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# BOUNDARY LAYERS IN INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH NAVIER BOUNDARY CONDITIONS FOR THE VANISHING VISCOSITY LIMIT*

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**Abstract.** In this paper, we study the vanishing viscosity limit for the incompressible Navier-Stokes equations with the Navier friction boundary condition. To simplify the expansion of solutions in terms of the viscosity, we shall only consider the case that the slip length $\alpha$ in the Navier boundary condition is a power of the viscosity $\epsilon$, $\alpha=\epsilon^\gamma$. First, by multi-scale analysis we formally deduce that $\gamma=\frac{1}{2}$ is critical in determining the boundary layer behavior. When $\gamma>\frac{1}{2}$, the boundary layer appears in the zero-th order terms of the expansion of solutions, and satisfies the same boundary value problem for the nonlinear Prandtl equations as in the non-slip case, when $\gamma=\frac{1}{2}$, the boundary layer also appears in the zero-th order terms of solutions, and satisfies the nonlinear Prandtl equations but with a Robin boundary condition for the tangential velocity profile, and when $\gamma<\frac{1}{2}$, the boundary layer appears in the order $O(\epsilon^{1-2\gamma})$ terms of solutions, and satisfies a boundary value problem for the linearized Prandtl equations. Secondly, we justify rigorously the asymptotic behavior of the vanishing viscosity limit for the incompressible Navier-Stokes equations with anisotropic viscosities by using the energy method, when the slip length is larger than the square root of the vertical viscosity. Even though the boundary layer appears in the lower order terms of solutions and satisfies a linear problem, the vorticity of flow is unbounded in the vanishing viscosity limit.

**Key words.** Incompressible Navier-Stokes equations, Navier friction boundary condition, boundary layers, anisotropic viscosities.

**AMS subject classifications.** 76D05, 76D10, 35K65.

## 1. Introduction

In this paper, we consider the vanishing viscosity limit for the following incompressible Navier-Stokes equations with the Navier boundary condition in $\{t>0, x\in\Omega\}$ with $\Omega$ being a domain of $I\!R^n$ ($n=2$ or 3):

$$\begin{cases} \partial_t u^\epsilon + (u^\epsilon\cdot\nabla)u^\epsilon + \nabla p^\epsilon = \epsilon\Delta u^\epsilon, \quad t>0, x\in\Omega \\ \nabla\cdot u^\epsilon = 0, \quad t>0, x\in\Omega \\ u^\epsilon\cdot\vec{n}=0, \quad 2(D(u^\epsilon)\vec{n})\cdot\vec{\tau}+\eta u^\epsilon\cdot\vec{\tau}=0, \quad \text{on } \partial\Omega \\ u^\epsilon|_{t=0}=u_0^\epsilon(x,y), \end{cases} \tag{1.1}$$

where $\epsilon$ is the viscosity and $D(u^\epsilon)=\frac{1}{2}(\nabla u^\epsilon+(\nabla u^\epsilon)^T)$ is the rate of the strain tensor, with $\vec{n}$ and $\vec{\tau}$ being unit normal and tangent vectors on the boundary $\partial\Omega$. The boundary condition given in (1.1) is the so-called Navier friction condition, which was first proposed by Navier [10] and derived for gases by Maxwell [9]. It means that the rate of strain on the boundary is proportional to the tangential slip velocity. This friction boundary condition was also justified rigorously as an effective boundary condition for flows over rough boundaries; see [5].

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The asymptotic behavior of solutions to the incompressible Navier-Stokes equations in the vanishing viscosity limit, in the case where there are physical boundaries, is a challenging problem due to the formation of boundary layers. The problem with the non-slip boundary condition was formally studied by Prandtl in [12], in which it was derived that the boundary layer can be described by an initial-boundary problem for a nonlinear degenerate parabolic-elliptic coupled system, which is now called the Prandtl equations. Under the monotonic assumption on the velocity of the outflow, Oleinik and her collaborators established the local existence of smooth solutions for boundary value problems of the Prandtl equations in the 1960's, and their works were surveyed in the monograph [11]. The existence of global weak solutions to the Prandtl equations was obtained by Xin and Zhang in [18]. Recently, it was announced that such a solution is in fact unique and is classical by Xin, Zhang and Zhao in [19]. In [15], Sammartino and Caflisch obtained the local existence of analytic solutions to the Prandtl equations, and a rigorous theory on the stability of boundary layers in incompressible fluids with analytic data in the frame of the abstract Cauchy-Kowaleskaya theory. Rather recently, a rigorous theory was obtained in [7] for the behavior of boundary layers in a circularly symmetric flow with non-slip boundary conditions in two space variables.

As in [1], by a simple computation it is known that the Navier friction boundary condition given in (1.1) can be rewritten as

$$\text{curl } u^\epsilon = (2\kappa - \eta) u^\epsilon \cdot \vec{\tau}, \quad \text{on } \partial\Omega \tag{1.2}$$

in two space variables, where curl $u^\epsilon$ is the vorticity, and $\kappa$ is the curvature of $\partial\Omega$.

The problem of the vanishing viscosity limit when the non-slip boundary condition is replaced by the Navier friction condition has been studied by many mathematicians since the 1960's. Yodovich [20] and Lions [6] studied the vanishing viscosity limit for the incompressible Navier-Stokes equations in two space variables with a free boundary condition, $u^\epsilon \cdot \vec{n} = 0$, and curl $u^\epsilon = 0$ on $\partial\Omega$. For the two-dimensional Navier-Stokes equations with the Navier friction condition, Clopeau, Mikelic, and Robert ([1]), Lopes Filho, Nussenzveig Lopes, and Planas [8] obtained that the solution $u^\epsilon$ to (1.1) converges to the solution of the corresponding Euler equations in $L^\infty([0,T], L^2(\Omega))$ under certain boundedness assumptions on the initial vorticity when the slip length $\eta$ is a constant. Recently, Xiao and Xin [17] studied the vanishing viscosity limit from the Navier-Stokes equations to the Euler equations in three space variables for the slip case, $u^\epsilon \cdot \vec{n} = 0$, and curl $u^\epsilon \cdot \tau = 0$ on $\partial\Omega$. Almost all of these results do not have any detail description of the boundary layer behavior when the viscosity goes to zero. Certainly, this is a very interesting problem from the physical point of view. Only recently, Iftimie and Sueur [4] investigated the boundary layer behavior for the problem (1.1) when the slip length $\eta$ is independent of the viscosity $\epsilon$.

As mentioned in [14], many interesting physical phenomena show that the slip length should depend on viscosities in general.

The main proposal of this work is to describe the asymptotic behavior of solutions to (1.1) in the vanishing viscosity limit, especially the behavior of boundary layers when the slip length $\eta$ depends on the viscosity $\epsilon$. In this paper, we shall first study the asymptotic behavior of solutions to the problem (1.1) when the viscosity $\epsilon$ goes to zero for different dependencies of $\eta$ on the viscosity, and derive problems of boundary layer profiles. Then, we study rigorously the stability of boundary layers.

For simplicity of presentation, we shall only consider the problem (1.1) in the half plane $\Omega = \{x \in I\!R, y > 0\}$. In the following sections, one will see that it is not difficult to generalize our discussion to multi-dimensional problems in an arbitrary bounded domain.

In order to have a complete expansion of solutions in terms of the viscosity, we shall only consider the case that the slip length in the Navier boundary condition is a power of the viscosity. Let $\eta=\frac{\beta}{\alpha^\epsilon}$ with $\beta$ independent of $\epsilon$ and $\alpha^\epsilon=\epsilon^\gamma$ for an index $\gamma\in I\!R$. Then the Navier boundary condition given in (1.1) can be simplified as

$$u_2^\epsilon=0, \qquad \beta u_1^\epsilon-\alpha^\epsilon\frac{\partial u_1^\epsilon}{\partial y}=0, \quad \text{on } y=0. \tag{1.3}$$

Obviously, when $\alpha^\epsilon\to 0$ the boundary conditions in (1.3) formally tend to the non-slip case, $u^\epsilon|_{y=0}=0$, while if $\alpha^\epsilon\to+\infty$ the boundary conditions in (1.3) tend to the complete slip case, $u_2^\epsilon|_{y=0}=0$ and $\partial_y u_1^\epsilon|_{y=0}=0$.

From the above discussion, we already knew that the behavior of boundary layers has completely different phenomena for the non-slip and slip boundary condition cases. Therefore, the behavior of the vanishing viscosity limit for the problem (1.1) with the boundary conditions (1.3) should be clearly influenced by the amplitude of the slip length. Indeed, in the following sections, by multi-scale analysis we shall deduce that $\gamma=\frac{1}{2}$ is critical in determining the boundary layer behavior. When $\gamma$ is super-critical, the leading boundary layer profile satisfies the same boundary problem for the nonlinear Prandtl equations as in the non-slip case, in the critical case $\gamma=\frac{1}{2}$, the boundary layer profile also satisfies the nonlinear Prandtl equations but with a Robin boundary condition for the tangential velocity profile, and when $\gamma$ is sub-critical, the boundary layer appears in the order $O(\epsilon^{1-2\gamma})$ terms of solutions, and satisfies a boundary value problem for linearized Prandtl equations.

The second goal of this paper is to study the stability of boundary layers rigorously. We shall justify the asymptotic behavior of the vanishing viscosity limit for the incompressible Navier-Stokes equations with anisotropic viscosities by using the energy method, when the slip length is larger than the square root of the vertical viscosity, in which even though the boundary layer appears in the lower order terms of solutions and obeys a linear law but it still produces an unbounded vorticity of flow in the vanishing viscosity limit. From the approach of this paper, one can easily deduce Iftimie and Sueur's results on the leading profile expansion of boundary layers hold not only in $L^\infty(0,T,L^2(\Omega))$ as given in [4], but even in $L^\infty([0,T]\times\Omega)$, moreover we have a complete expansion of $u^\epsilon$ with respect to the viscosity $\epsilon$.

The remainder of this paper is arranged as follows: In section 2, we study the asymptotic behavior of solutions to the problem (1.1) (1.3) in the vanishing viscosity limit by multi-scale analysis, from which we observe that the power $\gamma=\frac{1}{2}$ of the slip length $\alpha^\epsilon=\epsilon^\gamma$ is critical for the behavior of boundary layers. In section 3 and section 4, we justify rigorously the asymptotic behavior of the vanishing viscosity limit for the anisotropic Navier-Stokes equations by using the energy method, and obtain that boundary layer is stable when the slip length is larger than the square root of the vertical viscosity.

The preliminary version of the results given in this paper was announced in [16].

## 2. Formal asymptotic analysis

In this section, we study the vanishing viscosity limit for the following initial boundary value problem:

$$\begin{cases} \partial_t u^\epsilon + (u^\epsilon\cdot\nabla)u^\epsilon + \nabla p^\epsilon = \epsilon\Delta u^\epsilon, \quad t>0,\,(x,y)\in I\!R^2_+ \\ \nabla\cdot u^\epsilon = 0, \quad t>0,\,(x,y)\in I\!R^2_+ \\ u_2 = 0, \quad \beta u_1^\epsilon - \alpha^\epsilon \frac{\partial u_1^\epsilon}{\partial y} = 0, \quad \text{on } y=0 \\ u^\epsilon|_{t=0} = u_0(x,y) \end{cases} \tag{2.1}$$

by multi-scale analysis for different dependencies of $\alpha^\epsilon$ on the viscosity. In order to simplify the presentation, we shall only consider the case where the slip length is a power of the viscosity, $\alpha^\epsilon = \epsilon^\gamma$.

**2.1. The cases $\alpha^\epsilon = \epsilon$ and $\epsilon^{\frac{1}{2}}$.** In these cases, we take the following ansatz:

$$\begin{cases} u^\epsilon(t,x,y) = \sum_{j\ge 0}\epsilon^{\frac{j}{2}}(u^{I,j}(t,x,y) + u^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ p^\epsilon(t,x,y) = \sum_{j\ge 0}\epsilon^{\frac{j}{2}}(p^{I,j}(t,x,y) + p^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \end{cases} \tag{2.2}$$

for the solutions of (2.1), where $u^{B,j}(t,x,z)$ and $p^{B,j}(t,x,z)$ are rapidly decreasing when $z=\frac{y}{\sqrt{\epsilon}}\to+\infty$.

Plugging (2.2) into the divergence free condition given in $(2.1)_2$, it follows

$$\nabla\cdot u^{I,j} = 0, \qquad \forall j\ge 0, \tag{2.3}$$

and

$$\partial_z u_2^{B,0} = 0, \quad \partial_x u_1^{B,j} + \partial_z u_2^{B,j+1} = 0, \quad \forall j\ge 0 \tag{2.4}$$

which implies

$$u_2^{B,0} \equiv 0, \tag{2.5}$$

by noting that $u_2^{B,0}(t,x,z)$ is fast decay when $z\to+\infty$.

Plugging (2.2) into the equations given in $(2.1)_1$, it follows

$$\begin{aligned} &\sum_{j\ge 0}\epsilon^{\frac{j}{2}}\partial_t(u^{I,j}+u^{B,j}) + \sum_{j\ge 0}\epsilon^{\frac{j}{2}}\sum_{k=0}^{j}((u^{I,k}+u^{B,k})\cdot\nabla u^{I,j-k} + (u_1^{I,k}+u_1^{B,k})\partial_x u^{B,j-k}) \\ &+\epsilon^{-\frac{1}{2}}(u_2^{I,0}+u_2^{B,0})\partial_z u^{B,0} + \sum_{j\ge 0}\epsilon^{\frac{j}{2}}\sum_{k=0}^{j+1}(u_2^{I,k}+u_2^{B,k})\partial_z u^{B,j+1-k} \\ &+\sum_{j\ge 0}\epsilon^{\frac{j}{2}}\nabla p^{I,j} + \epsilon^{-\frac{1}{2}}\begin{pmatrix} 0 \\ \partial_z p^{B,0}\end{pmatrix} + \sum_{j\ge 0}\epsilon^{\frac{j}{2}}\begin{pmatrix}\partial_x p^{B,j} \\ \partial_z p^{B,j+1}\end{pmatrix} \\ =\ &\partial_z^2 u^{B,0} + \epsilon^{\frac{1}{2}}\partial_z^2 u^{B,1} + \sum_{j\ge 0}\epsilon^{1+\frac{j}{2}}(\Delta u^{I,j} + \partial_x^2 u^{B,j} + \partial_z^2 u^{B,j+2}). \end{aligned} \tag{2.6}$$

Letting $z$ go to $+\infty$ in (2.6), it gives

$$\partial_t u^{I,0} + (u^{I,0}\cdot\nabla)u^{I,0} + \nabla p^{I,0} = 0, \tag{2.7}$$

and

$$\partial_t u^{I,j}+(u^{I,0}\cdot\nabla)u^{I,j}+(u^{I,j}\cdot\nabla)u^{I,0}+\nabla p^{I,j}=\Delta u^{I,j-2}-\sum_{1\le k\le j-1}(u^{I,k}\cdot\nabla)u^{I,j-k} \tag{2.8}$$

for all $j\ge 1$, where we denote by $u^{I,-1}=0$.

In the following discussion, we shall always denote the trace of a function $u(t,x,y)$ on $\{y=0\}$ by

$$\overline{u}(t,x)=u(t,x,0).$$

The vanishing of the order $O(\epsilon^{-\frac{1}{2}})$ terms in (2.6) implies that

$$(\overline{u_2^{I,0}}+u_2^{B,0})\partial_z u^{B,0}+\begin{pmatrix}0\\ \partial_z p^{B,0}\end{pmatrix}=0 \tag{2.9}$$

which yields

$$p^{B,0}\equiv 0 \tag{2.10}$$

by using (2.5).

From the order $O(\epsilon^0)$ terms of (2.6) we obtain

$$\begin{aligned}\partial_t(\overline{u^{I,0}}+u^{B,0})+(\overline{u^{I,0}}+u^{B,0})\cdot\overline{\nabla u^{I,0}}+(\overline{u_1^{I,0}}+u_1^{B,0})\partial_x u^{B,0}+z\overline{\partial_y u_2^{I,0}}\partial_z u^{B,0}\\ +(\overline{u_2^{I,1}}+u_2^{B,1})\partial_z u^{B,0}+\overline{\nabla p^{I,0}}+\begin{pmatrix}\partial_x p^{B,0}\\ \partial_z p^{B,1}\end{pmatrix}=\partial_z^2 u^{B,0},\end{aligned} \tag{2.11}$$

whose second component reads as

$$\overline{\partial_y p^{I,0}}+\partial_z p^{B,1}=0$$

implying

$$p^{B,1}\equiv 0 \tag{2.12}$$

by noting that $\overline{\partial_y p^{I,0}}=0$ from (2.7).

For any $j\ge 0$, define

$$u^{p,j}(t,x,z)=u^{B,j}(t,x,z)+\sum_{k=0}^{j}\frac{z^k}{k!}\overline{\partial_y^k u^{I,j-k}}(t,x) \tag{2.13}$$

with $0!=1$.

From the first component of (2.11) and (2.4), (2.5), we know that

$$\begin{cases}u_1^{p,0}=u_1^{I,0}(t,x,0)+u^{B,0}(t,x,z)\\ u_2^{p,1}=u_2^{I,1}(t,x,0)+u_2^{B,1}(t,x,z)+z\partial_y u_2^{I,0}(t,x,0)\end{cases}$$

satisfy the following Prandtl equations:

$$\begin{cases}\partial_t u_1^{p,0}+u_1^{p,0}\partial_x u_1^{p,0}+u_2^{p,1}\partial_z u_1^{p,0}+\overline{\partial_x p^{I,0}}=\partial_z^2 u_1^{p,0}\\ \partial_x u_1^{p,0}+\partial_z u_2^{p,1}=0.\end{cases} \tag{2.14}$$

The vanishing of the order $O(\epsilon^{\frac{1}{2}})$ terms in (2.6) implies that

$$\begin{aligned}&(\partial_t+(\overline{u_1^{I,0}}+u_1^{B,0})\partial_x)(\overline{u^{I,1}}+u^{B,1}+z\overline{\partial_y u^{I,0}})+(\overline{u_1^{I,1}}+u_1^{B,1}+z\overline{\partial_y u_1^{I,0}})\partial_x(\overline{u^{I,0}}+u^{B,0})\\&+(\overline{u_2^{I,2}}+u_2^{B,2}+z\overline{\partial_y u_2^{I,1}}+\tfrac{z^2}{2}\overline{\partial_y^2 u_2^{I,0}})\partial_z u^{B,0}+(\overline{u_2^{I,1}}+u_2^{B,1}+z\overline{\partial_y u_2^{I,0}})(\partial_z u^{B,1}+\overline{\partial_y u^{I,0}})\\&+\overline{\nabla p^{I,1}}+z\overline{\nabla\partial_y p^{I,0}}+\begin{pmatrix}\partial_x p^{B,1}\\ \partial_z p^{B,2}\end{pmatrix}=\partial_z^2 u^{B,1}.\end{aligned}\tag{2.15}$$

Obviously, the second component of (2.15) can be written as

$$\partial_t u_2^{p,1}+u_1^{p,0}\partial_x u_2^{p,1}+u_2^{p,1}\partial_z u_2^{p,1}+\overline{\partial_y p^{I,1}}+z\overline{\partial_y^2 p^{I,0}}+\partial_z p^{B,2}=\partial_z^2 u_2^{B,1},\tag{2.16}$$

which is equivalent to

$$\partial_z p^{B,2}=\partial_z^2 u_2^{B,1}-(\partial_t+\overline{u_1^{I,0}}\partial_x+u_2^{p,1}\partial_z+\overline{\partial_y u_2^{I,0}})u_2^{B,1}-u_1^{B,0}\partial_x u_2^{p,1},\tag{2.17}$$

by using the facts

$$(\partial_t+\overline{u_1^{I,0}}\partial_x+\overline{\partial_y u_2^{I,0}})\overline{\partial_y u_2^{I,0}}+\overline{\partial_y^2 p^{I,0}}=0$$

and

$$(\partial_t+\overline{u_1^{I,0}}\partial_x+\overline{\partial_y u_2^{I,0}})\overline{u_2^{I,1}}+\overline{\partial_y p^{I,1}}=0$$

derived directly from (2.7) and (2.8).

The unknown $p^{B,2}(t,x,z)$ rapidly decreasing in $z\to+\infty$ can be easily determined uniquely from (2.17).

From the first component of (2.15) and (2.4), (2.5), (2.12) we know that

$$\begin{cases}u_1^{p,1}=u_1^{I,1}(t,x,0)+u_1^{B,1}(t,x,z)+z\partial_y u_1^{I,0}(t,x,0)\\ u_2^{p,2}=u_2^{I,2}(t,x,0)+u_2^{B,2}(t,x,z)+z\partial_y u_2^{I,1}(t,x,0)+\frac{z^2}{2}\partial_y^2 u_2^{I,0}(t,x,0)\end{cases}$$

satisfy the following linearized Prandtl equations:

$$\begin{cases}(\partial_t+u_1^{p,0}\partial_x+u_2^{p,1}\partial_z+\partial_x u_1^{p,0})u_1^{p,1}+u_2^{p,2}\partial_z u_1^{p,0}+\partial_x(\overline{p^{I,0}}+z\overline{\partial_y p^{I,0}})=\partial_z^2 u_1^{p,1}\\ \partial_x u_1^{p,1}+\partial_z u_2^{p,2}=0.\end{cases}\tag{2.18}$$

Similarly, for any $j\geq 2$, from the $O(\epsilon^{\frac{j}{2}})-$order terms of $(2.6)_2$ one can determine $p^{B,j+1}(t,x,z)$ uniquely provided that $\{u_1^{p,k}(t,x,z)\}_{k\leq j-1}$ and $\{u_2^{p,k}(t,x,z)\}_{k\leq j}$ are known already. From the $O(\epsilon^{\frac{j}{2}})-$order terms of $(2.6)_1$ and (2.4), (2.5) we deduce that $(u_1^{p,j},u_2^{p,j+1})$ satisfy a linearized Prandtl system similar to (2.18).

To solve the boundary layer profiles from equation (2.14) and (2.18), the boundary conditions must be determined.

First, from the first condition given in $(2.1)_3$ we have

$$u_2^{p,j}|_{z=0}=0\tag{2.19}$$

for all $j\geq 1$.

Substituting the ansatz (2.2) into the Navier boundary condition given in $(2.1)_3$, it follows that

$$\beta\sum_{j\geq 0}\epsilon^{\frac{j}{2}}(u_1^{I,j}+u_1^{B,j})=\alpha^\epsilon\{\epsilon^{-\frac{1}{2}}\partial_z u_1^{B,0}+\sum_{j\geq 0}\epsilon^{\frac{j}{2}}(\partial_y u_1^{I,j}+\partial_z u_1^{B,j+1})\} \tag{2.20}$$

on $\{y=z=0\}$.

Now, we study (2.20) for two cases.

**Case 1:** $\alpha^\epsilon=\epsilon$.

In this case, from (2.20) we immediately obtain

$$\begin{cases} u_1^{p,0}|_{z=0}=0 \\ \lim_{z\to+\infty}(u_1^{p,0}-\overline{u_1^{I,0}})=0 \quad \text{exponentially,} \end{cases} \tag{2.21}$$

and

$$\begin{cases} u_1^{p,j}=\frac{1}{\beta}\partial_z u_1^{p,j-1}, \qquad \text{on } z=0 \\ \lim_{z\to+\infty}(u_1^{p,j}-\sum_{k=0}^{j}\frac{z^k}{k!}\overline{\partial_y^k u_1^{I,j-k}})=0 \quad \text{exponentially} \end{cases} \tag{2.22}$$

for all $j\geq 1$.

Therefore, one concludes

Conclusion 2.1. *The solutions $(u^\epsilon,p^\epsilon)$ to the problem (2.1) with $\alpha^\epsilon=\epsilon$ formally have the following asymptotic expansions:*

$$\begin{cases} u_1^\epsilon(t,x,y)=\sum_{j\geq 0}\epsilon^{\frac{j}{2}}(u_1^{I,j}(t,x,y)+u_1^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ u_2^\epsilon(t,x,y)=u_2^{I,0}(t,x,y)+\sum_{j\geq 1}\epsilon^{\frac{j}{2}}(u_2^{I,j}(t,x,y)+u_2^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ p^\epsilon(t,x,y)=\sum_{j\geq 0}\epsilon^{\frac{j}{2}}p^{I,j}(t,x,y)+\sum_{j\geq 2}\epsilon^{\frac{j}{2}}p^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}}) \end{cases} \tag{2.23}$$

*for rapidly decreasing $(u^{B,j},p^{B,j})(t,x,z)$ in $z\to+\infty$, where*

*(1) $(u^{I,0},p^{I,0})$ are solutions to the following problem for the Euler equations:*

$$\begin{cases} \partial_t u^{I,0}+(u^{I,0}\cdot\nabla)u^{I,0}+\nabla p^{I,0}=0 \\ \nabla\cdot u^{I,0}=0 \\ u_2^{I,0}|_{y=0}=0 \end{cases} \tag{2.24}$$

*and for all $j\geq 1$, $(u^{I,j},p^{I,j})$ are solutions to problems for the linearized Euler equations (2.8) and (2.3),*

*(2) the leading boundary layer profiles $(u_1^{p,0},u_2^{p,1})=(u_1^{B,0}+\overline{u_1^{I,0}},u_2^{B,1}+\overline{u_2^{I,1}}+\overline{z\partial_y u_2^{I,0}})$ satisfy the following problem for the Prandtl equations:*

$$\begin{cases} \partial_t u_1^{p,0}+u_1^{p,0}\partial_x u_1^{p,0}+u_2^{p,1}\partial_z u_1^{p,0}+\overline{\partial_x p^{I,0}}=\partial_z^2 u_1^{p,0} \\ \partial_x u_1^{p,0}+\partial_z u_2^{p,1}=0 \\ u_1^{p,0}|_{z=0}=u_2^{p,1}|_{z=0}=0 \\ \lim_{z\to+\infty}(u_1^{p,0}-\overline{u_1^{I,0}})=0 \quad \text{exponentially,} \end{cases} \tag{2.25}$$

*$p^{B,2}(t,x,z)$ is uniquely determined by the equation (2.17), for all $j \geq 1$, $(u_1^{p,j}, u_2^{p,j+1})$ with $u^{p,j} = u^{B,j} + \sum_{k=0}^{j} \frac{z^k}{k!} \overline{\partial_y^k u_2^{I,j-k}}$ satisfy problems for the linearized Prandtl equations similar to (2.18) with boundary conditions given in (2.19) and (2.22), and $p^{B,j+2}(t,x,z)$ are uniquely determined by equations similar to (2.17).*

**Case 2:** $\alpha^\epsilon = \epsilon^{\frac{1}{2}}$.

In this case, from (2.20) we immediately obtain

$$\begin{cases} \partial_z u_1^{p,j} - \beta u_1^{p,j} = 0, & \text{on } z = 0 \\ \lim_{z \to +\infty} (u_1^{p,j} - \sum_{k=0}^{j} \frac{z^k}{k!} \overline{\partial_y^k u_1^{I,j-k}}) = 0 & \text{exponentially} \end{cases} \tag{2.26}$$

for all $j \geq 0$.

Therefore, we deduce

CONCLUSION 2.2. *The solutions $(u^\epsilon, p^\epsilon)$ to the problem (2.1) with $\alpha^\epsilon = \epsilon^{\frac{1}{2}}$ formally have the same expansions as given in Conclusion 2.1 except that the leading boundary layer profiles $(u_1^{p,0}, u_2^{p,1})$ satisfy the nonlinear Prandtl equations as given in (2.25) but with the boundary conditions*

$$\begin{cases} u_2^{p,1} = 0, \quad \partial_z u_1^{p,0} - \beta u_1^{p,0} = 0, & \text{on } z = 0 \\ \lim_{z \to +\infty} (u_1^{p,0} - \overline{u_1^{I,0}}) = 0 & \text{exponentially} \end{cases} \tag{2.27}$$

*and the lower order boundary layer profiles $(u_1^{p,j}, u_2^{p,j+1})$ $(j \geq 1)$ satisfy the linearized Prandtl equations as given in (2.18) with the boundary conditions (2.19) and (2.26).*

REMARK 2.3. From the above discussion, in general when the slip length $\alpha^\epsilon = \epsilon^\gamma$ for a fixed $\gamma > \frac{1}{2}$, for the solution $u^\epsilon$ to the problem (2.1) we can deduce that the boundary layer appears in the zero-th order terms of the expansion of $u^\epsilon$, and the leading boundary layer profiles satisfy the boundary value problem (2.25), which is the same as in the non-slip case [12]. Complete expansions of solutions can be derived as well in a way similar to the one given in sections 4.1 and 4.2.

**2.2. The case $\alpha^\epsilon = \epsilon^{\frac{1}{4}}$.** When $\alpha^\epsilon = \epsilon^{\frac{1}{4}}$, we take the following ansatz:

$$\begin{cases} u^\epsilon(t,x,y) = \sum_{j \geq 0} \epsilon^{\frac{j}{4}} (u^{I,j}(t,x,y) + u^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ p^\epsilon(t,x,y) = \sum_{j \geq 0} \epsilon^{\frac{j}{4}} (p^{I,j}(t,x,y) + p^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \end{cases} \tag{2.28}$$

for solutions of (2.1), where $u^{B,j}(t,x,z)$ and $p^{B,j}(t,x,z)$ are rapidly decreasing when $z = \frac{y}{\sqrt{\epsilon}} \to +\infty$.

Plugging (2.28) into $(2.1)_2$, we obtain

$$\nabla \cdot u^{I,j} = 0, \qquad \forall j \geq 0, \tag{2.29}$$

and

$$\partial_z u_2^{B,0} = \partial_z u_2^{B,1} = 0, \quad \partial_x u_1^{B,j} + \partial_z u_2^{B,j+2} = 0, \quad \forall j \geq 0, \tag{2.30}$$

which implies

$$u_2^{B,0} = u_2^{B,1} \equiv 0, \tag{2.31}$$

yielding

$$u_2^{I,0} = u_2^{I,1} = 0, \quad \text{on } y = 0 \tag{2.32}$$

from the boundary condition $u_2^\epsilon|_{y=0} = 0$.

Plugging (2.28) into $(2.1)_1$, it follows that

$$\begin{aligned}
&\sum_{j\geq 0} \epsilon^{\frac{j}{4}} \partial_t (u^{I,j} + u^{B,j}) + \sum_{j\geq 0} \epsilon^{\frac{j}{4}} \sum_{k=0}^{j} ((u^{I,k} + u^{B,k}) \cdot \nabla u^{I,j-k} + (u_1^{I,k} + u_1^{B,k}) \partial_x u^{B,j-k}) \\
&+ \sum_{j=0}^{1} \epsilon^{\frac{j-2}{4}} \sum_{k=0}^{j} (u_2^{I,k} + u_2^{B,k}) \partial_z u^{B,j-k} + \sum_{j\geq 0} \epsilon^{\frac{j}{4}} \sum_{k=0}^{j+2} (u_2^{I,k} + u_2^{B,k}) \partial_z u^{B,j+2-k} \\
&+ \sum_{j\geq 0} \epsilon^{\frac{j}{4}} \nabla p^{I,j} + \epsilon^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \partial_z p^{B,0} \end{pmatrix} + \epsilon^{-\frac{1}{4}} \begin{pmatrix} 0 \\ \partial_z p^{B,1} \end{pmatrix} + \sum_{j\geq 0} \epsilon^{\frac{j}{4}} \begin{pmatrix} \partial_x p^{B,j} \\ \partial_z p^{B,j+2} \end{pmatrix} \\
= &\sum_{k=0}^{3} \epsilon^{\frac{k}{4}} \partial_z^2 u^{B,k} + \sum_{j\geq 0} \epsilon^{1+\frac{j}{4}} (\Delta u^{I,j} + \partial_x^2 u^{B,j} + \partial_z^2 u^{B,j+4}).
\end{aligned} \tag{2.33}$$

Letting $z \to +\infty$ in (2.33), this yields

$$\begin{cases} \partial_t u^{I,0} + (u^{I,0} \cdot \nabla) u^{I,0} + \nabla p^{I,0} = 0 \\ \nabla \cdot u^{I,0} = 0 \\ u_2^{I,0}|_{y=0} = 0 \end{cases} \tag{2.34}$$

and

$$\begin{cases} \partial_t u^{I,j} + (u^{I,0} \cdot \nabla) u^{I,j} + (u^{I,j} \cdot \nabla) u^{I,0} + \nabla p^{I,j} = \Delta u^{I,j-4} - \sum\limits_{1\leq k\leq j-1} (u^{I,k} \cdot \nabla) u^{I,j-k} \\ \nabla \cdot u^{I,j} = 0 \end{cases} \tag{2.35}$$

for all $j \geq 1$, where we set $u^{I,k} = 0$ when $k \leq -1$.

By using (2.31), the vanishing of the order $O(\epsilon^{-\frac{1}{2}})$ and $O(\epsilon^{-\frac{1}{4}})$ terms in (2.33) implies that

$$\partial_z p^{B,0} = \partial_z p^{B,1} = 0, \tag{2.36}$$

which yields

$$p^{B,0} = p^{B,1} \equiv 0. \tag{2.37}$$

From the order $O(\epsilon^0)$ terms of (2.33), we obtain

$$\begin{aligned}
&\partial_t (\overline{u^{I,0}} + u^{B,0}) + (\overline{u^{I,0}} + u^{B,0}) \cdot \overline{\nabla u^{I,0}} + (\overline{u_1^{I,0}} + u_1^{B,0}) \partial_x u^{B,0} + z \overline{\partial_y u_2^{I,0}} \partial_z u^{B,0} \\
&+ \sum_{k=0}^{2} (\overline{u_2^{I,k}} + u_2^{B,k}) \partial_z u^{B,2-k} + \overline{\nabla p^{I,0}} + \begin{pmatrix} \partial_x p^{B,0} \\ \partial_z p^{B,2} \end{pmatrix} = \partial_z^2 u^{B,0},
\end{aligned} \tag{2.38}$$

whose second component reads as

$$\overline{\partial_y p^{I,0}} + \partial_z p^{B,2} = 0$$

implying

$$p^{B,2} \equiv 0 \tag{2.39}$$

by noting $\overline{\partial_y p^{I,0}} = 0$ from (2.34).

For any $j \geq 0$, define

$$u^{q,j}(t,x,z) = u^{B,j}(t,x,z) + \sum_{k=0}^{[\frac{j}{2}]} \frac{z^k}{k!} \overline{\partial_y^k u^{I,j-2k}}(t,x) \tag{2.40}$$

with $0! = 1$.

From the first component of (2.38) and (2.29), (2.30), we know that $(u_1^{q,0}, u_2^{q,2})$ satisfy the following Prandtl equations:

$$\begin{cases} \partial_t u_1^{q,0} + u_1^{q,0} \partial_x u_1^{q,0} + u_2^{q,2} \partial_z u_1^{q,0} + \overline{\partial_x p^{I,0}} = \partial_z^2 u_1^{q,0} \\ \partial_x u_1^{q,0} + \partial_z u_2^{q,2} = 0. \end{cases} \tag{2.41}$$

The vanishing of the order $O(\epsilon^{\frac{1}{4}})$ terms in (2.33) implies that

$$\begin{aligned} &\partial_t(\overline{u^{I,1}} + u^{B,1}) + \sum_{k=0}^{1} \left\{ (\overline{u^{I,k}} + u^{B,k}) \cdot \overline{\nabla u^{I,1-k}} + (\overline{u_1^{I,k}} + u_1^{B,k}) \partial_x u^{B,1-k} \right\} \\ &+ \sum_{k=0}^{1} z \overline{\partial_y u_2^{I,k}} \partial_z u^{B,1-k} + \sum_{k=0}^{3} (\overline{u_2^{I,k}} + u_2^{B,k}) \partial_z u^{B,3-k} + \overline{\nabla p^{I,1}} + \begin{pmatrix} \partial_x p^{B,1} \\ \partial_z p^{B,3} \end{pmatrix} = \partial_z^2 u^{B,1}. \end{aligned} \tag{2.42}$$

Obviously, the second component of (2.42) can be written as

$$\overline{\partial_y p^{I,1}} + \partial_z p^{B,3} = 0, \tag{2.43}$$

which implies that

$$p^{B,3}(t,x,z) \equiv 0 \tag{2.44}$$

by using the fact $\overline{\partial_y p^{I,1}} = 0$ from (2.35) with $j = 1$.

From the first component of (2.42) and (2.29), (2.30) we know that $(u_1^{q,1}, u_2^{q,3})$ satisfy the following linearized Prandtl equations:

$$\begin{cases} \partial_t u_1^{q,1} + u_1^{q,0} \partial_x u_1^{q,1} + u_1^{q,1} \partial_x u_1^{q,0} + u_2^{q,2} \partial_z u_1^{q,1} + u_2^{q,3} \partial_z u_1^{q,0} + \overline{\partial_x p^{I,1}} = \partial_z^2 u_1^{q,1} \\ \partial_x u_1^{q,1} + \partial_z u_2^{q,3} = 0, \end{cases} \tag{2.45}$$

From the order $O(\epsilon^{\frac{1}{2}})$ terms of (2.33), we deduce

$$\begin{aligned} &(\partial_t + u_1^{q,0} \partial_x) u^{q,2} + u_1^{q,1} \partial_x u^{q,1} + u_1^{q,2} \partial_x u^{B,0} + u^{q,2} \cdot \overline{\nabla u^{I,0}} + \sum_{k=2}^{4} u_2^{q,k} \partial_z u^{B,4-k} \\ &+ \overline{\nabla p^{I,2}} + z \overline{\nabla \partial_y p^{I,0}} + \begin{pmatrix} \partial_x p^{B,2} \\ \partial_z p^{B,4} \end{pmatrix} = \partial_z^2 u^{B,2}. \end{aligned} \tag{2.46}$$

By using (2.34) and (2.35), the second component of (2.46) can be written as

$$\partial_z p^{B,4} = \partial_z^2 u_2^{B,2} - (\partial_t + u_1^{q,0}\partial_x + u_2^{p,2}\partial_z + \overline{\partial_y u_2^{I,0}}) u_2^{B,2} - u_1^{B,0}\partial_x(\overline{u_2^{I,2}} + z\overline{\partial_y u_2^{I,0}}), \quad (2.47)$$

which determines $p^{B,4}(t,x,z)$ uniquely provided that $u_1^{B,0}$ and $u_2^{B,2}$ are known already.

To solve $(u_1^{q,0}, u_2^{q,2})$ and $(u_1^{q,1}, u_2^{q,3})$ from equations (2.41) and (2.45) respectively, we need to study their boundary conditions.

First, from $(2.1)_3$ we immediately have

$$u_2^{q,j}|_{z=0} = 0 \quad (2.48)$$

for all $j \geq 2$.

Substituting the ansatz (2.28) into the Navier boundary condition, it follows that

$$\beta \sum_{j\geq 0} \epsilon^{\frac{j}{4}} (u_1^{I,j} + u_1^{B,j}) = \epsilon^{-\frac{1}{4}} \partial_z u_1^{B,0} + \partial_z u_1^{B,1} + \sum_{j\geq 0} \epsilon^{\frac{j+1}{4}} (\partial_y u_1^{I,j} + \partial_z u_1^{B,j+2}) \quad (2.49)$$

on $\{y = z = 0\}$, which implies that

$$\begin{cases} \partial_z u_1^{B,0}|_{z=0} = 0 \\ \partial_z u_1^{B,1}|_{z=0} = \beta(u_1^{I,0} + u_1^{B,0})|_{y=z=0} \\ \partial_z u_1^{B,j} = \beta(u_1^{I,j-1} + u_1^{B,j-1}) - \partial_y u_1^{I,j-2} \qquad \text{on} \quad y = z = 0, \quad \forall j \geq 2. \end{cases} \quad (2.50)$$

Therefore, from (2.41) and (2.50) we know that

$$\begin{cases} u_1^{q,0}(t,x,z) = \overline{u_1^{I,0}} + u_1^{B,0} \\ u_2^{q,2}(t,x,z) = \overline{u_2^{I,2}} + u_2^{B,2} + z\overline{\partial_y u_2^{I,0}} \end{cases}$$

satisfy the following problem:

$$\begin{cases} \partial_t u_1^{q,0} + u_1^{q,0}\partial_x u_1^{q,0} + u_2^{q,2}\partial_z u_1^{q,0} + \overline{\partial_x p^{I,0}} = \partial_z^2 u_1^{q,0} \\ \partial_x u_1^{q,0} + \partial_z u_2^{q,2} = 0 \\ \partial_z u_1^{q,0}|_{z=0} = u_2^{q,2}|_{z=0} = 0 \\ \lim_{z\to+\infty}(u_1^{q,0} - \overline{u_1^{I,0}}) = 0 \quad \text{exponentially.} \end{cases} \quad (2.51)$$

By uniqueness of classical solutions to (2.51), it follows that

$$u_1^{q,0}(t,x,z) = \overline{u_1^{I,0}}(t,x), \quad \text{i.e.} \quad u_1^{B,0} \equiv 0, \qquad u_2^{B,2} \equiv 0. \quad (2.52)$$

Substituting (2.52) into (2.47) and (2.45), it follows immediately that

$$p^{B,4}(t,x,z) \equiv 0 \quad (2.53)$$

and

$$\begin{cases} u_1^{q,1}(t,x,z) = \overline{u_1^{I,1}}(t,x) + u_1^{B,1}(t,x,z) \\ u_2^{q,3}(t,x,z) = z\overline{\partial_y u_2^{I,1}}(t,x) + \overline{u_2^{I,3}}(t,x) + u_2^{B,3}(t,x,z) \end{cases} \quad (2.54)$$

satisfy the following problem for the linearized Prandtl equations:

$$\begin{cases} \partial_t u_1^{q,1} + \overline{u_1^{I,0}} \partial_x u_1^{q,1} + \overline{z\partial_y u_2^{I,0}} \partial_z u_1^{q,1} + u_1^{q,1} \partial_x \overline{u_1^{I,0}} + \overline{\partial_x p^{I,1}} = \partial_z^2 u_1^{q,1} \\ \partial_x u_1^{q,1} + \partial_z u_2^{q,3} = 0 \\ u_2^{q,3}|_{z=0} = 0, \qquad \partial_z u_1^{q,1}|_{z=0} = \beta \overline{u_1^{I,0}}(t,x) \\ \lim_{z\to+\infty} u_1^{q,1} = \overline{u_1^{I,1}}(t,x) \quad \text{exponentially.} \end{cases} \tag{2.55}$$

From the order $O(\epsilon^{\frac{3}{4}})$ terms of (2.33) we get the following equation for determining $p^{B,5}(t,x,z)$:

$$\partial_z p^{B,5} = \partial_z^2 u_2^{B,3} - (\partial_t + \overline{u_1^{I,0}} \partial_x + (\overline{u_2^{I,2}} + \overline{z\partial_y u_2^{I,0}})\partial_z + \overline{\partial_y u_2^{I,0}}) u_2^{B,3}. \tag{2.56}$$

Similar to the above discussion, for any $j \geq 2$, from the $O(\epsilon^{\frac{j}{4}})-$order terms of $(2.33)_1$ and (2.29), (2.30) we deduce that $(u_1^{q,j}, u_2^{q,j+2})$ satisfy a problem for the linearized Prandtl equations as given in (2.55) with the boundary conditions

$$\begin{cases} u_2^{q,j+2} = 0, \quad \partial_z u_1^{q,j} = \beta u_1^{q,j-1} \qquad \text{on } z = 0 \\ \lim_{z\to+\infty} \left( u_1^{q,j} - \sum\limits_{k=0}^{[j/2]} \frac{z^k}{k!} \overline{\partial_y^k u_1^{I,j-2k}} \right) = 0 \quad \text{exponentially.} \end{cases} \tag{2.57}$$

From the $O(\epsilon^{\frac{j}{4}})-$order terms of $(2.33)_2$ one can determine $p^{B,j+2}(t,x,z)$ uniquely provided that $\{u_1^{q,k}(t,x,z)\}_{k\leq j-2}$ and $\{u_2^{q,k}(t,x,z)\}_{k\leq j}$ are known already.

Therefore, we conclude:

Conclusion 2.4. *The solutions $(u^\epsilon, p^\epsilon)$ to the problem (2.1) with $\alpha^\epsilon = \epsilon^{\frac{1}{4}}$ formally have the following asymptotic expansions:*

$$\begin{cases} u_1^\epsilon(t,x,y) = u_1^{I,0}(t,x,y) + \sum\limits_{j\geq 1} \epsilon^{\frac{j}{4}} (u_1^{I,j}(t,x,y) + u_1^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ u_2^\epsilon(t,x,y) = \sum\limits_{j=0}^{2} \epsilon^{\frac{j}{4}} u_2^{I,0}(t,x,y) + \sum\limits_{j\geq 3} \epsilon^{\frac{j}{4}} (u_2^{I,j}(t,x,y) + u_2^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ p^\epsilon(t,x,y) = \sum\limits_{j=0}^{4} \epsilon^{\frac{j}{4}} p^{I,j} + \sum\limits_{j\geq 5} \epsilon^{\frac{j}{4}} (p^{I,j}(t,x,y) + p^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \end{cases} \tag{2.58}$$

*for rapidly decreasing $(u^{B,j}, p^{B,j})(t,x,z)$ in $z \to +\infty$, where $(u^{I,0}, p^{I,0})$ are solutions to the problem (2.34) for the Euler equations, for all $j \geq 1$, $(u^{I,j}, p^{I,j})$ are solutions to the linearized Euler equations (2.35), for all $j \geq 1$,*

$$\begin{cases} u_1^{q,j}(t,x,z) = u_1^{B,j}(t,x,z) + \sum\limits_{k=0}^{[j/2]} \frac{z^k}{k!} \partial_y^k u_1^{I,j-2k}(t,x,0) \\ u_2^{q,j+2}(t,x,z) = u_2^{B,j+2}(t,x,z) + \sum\limits_{k=0}^{[\frac{j+2}{2}]} \frac{z^k}{k!} \partial_y^k u_2^{I,j+2-2k}(t,x,0) \end{cases} \tag{2.59}$$

*satisfy a boundary value problem for linearized Prandtl equations similar to (2.55), and for all $j \geq 5$, $p^{B,j}(t,x,z)$ are given by equations similar to (2.56) directly.*

REMARK 2.5. As in the above discussion, in general when the slip length $\alpha^\epsilon = \epsilon^\gamma$ for a fixed $0 < \gamma < \frac{1}{2}$, for the solution $u^\epsilon$ to the problem (2.1) we can deduce that the boundary layer appears in the order $O(\epsilon^{1-2\gamma})$ terms of solutions, and satisfies a boundary value problem for linearized Prandtl equations, but it still yields the vorticity of flow being unbounded in the vanishing viscosity limit.

**2.3. The case $\alpha^\epsilon = 1$.** In the case $\alpha^\epsilon = 1$, we take the same ansatz as in (2.2). From the boundary condition (2.20) with $\alpha^\epsilon = 1$, we obtain

$$\begin{cases} \partial_z u_1^{p,0}|_{z=0} = 0 \\ \partial_z u_1^{p,j} - \beta u_1^{p,j-1} = 0 \quad \text{on } z=0,\ \forall j \geq 1 \\ \lim_{z\to+\infty} \left( u_1^{p,j} - \sum\limits_{k=0}^{j} \frac{z^k}{k!} \overline{\partial_y^k u_1^{I,j-k}} \right) = 0 \quad \text{exponentially},\ \forall j \geq 0. \end{cases} \tag{2.60}$$

Thus, from (2.14), (2.19), and (2.60) we know that

$$\begin{cases} u_1^{p,0}(t,x,z) = u_1^{B,0}(t,x,z) + \overline{u_1^{I,0}}(t,x) \\ u_2^{p,1}(t,x,z) = u_2^{B,1}(t,x,z) + \overline{u_2^{I,1}}(t,x) + z\overline{\partial_y u_2^{I,0}}(t,x) \end{cases}$$

satisfy the following problem:

$$\begin{cases} \partial_t u_1^{p,0} + u_1^{p,0}\partial_x u_1^{p,0} + u_2^{p,1}\partial_z u_1^{p,0} + \overline{\partial_x p^{I,0}} = \partial_z^2 u_1^{p,0} \\ \partial_x u_1^{p,0} + \partial_z u_2^{p,1} = 0 \\ \partial_z u_1^{p,0}|_{z=0} = u_2^{p,1}|_{z=0} = 0 \\ \lim_{z\to+\infty} \left( u_1^{p,0} - \overline{u_1^{I,0}} \right) = 0 \quad \text{exponentially}. \end{cases} \tag{2.61}$$

On the other hand, from (2.7) and $u_2^{I,0}|_{y=0} = 0$, we have

$$(\partial_t + \overline{u_1^{I,0}}\partial_x)\overline{u_1^{I,0}} + \overline{\partial_x p^{I,0}} = 0. \tag{2.62}$$

So, by uniqueness of solutions to (2.61), we deduce

$$u_1^{B,0}(t,x,z) = u_2^{B,1}(t,x,z) \equiv 0. \tag{2.63}$$

Substituting (2.63) into (2.17) and (2.18) respectively, it follows that

$$p^{B,2}(t,x,z) \equiv 0 \tag{2.64}$$

and

$$\begin{cases} u_1^{p,1}(t,x,z) = u_1^{B,1}(t,x,z) + \overline{u_1^{I,1}}(t,x) + z\overline{\partial_y u_1^{I,0}}(t,x) \\ u_2^{p,2}(t,x,z) = u_2^{B,2}(t,x,z) + \overline{u_2^{I,2}}(t,x) + z\overline{\partial_y u_2^{I,1}}(t,x) + \frac{z^2}{2}\overline{\partial_y^2 u_2^{I,0}}(t,x) \end{cases}$$

satisfy the following problem for the linearized Prandtl equations:

$$\begin{cases} \partial_t u_1^{p,1} + \overline{u_1^{I,0}}\partial_x u_1^{p,1} + z\overline{\partial_y u_2^{I,0}}\partial_z u_1^{p,1} + u_1^{p,1}\overline{\partial_x u_1^{I,0}} + \partial_x\left(\overline{p^{I,1}} + z\overline{\partial_y p^{I,0}}\right) = \partial_z^2 u_1^{p,1} \\ \partial_x u_1^{p,1} + \partial_z u_2^{p,2} = 0 \\ \partial_z u_1^{p,1}|_{z=0} = \beta\overline{u_1^{I,0}}(t,x), \quad u_2^{p,2}|_{z=0} = 0 \\ \lim_{z\to+\infty}\left(u_1^{p,1} - \overline{u_1^{I,1}} - z\overline{\partial_y u_1^{I,0}}\right) = 0 \quad \text{exponentially.} \end{cases} \tag{2.65}$$

The vanishing of the second component of the $O(\epsilon)-$order terms in that (2.6) implies

$$\begin{aligned} &\left(\partial_t + \overline{u_1^{I,0}}\partial_x + z\overline{\partial_y u_2^{I,0}}\partial_z + \overline{\partial_y u_2^{I,0}}\right)\left(u_2^{B,2} + \overline{u_2^{I,2}} + z\overline{\partial_y u_2^{I,1}} + \frac{z^2}{2}\overline{\partial_y^2 u_2^{I,0}}\right) \\ &+z\overline{\partial_{xy}^2 u_2^{I,0}}\left(\overline{u_1^{I,1}} + z\overline{\partial_y u_1^{I,0}}\right) + \overline{\partial_y p^{I,2}} + z\overline{\partial_y^2 p^{I,1}} + \frac{z^2}{2}\overline{\partial_y^3 p^{I,0}} + \partial_z p^{B,3} \\ =&\overline{\partial_y^2 u_2^{I,0}} + \partial_z^2 u_2^{B,2} \end{aligned} \tag{2.66}$$

which gives rise to

$$\partial_z p^{B,3} = \left(\partial_z^2 - \partial_t - \overline{u_1^{I,0}}\partial_x - z\overline{\partial_y u_2^{I,0}}\partial_z - \overline{\partial_y u_2^{I,0}}\right)u_2^{B,2} \tag{2.67}$$

by using

$$\overline{\partial_y^3 p^{I,0}} + \left(\partial_t + 3\overline{\partial_y u_2^{I,0}}\right)\overline{\partial_y^2 u_2^{I,0}} + \overline{u_1^{I,0}\partial_x\partial_y^2 u_2^{I,0}} + 2\overline{\partial_y u_1^{I,0}\partial_{xy}^2 u_2^{I,0}} = 0,$$

$$\overline{\partial_y^2 p^{I,1}} + \left(\partial_t + 2\overline{\partial_y u_2^{I,0}}\right)\overline{\partial_y u_2^{I,1}} + \overline{u_1^{I,0}\partial_{xy}^2 u_2^{I,1}} + \overline{u_1^{I,1}\partial_{xy}^2 u_2^{I,0}} = 0$$

and

$$\overline{\partial_y p^{I,2}} + \left(\partial_t + \overline{u_1^{I,0}}\partial_x + \overline{\partial_y u_2^{I,0}}\right)\overline{u_2^{I,2}} = \overline{\partial_y^2 u_2^{I,0}}$$

derived immediately from (2.7) and (2.8).

Therefore, one concludes

Conclusion 2.6. *The solutions $(u^\epsilon, p^\epsilon)$ to the problem (2.1) with $\alpha^\epsilon = 1$ formally have the following asymptotic expansions:*

$$\begin{cases} u_1^\epsilon(t,x,y) = u_1^{I,0}(t,x,y) + \sum_{j\geq 1}\epsilon^{\frac{j}{2}}(u_1^{I,j}(t,x,y) + u_1^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ u_2^\epsilon(t,x,y) = (u_2^{I,0} + \epsilon^{\frac{1}{2}}u_2^{I,0})(t,x,y) + \sum_{j\geq 2}\epsilon^{\frac{j}{2}}(u_2^{I,j}(t,x,y) + u_2^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ p^\epsilon(t,x,y) = \sum_{j=0}^{2}\epsilon^{\frac{j}{2}}p^{I,j}(t,x,y) + \sum_{j\geq 3}\epsilon^{\frac{j}{2}}(p^{I,j}(t,x,y) + p^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \end{cases} \tag{2.68}$$

*for rapidly decreasing $(u^{B,j}, p^{B,j})(t,x,z)$ in $z \to +\infty$, where $(u_1^{p,1}, u_2^{p,2}) = \left(u_1^{B,1} + \overline{u_1^{I,1}} + z\overline{\partial_y u_1^{I,0}}, u_2^{B,2} + \sum_{k=0}^{2}\frac{z^k}{k!}\overline{\partial_y^k u_2^{I,2-k}}\right)$ satisfy the problem (2.65) for the* *linearized Prandtl equations, and $p^{B,3}(t,x,z)$ is uniquely determined by equation (2.67).*

REMARK 2.7. Similar to the above discussion, one can deduce that when $\gamma \leq 0$, the amplitude of the boundary layer is at most of the order $O(\epsilon^{\frac{1}{2}})$, which yields that the convection term $(u^\epsilon \cdot \nabla)u^\epsilon$ is uniformly bounded in $\epsilon$. By using an approach similar to the one presented in section 3, one can justify rigorously the asymptotic expansions of solutions $(u^\epsilon, p^\epsilon)$ given in (2.68) in the vanishing viscosity limit. Rather recently, in [4] Iftimie and Sueur studied this expansion up to the order $o(\epsilon^{\frac{1}{2}})$ in $L^\infty(0,T,L^2(\Omega))$.

## 3. Stability of boundary layers with unbounded vorticity

In this section, we study rigorously the asymptotic behavior of solutions to the initial-boundary value problem for anisotropic Navier-Stokes equations with the Navier friction boundary condition for the vanishing viscosity limit.

Due to the degeneracy of the Prandtl equations, it is a challenging problem to rigorously justify the formal asymptotic expansions of solutions obtained in section 2 for the vanishing viscosity limit in the Sobolev spaces, except that one can verify these expansions when the data are analytic in the frame of the abstract Cauchy-Kowaleskaya theory as done by Sammartino and Caflisch in [15] in the case where the velocity field satisfies the non-slip condition on the boundary.

As we shall see, the crucial point in rigorously justifying the formal expansions of solutions obtained in section 2 in the Sobolev norms is estimating the convection term $u^\epsilon \cdot \nabla u^\epsilon$ by the viscous term in the Navier-Stokes equations. To do so, in this section, we shall first study a problem similar to (2.1) in the case where the slip length $\alpha^\epsilon = \epsilon^{\frac{1}{4}}$ with $\epsilon$ being the vertical viscosity, and the horizontal viscosity vanishes as well when $\epsilon$ goes to zero. As we have seen, from the formal analysis given in section 2, even though in this case the boundary layer profiles satisfy a linearized Prandtl system, but the vorticity of flow in the layer is not uniformly bounded in $\epsilon$, and the convection term is unbounded as well. In order to control the convection term by the viscous term, instead of (2.1) we study this problem for the anisotropic Navier-Stokes equations with the horizontal viscosity being $\epsilon^{\frac{1}{2}}$. The vanishing viscosity limit problem for the Navier-Stokes equation with the Navier boundary condition for general anisotropic viscosities will be considered later.

The anisotropic Navier-Stokes equations are widely used in geophysical fluid dynamics as a mathematical model for water flows in lakes and oceans, and also in the study of the Ekman boundary layers for rotating fluids; see [2, 13].

Similar to that mentioned in section 1, for simplicity of presentation we shall mainly study the problem in the two-dimensional half space, though it is not difficult to generalize our discussion to the problem in a smooth bounded domain in $\mathbb{R}^n$ ($n=2$ or 3). Consider the following problem:

$$\begin{cases} \partial_t u^\epsilon + (u^\epsilon \cdot \nabla)u^\epsilon + \nabla p^\epsilon = \epsilon^{\frac{1}{2}} \partial_x^2 u^\epsilon + \epsilon \partial_y^2 u^\epsilon, \qquad t>0,\, (x,y) \in \mathbb{R}^2_+ \\ \nabla \cdot u^\epsilon = 0, \qquad t>0,\, (x,y) \in \mathbb{R}^2_+ \\ u_2^\epsilon = 0, \quad \beta u_1^\epsilon - \epsilon^{\frac{1}{4}} \frac{\partial u_1^\epsilon}{\partial y} = 0, \quad \text{on} \quad y=0 \\ u^\epsilon|_{t=0} = u_0(x,y), \end{cases} \tag{3.1}$$

where $\beta$ is a positive constant.

For the solutions $(u^\epsilon, p^\epsilon)$ of (3.1), by taking the same ansatz as (2.28) one can formally derive problems for all order profiles in a way similar to that given in section 2.2, and conclude that

CONCLUSION 3.1. *The solutions $(u^\epsilon, p^\epsilon)$ to the problem (3.1) formally have the following asymptotic expansions:*

$$\begin{cases} u_1^\epsilon(t,x,y) = u_1^{I,0}(t,x,y) + \epsilon^{\frac14} u_1^{B,1}(t,x,\frac{y}{\sqrt{\epsilon}}) + \sum\limits_{j\geq 2} \epsilon^{\frac{j}{4}} (u_1^{I,j}(t,x,y) + u_1^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ u_2^\epsilon(t,x,y) = u_2^{I,0}(t,x,y) + \epsilon^{\frac12} u_2^{I,2}(t,x,y) + \sum\limits_{j\geq 3} \epsilon^{\frac{j}{4}} (u_2^{I,j}(t,x,y) + u_2^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ p^\epsilon(t,x,y) = \sum\limits_{j=0}^{4} \epsilon^{\frac{j}{4}} p^{I,j}(t,x,y) + \sum\limits_{j\geq 5} \epsilon^{\frac{j}{4}} (p^{I,j}(t,x,y) + p^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) \end{cases} \tag{3.2}$$

*for rapidly decreasing $(u^{B,j}, p^{B,j})(t,x,z)$ in $z \to +\infty$, where $(u^{I,0}, p^{I,0})$ are solutions to the problem (2.34) for the Euler equations with $u^{I,0}|_{t=0} = u_0(x,y)$, $p^{I,1}$ a constant, for all $j \geq 2$, $(u^{I,j}, p^{I,j})$ are solutions to the following problem for the linearized Euler equations:*

$$\begin{cases} \partial_t u^{I,j} + (u^{I,0}\cdot\nabla)u^{I,j} + (u^{I,j}\cdot\nabla)u^{I,0} + \nabla p^{I,j} \\ \qquad = \partial_x^2 u^{I,j-2} + \partial_y^2 u^{I,j-4} - \sum\limits_{1\leq k\leq j-1} (u^{I,k}\cdot\nabla)u^{I,j-k} \\ \nabla\cdot u^{I,j} = 0 \\ u_2^{I,j}|_{y=0} = -\int_0^{+\infty} \partial_x u_1^{B,j-2}(t,x,\xi)d\xi \\ u^{I,j}|_{t=0} = 0, \end{cases} \tag{3.3}$$

*and for all $j \geq 1$, $u_1^{B,j}$ satisfies the boundary value problem for a linear degenerate parabolic equation:*

$$\begin{cases} \partial_t u_1^{B,j} + \overline{u_1^{I,0}}\partial_x u_1^{B,j} + z\overline{\partial_y u_2^{I,0}}\partial_z u_1^{B,j} + \overline{\partial_x u_1^{I,0}} u_1^{B,j} - \partial_z^2 u_1^{B,j} = f_1^j \\ \partial_z u_1^{B,j} = \beta(\overline{u_1^{I,j-1}} + u_1^{B,j-1}) - \overline{\partial_y u_1^{I,j-2}}, \qquad \text{on } \{z=0\} \\ \lim_{z\to+\infty} u_1^{B,j}(t,x,z) = 0 \quad \text{exponentially} \\ u_1^{B,j}|_{t=0} = 0 \end{cases} \tag{3.4}$$

*where*

$$f_1^j = \partial_x^2 u_1^{B,j-2} - \sum_{k=1}^{j-1}\left(\sum_{n=0}^{[\frac{k}{2}]} \frac{z^n}{n!}\partial_x\left(\overline{\partial_y^n u_1^{I,k-2n}} u_1^{B,j-k}\right) + u_1^{B,k}\partial_x u_1^{B,j-k}\right) - \partial_x p^{B,j}$$
$$- \sum_{k=0}^{j-3}\sum_{n=1}^{[\frac{k}{2}]} \frac{z^n}{n!}\overline{\partial_y^{n+1} u_1^{I,k-2n}} u_2^{B,j-k} - \sum_{k=1}^{j-1}\sum_{n=1}^{[\frac{k}{2}]+1} \frac{z^n}{n!}\overline{\partial_y^n u_2^{I,k+2-2n}}\partial_z u_1^{B,j-k}.$$

*$u_2^{B,j+2}$ is given explicitly by*

$$u_2^{B,j+2}(t,x,z) = \int_z^{+\infty} \partial_x u_1^{B,j}(t,x,\xi)d\xi \tag{3.5}$$

*and for all $j \geq 5$, $p^{B,j}(t,x,z)$ is given by*

$$p^{B,j} = -\int_z^{+\infty} f_2^j(t,x,\xi)d\xi, \tag{3.6}$$

*where*

$$
\begin{aligned}
f_2^j=&\partial_z^2 u_2^{B,j-2}+\partial_x^2 u_2^{B,j-4}-\partial_t u_2^{B,j-2}-\sum_{k=0}^{j-5}\left(\sum_{n=0}^{[\frac{k}{2}]}\frac{z^n}{n!}\overline{\partial_y^n u_1^{I,k-2n}}+u_1^{B,k}\right)\partial_x u_2^{B,j-k-2}\\
&-\sum_{k=2}^{j-3}\sum_{n=0}^{[\frac{k}{2}]}\frac{z^n}{n!}\overline{\partial_x\partial_y^n u_2^{I,k-2n}}u_1^{B,j-k-2}-\sum_{k=0}^{j-5}\sum_{n=0}^{[\frac{k}{2}]}\frac{z^n}{n!}\overline{\partial_y^{n+1}u_2^{I,k-2n}}u_2^{B,j-k-2}\\
&-\sum_{k=2}^{j-3}\left(\sum_{n=0}^{[\frac{k}{2}]}\frac{z^n}{n!}\overline{\partial_y^n u_2^{I,k-2n}}+u_2^{B,k}\right)\partial_z u_2^{B,j-k}.
\end{aligned}
$$

Now, let us justify rigorously the above expansions (3.2).

First as in [4], for the problem (3.1) we have

PROPOSITION 3.2. *Assume that $u_0\in L^2(\Omega)$ with $\nabla\cdot u_0=0$. There exists a global weak solution $u^\epsilon\in C_w^0([0,+\infty),L^2(\Omega))\cap L^2_{loc}([0,+\infty),H^1(\Omega))$ to (3.1), moreover, we have the estimate*

$$
\begin{aligned}
\|u^\epsilon(t)\|_{L^2}^2 &+2\epsilon^{\frac12}\int_0^t\|\partial_x u^\epsilon(\tau)\|_{L^2}^2d\tau+2\epsilon\int_0^t\|\partial_y u^\epsilon(\tau)\|_{L^2}^2d\tau\\
&+2\epsilon^{\frac34}\int_0^t\int_{\partial\Omega}\beta|u_1^\epsilon|^2dxd\tau\le\|u_0\|_{L^2}^2
\end{aligned}
\tag{3.7}
$$

*for all $t\ge 0$, where $\|\cdot\|_{L^2}$ denotes the $L^2-$norm on $\Omega=\{x\in I\!R,y>0\}$ with the boundary $\partial\Omega=\{y=0\}$.*

REMARK 3.3. The estimate (3.7) can easily be obtained by multiplying equation (3.1) by $u^\epsilon$ and integrating over $\Omega$. When $\beta$ is non-negative, from (3.7) the a priori bound follows for $u^\epsilon$ in the space $L^\infty([0,+\infty),L^2(\Omega))\cap L^2_{loc}([0,+\infty),H^1(\Omega))$. As noted in [4], even if $\beta$ is not non-negative (3.7) also implies an a priori bound for $u^\epsilon$ in the space $L^\infty_{loc}([0,+\infty),L^2(\Omega))\cap L^2_{loc}([0,+\infty),H^1(\Omega))$. Indeed, the boundary term in (3.7) can be estimated as

$$
\epsilon^{\frac34}\int_{\partial\Omega}\beta|u_1^\epsilon|^2dx=-\epsilon^{\frac34}\int_\Omega\partial_y(\beta|u_1^\epsilon|^2)dxdy\ge -C\epsilon^{\frac12}\|u^\epsilon(t)\|_{L^2}^2-\frac{\epsilon}{2}\|\partial_y u^\epsilon(t)\|_{L^2}^2.
$$

Substituting this estimate into (3.7), it follows that

$$
\|u^\epsilon(t)\|_{L^2}^2+\int_0^t e^{C\sqrt{\epsilon}(t-\tau)}(2\epsilon^{\frac12}\|\partial_x u^\epsilon(\tau)\|_{L^2}^2+\epsilon\|\partial_y u^\epsilon(\tau)\|_{L^2}^2)d\tau\le e^{C\sqrt{\epsilon}t}\|u_0\|_{L^2}^2 \tag{3.8}
$$

for all $t\ge 0$. The existence of weak solutions to (3.1) can be obtained by using the argument of Theorem 3.1 in [4].

For the asymptotic behavior of the solution $u^\epsilon$ when $\epsilon$ goes to zero, first, similar to [4], we have:

PROPOSITION 3.4. *Assume that $u_0\in H^s(\Omega)$ with $\nabla\cdot u_0=0$ for a fixed $s>2$ and let $u^0\in C([0,T],H^s(\Omega))\cap C^1([0,T],H^{s-1}(\Omega))$ be a unique solution to the initial-boundary problem for the Euler equations (2.34). Then, for the solutions $u^\epsilon$ to (3.1), we have*

$$
\|u^\epsilon-u^0\|_{L^\infty(0,T;L^2(\Omega))}=O(\epsilon^{\frac14}) \tag{3.9}
$$

*when $\epsilon$ goes to zero.*

*Proof.* The proof of this proposition is similar to the one given in [4]. For completeness, let us sketch the main ideas.

Set $\tilde{u}^\epsilon = u^\epsilon - u^0$ and $\tilde{p}^\epsilon = p^\epsilon - p^0$. From (3.1) and (2.34) we know that $(\tilde{u}^\epsilon, \tilde{p}^\epsilon)$ satisfy the following problem:

$$\begin{cases} \partial_t \tilde{u}^\epsilon + (u^\epsilon \cdot \nabla)\tilde{u}^\epsilon + (\tilde{u}^\epsilon \cdot \nabla)u^0 + \nabla \tilde{p}^\epsilon - \epsilon^{\frac{1}{2}} \partial_x^2 u^\epsilon - \epsilon \partial_y^2 u^\epsilon = 0 \\ \nabla \cdot \tilde{u}^\epsilon = 0 \\ \tilde{u}_2^\epsilon|_{y=0} = 0 \\ \tilde{u}^\epsilon|_{t=0} = 0. \end{cases} \tag{3.10}$$

Multiplying the equations in (3.10) by $\tilde{u}^\epsilon$ and integrating in space variables, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}^\epsilon(t)\|_{L^2}^2 + \int_\Omega \tilde{u}^\epsilon \cdot (\tilde{u}^\epsilon \cdot \nabla)u^0 dxdy = \int_\Omega (\epsilon^{\frac{1}{2}} \tilde{u}^\epsilon \cdot \partial_x^2 u^\epsilon + \epsilon \tilde{u}^\epsilon \cdot \partial_y^2 u^\epsilon) dxdy. \tag{3.11}$$

Obviously, we have

$$\int_\Omega \tilde{u}^\epsilon \cdot \partial_x^2 u^\epsilon dxdy = -\int_\Omega \partial_x \tilde{u}^\epsilon \cdot \partial_x u^\epsilon dxdy \le -\frac{1}{2}(\|\partial_x \tilde{u}^\epsilon(t)\|_{L^2}^2 - \|\partial_x u^0(t)\|_{L^2}^2)$$

and

$$\begin{aligned} \textstyle\int_\Omega \tilde{u}^\epsilon \cdot \partial_y^2 u^\epsilon dxdy &= -\textstyle\int_\Omega \partial_y \tilde{u}^\epsilon \cdot \partial_y u^\epsilon dxdy - \epsilon^{-\frac{1}{4}} \int_{\partial\Omega} \beta \tilde{u}_1^\epsilon u_1^\epsilon dx \\ &\le -\tfrac{1}{2}(\|\partial_y \tilde{u}^\epsilon(t)\|_{L^2}^2 - \|\partial_y u^0(t)\|_{L^2}^2) - \epsilon^{-\frac{1}{4}} \textstyle\int_{\partial\Omega} \beta \tilde{u}_1^\epsilon u_1^\epsilon dx \end{aligned}$$

by using $(\beta u_1^\epsilon - \epsilon^{\frac{1}{4}} \frac{\partial u_1^\epsilon}{\partial y})|_{y=0} = 0$.

On the other hand, we have

$$\begin{aligned} |\epsilon^{-\frac{1}{4}} \textstyle\int_{\partial\Omega} \beta \tilde{u}_1^\epsilon u_1^\epsilon dx| &= |\epsilon^{-\frac{1}{4}} \textstyle\int_\Omega \frac{\partial}{\partial y}(\beta \tilde{u}_1^\epsilon u_1^\epsilon) dxdy| \\ &\le C_1 \epsilon^{-\frac{1}{4}} \|\tilde{u}_1^\epsilon(t)\|_{H^1} (\|\tilde{u}_1^\epsilon(t)\|_{L^2} + \|u_1^0(t)\|_{H^1}). \end{aligned}$$

So, from (3.11) we deduce

$$\begin{aligned} &\tfrac{d}{dt}\|\tilde{u}^\epsilon(t)\|_{L^2}^2 + \epsilon^{\frac{1}{2}} \|\partial_x \tilde{u}^\epsilon(t)\|_{L^2}^2 + \epsilon \|\partial_y \tilde{u}^\epsilon(t)\|_{L^2}^2 \\ &\le 2C_1 \epsilon^{\frac{3}{4}} \|\tilde{u}_1^\epsilon(t)\|_{H^1} (\|\tilde{u}_1^\epsilon(t)\|_{L^2} + \|u_1^0(t)\|_{H^1}) + 2\|\nabla u^0(t)\|_{L^\infty} \|\tilde{u}^\epsilon(t)\|_{L^2}^2 + 2\epsilon^{\frac{1}{2}} \|u^0(t)\|_{H^1}^2 \end{aligned}$$

which implies

$$\|\tilde{u}^\epsilon(t)\|_{L^2}^2 \le C_2 \epsilon^{\frac{1}{2}} \int_0^t \|u^0(\tau)\|_{H^1}^2 d\tau \exp(C_3 t + C_3 \int_0^t \|\nabla u^0(\tau)\|_{L^\infty} d\tau) \tag{3.12}$$

for all $0 \le t \le T$.

From (3.12) we immediately conclude the estimate (3.9). ∎

Remark 3.5. If the slip length and horizontal viscosity in the problem (3.1) are generalized as $\epsilon^\gamma$ and $\epsilon^\delta$, with $0 \le \gamma < \frac{1}{2}$ and $\delta > 0$, respectively, then by the same approach as above we can obtain

$$\|u^\epsilon - u^0\|_{L^\infty(0,T;L^2(\Omega))} = O(\epsilon^{\min(\frac{1}{2}-\gamma, \frac{\delta}{2})}) \tag{3.13}$$

under the same assumption as in Proposition 3.3. In particular, the case $\gamma=0$, $\delta=1$ is the one studied by Iftimie and Sueur in [4].

We are going to justify rigorously the asymptotic expansions (3.2). For simplicity, let $\beta$ be a positive constant in (3.1). Suppose that for a fixed $s>18$, $u_0\in H^s(\Omega)$ with $\nabla\cdot u_0=0$ and $u_{0,2}=0$ on $\{y=0\}$. Then, from the problems of profiles given in Conclusion 3.1, it is easy to have

$$\begin{cases} u^{I,k}\in\cap_{j=0}^2 C^j([0,T],H^{s-k-j}(\Omega)), \quad 0\le k\le 10 \\ u_1^{B,k}\in\cap_{j=0}^2 C^j([0,T],H^{s-k-2j-1}(\Omega)), \quad 1\le k\le 10 \\ u_2^{B,k}\in\cap_{j=0}^2 C^j([0,T],H^{s-k-2j}(\Omega)), \quad 3\le k\le 12. \end{cases} \tag{3.14}$$

Denote $(u^{\epsilon,a},p^{\epsilon,a})$ to be

$$\begin{cases} u_1^{\epsilon,a}(t,x,y)=\sum_{k=0}^{10}\epsilon^{\frac{k}{4}}u_1^{I,k}(t,x,y)+\sum_{k=1}^{10}\epsilon^{\frac{k}{4}}u_1^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}}) \\ u_2^{\epsilon,a}(t,x,y)=\sum_{k=0}^{10}\epsilon^{\frac{k}{4}}u_2^{I,k}(t,x,y)+\sum_{k=3}^{12}\epsilon^{\frac{k}{4}}u_2^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}}) \\ p^{\epsilon,a}(t,x,y)=\sum_{k=0}^{10}\epsilon^{\frac{k}{4}}p^{I,k}(t,x,y)+\sum_{k=5}^{10}\epsilon^{\frac{k}{4}}p^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}}) \end{cases} \tag{3.15}$$

the approximate solutions to (3.1), where all profiles are given in Conclusion 3.1, and let the solutions of the problem (3.1) have the expansions:

$$\begin{cases} u^\epsilon(t,x,y)=u^{\epsilon,a}(t,x,y)+\epsilon^{\frac{11}{4}}R^\epsilon(t,x,y) \\ p^\epsilon(t,x,y)=p^{\epsilon,a}(t,x,y)+\epsilon^{\frac{11}{4}}\pi^\epsilon(t,x,y). \end{cases} \tag{3.16}$$

Then, from Conclusion 3.1, we know that $(R^\epsilon,\pi^\epsilon)$ satisfy the following problem:

$$\begin{cases} \partial_t R^\epsilon+(u^\epsilon\cdot\nabla)R^\epsilon+\nabla\pi^\epsilon-(\epsilon^{\frac12}\partial_x^2+\epsilon\partial_y^2)R^\epsilon+(R^\epsilon\cdot\nabla)u^{\epsilon,a}=F^\epsilon \\ \nabla\cdot R^\epsilon=0 \\ R_2^\epsilon|_{y=0}=-(u_2^{B,11}+\epsilon^{\frac14}u_2^{B,12})(t,x,0) \\ \beta R_1^\epsilon-\epsilon^{\frac14}\frac{\partial R_1^\epsilon}{\partial y}=\partial_y(u_1^{I,10}+\epsilon^{-\frac14}u_1^{I,9})-\beta\epsilon^{-\frac14}(u_1^{I,10}+u_1^{B,10}), \quad \text{on } \{y=0\} \\ R^\epsilon|_{t=0}=0, \end{cases} \tag{3.17}$$

where $F^\epsilon$ is bounded in the space $L^\infty(0,T;H_\epsilon^{s-14}(I\!R_+^2))\cap W^{1,\infty}(0,T;H_\epsilon^{s-16}(I\!R_+^2))$ with the norm

$$\|F^\epsilon\|_{L^\infty(0,T;H_\epsilon^s(I\!R_+^2))}=\max_{0\le t\le T}\left\{\sum_{|\alpha|\le s}\|\partial_x^{\alpha_1}(\sqrt{\epsilon}\partial_y)^{\alpha_2}F^\epsilon(t)\|_{L^2(I\!R_+^2)}^2\right\}^{\frac12}.$$

By constructing $\overline{R}^\epsilon\in\cap_{j=0}^2 C^j([0,T],H^{s-12-2j}(I\!R_+^2))$ satisfying

$$\begin{cases} \nabla\cdot\overline{R}^\epsilon=0 \\ \overline{R_2}^\epsilon|_{y=0}=-(u_2^{B,11}+\epsilon^{\frac14}u_2^{B,12})(t,x,0), \end{cases}$$

we know that $\tilde{R}^\epsilon = R^\epsilon - \overline{R^\epsilon}$ satisfies the following problem:

$$\begin{cases} \partial_t R^\epsilon + (u^\epsilon \cdot \nabla) R^\epsilon + \nabla \pi^\epsilon - (\epsilon^{\frac{1}{2}} \partial_x^2 + \epsilon \partial_y^2) R^\epsilon + (R^\epsilon \cdot \nabla) u^{\epsilon,a} = F^\epsilon \\ \nabla \cdot R^\epsilon = 0 \\ R_2^\epsilon = 0, \quad \beta R_1^\epsilon - \epsilon^{\frac{1}{4}} \frac{\partial R_1^\epsilon}{\partial y} = r^\epsilon(t,x), \quad \text{on } \{y=0\} \\ R^\epsilon|_{t\le 0} = 0, \end{cases} \tag{3.18}$$

where we have dropped tilde notation for simplicity, $r^\epsilon = O(\epsilon^{-\frac{1}{4}})$ in $L^\infty(0,T; H_\epsilon^{s-\frac{27}{2}}(I\!R)) \cap W^{1,\infty}(0,T; H_\epsilon^{s-\frac{31}{2}}(I\!R))$, and $F^\epsilon = O(\epsilon^{-\frac{1}{4}})$ in the space $L^\infty(0,T; H_\epsilon^{s-14}(\Omega)) \cap W^{1,\infty}(0,T; H_\epsilon^{s-16}(\Omega))$.

PROPOSITION 3.6. *For the solution $R^\epsilon$ to the problem (3.18), we have the following estimate:*

$$\sup_{0\le t\le T} \|R^\epsilon(t)\|_{L^2}^2 + \int_0^T (\sqrt{\epsilon}\|\partial_x R^\epsilon(t)\|_{L^2}^2 + \epsilon \|\partial_y R^\epsilon(t)\|_{L^2}^2) dt \le C \epsilon^{-\frac{1}{2}}. \tag{3.19}$$

*Proof.* Multiplying the equations in (3.18) by $R^\epsilon$ and integrating on $\Omega$, it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|R^\epsilon(t)\|_{L^2}^2 &+ \int_\Omega R^\epsilon \cdot (R^\epsilon \cdot \nabla) u^{\epsilon,a} dxdy \\ &- \int_\Omega R^\epsilon \cdot (\epsilon^{\frac{1}{2}} \partial_x^2 + \epsilon \partial_y^2) R^\epsilon dxdy = \int_\Omega R^\epsilon \cdot F^\epsilon dxdy. \end{aligned} \tag{3.20}$$

Obviously, by using the boundary conditions given in (3.18), we have

$$\begin{aligned} \int_\Omega R^\epsilon \cdot (\epsilon^{\frac{1}{2}} \partial_x^2 + \epsilon \partial_y^2) R^\epsilon dxdy = &-\epsilon^{\frac{1}{2}} \|\partial_x R^\epsilon(t)\|_{L^2}^2 - \epsilon \|\partial_y R^\epsilon(t)\|_{L^2}^2 \\ &- \epsilon^{\frac{3}{4}} \int_{\partial\Omega} \beta |R_1^\epsilon(t)|^2 dx + \epsilon^{\frac{3}{4}} \int_{\partial\Omega} R_1^\epsilon r^\epsilon dx. \end{aligned} \tag{3.21}$$

It is easy to show that

$$\begin{aligned} &-\epsilon^{\frac{3}{4}} \int_{\partial\Omega} \beta |R_1^\epsilon(t)|^2 dx + \epsilon^{\frac{3}{4}} \int_{\partial\Omega} R_1^\epsilon r^\epsilon dx \\ &\le \frac{\epsilon}{2} \|\partial_y R_1^\epsilon(t)\|_{L^2}^2 + C \|R_1^\epsilon(t)\|_{L^2}^2 + C \epsilon^{\frac{1}{2}} \|r^\epsilon(t)\|_{L^2(\partial\Omega)}^2. \end{aligned} \tag{3.22}$$

On the other hand, by using (3.14), the divergence free part of $R^\epsilon$, and $R_2^\epsilon|_{y=0} = 0$, we obtain

$$\begin{aligned} \left| \int_\Omega R^\epsilon \cdot (R^\epsilon \cdot \nabla) u^{\epsilon,a} dxdy \right| &\le C_1 \|R^\epsilon(t)\|_{L^2}^2 + \left| \epsilon^{-\frac{1}{4}} \int_\Omega R_2^\epsilon R_1^\epsilon \partial_z u_1^{B,1} dxdy \right| \\ &\le C_1 \|R^\epsilon(t)\|_{L^2}^2 + C_2 \epsilon^{\frac{1}{4}} \|R_1^\epsilon(t)\|_{L^2} \|\partial_x R_1^\epsilon\|_{L^2}. \end{aligned} \tag{3.23}$$

Plugging (3.21), (3.22), and (3.23) into (3.20), it follows that

$$\begin{aligned} &\frac{d}{dt}\|R^\epsilon(t)\|_{L^2}^2 + \epsilon^{\frac{1}{2}} \|\partial_x R^\epsilon(t)\|_{L^2}^2 + \epsilon \|\partial_y R^\epsilon(t)\|_{L^2}^2 \\ &\le C(\|R^\epsilon(t)\|_{L^2}^2 + \|F^\epsilon(t)\|_{L^2}^2 + \epsilon^{\frac{1}{2}} \|r^\epsilon(t)\|_{L^2(\partial\Omega)}^2), \end{aligned} \tag{3.24}$$

which implies the estimate (3.19) by using the Gronwall inequality. ∎

PROPOSITION 3.7. *For the problem (3.18), we have the following estimate for* $\partial_t R^\epsilon$*:*

$$\sup_{0\le t\le T}\|\partial_t R^\epsilon(t)\|_{L^2}^2+\int_0^T(\sqrt{\epsilon}\|\partial_x\partial_t R^\epsilon(t)\|_{L^2}^2+\epsilon\|\partial_y\partial_t R^\epsilon(t)\|_{L^2}^2)dt\le C\epsilon^{-\frac{3}{2}}. \tag{3.25}$$

*Proof.* From (3.18), we know that $\partial_t R^\epsilon$ satisfies the following problem

$$\begin{cases}\partial_t(\partial_t R^\epsilon)+(u^\epsilon\cdot\nabla)(\partial_t R^\epsilon)+\nabla(\partial_t\pi^\epsilon)-(\epsilon^{\frac{1}{2}}\partial_x^2+\epsilon\partial_y^2)\partial_t R^\epsilon+((\partial_t R^\epsilon)\cdot\nabla)u^{\epsilon,a}\\ \qquad\qquad+(\partial_t u^\epsilon\cdot\nabla)R^\epsilon+(R^\epsilon\cdot\nabla)\partial_t u^{\epsilon,a}=\partial_t F^\epsilon\\ \nabla\cdot(\partial_t R^\epsilon)=0\\ \partial_t R_2^\epsilon=0,\quad \beta\partial_t R_1^\epsilon-\epsilon^{\frac{1}{4}}\frac{\partial(\partial_t R_1^\epsilon)}{\partial y}=\partial_t r^\epsilon(t,x),\quad \text{on }\{y=0\}\\ \partial_t R^\epsilon|_{t=0}=0.\end{cases} \tag{3.26}$$

Multiplying the equations in (3.26) by $\partial_t R^\epsilon$ and integrating on $\Omega$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\partial_t R^\epsilon(t)\|_{L^2}^2+\int_\Omega\partial_t R^\epsilon\cdot((\partial_t R^\epsilon\cdot\nabla)u^{\epsilon,a}+(\partial_t u^\epsilon\cdot\nabla)R^\epsilon+(R^\epsilon\cdot\nabla)\partial_t u^{\epsilon,a})dxdy$$
$$-\int_\Omega\partial_t R^\epsilon\cdot(\epsilon^{\frac{1}{2}}\partial_x^2+\epsilon\partial_y^2)\partial_t R^\epsilon dxdy=\int_\Omega\partial_t R^\epsilon\cdot\partial_t F^\epsilon dxdy. \tag{3.27}$$

As in (3.21), by using the boundary conditions given in (3.26), we have

$$\int_\Omega\partial_t R^\epsilon\cdot(\epsilon^{\frac{1}{2}}\partial_x^2+\epsilon\partial_y^2)\partial_t R^\epsilon dxdy=-\epsilon^{\frac{1}{2}}\|\partial_x\partial_t R^\epsilon(t)\|_{L^2}^2-\epsilon\|\partial_y\partial_t R^\epsilon(t)\|_{L^2}^2$$
$$-\epsilon^{\frac{3}{4}}\int_{\partial\Omega}\beta|\partial_t R_1^\epsilon(t)|^2dx+\epsilon^{\frac{3}{4}}\int_{\partial\Omega}\partial_t R_1^\epsilon\partial_t r^\epsilon dx. \tag{3.28}$$

and

$$-\epsilon^{\frac{3}{4}}\int_{\partial\Omega}\beta|\partial_t R_1^\epsilon(t)|^2dx+\epsilon^{\frac{3}{4}}\int_{\partial\Omega}\partial_t R_1^\epsilon\partial_t r^\epsilon dx$$
$$\le\frac{\epsilon}{2}\|\partial_y\partial_t R_1^\epsilon(t)\|_{L^2}^2+C\|\partial_t R_1^\epsilon(t)\|_{L^2}^2+C\epsilon^{\frac{1}{2}}\|\partial_t r^\epsilon(t)\|_{L^2(\partial\Omega)}^2. \tag{3.29}$$

By using (3.14), the divergence free of $R^\epsilon$ and $R_2^\epsilon|_{y=0}=\partial_t R_2^\epsilon|_{y=0}=0$, we obtain

$$|\textstyle\int_\Omega\partial_t R^\epsilon\cdot(\partial_t R^\epsilon\cdot\nabla)u^{\epsilon,a}dxdy|\le C_1\|\partial_t R^\epsilon(t)\|_{L^2}^2+|\epsilon^{-\frac{1}{4}}\int_\Omega\partial_t R_2^\epsilon\partial_t R_1^\epsilon\partial_z u_1^{B,1}dxdy|$$
$$\le C_1\|\partial_t R^\epsilon(t)\|_{L^2}^2+C_2\epsilon^{\frac{1}{4}}\|\partial_t R_1^\epsilon(t)\|_{L^2}\|\partial_x\partial_t R_1^\epsilon\|_{L^2} \tag{3.30}$$

and

$$\left|\int_\Omega\partial_t R^\epsilon\cdot(R^\epsilon\cdot\nabla)\partial_t u^{\epsilon,a}dxdy\right|$$
$$\le C_3\|\partial_t R^\epsilon(t)\|_{L^2}\|R^\epsilon(t)\|_{L^2}+|\epsilon^{-\frac{1}{4}}\int_\Omega\partial_t R_1^\epsilon R_2^\epsilon\partial_z\partial_t u_1^{B,1}dxdy|$$
$$\le C_3\|\partial_t R^\epsilon(t)\|_{L^2}\|R^\epsilon(t)\|_{L^2}+C_4\epsilon^{\frac{1}{4}}\|\partial_t R_1^\epsilon(t)\|_{L^2}\|\partial_x R_1^\epsilon\|_{L^2}. \tag{3.31}$$

On the other hand, we have

$$
\begin{aligned}
&\left|\int_\Omega \partial_t R^\epsilon\cdot(\partial_t u^\epsilon\cdot\nabla)R^\epsilon dxdy\right|\\
=&\left|\int_\Omega \partial_t R^\epsilon\cdot(\partial_t(u^{\epsilon,a}+\epsilon^{\frac{11}{4}}R^\epsilon)\cdot\nabla)R^\epsilon dxdy\right|\\
\le& C_5\|\nabla R^\epsilon(t)\|_{L^2}\|\partial_t R^\epsilon(t)\|_{L^2}+\epsilon^{\frac{11}{4}}\|\nabla R^\epsilon\|_{L^2}\|\partial_t R^\epsilon\|_{L^4}^2\\
\le& C_5\|\nabla R^\epsilon(t)\|_{L^2}\|\partial_t R^\epsilon(t)\|_{L^2}+\epsilon^{\frac{11}{4}}\|\nabla R^\epsilon\|_{L^2}\|\partial_t R^\epsilon\|_{L^2}\|\partial_t R^\epsilon\|_{H^1}\\
\le& \frac{\epsilon}{4}\|\nabla\partial_t R^\epsilon\|_{L^2}^2+C_6(\epsilon^{\frac{3}{2}}\|\nabla R^\epsilon\|_{L^2}^2+1)\|\partial_t R^\epsilon\|_{L^2}^2+\|\nabla R^\epsilon\|_{L^2}^2.
\end{aligned}\tag{3.32}
$$

Plugging (3.28)–(3.32) into (3.27), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_t R^\epsilon(t)\|_{L^2}^2+\epsilon^{\frac{1}{2}}\|\partial_x\partial_t R^\epsilon(t)\|_{L^2}^2+\epsilon\|\partial_y\partial_t R^\epsilon(t)\|_{L^2}^2\\
\le& C_7(\epsilon^{\frac{3}{2}}\|\nabla R^\epsilon\|_{L^2}^2+1)\|\partial_t R^\epsilon\|_{L^2}^2+\|\nabla R^\epsilon\|_{L^2}^2+\|R^\epsilon(t)\|_{L^2}^2\\
&+\|\partial_t F^\epsilon(t)\|_{L^2}^2+\epsilon^{\frac{1}{2}}\|\partial_t r^\epsilon(t)\|_{L^2(\partial\Omega)}^2,
\end{aligned}\tag{3.33}
$$

which implies that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\partial_t R^\epsilon(t)\|_{L^2}^2+\int_0^T(\sqrt{\epsilon}\|\partial_x\partial_t R^\epsilon(t)\|_{L^2}^2+\epsilon\|\partial_y\partial_t R^\epsilon(t)\|_{L^2}^2)dt\\
\le&\int_0^T e^{\int_t^T C_7(\epsilon^{\frac{3}{2}}\|\nabla R^\epsilon(s)\|_{L^2}^2+1)ds}(\|\nabla R^\epsilon(t)\|_{L^2}^2+\|R^\epsilon(t)\|_{L^2}^2\\
&+\|\partial_t F^\epsilon(t)\|_{L^2}^2+\epsilon^{\frac{1}{2}}\|\partial_t r^\epsilon(t)\|_{L^2(\partial\Omega)}^2)dt.
\end{aligned}\tag{3.34}
$$

By using (3.19) in (3.34), the estimate (3.25) follows immediately. □

Similarly, by differentiating the problem (3.18) with respect to the $x$-variable, and using the same argument as above, we can conclude

PROPOSITION 3.8. *For the problem (3.18), the following estimate holds for $\partial_x R^\epsilon$:*

$$
\sup_{0\le t\le T}\|\partial_x R^\epsilon(t)\|_{L^2}^2+\int_0^T(\sqrt{\epsilon}\|\partial_x^2 R^\epsilon(t)\|_{L^2}^2+\epsilon\|\partial_y\partial_x R^\epsilon(t)\|_{L^2}^2)dt\le C\epsilon^{-\frac{3}{2}}.\tag{3.35}
$$

Finally, let us study the estimate for $\partial_{tx}^2 R^\epsilon$. For this, we obtain

PROPOSITION 3.9. *For the problem (3.18), we have the following estimate:*

$$
\sup_{0\le t\le T}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2+\int_0^T(\sqrt{\epsilon}\|\partial_x\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2+\epsilon\|\partial_y\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2)dt\le C\epsilon^{-\frac{5}{2}}.\tag{3.36}
$$

*Proof.* From (3.18), we know that $\partial_{tx}^2 R^\epsilon$ satisfies the following problem:

$$\begin{cases} \partial_t(\partial_{tx}^2 R^\epsilon)+(u^\epsilon\cdot\nabla)(\partial_{tx}^2 R^\epsilon)+\nabla(\partial_{tx}^2\pi^\epsilon)-(\epsilon^{\frac12}\partial_x^2+\epsilon\partial_y^2)\partial_{tx}^2 R^\epsilon+((\partial_{tx}^2 R^\epsilon)\cdot\nabla)u^{\epsilon,a} \\ \qquad +(\partial_{tx}^2 u^\epsilon\cdot\nabla)R^\epsilon+(\partial_t u^\epsilon\cdot\nabla)\partial_x R^\epsilon+(\partial_x u^\epsilon\cdot\nabla)\partial_t R^\epsilon \\ \qquad +(R^\epsilon\cdot\nabla)\partial_{tx}^2 u^{\epsilon,a}+(\partial_x R^\epsilon\cdot\nabla)\partial_t u^{\epsilon,a}+(\partial_t R^\epsilon\cdot\nabla)\partial_x u^{\epsilon,a}=\partial_{tx}^2 F^\epsilon \\ \nabla\cdot(\partial_{tx}^2 R^\epsilon)=0 \\ \partial_{tx}^2 R_2^\epsilon=0, \quad \beta\partial_{tx}^2 R_1^\epsilon-\epsilon^{\frac14}\frac{\partial(\partial_{tx}^2 R_1^\epsilon)}{\partial y}=\partial_{tx}^2 r^\epsilon(t,x), \quad \text{on } \{y=0\} \\ \partial_{tx}^2 R^\epsilon|_{t=0}=0. \end{cases} \tag{3.37}$$

Multiplying the equations in (3.37) by $\partial_{tx}^2 R^\epsilon$ and integrating on $\Omega$, it follows that

$$\begin{aligned} &\frac12\frac{d}{dt}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2-\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\epsilon^{\frac12}\partial_x^2+\epsilon\partial_y^2)\partial_{tx}^2 R^\epsilon dxdy \\ =&\int_\Omega \partial_{tx}^2 R^\epsilon\cdot\partial_{tx}^2 F^\epsilon dxdy-\int_\Omega \partial_{tx}^2 R^\epsilon\cdot\big\{(\partial_{tx}^2 R^\epsilon\cdot\nabla)u^{\epsilon,a}+(\partial_t R^\epsilon\cdot\nabla)\partial_x u^{\epsilon,a}+(\partial_x R^\epsilon\cdot\nabla)\partial_t u^{\epsilon,a} \\ &+(R^\epsilon\cdot\nabla)\partial_{tx}^2 u^{\epsilon,a}+(\partial_{tx}^2 u^\epsilon\cdot\nabla)R^\epsilon+(\partial_t u^\epsilon\cdot\nabla)\partial_x R^\epsilon+(\partial_x u^\epsilon\cdot\nabla)\partial_t R^\epsilon\big\}dxdy. \end{aligned} \tag{3.38}$$

As in (3.21), by using the boundary conditions given in (3.37), we have

$$\begin{aligned} \textstyle\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\epsilon^{\frac12}\partial_x^2+\epsilon\partial_y^2)\partial_{tx}^2 R^\epsilon dxdy=&-\epsilon^{\frac12}\|\partial_x\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2-\epsilon\|\partial_y\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2 \\ &-\epsilon^{\frac34}\textstyle\int_{\partial\Omega}\beta|\partial_{tx}^2 R_1^\epsilon(t)|^2dx+\epsilon^{\frac34}\int_{\partial\Omega}\partial_{tx}^2 R_1^\epsilon\partial_{tx}^2 r^\epsilon dx \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} &-\epsilon^{\frac34}\int_{\partial\Omega}\beta|\partial_{tx}^2 R_1^\epsilon(t)|^2dx+\epsilon^{\frac34}\int_{\partial\Omega}\partial_{tx}^2 R_1^\epsilon\partial_{tx}^2 r^\epsilon dx \\ \le&\frac{\epsilon}{2}\|\partial_y\partial_{tx}^2 R_1^\epsilon(t)\|_{L^2}^2+C_1\|\partial_{tx}^2 R_1^\epsilon(t)\|_{L^2}^2+C_1\epsilon^{\frac12}\|\partial_{tx}^2 r^\epsilon(t)\|_{L^2(\partial\Omega)}^2. \end{aligned} \tag{3.40}$$

Now, let us estimate each term in the last integral of (3.38). By using (3.14), the divergence free part of $R^\epsilon$, and $R_2^\epsilon|_{y=0}=0$, we obtain

$$\begin{aligned} |\textstyle\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\partial_{tx}^2 R^\epsilon\cdot\nabla)u^{\epsilon,a}dxdy|\le& C_2\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2+|\epsilon^{-\frac14}\textstyle\int_\Omega \partial_{tx}^2 R_2^\epsilon\partial_{tx}^2 R_1^\epsilon\partial_z u_1^{B,1}dxdy| \\ \le& C_2\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2+C_3\epsilon^{\frac14}\|\partial_{tx}^2 R_1^\epsilon(t)\|_{L^2}\|\partial_x\partial_{tx}^2 R_1^\epsilon\|_{L^2}, \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} &\left|\int_\Omega \partial_{tx}^2 R^\epsilon\cdot((\partial_t R^\epsilon\cdot\nabla)\partial_x u^{\epsilon,a}+(\partial_x R^\epsilon\cdot\nabla)\partial_t u^{\epsilon,a}+(R^\epsilon\cdot\nabla)\partial_{tx}^2 u^{\epsilon,a})dxdy\right| \\ \le& C_4\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}(\|\partial_t R^\epsilon(t)\|_{L^2}+\|\partial_x R^\epsilon(t)\|_{L^2}+\|R^\epsilon(t)\|_{L^2}) \\ &+\epsilon^{-\frac14}\left|\int_\Omega \partial_{tx}^2 R_1^\epsilon(\partial_t R_2^\epsilon\partial_{zx}^2 u_1^{B,1}+\partial_x R_2^\epsilon\partial_{zt}^2 u_1^{B,1}+R_2^\epsilon\partial_z\partial_{tx}^2 u_1^{B,1})dxdy\right| \\ \le& C_4\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}(\|\partial_t R^\epsilon(t)\|_{L^2}+\|\partial_x R^\epsilon(t)\|_{L^2}+\|R^\epsilon(t)\|_{L^2}) \\ &+C_5\epsilon^{\frac14}\|\partial_{tx}^2 R_1^\epsilon(t)\|_{L^2}(\|\partial_{tx}^2 R_1^\epsilon(t)\|_{L^2}+\|\partial_x^2 R_1^\epsilon(t)\|_{L^2}+\|\partial_x R_1^\epsilon(t)\|_{L^2}). \end{aligned} \tag{3.42}$$

On the other hand, we have

$$\begin{aligned}
&\left|\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\partial_{tx}^2 u^\epsilon\cdot\nabla)R^\epsilon dxdy\right| \\
=&\left|\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\partial_{tx}^2 (u^{\epsilon,a}+\epsilon^{\frac{11}{4}}R^\epsilon)\cdot\nabla)R^\epsilon dxdy\right| \\
\le& C_6\|\nabla R^\epsilon(t)\|_{L^2}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}+\epsilon^{\frac{11}{4}}\|\nabla R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{L^4}^2 \\
\le& C_6\|\nabla R^\epsilon(t)\|_{L^2}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}+\epsilon^{\frac{11}{4}}\|\nabla R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{H^1} \\
\le& \frac{\epsilon}{8}\|\nabla\partial_{tx}^2 R^\epsilon\|_{L^2}^2+C_7(\epsilon^{\frac{3}{2}}\|\nabla R^\epsilon\|_{L^2}^2+1)\|\partial_{tx}^2 R^\epsilon\|_{L^2}^2+\|\nabla R^\epsilon\|_{L^2}^2, \qquad (3.43)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\partial_x u^\epsilon\cdot\nabla)\partial_t R^\epsilon dxdy\right| \\
=&\left|\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\partial_x (u^{\epsilon,a}+\epsilon^{\frac{11}{4}}R^\epsilon)\cdot\nabla)\partial_t R^\epsilon dxdy\right| \\
\le& C_8\|\nabla\partial_t R^\epsilon(t)\|_{L^2}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}+\epsilon^{\frac{11}{4}}\|\nabla\partial_t R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{L^4}\|\partial_x R^\epsilon\|_{L^4} \\
\le& C_8\|\nabla\partial_t R^\epsilon(t)\|_{L^2}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2} \\
&\qquad +C_9\epsilon^{\frac{11}{4}}\|\nabla\partial_t R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{L^2}^{\frac{1}{2}}\|\partial_{tx}^2 R^\epsilon\|_{H^1}^{\frac{1}{2}}\|\partial_x R^\epsilon\|_{L^2}^{\frac{1}{2}}\|\partial_x R^\epsilon\|_{H^1}^{\frac{1}{2}} \\
\le& C_8\|\nabla\partial_t R^\epsilon(t)\|_{L^2}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2} \\
&\qquad +C_{10}\epsilon^{\frac{19}{8}}\|\nabla\partial_t R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{L^2}^{\frac{1}{2}}\|\partial_{tx}^2 R^\epsilon\|_{H^1}^{\frac{1}{2}}\|\partial_x R^\epsilon\|_{H^1}^{\frac{1}{2}} \\
\le& \frac{\epsilon}{8}\|\nabla\partial_{tx}^2 R^\epsilon\|_{L^2}^2+C_{11}(\epsilon^{\frac{5}{2}}\|\partial_x R^\epsilon\|_{H^1}^2+1)\|\partial_{tx}^2 R^\epsilon\|_{L^2}^2+\|\nabla\partial_t R^\epsilon\|_{L^2}^2, \qquad (3.44)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\partial_t u^\epsilon\cdot\nabla)\partial_x R^\epsilon dxdy\right| \\
=&\left|\int_\Omega \partial_{tx}^2 R^\epsilon\cdot(\partial_t (u^{\epsilon,a}+\epsilon^{\frac{11}{4}}R^\epsilon)\cdot\nabla)\partial_x R^\epsilon dxdy\right| \\
\le& C_{12}\|\nabla\partial_x R^\epsilon(t)\|_{L^2}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}+\epsilon^{\frac{11}{4}}\|\nabla\partial_x R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{L^4}\|\partial_t R^\epsilon\|_{L^4} \\
\le& C_{12}\|\nabla\partial_x R^\epsilon(t)\|_{L^2}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2} \\
&\qquad +C_{13}\epsilon^{\frac{11}{4}}\|\nabla\partial_x R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{L^2}^{\frac{1}{2}}\|\partial_{tx}^2 R^\epsilon\|_{H^1}^{\frac{1}{2}}\|\partial_t R^\epsilon\|_{L^2}^{\frac{1}{2}}\|\partial_t R^\epsilon\|_{H^1}^{\frac{1}{2}} \\
\le& C_{12}\|\nabla\partial_x R^\epsilon(t)\|_{L^2}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2} \\
&\qquad +C_{14}\epsilon^{\frac{7}{8}}\|\nabla\partial_x R^\epsilon\|_{L^2}\|\partial_{tx}^2 R^\epsilon\|_{L^2}^{\frac{1}{2}}\|\partial_{tx}^2 R^\epsilon\|_{H^1}^{\frac{1}{2}}\|\partial_t R^\epsilon\|_{H^1}^{\frac{1}{2}} \\
\le& \frac{\epsilon}{8}\|\nabla\partial_{tx}^2 R^\epsilon\|_{L^2}^2+C_{15}(\epsilon^{\frac{5}{2}}\|\partial_t R^\epsilon\|_{H^1}^2+1)\|\partial_{tx}^2 R^\epsilon\|_{L^2}^2+\|\nabla\partial_x R^\epsilon\|_{L^2}^2. \qquad (3.45)
\end{aligned}$$

Plugging (3.39)–(3.45) into (3.38), it follows that

$$\frac{d}{dt}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2+\epsilon^{\frac{1}{2}}\|\partial_x\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2+\epsilon\|\partial_y\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2$$

$$\le C_{16}(\epsilon^{\frac{3}{2}}\|\nabla R^\epsilon\|_{L^2}^2+\epsilon^{\frac{5}{2}}\|\nabla_{t,x} R^\epsilon\|_{H^1}^2+1)\|\partial_{tx}^2 R^\epsilon\|_{L^2}^2+\|\nabla R^\epsilon,\partial_t R^\epsilon\|_{L^2}^2+\|\nabla\partial_t R^\epsilon\|_{L^2}^2$$

$$+\|\nabla\partial_x R^\epsilon\|_{L^2}^2+\|R^\epsilon(t)\|_{L^2}^2+\|\partial_{tx}^2 F^\epsilon(t)\|_{L^2}^2+\epsilon^{\frac{1}{2}}\|\partial_{tx}^2 r^\epsilon(t)\|_{L^2(\partial\Omega)}^2, \tag{3.46}$$

which implies that

$$\sup_{0\le t\le T}\|\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2+\int_0^T(\sqrt{\epsilon}\|\partial_x\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2+\epsilon\|\partial_y\partial_{tx}^2 R^\epsilon(t)\|_{L^2}^2)dt$$

$$\le\int_0^T e^{\int_t^T C_{16}(\epsilon^{\frac{3}{2}}\|\nabla R^\epsilon(s)\|_{L^2}^2+\epsilon^{\frac{5}{2}}\|\nabla_{t,x} R^\epsilon(s)\|_{H^1}^2+1)ds}(\|\nabla R^\epsilon(t),\partial_t R^\epsilon(t)\|_{L^2}^2+\|\nabla\partial_t R^\epsilon(t)\|_{L^2}^2$$

$$+\|\nabla\partial_x R^\epsilon(t)\|_{L^2}^2+\|R^\epsilon(t)\|_{L^2}^2+\|\partial_{tx}^2 F^\epsilon(t)\|_{L^2}^2+\epsilon^{\frac{1}{2}}\|\partial_{tx}^2 r^\epsilon(t)\|_{L^2(\partial\Omega)}^2)dt. \tag{3.47}$$

By using (3.19), (3.25), and (3.35) in (3.47), the estimate (3.36) follows immediately. ∎

Finally, by combining the estimates given in Propositions 3,6, 3.7, 3.8, and 3.9, with the classical Sobolev embedding theorem,

$$\|R^\epsilon\|_{L^\infty}^2\le C(\|R^\epsilon\|_{L^2}^2+\|\partial_t R^\epsilon,\nabla R^\epsilon\|_{L^2}^2+\|\partial_{tx}^2 R^\epsilon,\partial_{ty}^2 R^\epsilon,\partial_{xy}^2 R^\epsilon\|_{L^2}^2+\|\partial_t\partial_{xy}^2 R^\epsilon\|_{L^2}^2),$$

and with $\|\cdot\|_{L^2}$ being the norm in $L^2([0,T]\times I\!R_+^2)$, we obtain

Proposition 3.10. *For the solution $R^\epsilon$ to the problem (3.18), we have the estimate*

$$\|R^\epsilon\|_{L^\infty((0,T)\times I\!R_+^2)}\le C\epsilon^{-\frac{7}{4}}. \tag{3.48}$$

Thus, from the representation of solutions given in (3.15), we conclude:

Theorem 3.11. *For the problem (3.1), suppose that the initial data $u_0$ of (3.1) belongs to $H^s(\Omega)$ for a fixed $s>18$, with $\nabla\cdot u_0=0$ and $u_{0,2}=0$ on $\{y=0\}$. Then, in $L^\infty((0,T)\times I\!R_+^2)$ we have*

$$\begin{cases} u_1^\epsilon(t,x,y)=u_1^{I,0}(t,x,y)+\sum\limits_{j=1}^{3}\epsilon^{\frac{j}{4}}(u_1^{I,j}(t,x,y)+u_1^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}}))+O(\epsilon)\\ u_2^\epsilon(t,x,y)=\sum\limits_{j=0}^{3}\epsilon^{\frac{j}{4}}u_2^{I,j}(t,x,y)+\epsilon^{\frac{3}{4}}u_2^{B,3}(t,x,\frac{y}{\sqrt{\epsilon}})+O(\epsilon)\\ p^\epsilon(t,x,y)=\sum\limits_{j=0}^{3}\epsilon^{\frac{j}{4}}p^{I,j}(t,x,y)+O(\epsilon)\end{cases} \tag{3.49}$$

*with all profiles being given in Conclusion 3.1.*

Remark 3.12. For any fixed $J\ge 4$, under certain compatibility and regularity assumptions on initial data one can similarly conclude

$$\begin{cases} u_1^\epsilon(t,x,y)=u_1^{I,0}(t,x,y)+\sum\limits_{j=1}^{J}\epsilon^{\frac{j}{4}}(u_1^{I,j}(t,x,y)+u_1^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}}))+O(\epsilon^{\frac{J+1}{4}})\\ u_2^\epsilon(t,x,y)=\sum\limits_{j=0}^{2}\epsilon^{\frac{j}{4}}u_2^{I,j}(t,x,y)+\sum\limits_{j=3}^{J}\epsilon^{\frac{j}{4}}(u_2^{I,j}(t,x,y)+u_2^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}}))+O(\epsilon^{\frac{J+1}{4}})\\ p^\epsilon(t,x,y)=\sum\limits_{j=0}^{4}\epsilon^{\frac{j}{4}}p^{I,j}(t,x,y)+\sum\limits_{j=5}^{J}\epsilon^{\frac{j}{4}}(p^{I,j}(t,x,y)+p^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}}))+O(\epsilon^{\frac{J+1}{4}})\end{cases} \tag{3.50}$$

in $L^\infty((0,T)\times \mathbb{R}^2_+)$.

## 4. Remarks on some general cases

In this section, we are going to sketch the main idea for generalizing the above discussion to the case that the slip length $\epsilon^{\frac14}$ in the problem (2.2) is replaced by $\epsilon^\gamma$ for a fixed $0<\gamma<\frac12$.

### 4.1. The case $\epsilon^\gamma$ for a rational number $0<\gamma<\frac12$.

Consider the following problem in $\{t,y>0,x\in\mathbb{R}\}$:

$$\begin{cases}\partial_t u^\epsilon + (u^\epsilon\cdot\nabla)u^\epsilon + \nabla p^\epsilon = \epsilon^{1-2\gamma}\partial_x^2 u^\epsilon + \epsilon\partial_y^2 u^\epsilon\\ \nabla\cdot u^\epsilon = 0\\ u_2^\epsilon|_{y=0}=0\\ (\beta u_1^\epsilon - \epsilon^\gamma\frac{\partial u_1^\epsilon}{\partial y})|_{y=0}=0\\ u^\epsilon|_{t=0}=u_0(x,y),\end{cases}\tag{4.1}$$

for a rational number $0<\gamma<\frac12$.

Let $r>0$ be such that $a=\frac{\gamma}{r}$ and $b=\frac{1}{2r}$ are co-prime integers. Take the following ansatz for the solutions to (4.1):

$$\begin{cases}u^\epsilon(t,x,y)=\sum\limits_{j\ge0}\epsilon^{rj}(u^{I,j}(t,x,y)+u^{B,j}(t,x,\frac{y}{\sqrt\epsilon}))\\ p^\epsilon(t,x,y)=\sum\limits_{j\ge0}\epsilon^{rj}(p^{I,j}(t,x,y)+p^{B,j}(t,x,\frac{y}{\sqrt\epsilon})),\end{cases}\tag{4.2}$$

where $u^{B,j}(t,x,z)$ and $p^{B,j}(t,x,z)$ are rapidly decreasing when $z=\frac{y}{\sqrt\epsilon}\to+\infty$.

Plugging (4.2) into $(4.1)_2$, it follows that

$$\begin{cases}\nabla\cdot u^{I,j}=0, \qquad \forall j\ge0\\ \partial_z u_2^{B,j}=0, \qquad \forall j\le b-1\\ \partial_x u_1^{B,j}+\partial_z u_2^{B,j+b}=0, \qquad \forall j\ge 0\end{cases}\tag{4.3}$$

which implies that

$$u_2^{B,j}\equiv0,\quad u_2^{I,j}|_{y=0}=0,\qquad \forall j\le b-1.\tag{4.4}$$

Plugging (4.2) into $(4.1)_1$ and $(4.1)_4$ respectively, it follows that

$$\begin{aligned}&\sum_{j\ge0}\epsilon^{rj}\partial_t(u^{I,j}+u^{B,j})+\sum_{j\ge0}\epsilon^{rj}\sum_{k=0}^{j}((u^{I,k}+u^{B,k})\cdot\nabla u^{I,j-k}+(u_1^{I,k}+u_1^{B,k})\partial_x u^{B,j-k})\\ &+\sum_{j=0}^{b-1}\epsilon^{rj-\frac12}\sum_{k=0}^{j}(u_2^{I,k}+u_2^{B,k})\partial_z u^{B,j-k}+\sum_{j\ge0}\epsilon^{rj}\sum_{k=0}^{j+b}(u_2^{I,k}+u_2^{B,k})\partial_z u^{B,j+b-k}\\ &+\sum_{j\ge0}\epsilon^{rj}\nabla p^{I,j}+\sum_{j=0}^{b-1}\epsilon^{rj-\frac12}\begin{pmatrix}0\\ \partial_z p^{B,j}\end{pmatrix}+\sum_{j\ge0}\epsilon^{rj}\begin{pmatrix}\partial_x p^{B,j}\\ \partial_z p^{B,j+b}\end{pmatrix}\\ =&\sum_{k=0}^{2b-1}\epsilon^{rk}\partial_z^2u^{B,k}+\sum_{j\ge0}\epsilon^{1+r(j-2a)}\partial_x^2(u^{I,j}+u^{B,j})+\sum_{j\ge0}\epsilon^{1+rj}(\partial_y^2u^{I,j}+\partial_z^2u^{B,j+2b})\end{aligned}\tag{4.5}$$

and

$$\beta\sum_{j\geq 0}\epsilon^{rj}(u_1^{I,j}+u_1^{B,j})=\epsilon^{ar}\left\{\sum_{j\geq 0}\epsilon^{rj}\partial_y u_1^{I,j}+\sum_{j\geq 0}\epsilon^{rj-\frac{1}{2}}\partial_z u_1^{B,j}\right\} \tag{4.6}$$

on $\{y=z=0\}$.

Set

$$u^{p,j}(t,x,z)=u^{B,j}(t,x,z)+\sum_{k=0}^{[j/b]}\frac{z^k}{k!}\overline{\partial_y^k u^{I,j-bk}}. \tag{4.7}$$

From (4.6) we immediately obtain

$$\begin{cases}\partial_z u_1^{B,j}|_{z=0}=0, \quad \forall j\leq b-a-1\\ (\partial_z u_1^{p,j}-\beta u_1^{p,j-b+a})|_{z=0}=0, \quad \forall j\geq b-a.\end{cases} \tag{4.8}$$

From (4.5) and (4.8), we can deduce that

$$u_1^{B,j}(t,x,z)\equiv 0, \quad \forall j\leq b-a-1 \tag{4.9}$$

which implies

$$u_2^{B,j}(t,x,z)\equiv 0, \quad \forall j\leq 2b-a-1 \tag{4.10}$$

from $(4.3)_3$.

The vanishing of the order $O(\epsilon^{r(b-a)})$ terms in the first component of (4.5) implies that $u_1^{p,b-a}(t,x,z)$ satisfies the following problem for a linear degenerate parabolic equation:

$$\begin{cases}\partial_t u_1^{p,b-a}+\overline{u_1^{I,0}}\partial_x u_1^{p,b-a}+u_1^{p,b-a}\overline{\partial_x u_1^{I,0}}+z\overline{\partial_y u_2^{I,0}}\partial_z u_1^{p,b-a}\\ \qquad\qquad +\sum_{k=1}^{b-a-1}\overline{u_1^{I,k}\partial_x u_1^{I,b-a-k}}+\overline{\partial_x p^{I,b-a}}=\partial_z^2 u_1^{p,b-a}\\ \partial_z u_1^{p,b-a}|_{z=0}=\beta\overline{u_1^{I,0}}\\ \lim_{z\to+\infty}u_1^{p,b-a}(t,x,z)=\overline{u_1^{I,b-a}}(t,x) \qquad \text{exponentially.}\end{cases} \tag{4.11}$$

From (4.3), we immediately get that $u_2^{p,2b-a}(t,x,z)$ satisfies

$$\begin{cases}\partial_z u_2^{p,2b-a}+\partial_x u_1^{p,b-a}=0\\ u_2^{p,2b-a}|_{z=0}=0.\end{cases} \tag{4.12}$$

The vanishing of the order $O(\epsilon^{r(2b-a)})$ terms in the second component of (4.5) gives rise to an equation for determining $p^{B,3b-a}(t,x,z)$ by using $u_1^{B,b-a}$ and $u_2^{B,2b-a}$.

Similarly, for any $j\geq b-a+1$, from the $O(\epsilon^{rj})-$order terms of the first component of (4.5) we deduce an equation for solving $u_1^{p,j}$. Then, we can determine $u_2^{p,j+b}$ from $(4.3)_3$. The vanishing of the $O(\epsilon^{r(j+b)})-$order terms of the second component of (4.5) gives an explicit formula for determining $p^{B,j+2b}(t,x,z)$ by using $\{u_1^{B,k}\}_{k\leq j}$ and $\{u_2^{B,k}\}_{k\leq j+b}$.

In the same way as in the proof of Theorem 3.11, for the problem (4.1), we can conclude

THEOREM 4.1. *For the problem (4.1), under certain assumptions on the regularity and compatibility conditions on the initial data $u_0$, the solutions to (4.1) have the following expansions:*

$$\begin{cases} u_1^\epsilon(t,x,y) = \sum_{k=0}^{b-a-1} \epsilon^{rk} u_1^{I,k}(t,x,y) + \sum_{j=b-a}^{J} \epsilon^{rj}(u_1^{I,j}(t,x,y) + u_1^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) + o(\epsilon^{rJ}) \\ u_2^\epsilon(t,x,y) = \sum_{k=0}^{2b-a-1} \epsilon^{rk} u_2^{I,k}(t,x,y) + \sum_{j\geq 2b-a}^{J} \epsilon^{rj}(u_2^{I,j}(t,x,y) + u_2^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) + o(\epsilon^{rJ}) \\ p^\epsilon(t,x,y) = \sum_{k=0}^{3b-a-1} \epsilon^{rk} p^{I,k}(t,x,y) + \sum_{j\geq 3b-a}^{J} \epsilon^{rj}(p^{I,j}(t,x,y) + p^{B,j}(t,x,\frac{y}{\sqrt{\epsilon}})) + o(\epsilon^{rJ}) \end{cases} \tag{4.13}$$

*in $L^\infty((0,T)\times \mathbb{R}_+^2)$ for a fixed $J\geq 1$. Moreover, the first components of the boundary layer profiles*

$$u^{p,j}(t,x,z) = u^{B,j}(t,x,z) + \sum_{k=0}^{[j/b]} \frac{z^k}{k!} \overline{\partial_y^k u^{I,j-bk}}$$

*satisfy linear degenerate parabolic problems similar to (4.11) when $j\geq b-a$, the second components are explicitly given by the problem (4.12) for all $j\geq 2b-a$, and for all $j\geq 3b-a$, $p^{B,j}(t,x,z)$ are given explicitly by $\{u_1^{p,k}(t,x,z)\}_{k\leq j-2b}$ and $\{u_2^{p,k}(t,x,z)\}_{k\leq j-b}$ as in (3.6).*

**4.2. The case $\epsilon^\gamma$ for a irrational number $0<\gamma<\frac{1}{2}$.** Consider the following problem in $\{t,y>0,x\in\mathbb{R}\}$:

$$\begin{cases} \partial_t u^\epsilon + (u^\epsilon\cdot\nabla)u^\epsilon + \nabla p^\epsilon = \epsilon^{1-2\gamma}\partial_x^2 u^\epsilon + \epsilon\partial_y^2 u^\epsilon \\ \nabla\cdot u^\epsilon = 0 \\ u_2^\epsilon|_{y=0} = 0 \\ (\beta u_1^\epsilon - \epsilon^\gamma \frac{\partial u_1^\epsilon}{\partial y})|_{y=0} = 0 \\ u^\epsilon|_{t=0} = u_0(x,y), \end{cases} \tag{4.14}$$

for a irrational number $0<\gamma<\frac{1}{2}$.

For the solutions to the problem (4.14), we take the following ansatz:

$$\begin{cases} u^\epsilon(t,x,y) = u^0(t,x,y) + \sum_{k\geq 1} \epsilon^{\frac{k}{2}}(v^{I,k}(t,x,y) + v^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ \qquad + \sum_{k\geq 1}\epsilon^{(\frac{1}{2}-\gamma)k}[(u^{I,k}(t,x,y) + u^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ \qquad + \sum_{j\geq 1}\epsilon^{\frac{j}{2}}(w^{I,k,j}(t,x,y) + w^{B,k,j}(t,x,\frac{y}{\sqrt{\epsilon}}))] \\ p^\epsilon(t,x,y) = p^0(t,x,y) + \sum_{k\geq 1}\epsilon^{\frac{k}{2}}(p_v^{I,k}(t,x,y) + p_v^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ \qquad + \sum_{k\geq 1}\epsilon^{(\frac{1}{2}-\gamma)k}[(p_u^{I,k}(t,x,y) + p_u^{B,k}(t,x,\frac{y}{\sqrt{\epsilon}})) \\ \qquad + \sum_{j\geq 1}\epsilon^{\frac{j}{2}}(p_w^{I,k,j}(t,x,y) + p_w^{B,k,j}(t,x,\frac{y}{\sqrt{\epsilon}}))], \end{cases} \tag{4.15}$$

where all profiles with the index "B" decay exponentially when $z=\frac{y}{\sqrt{\epsilon}}\to+\infty$.

From the following discussion, one can see the motivation for taking the ansatz (4.15) is that first the term $u^{B,1}(t,x,\frac{y}{\sqrt{\epsilon}})$ is to cancel $\beta u_1^0$ in the Navier boundary condition, secondly, the term $v^{B,1}(t,x,\frac{y}{\sqrt{\epsilon}})$ is to cancel $\partial_y u_1^0$ in the Navier boundary condition, thirdly, the term $w^{B,1,1}(t,x,\frac{y}{\sqrt{\epsilon}})$ is for satisfying the divergence free condition associated with $u^{B,1}(t,x,\frac{y}{\sqrt{\epsilon}})$, fourthly, the term $p_w^{B,1,1}(t,x,\frac{y}{\sqrt{\epsilon}})$ in the pressure is to to cancel the second component of the equations for $w^{B,1,1}(t,x,\frac{y}{\sqrt{\epsilon}})$, and finally all higher order terms come from the nonlinear interaction in the Navier-Stokes equations.

Now, we are going to describe how to determine each order profile in (4.15) step by step.

First, by plugging (4.15) into the divergence free relation, we get that the vanishing of the order $O(\epsilon^{(\frac{1}{2}-\gamma)k-\frac{1}{2}})$ terms for each $k\geq 1$ implies that

$$\partial_z u_2^{B,k}=0, \tag{4.16}$$

yielding

$$u_2^{B,k}(t,x,z)\equiv 0, \qquad \text{for all } k\geq 1. \tag{4.17}$$

Therefore, the vanishing of the order $O(\epsilon^{(\frac{1}{2}-\gamma)k})$ terms for the second component of the equations in $(4.14)_1$ gives

$$\partial_z p_w^{B,k,1}=0, \tag{4.18}$$

which implies that

$$p_w^{B,k,1}(t,x,z)\equiv 0, \qquad \text{for all } k\geq 1. \tag{4.19}$$

Obviously, the vanishing of the order $O(\epsilon^{(\frac{1}{2}-\gamma)k-\frac{1}{2}})$ terms for the second component of the equations in $(4.14)_1$ gives

$$\partial_z p_u^{B,k}=0, \tag{4.20}$$

which implies that

$$p_u^{B,k}(t,x,z)\equiv 0, \qquad \text{for all } k\geq 1. \tag{4.21}$$

The divergence free condition implies that

$$\begin{cases} \mathrm{div}u^0=\mathrm{div}v^{I,k}=\mathrm{div}u^{I,k}=\mathrm{div}w^{I,k,j}=0 \quad (\forall k,j\geq 1)\\ \partial_z v_2^{B,1}=0\\ \partial_x v_1^{B,k}+\partial_z v_2^{B,k+1}=0 \quad (\forall k\geq 1)\\ \partial_x u_1^{B,k}+\partial_z w_2^{B,k,1}=0 \quad (\forall k\geq 1)\\ \partial_x w_1^{B,k,j}+\partial_z w_2^{B,k,j+1}=0 \quad (\forall k,j\geq 1), \end{cases} \tag{4.22}$$

which leads to

$$\begin{cases} v_2^{B,1} \equiv 0 \\ v_2^{B,k+1}(t,x,z) = \int_z^\infty \partial_x v_1^{B,k}(t,x,\xi)d\xi \quad (\forall k \geq 1) \\ w_2^{B,k,1}(t,x,z) = \int_z^\infty \partial_x u_1^{B,k}(t,x,\xi)d\xi \quad (\forall k \geq 1) \\ w_2^{B,k,j+1}(t,x,z) = \int_z^\infty \partial_x w_1^{B,k,j}(t,x,\xi)d\xi \quad (\forall k,j \geq 1). \end{cases} \tag{4.23}$$

By a direct computation, we have

$$\begin{aligned} &\beta u_1^\epsilon - \epsilon^\gamma \partial_y u_1^\epsilon \\ =& \beta u_1^0 - \partial_z u_1^{B,1} - \epsilon^\gamma(\partial_y u_1^0 + \partial_z v_1^{B,1}) \\ &+ \sum_{k\geq 1} \epsilon^{(\frac{1}{2}-\gamma)k}(\beta(u_1^{I,k} + u_1^{B,k}) - \partial_z u_1^{B,k+1}) \\ &+ \epsilon^{\frac{1}{2}}(\beta(v_1^{I,1} + v_1^{B,1}) - \partial_y u_1^{I,1} - \partial_z w_1^{B,1,1}) + \sum_{k\geq 1} \epsilon^{\gamma+\frac{k}{2}}(\partial_y v_1^{I,k} + \partial_z v_1^{B,k+1}) \\ &+ \sum_{k\geq 2} \epsilon^{\frac{k}{2}}(\beta(v_1^{I,k} + v_1^{B,k}) - \partial_y w_1^{I,1,k-1} - \partial_z w_1^{B,1,k}) \\ &+ \sum_{k\geq 1} \epsilon^{(\frac{1}{2}-\gamma)k+\frac{1}{2}}(\beta(w_1^{I,k,1} + w_1^{B,k,1}) - \partial_y u_1^{I,k+1} - \partial_z w_1^{B,k+1,1}) \\ &+ \sum_{k\geq 1}\sum_{j\geq 2} \epsilon^{(\frac{1}{2}-\gamma)k+\frac{j}{2}}(\beta(w_1^{I,k,j} + w_1^{B,k,j}) - \partial_y w_1^{I,k+1,j-1} - \partial_z w_1^{B,k+1,j}). \end{aligned} \tag{4.24}$$

So, the Navier boundary condition implies that

$$\begin{cases} \partial_z u_1^{B,1}|_{z=0} = \beta u_1^0|_{y=0} \\ \partial_z v_1^{B,1}|_{z=0} = -\partial_y u_1^0|_{y=0} \\ \partial_z w_1^{B,1,1}|_{z=0} = (\beta(v_1^{I,1} + v_1^{B,1}) - \partial_y u_1^{I,1})|_{y=z=0} \\ \partial_z u_1^{B,k}|_{z=0} = \beta(u_1^{I,k-1} + u_1^{B,k-1})|_{y=z=0} \qquad (\forall k \geq 2) \\ \partial_z v_1^{B,k}|_{z=0} = -\partial_y v_1^{I,k-1}|_{y=0} \qquad (\forall k \geq 2) \\ \partial_z w_1^{B,1,k}|_{z=0} = (\beta(v_1^{I,k} + v_1^{B,k}) - \partial_y w_1^{I,1,k-1})|_{y=z=0} \qquad (\forall k \geq 2) \\ \partial_z w_1^{B,k,1}|_{z=0} = (\beta(w_1^{I,k-1,1} + w_1^{B,k-1,1}) - \partial_y u_1^{I,k})|_{y=z=0} \qquad (\forall k \geq 2) \\ \partial_z w_1^{B,k,j}|_{z=0} = (\beta(w_1^{I,k-1,j} + w_1^{B,k-1,j}) - \partial_y w_1^{I,k,j-1})|_{y=z=0} \qquad (\forall k,j \geq 2). \end{cases} \tag{4.25}$$

Obviously, the boundary condition $u_2^\epsilon|_{y=0} = 0$ implies that

$$\begin{cases} u_2^0|_{y=0} = 0 \\ v_2^{I,k}|_{y=0} = -v_2^{B,k}|_{z=0} \qquad (\forall k \geq 1) \\ u_2^{I,k}|_{y=0} = -u_2^{B,k}|_{z=0} = 0 \qquad (\forall k \geq 1) \\ w_2^{I,k,j}|_{y=0} = -w_2^{B,k,j}|_{z=0} \qquad (\forall k,j \geq 1). \end{cases} \tag{4.26}$$

Now, we determine each order profile in the expansions (4.15) as the following steps.

(1) First, we solve $(u^0,p^0)$ from the Euler equations (2.7).

(2) Secondly, the vanishing of the order $O(\epsilon^{\frac{1}{2}-\gamma})$ terms in the first component of the equations $(4.14)_1$ gives that $u_1^{B,1}(t,x,z)$ satisfies a linear degenerate parabolic equation with the boundary condition $(4.25)_1$, which has a unique solution $u_1^{B,1}(t,x,z)$ which decays quickly when $z\to+\infty$. The observation (4.17) gives that $u_2^{I,1}=0$ on $\{y=0\}$, so we can determine $(u^{I,1},p_u^{I,1})$ by solving the linearized Euler equations. From the divergence free condition $(4.22)_4$ with $k=1$, we get $w_2^{B,1,1}(t,x,z)$, from which one can determine $p_w^{B,1,2}$ by using the vanishing of the order $O(\epsilon^{1-\gamma})$ terms in the second component of the equations $(4.14)_1$. This also gives the boundary condition of $w_2^{I,1,1}(t,x,z)$, which can be used to determine $(w^{I,1,1},p_w^{I,1,1})$ by solving the linearized Euler equations.

(3) Thirdly, the vanishing of the order $O(\epsilon^{\frac{1}{2}})$ terms in the first component of the equations $(4.14)_1$ gives that $v_1^{B,1}(t,x,z)$ satisfies a linear degenerate parabolic equation with the boundary condition $(4.25)_2$, which has a unique solution $v_1^{B,1}(t,x,z)$ which decays quickly when $z\to+\infty$. From $(4.23)_1$ and the second component of the equations $(4.14)_1$ at the order $O(\epsilon^{\frac{1}{2}})$, we get $p_v^{B,1}\equiv 0$. The fact $(4.23)_1$ also gives that $v_2^{I,1}=0$ on $\{y=0\}$, so we can determine $(v^{I,1},p_v^{I,1})$ by solving the linearized Euler equations. From the divergence free condition $(4.22)_3$ with $k=1$, we get $v_2^{B,2}(t,x,z)$, from which one can determine $p_v^{B,3}$ by using the vanishing of the order $O(\epsilon)$ terms in the second component of the equations $(4.14)_1$. This also gives the boundary condition of $v_2^{I,2}(t,x,z)$, which can be used to determine $(v^{I,2},p_v^{I,2})$ by solving the linearized Euler equations.

(4) Fourthly, the vanishing of the order $O(\epsilon^{1-2\gamma})$ terms in the first component of the equations $(4.14)_1$ gives that $u_1^{B,2}(t,x,z)$ satisfies a linear degenerate parabolic equation with the boundary condition $(4.25)_4$ with $k=2$, which has a unique solution $u_1^{B,2}(t,x,z)$ which decays quickly when $z\to+\infty$. From the divergence free condition $(4.22)_4$ with $k=2$, we get $w_2^{B,2,1}(t,x,z)$, from which one can determine $p_w^{B,2,2}$ by using the vanishing of the order $O(\epsilon^{\frac{3}{2}-2\gamma})$ terms in the second component of the equations $(4.14)_1$. This also gives the boundary condition of $w_2^{I,2,1}(t,x,z)$, which can be used to determine $(w^{I,2,1},p_w^{I,2,1})$ by solving the linearized Euler equations.

(5) Fifthly, the vanishing of the order $O(\epsilon^{1-\gamma})$ terms in the first component of the equations $(4.14)_1$ gives that $w_1^{B,1,1}(t,x,z)$ satisfies a linear degenerate parabolic equation with the boundary condition $(4.25)_3$, which has a unique solution $w_1^{B,1,1}(t,x,z)$ which decays quickly when $z\to+\infty$. From the divergence free condition $(4.22)_5$ with $k=j=1$, we get $w_2^{B,1,2}(t,x,z)$, from which one can determine $p_w^{B,1,3}$ by using the vanishing of the order $O(\epsilon^{\frac{3}{2}-\gamma})$ terms in the second component of the equations $(4.14)_1$. This also gives the boundary condition of $w_2^{I,1,2}(t,x,z)$, which can be used to determine $(w^{I,1,2},p_w^{I,1,2})$ by solving the linearized Euler equations.

(6) Similarly, for any fixed $k\geq 2$, the vanishing of the order $O(\epsilon^{\frac{k}{2}})$ terms in the first component of the equations $(4.14)_1$ gives that $v_1^{B,k}(t,x,z)$ satisfies a linear

degenerate parabolic equation with the boundary condition $(4.25)_5$, which has a unique solution $v_1^{B,k}(t,x,z)$ which decays quickly when $z\to+\infty$. From the divergence free condition $(4.22)_3$, we get $v_2^{B,k+1}(t,x,z)$, from which one can determine $p_v^{B,k+2}$ by using the vanishing of the order $O(\epsilon^{\frac{k+1}{2}})$ terms in the second component of the equations $(4.14)_1$. This also gives the boundary condition of $v_2^{I,k+1}(t,x,z)$, which can be used to determine $(v^{I,k+1},p_v^{I,k+1})$ by solving the linearized Euler equations. The observation (4.17) gives that $u_2^{I,k}=0$ on $\{y=0\}$, so we can determine $(u^{I,k},p_u^{I,k})$ by solving the linearized Euler equations.

(7) The vanishing of the order $O(\epsilon^{(\frac{1}{2}-\gamma)k})$ terms in the first component of the equations $(4.14)_1$ gives that $u_1^{B,k}(t,x,z)$ satisfies a linear degenerate parabolic equation with the boundary condition $(4.25)_4$, which has a unique solution $u_1^{B,k}(t,x,z)$ which decays quickly when $z\to+\infty$. From the divergence free condition $(4.22)_4$, we get $w_2^{B,k,1}(t,x,z)$, from which one can determine $p_w^{B,k,2}$ by using the vanishing of the order $O(\epsilon^{\frac{k+1}{2}-\gamma k})$ terms in the second component of the equations $(4.14)_1$. This also gives the boundary condition of $w_2^{I,k,1}(t,x,z)$, which can be used to determine $(w^{I,k,1},p_w^{I,k,1})$ by solving the linearized Euler equations.

(8) The vanishing of the order $O(\epsilon^{(\frac{1}{2}-\gamma)k+\frac{1}{2}})$ terms in the first component of the equations $(4.14)_1$ gives that $w_1^{B,k,1}(t,x,z)$ satisfies a linear degenerate parabolic equation with the boundary condition $(4.25)_7$, which has a unique solution $w_1^{B,k,1}(t,x,z)$ which decays quickly when $z\to+\infty$. From the divergence free condition $(4.22)_5$, we get $w_2^{B,k,2}(t,x,z)$, from which one can determine $p_w^{B,k,3}$ by using the vanishing of the order $O(\epsilon^{(\frac{1}{2}-\gamma)k+1})$ terms in the second component of the equations $(4.14)_1$. This also gives the boundary condition of $w_2^{I,k,2}(t,x,z)$, which can be used to determine $(w^{I,k,2},p_w^{I,k,2})$ by solving the linearized Euler equations.

(9) For any fixed $k,j\geq 2$, the vanishing of the order $O(\epsilon^{(\frac{1}{2}-\gamma)k+\frac{j}{2}})$ terms in the first component of the equations $(4.14)_1$ gives that $w_1^{B,k,j}(t,x,z)$ satisfies a linear degenerate parabolic equation with the boundary condition $(4.25)_8$, which has a unique solution $w_1^{B,k,j}(t,x,z)$ which decays quickly when $z\to+\infty$. From the divergence free condition $(4.22)_5$, we get $w_2^{B,k,j+1}(t,x,z)$, from which one can determine $p_w^{B,k,j+2}$ by using the vanishing of the order $O(\epsilon^{(\frac{1}{2}-\gamma)k+\frac{j+1}{2}})$ terms in the second component of the equations $(4.14)_1$. This also gives the boundary condition of $w_2^{I,k,j+1}(t,x,z)$, which can be used to determine $(w^{I,k,j+1},p_w^{I,k,j+1})$ by solving the linearized Euler equations.

Remark 4.2.

(1) As in Theorem 4.1, one can justify rigorously the above formal analysis in a way similar to section 3.

(2) Now, we give a remark on the choice of the $x-$directional viscosity coefficient $\epsilon^{1-2\gamma}$ given in (4.14). If we study the expansions (4.15) rigorously as in section 3 by introducing an approximate solution up to certain order and investigate the problem of the remainders $(R^\epsilon,\pi^\epsilon)$ as before, then $(R^\epsilon,\pi^\epsilon)$ satisfy a problem similar to (3.17). Doing energy estimates for $R^\epsilon$ as in Proposition 3.6, we need to study term

$$\int_\Omega R^\epsilon\cdot(R^\epsilon\cdot\nabla)u^{\epsilon,a}dxdy,$$

in which the leading term can be estimated as

$$\left|\int_\Omega \epsilon^{-\gamma} R_1^\epsilon R_2^\epsilon \partial_z u_1^{B,1} dxdy\right| \leq C\epsilon^{\frac{1}{2}-\gamma} \|\partial_x R^\epsilon(t)\|_{L^2} \|R^\epsilon(t)\|_{L^2} \tag{4.27}$$

On the other hand, if in the problem (4.14), the $x-$directional viscosity coefficient equals to $\epsilon^\alpha$ for a fixed $\alpha>0$, then the viscous term in the problem of $R^\epsilon$ gives an estimate on

$$\epsilon^\alpha \|\partial_x R^\epsilon(t)\|_{L^2}^2$$

which can control the term given in (4.27) if and only if

$$0<\alpha\leq 1-2\gamma. \tag{4.28}$$

To make the above formal analysis work, we further require that $\alpha$ equals to $\frac{k}{2}$, $(\frac{1}{2}-\gamma)k$, or $(\frac{1}{2}-\gamma)k+\frac{j}{2}$ for certain $k,j\geq 1$. Therefore, the possibilities for $\alpha$ are

$$\alpha=\frac{1}{2}-\gamma, \quad 1-2\gamma, \quad \text{or } \frac{1}{2} \quad (\text{when } \gamma\leq\frac{1}{4}). \tag{4.29}$$

So, for simplicity and consistency with the discussion in section 3, we have set $\alpha=1-2\gamma$ in (4.14).

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