\|\boldsymbol{v}\|^{1/2}\|\nabla\boldsymbol{v}\|^{1/2}\|\nabla\boldsymbol{u}\|^2\\ &\leqslant\varepsilon\nu\|\nabla\boldsymbol{v}\|^2+\frac{C_\varepsilon}{\nu^{1/3}}\|\boldsymbol{v}\|^{2/3}\|\nabla\boldsymbol{u}\|^{8/3}\\ &\leqslant\varepsilon\nu\|\nabla\boldsymbol{v}\|^2+\varepsilon\nu\lambda_1\|\boldsymbol{v}\|^2+\frac{C_\varepsilon}{\nu\lambda_1^{1/2}}\|\nabla\boldsymbol{u}\|^4,\end{aligned}$$

having used the Hölder inequality, estimate (2.5), the identities

$$\|\boldsymbol{v}\|^2=\|\boldsymbol{u}\|^2+2\alpha^2\|\nabla\boldsymbol{u}\|^2+\alpha^4\|\Delta\boldsymbol{u}\|^2, \tag{3.8}$$

$$\|\nabla\boldsymbol{v}\|^2=\|\nabla\boldsymbol{u}\|^2+2\alpha^2\|\Delta\boldsymbol{u}\|^2+\alpha^4\|\nabla\Delta\boldsymbol{u}\|^2, \tag{3.9}$$

and Young's inequality. An application of the Poincaré inequality yields

$$\alpha^2\int|\partial\boldsymbol{u}|^3\leqslant 2\varepsilon\nu\|\nabla\boldsymbol{v}\|^2+\frac{C_\varepsilon}{\nu\lambda_1^{1/2}}\|\nabla\boldsymbol{u}\|^4. \tag{3.10}$$

Let us note that we can not follow a similar approach for terms with $\partial\boldsymbol{B}$, since we can not handle the quantity $\|\nabla\boldsymbol{B}\|^4$.

Nonetheless, using Hölder inequality, estimate (2.8) and Young's inequality, we easily deduce

$$\begin{aligned}\alpha^2\int|\partial\boldsymbol{u}||\partial\boldsymbol{B}|^2 &\leqslant C\alpha^2\|\nabla\boldsymbol{u}\|\,\|\nabla\boldsymbol{B}\|^2_{\mathrm{L}^4}\\ &\leqslant C\alpha^2\|\nabla\boldsymbol{u}\|\,\|\Delta\boldsymbol{B}\|^{7/4}\|\boldsymbol{B}\|^{1/4}\\ &\leqslant\varepsilon\mu\alpha^2\|\Delta\boldsymbol{B}\|^2+\frac{C_\varepsilon\alpha^2}{\mu^7}\|\nabla\boldsymbol{u}\|^8\|\boldsymbol{B}\|^2.\end{aligned} \tag{3.11}$$

It remains to estimate

$$\int\boldsymbol{f}\cdot\boldsymbol{v}=\int\boldsymbol{f}\cdot\boldsymbol{u}-\alpha^2\int\boldsymbol{f}\cdot\Delta\boldsymbol{u}.$$

By slightly modifying (3.4), we have

$$|(\boldsymbol{f},\boldsymbol{u})|\leqslant\begin{cases}\frac{\|A^{-1}\boldsymbol{f}\|^2}{\nu\alpha^2}+\frac{\nu\alpha^2}{4}\|\Delta\boldsymbol{u}\|^2\\ \frac{\|A^{-1/2}\boldsymbol{f}\|^2}{\nu}+\frac{\nu}{4}\|\nabla\boldsymbol{u}\|^2\end{cases}$$

and hence, recalling (3.9),

$$|(\boldsymbol{f},\boldsymbol{u})|\leqslant K_1+\frac{\nu\alpha^2}{4}\|\Delta\boldsymbol{u}\|^2+\frac{\nu}{4}\|\nabla\boldsymbol{u}\|^2\leqslant K_1+\frac{3\nu}{8}\|\nabla\boldsymbol{v}\|^2.$$

Similarly,

$$\alpha^2|(\boldsymbol{f},\Delta\boldsymbol{u})|\leqslant\begin{cases}\alpha^2\|A^{-1/2}\boldsymbol{f}\|\,\|\nabla\Delta\boldsymbol{u}\|\\ \alpha^2\|\boldsymbol{f}\|\,\|\Delta\boldsymbol{u}\|\end{cases}\leqslant\begin{cases}\frac{\|A^{-1/2}\boldsymbol{f}\|^2}{\nu}+\frac{\nu\alpha^4}{4}\|\nabla\Delta\boldsymbol{u}\|^2\\ \frac{\alpha^2\|\boldsymbol{f}\|^2}{\nu}+\frac{\nu\alpha^2}{4}\|\Delta\boldsymbol{u}\|^2\end{cases} \tag{3.12}$$

by Young inequality.

If we set

$$K_2 = \min\left\{ \frac{\|A^{-1/2}\boldsymbol{f}\|^2}{\nu}, \frac{\alpha^2\|\boldsymbol{f}\|^2}{\nu} \right\},$$

we have

$$\alpha^2|(\boldsymbol{f},\Delta\boldsymbol{u})| \leqslant K_2 + \frac{\nu\alpha^4}{4}\|\nabla\Delta\boldsymbol{u}\|^2 + \frac{\nu\alpha^2}{4}\|\Delta\boldsymbol{u}\|^2 \leqslant K_2 + \frac{3\nu}{8}\|\nabla\boldsymbol{v}\|^2.$$

In conclusion,

$$\left|\int \boldsymbol{f}\cdot\boldsymbol{v}\right| \leqslant K_1 + K_2 + \frac{3\nu}{4}\|\nabla\boldsymbol{v}\|^2. \tag{3.13}$$

Combining (3.10), (3.11), (3.13), and (3.6), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2 + \alpha^2\|\nabla\boldsymbol{B}\|^2\right) + \nu\|\nabla\boldsymbol{v}\|^2 + \mu\|\nabla\boldsymbol{B}\|^2 + \mu\alpha^2\|\Delta\boldsymbol{B}\|^2 \\
&\leqslant \left(\frac{3}{4} + 2\varepsilon\right)\nu\|\nabla\boldsymbol{v}\|^2 + \varepsilon\mu\alpha^2\|\Delta\boldsymbol{B}\|^2 + K_1 + K_2 \\
&\qquad + C_\varepsilon\|\nabla\boldsymbol{u}\|^4\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{\alpha^2}{\mu^7}\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{B}\|^2\right) \\
&\leqslant \frac{7}{8}\left(\nu\|\nabla\boldsymbol{v}\|^2 + \mu\|\nabla\boldsymbol{B}\|^2 + \mu\alpha^2\|\Delta\boldsymbol{B}\|^2\right) + K_1 + K_2 + \frac{Ck_1^2}{\alpha^4}\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{k_1^3}{\mu^7\alpha^2}\right),
\end{aligned} \tag{3.14}$$

which implies, thanks to (2.3),

$$y'(t) + \frac{\eta\lambda_1}{4}y(t) \leqslant k_2', \tag{3.15}$$

having set

$$\begin{aligned}
y(t) &= \|\boldsymbol{v}(t)\|^2 + \|\boldsymbol{B}(t)\|^2 + \alpha^2\|\nabla\boldsymbol{B}(t)\|^2, \\
k_2' &= 2(K_1 + K_2) + \frac{2Ck_1^2}{\alpha^4}\left(\frac{1}{\nu\lambda_1^{1/2}} + \frac{k_1^3}{\mu^7\alpha^2}\right).
\end{aligned}$$

Hence we deduce

$$\|\boldsymbol{v}(t)\|^2 + \|\boldsymbol{B}(t)\|^2 + \alpha^2\|\nabla\boldsymbol{B}(t)\|^2 \leqslant k_0'\mathrm{e}^{-\eta\lambda_1 t/4} + \frac{4k_2'}{\eta\lambda_1}(1 - \mathrm{e}^{-\eta\lambda_1 t/4}) \tag{3.16}$$

and finally

$$\|\boldsymbol{v}(t)\|^2 + \|\boldsymbol{B}(t)\|^2 + \alpha^2\|\nabla\boldsymbol{B}(t)\|^2 \leqslant k_2, \tag{3.17}$$

where

$$\begin{aligned}
k_0' &= \|\boldsymbol{v}(0)\|^2 + \|\boldsymbol{B}(0)\|^2 + \alpha^2\|\nabla\boldsymbol{B}(0)\|^2, \\
k_2 &= k_0' + \frac{4k_2'}{\eta\lambda_1}.
\end{aligned}$$

∎

4. Existence of a unique compact global attractor

From the existence and uniqueness properties of the solution to (1.4), we get a semigroup of solution operators, which we will denote by $(S(t))_{t\geqslant 0}$, that associates to each couple of initial data $(\boldsymbol{u}_0,\boldsymbol{B}_0)\in \mathrm{H}^1_\sigma(\Omega)\times \mathrm{L}^2_\sigma(\Omega)$ the semiflow for time $t\geqslant 0$, i.e., $S(t)(\boldsymbol{u}_0,\boldsymbol{B}_0)=(\boldsymbol{u}(t),\boldsymbol{B}(t))$.

Let us note that, following the computations performed for instance in Babin–Vishik [2, Chapter 7, Section 5], one can prove that $S(t)$ is differentiable with respect to the initial data. This property is needed in order to apply the techniques presented in section 5.

First, we show the existence of an absorbing ball in $D_1\doteq \mathrm{H}^1_\sigma\times \mathrm{L}^2_\sigma$. From (3.1), we choose t' large enough so that

$$\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla \boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2\leqslant \frac{2K_1}{\eta\lambda_1} \qquad \forall t\geqslant t'; \tag{4.1}$$

in particular, we have

$$\limsup_{t\to\infty}(\|\boldsymbol{u}(t)\|^2+\alpha^2\|\nabla \boldsymbol{u}(t)\|^2+\|\boldsymbol{B}(t)\|^2)\leqslant \frac{2K_1}{\eta\lambda_1}\doteq r_1^2.$$

Hence the ball B_1 of radius r_1 and centered at the origin is an absorbing ball in D_1 for system (1.4). Note that r_1 is independent of the initial data.

Now, let us note that we can modify inequality (3.15). Indeed, from (3.14), we also obtain

$$\begin{aligned} y'(t)+\frac{\eta\lambda_1}{4}y(t)&\leqslant 2(K_1+K_2)+2C\|\nabla \boldsymbol{u}\|^4\left(\frac{1}{\nu\lambda_1^{1/2}}+\frac{\alpha^2}{\mu^7}\|\nabla \boldsymbol{u}\|^4\|\boldsymbol{B}\|^2\right)\\ &\leqslant 2(K_1+K_2)+\frac{8CK_1^2}{\alpha^4\eta^2\lambda_1^2}\left(\frac{1}{\nu\lambda_1^{1/2}}+\frac{8K_1^3}{\alpha^2\mu^7\eta^3\lambda_1^3}\right)\doteq K_2' \end{aligned}$$

for each $t\geqslant t'$, where we have used (4.1). We multiply both members by $\mathrm{e}^{\eta\lambda_1 t/4}$ and integrate from t' to t to obtain

$$y(t)\leqslant y(t')\mathrm{e}^{-\eta\lambda_1(t-t')/4}+\frac{4K_2'}{\eta\lambda_1}(1-\mathrm{e}^{-\eta\lambda_1(t-t')/4}) \qquad \forall t\geqslant t'.$$

Thus we can choose $t''>t'$ large enough so that

$$\|\boldsymbol{v}(t)\|^2+\|\boldsymbol{B}(t)\|^2+\alpha^2\|\nabla \boldsymbol{B}(t)\|^2\leqslant \frac{5K_2'}{\eta\lambda_1} \qquad \forall t\geqslant t''.$$

In particular, we have

$$\limsup_{t\to\infty}(\|\boldsymbol{v}(t)\|^2+\|\boldsymbol{B}(t)\|^2+\alpha^2\|\nabla \boldsymbol{B}(t)\|^2)\leqslant \frac{5K_2'}{\eta\lambda_1}\doteq r_2^2;$$

this implies that the ball B_2 of radius r_2 and centered at the origin is an absorbing ball in $D_2\doteq \mathrm{H}^2_\sigma\times \mathrm{H}^1_\sigma$ for problem (1.4). Observe that r_2 as well is independent of the initial data.

From Rellich lemma (see [1]), D_2 is compactly imbedded in D_1, hence we have that $S(t):D_1\to D_2$, $t>0$, is a compact semigroup from D_1 to itself (bounded sets in

D_1 are mapped in bounded sets in D_2 that are compactly imbedded in D_1). This implies that $\overline{S(t)B_1} \neq \varnothing$ is compact in D_1 for each $t>0$.

Moreover, $S(t)B_1 \subseteq B_2$ if t is sufficiently large, thus, if we assume $s>0$ large enough, the set $\mathcal{C}_s \doteq \overline{\cup_{t\geqslant s} S(t)B_1}$ is nonempty and compact, since it is closed and contained in $\overline{B_2}$, which is compact in D_1. By monotonicity of $\mathcal{C}_s$ for $s>0$ and by the finite intersection property of compact sets, we deduce that

$$\mathscr{A} \doteq \bigcap_{s>0} \overline{\bigcup_{t\geqslant s} S(t)B_1}$$

is a nonempty compact set in D_1. $\mathscr{A}$ is a global attractor, since it attracts bounded sets of the whole space D_1, therefore it is the unique global attractor in D_1.

5. Estimate for the finite dimension of the attractor

First of all, we linearize the model about a solution $(\boldsymbol{v}(t),\boldsymbol{B}(t))$, where $\boldsymbol{v}(t)= \boldsymbol{u}(t)-\alpha^2\Delta\boldsymbol{u}(t)$. We denote by $\delta\boldsymbol{v}=\delta\boldsymbol{u}-\alpha^2\Delta\delta\boldsymbol{u}$ and $\delta\boldsymbol{B}$ perturbations satisfying

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{v}-\nu\Delta\delta\boldsymbol{v}+\mathscr{B}(\delta\boldsymbol{u},\boldsymbol{u})+\mathscr{B}(\boldsymbol{u},\delta\boldsymbol{u})-\mathscr{B}(\delta\boldsymbol{B},\boldsymbol{B})-\mathscr{B}(\boldsymbol{B},\delta\boldsymbol{B})=\mathbf{0}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{B}-\mu\Delta\delta\boldsymbol{B}+\mathscr{B}(\delta\boldsymbol{u},\boldsymbol{B})+\mathscr{B}(\boldsymbol{u},\delta\boldsymbol{B})-\mathscr{B}(\delta\boldsymbol{B},\boldsymbol{u})-\mathscr{B}(\boldsymbol{B},\delta\boldsymbol{u})=\mathbf{0}, \\ (\delta\boldsymbol{v}(0),\delta\boldsymbol{B}(0))=(\delta\boldsymbol{v}_0,\delta\boldsymbol{B}_0), \end{cases}$$

where, by definition, $\mathscr{B}(\boldsymbol{u},\boldsymbol{B})=(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$. Let us note that the first equation can be recast in terms of $\boldsymbol{u}$ and $\boldsymbol{B}$ only:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{u}-\nu\Delta\delta\boldsymbol{u}+(I-\alpha^2\Delta)^{-1}[\mathscr{B}(\delta\boldsymbol{u},\boldsymbol{u})+\mathscr{B}(\boldsymbol{u},\delta\boldsymbol{u})-\mathscr{B}(\delta\boldsymbol{B},\boldsymbol{B})-\mathscr{B}(\boldsymbol{B},\delta\boldsymbol{B})]=\mathbf{0}.$$

The above system has the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{w}+T(t)\delta\boldsymbol{w}=\mathbf{0}, \\ \delta\boldsymbol{w}(0)=\delta\boldsymbol{w}_0=(\delta\boldsymbol{u}_0,\delta\boldsymbol{B}_0) \end{cases} \tag{5.1}$$

if we set $\delta\boldsymbol{w}=\binom{\delta\boldsymbol{u}}{\delta\boldsymbol{B}}$, $A=-\Delta$, $T(t)=\begin{pmatrix}\nu I_3 & 0\\ 0 & \mu I_3\end{pmatrix}A+T_0(t)$, where I_3 is the 3×3 identity matrix, and

$$T_0(t)\begin{pmatrix}\delta\boldsymbol{u}\\ \delta\boldsymbol{B}\end{pmatrix} = \begin{pmatrix}(I-\alpha^2\Delta)^{-1}[\mathscr{B}(\delta\boldsymbol{u},\boldsymbol{u})+\mathscr{B}(\boldsymbol{u},\delta\boldsymbol{u})-\mathscr{B}(\delta\boldsymbol{B},\boldsymbol{B})-\mathscr{B}(\boldsymbol{B},\delta\boldsymbol{B})] \\ \mathscr{B}(\delta\boldsymbol{u},\boldsymbol{B})+\mathscr{B}(\boldsymbol{u},\delta\boldsymbol{B})-\mathscr{B}(\delta\boldsymbol{B},\boldsymbol{u})-\mathscr{B}(\boldsymbol{B},\delta\boldsymbol{u})\end{pmatrix}.$$

Now, let $E_0=\{\delta\boldsymbol{w}_i(0):i=1,2,\dots,N\}$ be a set of linearly independent vectors in $\mathrm{H}^1_\sigma(\Omega)\times\mathrm{L}^2_\sigma(\Omega)$ and let $E=\{\delta\boldsymbol{w}_i(t):i=1,2,\dots,N\}$ be the set of the solutions to (5.1) with initial data in E_0. Moreover, we set $E_1=\{\binom{\delta\boldsymbol{v}_i(t)}{\delta\boldsymbol{B}_i(t)}:i=1,2,\dots,N\}$ and

$$\mathcal{T}_N(t)=\mathrm{Trace}(P_N(t)\circ T(t)\circ P_N(t)), \tag{5.2}$$

where $P_N(t)$ is the orthogonal projection of $\mathrm{H}^1_\sigma(\Omega)\times\mathrm{L}^2_\sigma(\Omega)$ onto the span of E_1. Finally, let $\{\boldsymbol{\Phi}_i=\binom{\boldsymbol{\phi}_i}{\boldsymbol{\psi}_i}:i=1,2,\dots,N\}$ be an orthonormal basis for $P_N(\mathrm{H}^1_\sigma\times\mathrm{L}^2_\sigma)= \mathrm{span}(E_1)$ with respect to the inner product

$$\left[\begin{pmatrix}\boldsymbol{\phi}_i\\ \boldsymbol{\psi}_i\end{pmatrix},\begin{pmatrix}\boldsymbol{\phi}_j\\ \boldsymbol{\psi}_j\end{pmatrix}\right]=(\boldsymbol{\phi}_i,\boldsymbol{\phi}_j)+\alpha^2(\nabla\boldsymbol{\phi}_i,\nabla\boldsymbol{\phi}_j)+(\boldsymbol{\psi}_i,\boldsymbol{\psi}_j), \qquad (\cdot,\cdot)=(\cdot,\cdot)_{\mathrm{L}^2}.$$

Let us note that $[\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_j]=\delta_{ij}$ (Kronecker symbol) and hence $\|\boldsymbol{\phi}_i\|^2+\alpha^2\|\nabla\boldsymbol{\phi}_i\|^2+\|\boldsymbol{\psi}_i\|^2=1$ for each $i=1,2,\dots,N$.

Thanks to the trace representation (5.2), we obtain

$$\begin{aligned}\mathcal{T}_N(t)&=\sum_{i=1}^N[T(t)\boldsymbol{\Phi}_i(\cdot,t),\boldsymbol{\Phi}_i(\cdot,t)]\\&=\sum_{i=1}^N\left(\left[\begin{pmatrix}\nu I_3&0\\0&\mu I_3\end{pmatrix}A\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_i\right]+(\mathscr{B}(\boldsymbol{\phi}_i,\boldsymbol{u}),\boldsymbol{\phi}_i)-(\mathscr{B}(\boldsymbol{\psi}_i,\boldsymbol{B}),\boldsymbol{\phi}_i)\right.\\&\qquad\left.+(\mathscr{B}(\boldsymbol{\phi}_i,\boldsymbol{B}),\boldsymbol{\psi}_i)-(\mathscr{B}(\boldsymbol{\psi}_i,\boldsymbol{u}),\boldsymbol{\psi}_i)\right),\end{aligned}$$

having simplified terms according to the definition of the inner product $[\cdot,\cdot]$, which implies that

$$\left[\begin{pmatrix}(I-\alpha^2\Delta)^{-1}\boldsymbol{u}_1\\\boldsymbol{B}_1\end{pmatrix},\begin{pmatrix}\boldsymbol{u}_2\\\boldsymbol{B}_2\end{pmatrix}\right]=(\boldsymbol{u}_1,\boldsymbol{u}_2)+(\boldsymbol{B}_1,\boldsymbol{B}_2),$$

and (2.1).

We have

$$\begin{aligned}\sum_{i=1}^N\left[\begin{pmatrix}\nu I_3&0\\0&\mu I_3\end{pmatrix}A\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_i\right]&=\sum_{i=1}^N\Big(\big(M^2\nabla\boldsymbol{\Phi}_i,\nabla\boldsymbol{\Phi}_i\big)+\nu\alpha^2\|\Delta\boldsymbol{\phi}_i\|^2\Big)\\=\sum_{i=1}^N\big(\|M\nabla\boldsymbol{\Phi}_i\|^2+\nu\alpha^2\|\Delta\boldsymbol{\phi}_i\|^2\big)&\doteq Q_N,\end{aligned}$$

where

$$M=\begin{pmatrix}\sqrt{\nu}I_9&0\\0&\sqrt{\mu}I_9\end{pmatrix},\qquad\nabla\boldsymbol{\Phi}_i=(\partial_{x_1}\phi_i^{(1)},\partial_{x_2}\phi_i^{(1)},\dots,\partial_{x_2}\psi_i^{(3)},\partial_{x_3}\psi_i^{(3)})^{\mathrm{T}}$$

and $\boldsymbol{\phi}_i=(\phi_i^{(1)},\phi_i^{(2)},\phi_i^{(3)})^{\mathrm{T}}$, where T denotes the transposition of a vector. Thus we obtain

$$\mathcal{T}_N(t)=Q_N(t)+\mathcal{R}_N(t),\tag{5.3}$$

where

$$\mathcal{R}_N=\sum_{i=1}^N\Big((\mathscr{B}(\boldsymbol{\phi}_i,\boldsymbol{u}),\boldsymbol{\phi}_i)-(\mathscr{B}(\boldsymbol{\psi}_i,\boldsymbol{B}),\boldsymbol{\phi}_i)+(\mathscr{B}(\boldsymbol{\phi}_i,\boldsymbol{B}),\boldsymbol{\psi}_i)-(\mathscr{B}(\boldsymbol{\psi}_i,\boldsymbol{u}),\boldsymbol{\psi}_i)\Big).$$

Let us set

$$\boldsymbol{\Psi}_i=(\boldsymbol{\phi}_i,\alpha\partial_{x_1}\boldsymbol{\phi}_i,\alpha\partial_{x_2}\boldsymbol{\phi}_i,\alpha\partial_{x_3}\boldsymbol{\phi}_i)^{\mathrm{T}}$$

and note that $(\boldsymbol{\Psi}_i,\boldsymbol{\Psi}_i)\leqslant[\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_i]=1$.

We need the following estimate (see Cao–Lunasin–Titi [3]):

$$\|\boldsymbol{\phi}\|_{\mathrm{L}^\infty}^2\leqslant\frac{C}{\alpha^2}\left(\sum_{i=1}^N\|\nabla\boldsymbol{\Psi}_i\|^2\right)^{1/2},\tag{5.4}$$

where $\boldsymbol{\phi}^2 = \sum_{i=1}^N (\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_i)$ and $C > 0$ is a constant independent of N. Therefore, remembering the definition of Q_N, we deduce:

$$\|\boldsymbol{\phi}\|_{\mathrm{L}^\infty} \leqslant \frac{CQ_N^{1/4}}{\alpha\nu^{1/4}}, \qquad \left(\sum_{i=1}^N \|\nabla\boldsymbol{\phi}_i\|^2\right)^{1/2} \leqslant \frac{Q_N^{1/2}}{\nu^{1/2}},$$

$$\left(\sum_{i=1}^N \|\nabla\boldsymbol{\psi}_i\|^2\right)^{1/2} \leqslant \frac{Q_N^{1/2}}{\mu^{1/2}}, \qquad \left(\sum_{i=1}^N \|\Delta\boldsymbol{\phi}_i\|^2\right)^{1/2} \leqslant \frac{Q_N^{1/2}}{\alpha\nu^{1/2}}.$$

Another tool that we will exploit is the Lieb–Thirring inequality (see [7, 12, 15]). Let us assume that $\{\boldsymbol{\Theta}_i\}_{i=1}^N$ is an orthonormal set of functions in $(\mathrm{L}^2_\sigma)^k$. Then there exists a positive constant $C = C(k)$, independent of N, such that

$$\int_\Omega \left(\sum_{i=1}^N \boldsymbol{\Theta}_i(\boldsymbol{x}) \cdot \boldsymbol{\Theta}_i(\boldsymbol{x})\right)^{5/3} \mathrm{d}\boldsymbol{x} \leqslant C \sum_{i=1}^N \int_\Omega (\nabla\boldsymbol{\Theta}_i(\boldsymbol{x}) : \nabla\boldsymbol{\Theta}_i(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\ .$$

We can take

$$\boldsymbol{\Theta}_i = (\boldsymbol{\phi}_i, \boldsymbol{\psi}_i, \alpha\partial_{x_1}\boldsymbol{\phi}_i, \alpha\partial_{x_2}\boldsymbol{\phi}_i, \alpha\partial_{x_3}\boldsymbol{\phi}_i)^{\mathrm{T}},$$

since $(\boldsymbol{\Theta}_i, \boldsymbol{\Theta}_i) = [\boldsymbol{\Phi}_i, \boldsymbol{\Phi}_i] = 1$, and set $\boldsymbol{\Theta}^2(\boldsymbol{x},t) \doteq \sum_{i=1}^N \boldsymbol{\Theta}_i(\boldsymbol{x}) \cdot \boldsymbol{\Theta}_i(\boldsymbol{x})$; hence we have

$$\int_\Omega \boldsymbol{\Theta}^{10/3} \leqslant \frac{CQ_N}{\eta}. \tag{5.5}$$

We want to estimate $\mathcal{R}_N$.

$$\begin{aligned}
\sum_{i=1}^N |(\mathscr{B}(\boldsymbol{\phi}_i, \boldsymbol{u}), \boldsymbol{\phi}_i)| &\leqslant \int_\Omega \sum_{i=1}^N |(\boldsymbol{\phi}_i \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{\phi}_i|\,\mathrm{d}\boldsymbol{x} \\
&\leqslant \int \left(\sum_{i=1}^N |\boldsymbol{\phi}_i|^2\right) |\nabla\boldsymbol{u}| \cdot 1 \leqslant C\|\boldsymbol{\phi}\|^2_{\mathrm{L}^\infty} \|\nabla\boldsymbol{u}\|\,|\Omega|^{1/2} \\
&\leqslant \frac{CQ_N^{1/2}}{\alpha^2\nu^{1/2}} \|\nabla\boldsymbol{u}\|\,|\Omega|^{1/2} \leqslant \frac{1}{8}Q_N + \frac{C}{\alpha^4\nu}\|\nabla\boldsymbol{u}\|^2|\Omega|
\end{aligned} \tag{5.6}$$

by applying the Hölder inequality and then Young's inequality.

$$\begin{aligned}
&\sum_{i=1}^N |-(\mathscr{B}(\boldsymbol{\psi}_i, \boldsymbol{B}), \boldsymbol{\phi}_i)| = \sum_{i=1}^N |(\mathscr{B}(\boldsymbol{\psi}_i, \boldsymbol{\phi}_i), \boldsymbol{B})| \\
&\leqslant \sum_{i=1}^N \int |\boldsymbol{\psi}_i|\,|\nabla\boldsymbol{\phi}_i|\,|\boldsymbol{B}| \leqslant \frac{1}{\alpha}\int \left(\sum_{i=1}^N |\boldsymbol{\psi}_i|^2\right)^{1/2} \left(\sum_{i=1}^N |\alpha\nabla\boldsymbol{\phi}_i|^2\right)^{1/2} |\boldsymbol{B}| \\
&\leqslant \frac{1}{\alpha}\int \boldsymbol{\Theta}^2|\boldsymbol{B}| \leqslant \frac{1}{\alpha}\|\boldsymbol{\Theta}^2\|_{\mathrm{L}^{5/3}}\|\boldsymbol{B}\|_{\mathrm{L}^{5/2}} \\
&\leqslant \frac{CQ_N^{3/5}}{\alpha\eta^{3/5}}\|\boldsymbol{B}\|^{7/10}\|\nabla\boldsymbol{B}\|^{3/10} \\
&\leqslant \frac{1}{8}Q_N + \frac{C\|\boldsymbol{B}\|^{1/2}}{\alpha^{5/2}\eta^{3/2}}\|\boldsymbol{B}\|^{5/4}\|\nabla\boldsymbol{B}\|^{3/4} \\
&\leqslant \frac{1}{8}Q_N + \frac{CL^{5/4}\|\boldsymbol{B}\|^{1/2}}{\alpha^{5/2}\eta^{3/2}}\|\nabla\boldsymbol{B}\|^2,
\end{aligned} \tag{5.7}$$

having applied (5.5), (2.7), Young's inequality and then the Poincaré inequality.

$$
\begin{aligned}
&\sum_{i=1}^{N}|(\mathscr{B}(\boldsymbol{\phi}_i,\boldsymbol{B}),\boldsymbol{\psi}_i)| = \sum_{i=1}^{N}|-(\mathscr{B}(\boldsymbol{\phi}_i,\boldsymbol{\psi}_i),\boldsymbol{B})| \\
&\leqslant \sum_{i=1}^{N}\int|\boldsymbol{\phi}_i||\nabla\boldsymbol{\psi}_i||\boldsymbol{B}|\cdot 1 \leqslant \int\left(\sum_{i=1}^{N}|\boldsymbol{\phi}_i|^2\right)^{1/2}\left(\sum_{i=1}^{N}|\nabla\boldsymbol{\psi}_i|^2\right)^{1/2}|\boldsymbol{B}|\cdot 1 \\
&\leqslant \|\boldsymbol{\phi}\|_{\mathrm{L}^\infty}\left\|\left(\sum_{i=1}^{N}|\nabla\boldsymbol{\psi}_i|^2\right)^{1/2}\right\|\|\boldsymbol{B}\|_{\mathrm{L}^3}\|1\|_{\mathrm{L}^6} \\
&\leqslant \frac{CQ_N^{1/4}}{\alpha\nu^{1/4}}\left(\sum_{i=1}^{N}\|\nabla\boldsymbol{\psi}_i\|^2\right)^{1/2}\|\nabla\boldsymbol{B}\|^{1/2}\|\boldsymbol{B}\|^{1/2}|\Omega|^{1/6} \\
&\leqslant \frac{CQ_N^{3/4}}{\alpha\nu^{1/4}\mu^{1/2}}\|\nabla\boldsymbol{B}\|^{1/2}\|\boldsymbol{B}\|^{1/2}|\Omega|^{1/6} \\
&\leqslant \frac{1}{8}Q_N + \frac{C}{\alpha^4\nu\mu^2}\|\nabla\boldsymbol{B}\|^2\|\boldsymbol{B}\|^2|\Omega|^{2/3}. \qquad (5.8)
\end{aligned}
$$

Finally, proceeding similarly as for (5.7), we have

$$
\begin{aligned}
&\sum_{i=1}^{N}|(\mathscr{B}(\boldsymbol{\psi}_i,\boldsymbol{u}),\boldsymbol{\psi}_i)| \leqslant \int\sum_{i=1}^{N}|\boldsymbol{\psi}_i|^2|\nabla\boldsymbol{u}| \\
&\leqslant \int\boldsymbol{\Theta}^2|\nabla\boldsymbol{u}| \leqslant \frac{1}{8}Q_N + \frac{CL^{5/4}\|\nabla\boldsymbol{u}\|^{1/2}}{\eta^{3/2}}\|\Delta\boldsymbol{u}\|^2. \qquad (5.9)
\end{aligned}
$$

Combining (5.6), (5.7), (5.8), and (5.9) into (5.3) we can deduce

$$
\mathcal{T}_N(t) \geqslant \frac{Q_N(t)}{2} - CR(t), \qquad (5.10)
$$

where

$$
\begin{aligned}
R(t) = &\frac{L^3}{\alpha^4\nu}\|\nabla\boldsymbol{u}\|^2 + \frac{L^2}{\alpha^4\nu\mu^2}\|\nabla\boldsymbol{B}\|^2\|\boldsymbol{B}\|^2 + \frac{L^{5/4}\|\boldsymbol{B}\|^{1/2}}{\alpha^{5/2}\eta^{3/2}}\|\nabla\boldsymbol{B}\|^2 \\
&+ \frac{L^{5/4}\|\nabla\boldsymbol{u}\|^{1/2}}{\eta^{3/2}}\|\Delta\boldsymbol{u}\|^2 \doteq R_1(t)+R_2(t)+R_3(t)+R_4(t),
\end{aligned}
$$

since $|\Omega| = (2\pi L)^3$.

We will use the following result (see, for example, [7] and [15]). By the trace formula, if N is large enough so that

$$
X_N \doteq \liminf_{T\to\infty}\frac{1}{T}\int_0^T\mathcal{T}_N(t)\,\mathrm{d}t > 0,
$$

then N is an upper bound for the Hausdorff dimension $d_{\mathrm{H}}(\mathscr{A})$ and the fractal dimension $d_{\mathrm{F}}(\mathscr{A})$ of the global attractor $\mathscr{A}$:

$$
d_{\mathrm{H}}(\mathscr{A}) \leqslant d_{\mathrm{F}}(\mathscr{A}) \leqslant N.
$$

Now, the asymptotic behavior of the eigenvalues of the operator A is such that

$$\lambda_i \geqslant \frac{\lambda_1 i^{2/3}}{C},$$

hence we obtain (we refer to [7] and [15] for both the previous estimate and the middle part of the following one)

$$Q_N \geqslant \eta \sum_{i=1}^{N} \|\nabla \boldsymbol{\Psi}_i\|^2 \geqslant \eta \sum_{i=1}^{N} \lambda_i \geqslant \frac{\eta \lambda_1 N^{5/3}}{C},$$

since, by induction, one can easily prove that $2\sum_{i=1}^{N} i^{2/3} \geqslant N^{5/3}$. This implies that

$$X_N \geqslant \frac{\eta\lambda_1 N^{5/3}}{C} - C \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(R_1(t)+R_2(t)+R_3(t)+R_4(t)\big)\,\mathrm{d}t. \tag{5.11}$$

For R_1, thanks to (3.2), we have the estimate

$$\begin{aligned} \limsup_{T\to\infty} \frac{1}{T}\int_0^T R_1(t)\,\mathrm{d}t &= \limsup_{T\to\infty} \frac{L^3}{\alpha^4\nu^2 T}\int_0^T \nu\|\nabla \boldsymbol{u}(t)\|^2\,\mathrm{d}t \\ &\leqslant \frac{L^3}{\alpha^4\nu^2}\limsup_{T\to\infty}\frac{TK_1+k_1}{T} = \frac{L^3K_1}{\alpha^4\nu^2}. \end{aligned} \tag{5.12}$$

Let us note that the last term in the previous inequality is independent of the initial data.

As to R_2, we have

$$\begin{aligned} \limsup_{T\to\infty}\frac{1}{T}\int_0^T R_2(t)\,\mathrm{d}t &= \frac{L^2}{\alpha^4\nu\mu^2}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|\boldsymbol{B}(t)\|^2\|\nabla\boldsymbol{B}(t)\|^2\,\mathrm{d}t \\ &\leqslant \frac{L^2}{\alpha^4\nu\mu^2}\limsup_{T\to\infty}\frac{k_0}{T}\int_0^T \mathrm{e}^{-\eta\lambda_1 t}\|\nabla\boldsymbol{B}(t)\|^2\,\mathrm{d}t + \frac{L^2K_1}{\alpha^4\nu\mu^2\eta\lambda_1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|\nabla\boldsymbol{B}(t)\|^2\,\mathrm{d}t, \end{aligned}$$

since $\|\boldsymbol{B}\|^2 \leqslant k_0\mathrm{e}^{-\eta\lambda_1 t} + K_1/(\eta\lambda_1)$ because of (3.1). Now we use (3.3) and (3.2) for the first and the second integral respectively, and deduce

$$\begin{aligned} \limsup_{T\to\infty}\frac{1}{T}\int_0^T R_2(t)\,\mathrm{d}t &\leqslant \frac{L^2k_0}{\alpha^4\nu\mu^3}\limsup_{T\to\infty}\frac{1}{T}\left(\frac{4K_1}{\eta\lambda_1}+k_0\right) \\ &\quad + \frac{L^2K_1}{\alpha^4\nu\mu^3\eta\lambda_1}\limsup_{T\to\infty}\frac{1}{T}(TK_1+k_1) = \frac{L^4K_1^2}{\alpha^4\nu\mu^3\eta}, \end{aligned} \tag{5.13}$$

since $\lambda_1 = L^{-2}$.

As to R_3, recalling (3.1) we have

$$\begin{aligned} \limsup_{T\to\infty}\frac{1}{T}\int_0^T R_3(t)\,\mathrm{d}t &\leqslant \frac{L^{5/4}}{\alpha^{5/2}\eta^{3/2}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|\boldsymbol{B}(t)\|^{1/2}\|\nabla\boldsymbol{B}(t)\|^2\,\mathrm{d}t \\ &\leqslant \frac{L^{5/4}}{\alpha^{5/2}\eta^{3/2}}\limsup_{T\to\infty}\frac{k_0^{1/4}}{T}\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\|\nabla\boldsymbol{B}(t)\|^2\,\mathrm{d}t \\ &\qquad + \frac{L^{5/4}}{\alpha^{5/2}\eta^{3/2}}\limsup_{T\to\infty}\frac{K_1^{1/4}}{\lambda_1^{1/4}\eta^{1/4}T}\int_0^T\|\nabla\boldsymbol{B}(t)\|^2\,\mathrm{d}t \\ &= \frac{L^{7/4}K_1^{1/4}}{\alpha^{5/2}\eta^{7/4}}\limsup_{T\to\infty}\frac{TK_1+k_1}{\mu T} = \frac{L^{7/4}K_1^{5/4}}{\alpha^{5/2}\mu\eta^{7/4}}, \end{aligned} \tag{5.14}$$

having used (3.3) and (3.2).

Similarly, we obtain

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\int_0^T R_4(t)\,\mathrm{d}t \leqslant \frac{L^{5/4}}{\eta^{3/2}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|\nabla \boldsymbol{u}(t)\|^{1/2}\|\Delta \boldsymbol{u}(t)\|^2\,\mathrm{d}t\\
&\leqslant \frac{L^{5/4}}{\eta^{3/2}}\limsup_{T\to\infty}\frac{k_0^{1/4}}{\alpha^{1/2}T}\int_0^T \mathrm{e}^{-\eta\lambda_1 t/4}\|\Delta \boldsymbol{u}(t)\|^2\,\mathrm{d}t\\
&\qquad+\frac{L^{5/4}}{\eta^{3/2}}\limsup_{T\to\infty}\frac{K_1^{1/4}}{\alpha^{1/2}\lambda_1^{1/4}\eta^{1/4}T}\int_0^T\|\Delta \boldsymbol{u}(t)\|^2\,\mathrm{d}t\\
&=\frac{L^{7/4}K_1^{1/4}}{\alpha^{1/2}\eta^{7/4}}\limsup_{T\to\infty}\frac{TK_1+k_1}{\nu\alpha^2 T}=\frac{L^{7/4}K_1^{5/4}}{\alpha^{5/2}\nu\eta^{7/4}}. \qquad (5.15)
\end{aligned}$$

We conclude that

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T R(t)\,\mathrm{d}t\leqslant C\left[\frac{L^3K_1}{\alpha^4\eta^2}+\frac{L^4K_1^2}{\alpha^4\eta^5}+\frac{L^{7/4}K_1^{5/4}}{\alpha^{5/2}\eta^{11/4}}\right]. \qquad (5.16)$$

Now, since

$$K_1\leqslant\frac{\|A^{-1}\boldsymbol{f}\|^2}{\nu\alpha^2}\leqslant\frac{(\lambda_1^{-1}\|\boldsymbol{f}\|)^2}{\nu\alpha^2}\leqslant\frac{L^4\|\boldsymbol{f}\|^2}{\eta\alpha^2},$$

we have

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T R(t)\,\mathrm{d}t\leqslant C\frac{L^6\|\boldsymbol{f}\|^2}{\alpha^5\eta^3}\left[\left(\frac{L}{\alpha}\right)^{\frac{3}{5}}+\left(\frac{L^6\|\boldsymbol{f}\|^2}{\alpha^3\eta^4}\right)^{\frac{3}{5}}+\left(\frac{L^{3/4}\|\boldsymbol{f}\|^{1/2}}{\eta}\right)^{\frac{3}{5}}\right]^{\frac{5}{3}}.$$

In view of (5.11), we look for an N such that

$$\frac{\eta N^{5/3}}{CL^2}-C\frac{L^6\|\boldsymbol{f}\|^2}{\alpha^5\eta^3}\left[\left(\frac{L}{\alpha}\right)^{\frac{3}{5}}+\left(\frac{L^6\|\boldsymbol{f}\|^2}{\alpha^3\eta^4}\right)^{\frac{3}{5}}+\left(\frac{L^{3/4}\|\boldsymbol{f}\|^{1/2}}{\eta}\right)^{\frac{3}{5}}\right]^{\frac{5}{3}}\geqslant 0.$$

Introducing the modified Grashoff number

$$G=\frac{L^{3/2}\|\boldsymbol{f}\|}{\eta^2}$$

(observe that this is a nondimensional quantity), this request is satisfied if

$$N\geqslant CG^{6/5}\left(\frac{L}{\alpha}\right)^3\left[\left(\frac{L}{\alpha}\right)^{\frac{3}{5}}+G^{6/5}\left(\frac{L}{\alpha}\right)^{\frac{9}{5}}+G^{3/10}\right],$$

hence we have Theorem 1.2.

It remains to prove Theorem 1.3. With this aim, we prove an alternative estimate for $\mathcal{R}_N$. From (5.6), we have

$$\sum_{i=1}^N|(\mathscr{B}(\boldsymbol{\phi}_i,\boldsymbol{u}),\boldsymbol{\phi}_i)|\leqslant\frac{1}{8}Q_N+\frac{CL^3}{\alpha^4\eta}\|\nabla\boldsymbol{u}\|^2. \qquad (5.17)$$

From the computations for (5.7), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}|-(\mathscr{B}(\boldsymbol{\psi}_i,\boldsymbol{B}),\boldsymbol{\phi}_i)|\leqslant\frac{1}{\alpha}\int\boldsymbol{\Theta}^2|\boldsymbol{B}|\leqslant\frac{1}{\alpha}\|\boldsymbol{B}\|_{\mathrm{L}^6}\|\boldsymbol{\Theta}^2\|_{\mathrm{L}^{6/5}}\\
&\leqslant\frac{C}{\alpha}\|\nabla\boldsymbol{B}\|\left(\int\boldsymbol{\Theta}\cdot\boldsymbol{\Theta}^{7/5}\right)^{5/6}\leqslant\frac{C}{\alpha}\|\nabla\boldsymbol{B}\|\left(\int\boldsymbol{\Theta}^{10/3}\right)^{\frac{1}{4}}\left(\int\boldsymbol{\Theta}^2\right)^{\frac{7}{12}}\\
&\leqslant\frac{C}{\alpha}\|\nabla\boldsymbol{B}\|\frac{Q_N^{1/4}N^{7/12}}{\eta^{1/4}}\leqslant\frac{1}{8}Q_N+\frac{CN^{7/9}\|\nabla\boldsymbol{B}\|^{4/3}}{\alpha^{4/3}\eta^{1/3}},
\end{aligned}\tag{5.18}$$

since

$$\int_\Omega\boldsymbol{\Theta}^2(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}=N.$$

Similarly, from (5.9) we deduce

$$\sum_{i=1}^{N}|(\mathscr{B}(\boldsymbol{\psi}_i,\boldsymbol{u}),\boldsymbol{\psi}_i)|\leqslant\int\boldsymbol{\Theta}^2|\nabla\boldsymbol{u}|\leqslant\frac{1}{8}Q_N+\frac{CN^{7/9}\|\Delta\boldsymbol{u}\|^{4/3}}{\eta^{1/3}}.\tag{5.19}$$

Finally, we have

$$\begin{aligned}
&\sum_{i=1}^{N}|(\mathscr{B}(\boldsymbol{\phi}_i,\boldsymbol{B}),\boldsymbol{\psi}_i)|\leqslant\sum_{i=1}^{N}\int|\boldsymbol{\phi}_i||\boldsymbol{\psi}_i||\nabla\boldsymbol{B}|\\
&\leqslant\int\left(\sum_{i=1}^{N}|\boldsymbol{\phi}_i|^2\right)^{1/2}\left(\sum_{i=1}^{N}|\boldsymbol{\psi}_i|^2\right)^{1/2}|\nabla\boldsymbol{B}|\\
&\leqslant\|\boldsymbol{\phi}\|_{\mathrm{L}^\infty}\left\|\left(\sum_{i=1}^{N}|\boldsymbol{\psi}_i|^2\right)^{1/2}\right\|\|\nabla\boldsymbol{B}\|\\
&\leqslant\frac{CQ_N^{1/4}}{\alpha\nu^{1/4}}\left(\int\sum_{i=1}^{N}|\boldsymbol{\psi}_i|^2\right)^{1/2}\|\nabla\boldsymbol{B}\|\leqslant\frac{CQ_N^{1/4}}{\alpha\nu^{1/4}}\left(\int\boldsymbol{\Theta}^2\right)^{1/2}\|\nabla\boldsymbol{B}\|\\
&=\frac{CQ_N^{1/4}N^{1/2}}{\alpha\nu^{1/4}}\|\nabla\boldsymbol{B}\|\leqslant\frac{1}{8}Q_N+\frac{CN^{2/3}}{\alpha^{4/3}\eta^{1/3}}\|\nabla\boldsymbol{B}\|^{4/3}.
\end{aligned}\tag{5.20}$$

Combining (5.17), (5.18), (5.19) and (5.20), we conclude that

$$\begin{aligned}
\mathcal{R}_N(t)\leqslant\frac{1}{2}Q_N+\frac{CL^3}{\alpha^4\eta}\|\nabla\boldsymbol{u}(t)\|^2+\frac{CN^{7/9}}{\alpha^{4/3}\eta^{1/3}}\|\nabla\boldsymbol{B}(t)\|^{4/3}\\
+\frac{CN^{7/9}}{\eta^{1/3}}\|\Delta\boldsymbol{u}(t)\|^{4/3}.
\end{aligned}\tag{5.21}$$

Now, using (1.5), we have

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\int_0^T\frac{L^3}{\alpha^4\eta}\|\nabla\boldsymbol{u}(t)\|^2\,\mathrm{d}t=\frac{L^3}{\alpha^4\eta\nu}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\nu\|\nabla\boldsymbol{u}(t)\|^2\,\mathrm{d}t\\
&\leqslant\frac{L^6\eta}{\alpha^4\ell_d^4}.
\end{aligned}\tag{5.22}$$

On the other hand, also using the Hölder inequality,

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\int_0^T \frac{N^{7/9}}{\alpha^{4/3}\eta^{1/3}}\|\nabla \boldsymbol{B}(t)\|^{4/3}\,\mathrm{d}t\\
&=\frac{N^{7/9}}{\alpha^{4/3}\eta^{1/3}\mu^{2/3}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \mu^{2/3}\|\nabla \boldsymbol{B}(t)\|^{4/3}\cdot 1\,\mathrm{d}t\\
&\leqslant \frac{N^{7/9}}{\alpha^{4/3}\eta}\left(\limsup_{T\to\infty}\frac{1}{T}\int_0^T \mu\|\nabla \boldsymbol{B}(t)\|^{2}\,\mathrm{d}t\right)^{2/3}\leqslant \frac{N^{7/9}L^2\eta}{\alpha^{4/3}\ell_d^{8/3}}.
\end{aligned} \tag{5.23}$$

Similarly, we also have

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \frac{CN^{7/9}}{\eta^{1/3}}\|\Delta \boldsymbol{u}(t)\|^{4/3}\,\mathrm{d}t\leqslant \frac{N^{7/9}L^2\eta}{\alpha^{4/3}\ell_d^{8/3}}. \tag{5.24}$$

Therefore, in order to have $X_N>0$, it is sufficient to have

$$\frac{\eta N^{5/3}}{CL^2}\geqslant C\frac{L^6\eta}{\alpha^4\ell_d^4}+C\frac{N^{7/9}L^2\eta}{\alpha^{4/3}\ell_d^{8/3}},$$

or

$$N\geqslant C\left(\frac{L}{\alpha}\right)^{12/5}\left(\frac{L}{\ell_d}\right)^{12/5}+CN^{7/15}\left(\frac{L}{\alpha}\right)^{4/5}\left(\frac{L}{\ell_d}\right)^{8/5}.$$

Hence we need to have

$$N\geqslant C\max\left\{\left(\frac{L}{\alpha}\right)^{12/5}\left(\frac{L}{\ell_d}\right)^{12/5},\left(\frac{L}{\alpha}\right)^{3/2}\left(\frac{L}{\ell_d}\right)^{3}\right\},$$

which is Theorem 1.3.

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