ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE FULL COMPRESSIBLE NAVIER-STOKES EQUATIONS IN THE HALF SPACE*

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Abstract. The one-dimensional motion of compressible viscous and heat-conductive fluid is investigated in the half space. By examining the sign of fluid velocity prescribed on the boundary, initial boundary value problems with Dirichlet type boundary conditions are classified into three cases: impermeable wall problem, inflow problem and outflow problem. In this paper, the asymptotic stability of the rarefaction wave, boundary layer solution, and their combination is established for both the impermeable wall problem and the inflow problem under some smallness conditions. The proof is given by an elementary energy method.

Key words. Asymptotic behavior of solutions, Navier-Stokes equations, boundary layer, rarefaction wave.

AMS subject classifications. 35L65.

1. Introduction

The one-dimensional full compressible Navier-Stokes equations in Eulerian coordinate is given by

$$
\begin{cases}
\tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = 0, \\
(\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p})_{\tilde{x}} = \mu \tilde{u}_{\tilde{x}\tilde{x}}, \\
\left(\tilde{\rho}\left(\tilde{e} + \dfrac{\tilde{u}^2}{2} \right) \right)_t + \left(\tilde{\rho}\tilde{u}\left(\tilde{e} + \dfrac{\tilde{u}^2}{2} \right) + \tilde{p}\tilde{u} \right)_{\tilde{x}} = \kappa \tilde{\theta}_{\tilde{x}\tilde{x}} + (\mu \tilde{u}\tilde{u}_{\tilde{x}})_{\tilde{x}},
\end{cases}
\tag{1.1}
$$

where $\tilde{u}(\tilde{x},t)$ denotes the velocity, $\tilde{\rho}(\tilde{x},t) > 0$ the density, $\tilde{\theta}(\tilde{x},t) > 0$ the absolute temperature, $\mu > 0$ the viscosity constant, and $\kappa > 0$ the coefficient of heat conduction. The pressure $\tilde{p}$, the internal energy $\tilde{e}$, and the entropy $\tilde{s}$ are functions of $\tilde{\rho}$ and $\tilde{\theta}$ or functions of $\tilde{v}$ and $\tilde{\theta}$, where $\tilde{v} = \frac{1}{\tilde{\rho}}$ is the specific volume. Since there are only two independent variables, $\tilde{p}$ and $\tilde{e}$ are also regarded as functions of $\tilde{v}$ and $\tilde{s}$. We denote

$$
\tilde{p} = \tilde{p}(\tilde{v},\tilde{\theta}) = \hat{p}(\tilde{v},\tilde{s}), \tilde{e} = \tilde{e}(\tilde{v},\tilde{\theta}) = \hat{e}(\tilde{v},\tilde{s}).
$$

We assume that

$$
\begin{aligned}
&\tilde{p}_{\tilde{v}} < 0, \tilde{e}_{\tilde{\theta}} > 0, \hat{p}_{\tilde{v}\tilde{v}} > 0, \\
&\hat{p}(\tilde{v},\tilde{s}) \text{ is a convex function of } (\tilde{v},\tilde{s}),
\end{aligned}
\tag{1.2}
$$

which together with the thermodynamic relation

$$
d\tilde{e} = \tilde{\theta} d\tilde{s} - \tilde{p} d\tilde{v}
\tag{1.3}
$$

ensures the inviscid system of (1.1) is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. It is noted that the ideal polytropic gas satisfies this assumption (1.2).

We consider the motion of a gas in the half space $\tilde{x} > 0$. Our initial data is

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta})(\tilde{x}, 0) = (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)(\tilde{x}) \rightarrow (\rho_+, u_+, \theta_+), \; as \; \tilde{x} \rightarrow \infty. \tag{1.4}$$

As pointed out by [8], boundary conditions of Dirichet type for (1.1) are classified into three cases:

Case 1. (zero velocity on the boundary):

$$\tilde{u}(0, t) = u_- = 0, \tilde{\theta}(0, t) = \theta_-, t > 0, \tag{$1.5)_1$}$$

Case 2. (negative velocity on the boundary):

$$\tilde{u}(0, t)) = u_- < 0, \tilde{\theta}(0, t) = \theta_-, t > 0, \tag{$1.5)_2$}$$

Case 3. (positive velocity on the boundary):

$$\tilde{u}(0, t) = u_- > 0, \tilde{\rho}(0, t) = \rho_-, \tilde{\theta}(0, t) = \theta_-, t > 0. \tag{$1.5)_3$}$$

We assume that the initial data satisfies one of (1.5) as compatibility condition. It is noted that in cases 1 and 2, the density on the boundary can not be imposed, but in case 3 it has to be imposed so that the first equation of (1.1) is well posed. The initial boundary value problems $(1.1) - (1.5)_1$, $(1.1) - (1.5)_2$, and $(1.1) - (1.5)_3$ are called the impermeable wall problem, outflow problem, and inflow problem respectively.

There have been many works on the asymptotic behavior of solutions to the Cauchy problem of the system (1.1). We refer to [3-4, 9-11] and the references therein. All these results showed that the large time behavior of the solutions are described by the corresponding Riemann solutions to the hyperbolic part of (1.1). Recently, the initial boundary value problem has attracted interest because it is more physical than the Cauchy problem. We refer to [1-2, 7-8, 12-16]. Most of these results are concerned with isentropic gas, where only the conservation of mass and momentum of (1.1) are considered. It was found that a new wave, boundary layer solution, appears in the solutions due to the boundary effect. To be more physical, we should consider the full compressible Navier-Stokes equation (1.1). Comparing with isentropic gas, the boundary conditions for the full compressible Navier-Stokes system become more complicated and analytically it is more difficult to control the terms coming from the boundary. More recently, Kawashima and Zhu [16] investigated the existence of the boundary layer solution and its asymptotic stability for the outflow problem of (1.1) in which the sign of the velocity u_- on the boundary is negative, which plays a key role in the a priori estimate. In this paper we focus our attention on the impermeable wall problem and inflow problem. We shall investigate the asymptotic stability of the rarefaction wave, the boundary layer solution and their combination. Unlike the outflow problem, the velocity u_- on the boundary for the inflow problem is positive which is not good for the analysis, we have to estimate more boundary terms than those of [16].

Let us turn to the original problem. To handle the system (1.1) more easily, we transform (1.1) into the Lagrangian coordinate (x, t) for the impermeable wall

problem and inflow problem, i.e.,

$$\begin{cases} v_t - u_x = 0, & x > k_- t, t > 0, \\ u_t + p(v,\theta)_x = \mu \left(\dfrac{u_x}{v} \right)_x, & x > k_- t, t > 0, \\ \left(e(v,\theta) + \dfrac{u^2}{2} \right)_t + \big(p(v,\theta) u \big)_x = \left(\kappa \dfrac{\theta_x}{v} + \mu \dfrac{u u_x}{v} \right)_x, & x > k_- t, t > 0, \end{cases} \tag{1.6}$$

where $\rho(x,t) = \tilde{\rho}(\tilde{x},t)$, etc., $\hat{p} = p(v,s)$, etc., $k_- = -\frac{u_-}{v_-}$, and the boundary condition $(1.5)_1$ and $(1.5)_3$ become, respectively:

Impermeable wall problem:

$$u(0,t) = u_- = 0, \theta(0,t) = \theta_-, t > 0, \tag{1.7}_1$$

Inflow problem:

$$u(k_- t, t) = u_- > 0, v(k_- t, t) = v_-, \theta(k_- t, t) = \theta_-, t > 0, \tag{1.7}_2$$

and the initial data is

$$(v,u,\theta)(x,0) = (v_0, u_0, \theta_0)(x) \rightarrow (v_+, u_+, \theta_+) \text{ as } x \rightarrow \infty. \tag{1.8}$$

Before stating our main results, we recall some properties of p, which is useful in the following. The second law of thermodynamics (1.3) yields if $p = p(v,\theta), e = e(v,\theta), s = s(v,\theta)$,

$$e_v = -(p - \theta p_\theta), \quad s_v = p_\theta, \quad s_\theta = \dfrac{e_\theta}{\theta}, \tag{1.9}$$

and if $p = \hat{p}(v,s), \theta = \hat{\theta}(v,s), e = \hat{e}(v,s)$,

$$\hat{e}_v = -p, \ \hat{e}_s = \theta, \ \hat{p}_v = p_v - \dfrac{\theta p_\theta^2}{e_\theta}, \ \hat{p}_s = \dfrac{\theta p_\theta}{e_\theta}, \ \hat{\theta}_v = -\dfrac{\theta p_\theta}{e_\theta}, \quad \hat{\theta}_s = \dfrac{\theta}{e_\theta}. \tag{1.10}$$

We now formulate our main results. It is known that the characteristic speeds of hyperbolic part of (1.6) are

$$\lambda_1 = -\sqrt{-\hat{p}_v(v,s)}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{-\hat{p}_v(v,s)}, \tag{1.11}$$

and the sound speed $c(v,s)$ is defined by

$$c(v,s) = v\sqrt{-\hat{p}_v(v,s)}. \tag{1.12}$$

We divide the half space $(v,u,\theta), u > 0$ into three regions:

$$\begin{aligned} \Omega_{\text{sub}} &= \big\{ (v,u,\theta) \big| u < c(v,s), v > 0, u > 0, \theta > 0 \big\}, \\ \Gamma_{\text{trans}} &= \big\{ (v,u,\theta) \big| u = c(v,s), v > 0, u > 0, \theta > 0 \big\}, \\ \Omega_{\text{super}} &= \big\{ (v,u,\theta) \big| u > c(v,s), v > 0, u > 0, \theta > 0 \big\}. \end{aligned} \tag{1.13}$$

We call them the subsonic, transonic, and supersonic regions, respectively. For the inflow problem, we mainly consider the situation when $(v_+, u_+, \theta_+) \in \Omega_{sub}$. If the left state (v_-, u_-, θ_-) is sufficiently close to (v_+, u_+, θ_+) such that $(v_-, u_-, \theta_-) \in \Omega_{sub}$ also

holds, then the first characteristic speed $\lambda_1(v_-,u_-,\theta_-)$ is less than k_- due to (1.13). Thus a travelling wave solution

$$(V_B,U_B,\Theta_B)(\xi),\xi=x-k_-t,$$
$$(V_B,U_B,\Theta_B)(0)=(v_-,u_-,\theta_-), \quad (V_B,U_B,\Theta_B)(+\infty)=(v_+,u_+,\theta_+)$$

is expected. Substituting this into (1.6), we have

$$\begin{cases} -k_-V_B'-U_B'=0, \quad '=\dfrac{d}{d\xi}, \quad \xi=x-k_-t>0, \\[2mm] -k_-U_B'+p(V_B,\Theta_B)'=\mu\left(\dfrac{U_B'}{V_B}\right)', \\[2mm] -k_-\left(e(V_B,\Theta_B)+\dfrac{U_B^2}{2}\right)'+\big(p(V_B,\Theta_B)U_B\big)' \\[2mm] =\left(\kappa\dfrac{\Theta_B'}{V_B}+\mu\dfrac{U_BU_B'}{V_B}\right)', \\[2mm] (V_B,U_B,\Theta_B)(0)=(v_-,u_-,\theta_-), \\[2mm] (V_B,U_B,\Theta_B)(+\infty)=(v_+,u_+,\theta_+). \end{cases} \qquad (1.14)$$

We call the solution (V_B,U_B,Θ_B) the boundary layer solution, or BL-solution simply. If (V_B,U_B,Θ_B) exists, integration of (1.14) over $(\xi,+\infty)$ yields

$$\begin{cases} -k_-(V_B-v_+)-(U_B-u_+)=0, \\[2mm] -k_-(U_B-u_+)+(p(V_B,\Theta_B)-p(v_+,\theta_+))=\mu\dfrac{U_B'}{V_B}, \\[2mm] -k_-\left(e(V_B,\Theta_B)-e(v_+,\theta_+)+\dfrac{U_B^2-u_+^2}{2}\right) \\[2mm] +\big(p(V_B,\Theta_B)U_B-p(v_+,\theta_+)u_+\big)=\left(\kappa\dfrac{\Theta_B'}{V_B}+\mu\dfrac{U_BU_B'}{V_B}\right). \end{cases} \qquad (1.15)$$

Let $\xi=0$ in (1.15), then the first equation yields

$$k_-=-\frac{U_B(\xi)}{V_B(\xi)}=-\frac{u_-}{v_-}=-\frac{u_+}{v_+}. \qquad (1.16)$$

Substituting (1.16) into $(1.15)_2$ and $(1.15)_3$, we have

$$\begin{cases} V_B'=-\dfrac{V_B}{\mu k_-}\left[k_-^2(V_B-v_+)+\big(p(V_B,\Theta_B)-p(v_+,\theta_+)\big)\right] \\[2mm] \quad :=f(V_B,\Theta_B), \\[2mm] \Theta_B'=\dfrac{V_B}{\kappa}\big[-k_-\big(e(V_B,\Theta_B)-e(v_+,\theta_+)\big) \\[2mm] \quad -k_-p(v_+,\theta_+)(V_B-v_+)+\dfrac{k_-^3}{2}(V_B-v_+)^2\big] \\[2mm] \quad :=g(V_B,\Theta_B). \end{cases} \qquad (1.17)$$

Let

$$W_B = \begin{pmatrix} V_B \\ \Theta_B \end{pmatrix}, F(W_B) = \begin{pmatrix} f(V_B,\Theta_B) \\ g(V_B,\Theta_B) \end{pmatrix},$$

$$W^- = \begin{pmatrix} v_- \\ \theta_- \end{pmatrix}, W^+ = \begin{pmatrix} v_+ \\ \theta_+ \end{pmatrix}, \tag{1.18}$$

and consider the system below

$$\begin{cases} W_B' = F(W_B),\ F(W^+) = 0, \\ W_B(0) = W^-,\ W_B(+\infty) = W^+. \end{cases} \tag{1.19}$$

The Jacobian matrix of (1.19) at $\xi = +\infty$ is

$$J_+ = \begin{pmatrix} -\dfrac{v_+}{\mu k_-}\left(k_-^2 + p_v(v_+,\theta_+)\right) & -\dfrac{v_+}{\mu k_-}p_\theta(v_+,\theta_+) \\ -\dfrac{k_- v_+}{\kappa}\left(e_v(v_+,\theta_+) + p(v_+,\theta_+)\right) & -\dfrac{k_- v_+}{\kappa}e_\theta(v_+,\theta_+) \end{pmatrix}. \tag{1.20}$$

Since $(v_+,u_+,\theta_+) \in \Omega_{sub}$, (1.9), (1.10), and (1.13) yield

$$\det J_+ = \frac{v_+^2}{\mu\kappa}\left[(k_-^2 + p_v)e_\theta - (e_v + p)p_\theta\right]\big|_{(v_+,\theta_+)}$$

$$= \frac{v_+^2}{\mu\kappa}\left[(k_-^2 + \hat{p}_v)e_\theta\right]\big|_{(v_+,\theta_+)} < 0. \tag{1.21}$$

Thus J_+ has two eigenvalues $\Lambda_1 < 0 < \Lambda_2$. On the contrary, by [5] and (1.16) there exists a local (unique) one dimensional stable manifold $M_+ = M_+(v_+,u_+,\theta_+)$ around (v_+,u_+,θ_+) such that for any $(v_-,u_-,\theta_-) \in M_+$ there exists a unique solution to (1.14) satisfying

$$(|V_B - v_+|,|U_B - u_+|,|\Theta_B - \theta_+|)(\xi) = O(\delta)e^{-c\xi}, \tag{1.22}$$

$$(|V_B'|,|V_B''|,|U_B'|,|U_B''|,|\Theta_B'|,|\Theta_B''|)(\xi) = O(\delta)e^{-c\xi}, \tag{1.23}$$

where $\delta = |v_- - v_+| + |u_- - u_+| + |\theta_- - \theta_+|$ and the positive constant c is independent of δ.

On the other hand, for any given right state (v_+,u_+,θ_+), the i-rarefaction wave curve $(i = 1,3)$ through (v_+,u_+,θ_+) is known as

$$R_1(v_+,u_+,\theta_+) = \left\{(v,u,\theta)\big|s = s_+, u = u_+ - \int_{v_+}^v \lambda_1(\eta,s_+)d\eta, v < v_+\right\}, \tag{1.24}$$

$$R_3(v_+,u_+,\theta_+) = \left\{(v,u,\theta)\big|s = s_+, u = u_+ - \int_{v_+}^v \lambda_3(\eta,s_+)d\eta, v > v_+\right\}, \tag{1.25}$$

where $s_+ = s(v_+,\theta_+)$. When $(v_+,u_+,\theta_+) \in \Omega_{sub}$, we define

$$BLR_3(v_+,u_+,\theta_+) = \bigcup M_+(v_*,u_*,\theta_*),$$

$$(v_*,u_*,\theta_*) \in R_3(v_+,u_+,\theta_+) \cap \Omega_{sub}. \tag{1.26}$$

If $(v_-,u_-,\theta_-) \in BLR_3(v_+,u_+,\theta_+)$, there exists a unique point $(v_*,u_*,\theta_*) \in \Omega_{sub}$ such that $(v_*,u_*,\theta_*) \in R_3(v_+,u_+,\theta_+)$ and $(v_-,u_-,\theta_-) \in M_+(v_*,u_*,\theta_*)$. In fact, by (1.16), we have $u_* = -k_- v_*$. Substituting into (1.25) yields that

$$-k_- v_* = u_+ - \int_{v_+}^{v_*} \lambda_3(\eta, s_+) d\eta. \qquad (1.27)$$

To the contrary, it is easy to see that v_* is uniquely determined by (1.27) due to the fact that $-k_- v$ is increasing and the right hand side is decreasing with respect to $v > v_+$. After v_* is obtained, we obtain $\theta_* = \theta(v_*, s_+)$.

When $(v_-,u_-,\theta_-) \in \Omega_{super}$, then $0 > \lambda(v_-,u_-,\theta_-) > k_-$. Thus the behavior of the solutions is expected to be the same as that for the Cauchy problem. We define

$$R_1 R_3(v_+,u_+,\theta_+) = \bigcup R_1(v_*,u_*,\theta_*), \quad (v_*,u_*,\theta_*) \in R_3(v_+,u_+,\theta_+). \qquad (1.28)$$

Our aim is to investigate the stability of the rarefaction wave, the BL-solution and their combination for the impermeable wall problem and inflow problem. Our results are, roughly speaking, as follows:

(1) **Impermeable problem**:

For any given right state (v_+,u_+,θ_+) with $u_+ > 0$, there exist unique v_- and θ_- such that $(v_-,0,\theta_-) \in R_3(v_+,u_+,\theta_+)$ and the 3-rarefaction wave connecting $(v_-,0,\theta_-)$ and (v_+,u_+,θ_+) is stable provided that $|v_+ - v_-| + u_+ + |\theta_+ - \theta_-|$ is small.

(2) **Inflow problem**:

(I) For any $(v_+,u_+,\theta_+) \in \Omega_{sub}$, if $(v_-,u_-,\theta_-) \in M_+(v_+,u_+,\theta_+)$, then the BL-solution, connecting (v_-,u_-,θ_-) and (v_+,u_+,θ_+) is stable provided that $|v_+ - v_-| + |u_+ - u_-| + |\theta_+ - \theta_-|$ is small.

(II) For any $(v_+,u_+,\theta_+) \in \Omega_{sub}$, if $(v_-,u_-,\theta_-) \in BLR_3(v_+,u_+,\theta_+)$ then there exists a unique point $(v_*,u_*,\theta_*) \in R_3(v_+, u_+,\theta_+)$ such that the superposition of the BL-solution connecting (v_-,u_-,θ_-) with (v_*,u_*,θ_*) and the 3-rarefaction wave connecting (v_*,u_*,θ_*) with (v_+,u_+,θ_+) is stable provided that $|v_* - v_-| + |u_* - u_-| + |\theta_* - \theta_-|$ and $|v_+ - v_*| + |u_+ - u_*| + |\theta_+ - \theta_*|$ are small.

(III) If $(v_-,u_-,\theta_-) \in R_1 R_3(v_+,u_+,\theta_+) \cap \Omega_{super}$, then there exists a unique point $(v_*,u_*,\theta_*) \in R_3(v_+, u_+,\theta_+)$ such that the superposition of the 1-rarefaction wave connecting (v_-,u_-,θ_-) with (v_*,u_*,θ_*) and the 3-rarefaction wave connecting (v_*,u_*,θ_*) with (v_+,u_+,θ_+) is stable provided that $|v_* - v_-| + |u_* - u_-| + |\theta_* - \theta_-|$ and $|v_+ - v_*| + |u_+ - u_*| + |\theta_+ - \theta_*|$ are small.

Our plan of this paper is as follows. In section 2, we discuss the impermeable wall problem. In section 3 the inflow problem is treated.

Notations. Throughout this paper, several positive generic constants are denoted by C without confusions. For function spaces, $H^l(\Omega)$ denotes the l-th order Sobolev space with its norm

$$\|f\|_l = \left(\sum_{j=0}^{l} \|\partial_x^j f\|^2 \right)^{\frac{1}{2}}, \quad \text{when } \|\cdot\| := \|\cdot\|_{L^2(\Omega)}. \qquad (1.29)$$

The domain Ω will be often abbreviated without confusion.

2. Impermeable wall problem

For any right state (v_+, u_+, θ_+) with $u_+ > 0$, it is obvious that there exist unique $v_- > 0$ and $\theta_- > 0$ such that $(v_-, 0, \theta_-) \in R_3(v_+, u_+, \theta_+)$. The 3-rarefaction wave $(v^r, u^r, \theta^r)(\frac{x}{t})$ connecting $(v_-, 0, \theta_-)$ and (v_+, u_+, θ_+) is the weak solution of the Riemann problem

$$
\begin{cases}
v_t - u_x = 0, \\
u_t + p(v, \theta)_x = 0, \\
(e(v, \theta) + \dfrac{u^2}{2})_t + (p(v, \theta)u)_x = 0, \\
(v_0, u_0, \theta_0)(x) = \begin{cases} (v_-, 0, \theta_-), \, x < 0, \\ (v_+, u_+, \theta_+), \, x > 0. \end{cases}
\end{cases}
\tag{2.1}
$$

To study the large time behavior of the solutions, it is necessary to construct a smooth approximate rarefaction wave $(V, U, \Theta)(x, t)$ of $(v^r, u^r, \theta^r)(\frac{x}{t})$ in $\mathbb{R}_+ \times (0, +\infty)$. To this end, we apply the idea of [1]. We first construct the solution $w(x, t)$ of the following problem

$$
\begin{cases}
w_t + w w_x = 0, \quad (x, t) \in \mathbb{R} \times (0, +\infty), \\
w|_{t=0} = \begin{cases} w_-, & x < 0, \\ w_- + \tilde{w} K_q \displaystyle\int_0^{\varepsilon x} z^q e^{-z} dz, & x \geq 0, \end{cases}
\end{cases}
\tag{2.2}
$$

where $w_\pm = \lambda_3(v_\pm, s_+)$, $s_+ = s(v_+, \theta_+)$, $\tilde{w} = w_+ - w_-$, K_q is a constant such that $K_q \int_0^{+\infty} z^q e^{-z} dz = 1$ for large constant $q \geq 8$ and a small positive constant ε. We have the following properties of $w(x, t)$ due to [1].

LEMMA 2.1. [1] *Let $0 < w_- < w_+$. Then the problem (2.2) has a unique smooth solution $w(x, t)$ satisfying*

  i. $w_- \leq w(x, t) < w_+$, $w_x \geq 0$, *for $x \geq 0, t \geq 0$.*

  ii. *For any $p(1 \leq p \leq +\infty)$, there exists a constant $C_{p,q}$ such that for $t \geq 0$,*

$$
\|w_x(\cdot, t)\|_{L^p} \leq C_{p,q} \min\left(\tilde{w} \varepsilon^{1-\frac{1}{p}}, \tilde{w}^{\frac{1}{p}} t^{-1+\frac{1}{p}} \right),
$$
$$
\|w_{xx}(\cdot, t)\|_{L^p} \leq C_{p,q} \min\left(\tilde{w} \varepsilon^{2-\frac{1}{p}}, \tilde{w}^{\frac{1}{q}} \varepsilon^{1-\frac{1}{p}+\frac{1}{q}} t^{-1+\frac{1}{q}} \right).
$$

  iii. *when $x \leq 0$, $w(x, t) - w_- = w_x(x, t) = w_{xx}(x, t) = 0$.*

  iv. $\limsup_{t \to +\infty, x \in \Re} |w(x, t) - w^R(x, t)| = 0$.

Here $w^R(x, t)$ is the Riemann solution of the scalar equation (2.2) with the initial data $w_0(x) = w_-$, if $x < 0$, and $w_0(x) = w_+$, if $x > 0$.

Then the smooth approximation (V^r, U^r, Θ^r) to $(v^r, u^r, \theta^r)(x, t)$ is given by

$$
\begin{cases}
S^r(x, t) = s_+, \\
\lambda_3\big(V^r(x, t), s_+\big) = w(x, t), \\
U^r(x, t) = u_+ - \displaystyle\int_{v_+}^{V^r(x,t)} \lambda_3\big(\eta, s_+\big) d\eta.
\end{cases}
\tag{2.3}
$$

Setting

$$
(V, U, \Theta)(x, t) := (V^r, U^r, \Theta^r)(x, t)\big|_{x \geq 0},
\tag{2.4}
$$

then one easily has

$$\begin{cases} V_t - U_x = 0, & x > 0, t > 0, \\ U_t + p(V,\Theta)_x = 0, & x > 0, t > 0, \\ \left(e(V,\Theta) + \frac{1}{2}U^2\right)_t + (p(V,\Theta)U)_x = 0, & x > 0, t > 0, \\ (V,U,\Theta)\big|_{x=0} = (v_-,0,\theta_-), \\ (V,U,\Theta)\big|_{t=0} = (V_0,U_0,\Theta_0)(x) = (V,U,\Theta)(x,0). \end{cases} \qquad (2.5)$$

Due to Lemma 2.1, (V,U,Θ) has the following properties.

LEMMA 2.2.   *Let $\delta_1 = |v_+ - v_-| + u_+ + |\theta_+ - \theta_-|$. Then $(V,U,\Theta)(x,t)$ satisfies, if $q \geq p$,*

    i. $U_x(x,t) \geq 0, |U_x| \leq C\varepsilon\delta_1, \ for \ t \geq 0, x \geq 0,$

    ii.   *For any $p(1 \leq p \leq +\infty)$, there exists a constant $C_{p,q}$ such that*

$$\|(V_x,U_x,\Theta_x)\|_{L^p(x \geq 0)} \leq C_{p,q}\min\{\delta_1\varepsilon^{1-\frac{1}{p}}, \delta_1^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\},$$

$$\|(V_{xx},U_{xx},\Theta_{xx})\|_{L^p(x \geq 0)} \leq C_{p,q}\min\{\delta_1\varepsilon^{2-\frac{1}{p}}, \delta_1^{\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\}, \quad t \geq 0.$$

    iii. $(V,U,\Theta)|_{x \leq 0} = (v_-,0,\theta_-), \quad (V_x,U_x,\Theta_x,V_{xx},U_{xx},\Theta_{xx})|_{x \leq 0} = 0.$

    iv. $\displaystyle\limsup_{t \to +\infty, x \in \{x \geq 0\}} |(V,U,\Theta)(x,t) - (v^r,u^r,\theta^r)\left(\frac{x}{t}\right)| = 0.$

We assume

$$(v_0 - V_0, u_0 - U_0, \theta_0 - \Theta_0) \in L^2(\mathbb{R}_+), \quad (v_0,u_0,\theta_0)_x \in L^2(\mathbb{R}_+), \qquad (2.6)$$

and set

$$N_0 = \|v_0 - V_0\|_1 + \|u_0 - U_0\|_1 + \|\theta_0 - \Theta_0\|_1. \qquad (2.7)$$

It is noted that on the boundary $x = 0$, $(V,U,\Theta)(0,t) = (v_-,0,\theta_-)$ and $(u - U, \theta - \Theta)(0,t) = (0,0)$. That is, the boundary effect is exactly eliminated by our approximation because $(v - V)(0,t)$ is not important in the energy estimates. Thus the impermeable wall problem is similar to the Cauchy problem for the rarefaction wave case. Using the same lines as in [4], we have

THEOREM 2.3.   *For any right state (v_+,u_+,θ_+) with $u_+ > 0$, there exist unique v_- and θ_- such that $(v_-,0,\theta_-) \in R_3(v_+,u_+,\theta_+)$. Suppose that the Assumption (2.6) holds. Then there exists a positive constant δ_0 such that if $N_0 + \delta_1 < \delta_0$, the impermeable wall problem (1.6), (1.7)$_1$, and (1.8) has a unique global solution in time satisfying*

$$\begin{cases} (v - V, u - U, \theta - \Theta) \in C^0(0, +\infty; H^1), \\ (v,u,\theta)_x \in L^2(0, +\infty; L^2), \\ (u,\theta)_{xx} \in L^2(0, +\infty; L^2), \end{cases} \qquad (2.8)$$

and

$$\limsup_{t \to +\infty, \ x > 0} \left|(v,u,\theta)(x,t) - (v^r,u^r,\theta^r)(\frac{x}{t})\right| \to 0. \qquad (2.9)$$

3. Inflow problem

3.1. BL-solution. Assume that $(v_+, u_+, \theta_+) \in \Omega_{sub}$. Then there exists a local (unique) stable manifold $M_+ = M_+(v_+, u_+, \theta_+)$ such that for any $(v_-, u_-, \theta_-) \in M_+$, there exists a unique travelling wave solution $(V_B, U_B, \Theta_B)(\xi), \xi = x - k_- t$, satisfying (1.14). For simplicity, let $(V, U, \Theta)(\xi) = (V_B, U_B, \Theta_B)(\xi)$. We define the perturbation by

$$(\phi, \psi, \zeta)(\xi, t) = (v, u, \theta)(\xi, t) - (V, U, \Theta)(\xi). \tag{3.1}$$

On the other hand, by (1.9) and (1.10) the third equation of (1.6) can be reduced to

$$e_\theta \theta_t + \theta p_\theta u_x = \theta s_t = \kappa \left(\frac{\theta_x}{v} \right)_x + \mu \frac{u_x^2}{v}. \tag{3.2}$$

Thus (ϕ, ψ, ζ) satisfies,

$$\begin{cases}
\phi_t - k_- \phi_\xi - \psi_\xi = 0, \\[2mm]
\psi_t - k_- \psi_\xi + [p(v, \theta) - p(V, \Theta)]_\xi = \mu \left(\frac{u_\xi}{v} \right)_\xi - \mu \left(\frac{U_\xi}{V} \right)_\xi, \\[2mm]
\zeta_t - k_- \zeta_\xi + \frac{\theta p_\theta}{e_\theta}(v, \theta)\psi_\xi + \left\{ \frac{\theta p_\theta}{e_\theta}(v, \theta) - \frac{\Theta p_\theta}{e_\theta}(V, \Theta) \right\} U_\xi \\[2mm]
\quad = \frac{1}{e_\theta(v, \theta)} \left\{ \left(\frac{\kappa \theta_\xi}{v} \right)_\xi + \frac{\mu u_\xi^2}{v} \right\} - \frac{1}{e_\theta(V, \Theta)} \left\{ \left(\frac{\kappa \Theta_\xi}{V} \right)_\xi + \frac{\mu U_\xi^2}{V} \right\}, \\[2mm]
(\phi, \psi, \zeta)|_{\xi=0} = 0, \\[2mm]
(\phi, \psi, \zeta)|_{t=0} = (\phi_0, \psi_0, \zeta_0) = (v_0(\xi) - V, u_0 - U, \theta_0 - \Theta)(\xi).
\end{cases} \tag{3.3}$$

We have the following stability theorem for the BL-solution.

THEOREM 3.1. *Suppose that $(\phi_0, \psi_0, \zeta_0) \in H_0^1(R_+)$. Then there exists a positive constant δ_0 such that if*

$$\|(\phi_0, \psi_0, \zeta_0)\|_1 + |v_+ - v_-| + |u_+ - u_-| + |\theta_+ - \theta_-| \le \delta_0, \tag{3.4}$$

then the initial boundary value problem (3.3) has a unique global solution in time satisfying

$$\begin{cases}
(\phi, \psi, \zeta) \in C^0(0, +\infty; H_0^1), \\[2mm]
\phi_\xi \in L^2(0, +\infty; L^2), \\[2mm]
(\psi, \zeta)_\xi \in L^2(0, +\infty; H^1),
\end{cases} \tag{3.5}$$

and

$$\limsup_{t \to +\infty, \ \xi > 0} |(\phi, \psi, \zeta)(\xi, t)| \to 0. \tag{3.6}$$

To show the stability Theorem 3.1, we seek the solution (ϕ, ψ, ζ) in the solution space

$$\begin{aligned}
X(0, T) = \big\{ (\phi, \psi, \zeta) \in C^0(0, T; H_0^1) \big| \phi_\xi \in L^2(0, T; L^2), \\
(\psi_\xi, \zeta_\xi) \in L^2(0, T; H^1), \sup_{[0,T]} \|(\phi, \psi, \zeta)(t)\|_1 \le \varepsilon_1 \big\},
\end{aligned} \tag{3.7}$$

where $\varepsilon_1 = \frac{1}{4}\min\{v_-, v_+\}$. Let $\delta = |v_+ - v_-| + |u_+ - u_-| + |\theta_+ - \theta_-|$ and

$$N(T) = \sup_{0 \le \tau \le T} (\|\phi(\tau)\|_1 + \|\psi(\tau)\|_1 + \|\zeta(\tau)\|_1). \tag{3.8}$$

Then $N(T) \le \varepsilon_1$ which ensures the system (3.3) is nonsingular. By using the same lines as in the previous papers [12-13], it is easy to prove the local existence. We omit the proofs here. To justify Theorem 3.1, it is necessary to establish the a priori estimates. We have

PROPOSITION 3.2. *There exist positive constants δ_0, $\varepsilon_2 (\le \varepsilon_1)$ and C_0 such that, if $\delta_1 \le \delta_0$, $N(T) \le \varepsilon_2$ and $(\phi, \psi, \zeta)(\xi, t) \in X(0, T)$ is a solution of (3.3) for some $T > 0$, then (ϕ, ψ, ζ) satisfies*

$$\|(\phi, \psi, \zeta)(t)\|_1^2 + \int_0^t \left\{ \|\phi_\xi(\tau)\|^2 + \|(\psi, \zeta)_\xi(\tau)\|_1^2 \right\} d\tau$$
$$\le C_0 \|(\phi, \psi, \zeta)(0)\|^2. \tag{3.9}$$

Proposition 3.2 is proved by a series lemmas. We first give the following key lemma.

LEMMA 3.3. *It follows that*

$$\|(\phi, \psi, \zeta)(t)\|^2 + \int_0^t \left\{ \|\psi_\xi(\tau)\|^2 + \|\zeta_\xi(\tau)\|^2 \right\} d\tau$$
$$\le C \|(\phi, \psi, \zeta)(0)\|^2 + C\delta \int_0^t \|\phi_\xi\|^2(\tau) d\tau. \tag{3.10}$$

Proof. Let

$$\mathcal{E} = e(v, s) - e(V, S) + \frac{\psi^2}{2} + p(V, \Theta)(v - V) - \Theta(s - S). \tag{3.11}$$

Then by (1.9)–(1.10), (3.2)–(3.3), we have

$$\mathcal{E}_t + \frac{\mu\psi_\xi^2}{v} + \frac{\kappa\zeta_\xi^2}{v\theta} + \Delta_1 U_\xi + k_- \Delta_2 S_\xi$$
$$= \left(k_- \mathcal{E} + \mu\psi\left(\frac{u_\xi}{v} - \frac{U_\xi}{V}\right) + \kappa\zeta\left(\frac{\theta_\xi}{v\theta} - \frac{\Theta_\xi}{V\Theta}\right) - (p(v,\theta) - p(V,\Theta))\psi\right)_\xi$$
$$+ \frac{\mu\phi\psi_\xi U_\xi}{vV} + \mu\zeta\left(\frac{u_\xi^2}{v\theta} - \frac{U_\xi^2}{V\Theta}\right) + \kappa\zeta\left(\frac{\theta_\xi^2}{v\theta^2} - \frac{\Theta_\xi^2}{V\Theta^2}\right) - \kappa\zeta_\xi\Theta_\xi\left(\frac{1}{v\theta} - \frac{1}{V\Theta}\right), \tag{3.12}$$

where

$$\Delta_1 = \hat{p}(v, s) - \hat{p}(V, S) - \hat{p}_v(V, S)\phi - \hat{p}_s(V, S)(s - S),$$
$$\Delta_2 = \theta - \Theta - \hat{\theta}_v(V, S)\phi - \hat{\theta}_s(V, S)(s - S).$$

satisfy $|\Delta_i| \le C(\phi^2 + \zeta^2), i = 1, 2$. It is noted that $e(v, s)$ is a strictly convex function of v and s due to (1.2), (1.9), and (1.10). Since $s_\theta = \frac{e_\theta}{\theta} > 0$, the Cauchy inequality yields that there exists a positive constant $c_1 > 0$ such that

$$\mathcal{E} \ge c_1(\phi^2 + \psi^2 + \zeta^2). \tag{3.13}$$

Note that $(\phi,\psi,\zeta)(0,t)=(0,0,0)$. Integrating (3.12) over $[0,t]\times\mathbb{R}_+$, we have

$$
\|(\phi,\psi,\zeta)(t)\|^2+\int_0^t\|\partial_\xi(\psi,\zeta)(\tau)\|^2d\tau
$$

$$
\leq C_1\|(\phi,\psi,\zeta)(0)\|^2+C_1\left|\int_0^t\int_0^{+\infty}\frac{\mu\phi\psi_\xi U_\xi}{vV}+\mu\zeta(\frac{u_\xi^2}{v\theta}-\frac{U_\xi^2}{V\Theta})-\Delta_1 U_\xi\,d\xi dt\right|
$$

$$
+C_1\left|\int_0^t\int_0^{+\infty}\kappa\zeta\left(\frac{\theta_\xi^2}{v\theta^2}-\frac{\Theta_\xi^2}{V\Theta^2}\right)-\kappa\zeta_\xi\Theta_\xi\left(\frac{1}{v\theta}-\frac{1}{V\Theta}\right)-k_-\Delta_2 S_\xi\,d\xi dt\right| \tag{3.14}
$$

$$
=:C_1\|(\phi,\psi,\zeta)(0)\|^2+I_1+I_2.
$$

We use the idea of [6] to estimate the right hand side of (3.14). Since

$$
|(\phi^2,\psi^2,\zeta^2)(\xi)|\leq\xi\|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2,\ \forall\xi>0, \tag{3.15}
$$

the Cauchy inequality and (1.23) yield that

$$
I_1\leq C\delta\int_0^t\|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2dt+CN(T)\int_0^t\|\psi_\xi\|^2dt \tag{3.16}
$$

and

$$
I_2\leq C\delta\int_0^t\|(\phi_\xi,\psi_\xi,\zeta_\xi)\|^2dt+CN(T)\int_0^t\|\zeta_\xi\|^2dt. \tag{3.17}
$$

Thus there exist positive small constants δ_0 and ε_2. When $\delta\leq\delta_0$ and $N(T)\leq\varepsilon_2$, substituting (3.16)–(3.17) into (3.14) yields the desired estimate (3.10). $\qquad\square$

LEMMA 3.4.   *It follows that*

$$
\|\phi_\xi(t)\|^2+\int_0^t\|\phi_\xi\|^2d\tau\leq C\|(\phi,\psi,\zeta)(0)\|_1^2+C\int_0^t\check{v}_\xi^2(0,t)dt. \tag{3.18}
$$

Proof.   Following [11], we introduce a new variable $\check{v}=\frac{v}{V}$. Then $(3.3)_2$ is rewritten as

$$
(\mu\frac{\check{v}_\xi}{\check{v}}-\psi)_t=k_-(\mu\frac{\check{v}_\xi}{\check{v}}-\psi)_\xi+(p(v,\theta)-p(V,\Theta))_\xi. \tag{3.19}
$$

Multiplying (3.19) by $\frac{\check{v}_\xi}{\check{v}}$, we have

$$
\left\{\frac{\mu}{2}(\frac{\check{v}_\xi}{\check{v}})^2-\psi\frac{\check{v}_\xi}{\check{v}}\right\}_t-p_v(v,\theta)V\frac{\check{v}_\xi^2}{\check{v}}+\left(\frac{\check{v}_t}{\check{v}}\psi-\frac{k_-\mu}{2}\left(\frac{\check{v}_\xi}{\check{v}}\right)^2\right)_\xi
$$

$$
=\mu\frac{\psi_\xi^2}{v}-\frac{\mu\phi\psi_\xi U_\xi}{vV}+(p_v(v,\theta)\check{v}-p_v(V,\Theta))V_\xi\frac{\check{v}_\xi}{\check{v}} \tag{3.20}
$$

$$
+(p_\theta(v,\theta)\theta_\xi-p_\theta(V,\Theta)\Theta_\xi)\frac{\check{v}_\xi}{\check{v}}.
$$

We compute

$$
\left|(p_v(v,\theta)\check{v}-p_v(V,\Theta))V_\xi\frac{\check{v}_\xi}{\check{v}}\right|\leq\tilde{\delta}\check{v}_\xi^2+C_{\tilde{\delta}}(\phi^2+\zeta^2)V_\xi^2, \tag{3.21}
$$

$$\left|(p_\theta(v,\theta)\theta_\xi - p_\theta(V,\Theta)\Theta_\xi)\frac{\check{v}_\xi}{\check{v}}\right| \leq \tilde{\delta}\check{v}_\xi^2 + C_{\tilde{\delta}}\zeta_\xi^2 + C_{\tilde{\delta}}(\phi^2+\zeta^2)\Theta_\xi^2, \qquad (3.22)$$

$$\left|\frac{\mu\phi\psi_\xi U_\xi}{vV}\right| \leq C\phi^2 U_\xi^2 + C\psi_\xi^2, \qquad (3.23)$$

$$\int_0^\infty (\phi^2+\zeta^2)(V_\xi^2+U_\xi^2+\Theta_\xi^2)\,dx \leq C\delta^2(\|\phi_\xi\|^2+\|\zeta_\xi\|^2), \qquad (3.24)$$

and

$$c_2\phi_\xi^2 - c_3\phi^2 V_\xi^2 \leq \check{v}_\xi^2 \leq C_2\phi_\xi^2 + C_3\phi^2 V_\xi^2, \qquad (3.25)$$

where $c_i, C_i, i=2,3$ are positive constants which only depend on v_- and v_+. Note that δ is small. Thus integrating (3.20) on $R_+ \times [0,t]$, using (3.21)–(3.25) and Lemma 3.3 and choosing $\tilde{\delta}$ is suitably small, we get (3.18). Thus Lemma 3.4 is proved. $\square$

LEMMA 3.5.   *It follows that*

$$\|(\psi_\xi,\zeta_\xi)(t)\|^2 + \int_0^t \|(\psi_{\xi\xi},\zeta_{\xi\xi})(\tau)\|^2 d\tau \leq C\|(\phi,\psi,\zeta)(0)\|_1^2. \qquad (3.26)$$

Proof.   Multiplying $(3.3)_2$ by $-\psi_{\xi\xi}$ and $(3.3)_3$ by $-\zeta_{\xi\xi}$, and adding the resulting equalities, we have

$$\begin{aligned}
&\left(\frac{1}{2}\psi_\xi^2+\frac{1}{2}\zeta_\xi^2\right)_t + \left(\frac{k_-}{2}(\psi_\xi^2+\zeta_\xi^2) - \psi_t\psi_\xi - \zeta_t\zeta_\xi\right)_\xi + \mu\frac{\psi_{\xi\xi}^2}{v} + \kappa\frac{\zeta_{\xi\xi}^2}{ve_\theta(v,\theta)} \\
&= (p(v,\theta)-p(V,\Theta))_\xi\psi_{\xi\xi} + \left(\frac{\theta p_\theta}{e_\theta}(v,\theta)u_\xi - \frac{\Theta p_\theta}{e_\theta}(V,\Theta)U_\xi\right)\zeta_{\xi\xi} \\
&\quad -\mu\psi_{\xi\xi}U_{\xi\xi}\left(\frac{1}{v}-\frac{1}{V}\right) + \mu\psi_{\xi\xi}\left(\frac{u_\xi v_\xi}{v^2}-\frac{U_\xi V_\xi}{V^2}\right) \\
&\quad -\kappa\zeta_{\xi\xi}\Theta_{\xi\xi}\left(\frac{1}{e_\theta v}(v,\theta)-\frac{1}{e_\theta V}(V,\Theta)\right) + \kappa\zeta_{\xi\xi}\left(\frac{\theta_\xi v_\xi}{e_\theta v^2}(v,\theta)-\frac{\Theta_\xi V_\xi}{e_\theta V^2}(V,\Theta)\right) \\
&\quad -\mu\zeta_{\xi\xi}\left(\frac{u_\xi^2}{e_\theta v}(v,\theta)-\frac{U_\xi^2}{e_\theta V}(V,\Theta)\right).
\end{aligned} \qquad (3.27)$$

We estimate

$$\begin{aligned}
\text{RHS} \leq &\ \tilde{\delta}(\psi_{\xi\xi}^2+\zeta_{\xi\xi}^2) + C_{\tilde{\delta}}(\phi_\xi^2+\psi_\xi^2+\zeta_\xi^2) + C_{\tilde{\delta}}(\phi^2+\zeta^2)(V_\xi^2+U_\xi^2 \\
&+ \Theta_\xi^2+U_{\xi\xi}^2+\Theta_{\xi\xi}^2) + C(|\psi_{\xi\xi}\psi_\xi\phi_\xi|+|\zeta_{\xi\xi}\zeta_\xi\phi_\xi|+|\zeta_{\xi\xi}\psi_\xi^2|).
\end{aligned} \qquad (3.28)$$

It is noted that

$$\int_0^\infty |\psi_{\xi\xi}\psi_\xi\phi_\xi|d\xi \leq C\|\psi_{\xi\xi}\|^{\frac{3}{2}}\|\psi_\xi\|^{\frac{1}{2}}\|\phi_\xi\| \leq CN(T)(\|\psi_\xi\|^2+\|\psi_{\xi\xi}\|^2). \qquad (3.29)$$

The last two terms can be treated by the same way. Note that $k_- < 0$ and $\psi_t(0,t) = \zeta_t(0,t) = 0$. Thus, integrating (3.27) over $R_+ \times [0,T]$, choosing $\tilde{\delta}$ suitably small and using Lemmas 3.3–3.4, we have

$$\begin{aligned}
&\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|^2 + \int_0^t (\|\phi_\xi(\tau)\|^2+\|(\psi_{\xi\xi},\zeta_{\xi\xi})(\tau)\|^2)d\tau \\
&\leq C\|(\phi,\psi,\zeta)(0)\|_1^2 + C\int_0^t \|\psi_\xi\|^2 d\tau,
\end{aligned} \qquad (3.30)$$

where we have used the fact that

$$\check{v}_\xi^2(0,t) \le C\phi_\xi^2(0,t) \le C\psi_\xi^2(0,t) \le \tilde{\delta}\|\psi_{\xi\xi}\|^2 + C_{\tilde{\delta}}\|\psi_\xi\|^2. \qquad (3.31)$$

Adding $(3.30)\times\lambda$ and (3.10), we have

$$
\begin{aligned}
&\|(\phi,\psi,\zeta)(t)\|^2 + \lambda\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|^2 + (1-C\lambda)\int_0^t \|\psi_\xi\|^2 d\tau \\
&+ \int_0^t \|\zeta_\xi\|^2 d\tau + (\lambda-C\delta)\int_0^t \|\phi_\xi\|^2 d\tau + \lambda\int_0^t \|(\psi_{\xi\xi},\zeta_{\xi\xi})\|^2 d\tau \\
&\le C\|(\phi,\psi,\zeta)(0)\|_1^2.
\end{aligned}
\qquad (3.32)
$$

We now choose $\lambda = \frac{1}{2C} > 0$ such that $1-C\lambda = \frac{1}{2}$. Let δ be suitably small so that $\lambda - C\delta > \frac{1}{2}\lambda$. Therefore Lemma 3.5 is proved from (3.32). □

Proof of Proposition 3.2. Proposition 3.2 is obtained at once from Lemmas 3.3–3.5.

3.2. The superposition of the BL-solution and the rarefaction wave.
In this section, we investigate the case $(v_-, u_-, \theta_-) \in BLR_3(v_+, u_+, \theta_+)$. According to the previous arguments, there exists a unique point $(v_*, u_*, \theta_*) \in \Omega_{sub}$ such that $(v_*, u_*, \theta_*) \in R_3(v_+, u_+, \theta_+)$. We study the stability of the superposition of the BL-solution $(V_B, U_B, \Theta_B)(\xi), \xi = x - k_- t$, connecting (v_-, u_-, θ_-) with (v_*, u_*, θ_*) and the rarefaction wave $(v^r, u^r, \theta^r)(\frac{x}{t})$ connecting (v_*, u_*, θ_*) with (v_+, u_+, θ_+). The BL-solution (V_B, U_B, Θ_B) satisfies (1.14), (1.22), and (1.23), where the right state is replaced by (v_*, u_*, θ_*) and $\delta = |v_* - v_-| + |u_* - u_-| + |\theta_* - \theta_-|$.

For the rarefaction wave, we use the same smooth approximation (V^r, U^r, Θ^r) to $(v^r, u^r, \theta^r)(\frac{x}{t})$ as in (2.2) and (2.3), where the left state is replaced by (v_*, u_*, θ_*). Let

$$(V_3, U_3, \Theta_3)(x,t) := (V^r, U^r, \Theta^r)|_{x \ge k_- t}. \qquad (3.33)$$

If we introduce a new coordinate $t = t, \xi = x - k_- t$, then we have

$$
\begin{cases}
V_{3t} - k_- V_{3\xi} - U_\xi = 0, & \xi > 0, t > 0, \\
U_{3t} - k_- U_{3\xi} + p(V_3, \Theta_3)_\xi = 0, & \xi > 0, t > 0, \\
\Theta_{3t} - k_- \Theta_{3\xi} + \dfrac{\Theta_3 p_\theta}{e_\theta}(\Theta_3, V_3) U_{3\xi} = 0, & \xi > 0, t > 0, \\
(V_3, U_3, \Theta_3)\big|_{\xi=0} = (v_*, u_*, \theta_*), \quad (V_3, U_3, \Theta_3)\big|_{\xi \to +\infty} = (v_+, u_+, \theta_+), \\
(V_3, U_3, \Theta_3)\big|_{t=0} = (V_3, U_3, \Theta_3)(\xi, 0).
\end{cases}
\qquad (3.34)
$$

Similar to Lemma 2.2, $(V_3, U_3, \Theta_3)(\xi, t)$ has the following properties.

LEMMA 3.6. *Let* $\delta_1 = |v_+ - v_*| + |u_+ - u_*| + |\theta_+ - \theta_*|$, *then* $(V_3, U_3, \Theta_3)(\xi, t)$ *satisfies, if* $q \ge p$,

   i. $U_{3\xi}(x,t) \ge 0, |U_{3\xi}| \le C\varepsilon\delta_1$, *for* $t \ge 0, \xi > 0$,

   ii.   *For any* $p(1 \le p \le +\infty)$, *there exists a constant* $C_{p,q}$ *such that*

$$\|(V_{3\xi}, U_{3\xi}, \Theta_{3\xi})\|_{L^p(\xi \ge 0)} \le C_{p,q} \min\{\delta_1 \varepsilon^{1-\frac{1}{p}}, \delta_1^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\},$$

$$\|(V_{3\xi\xi}, U_{3\xi\xi}, \Theta_{3\xi\xi})\|_{L^p(\xi \ge 0)} \le C_{p,q} \min\{\delta_1 \varepsilon^{2-\frac{1}{p}}, \delta_1^{\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\}, \quad t \ge 0.$$

   iii. $(V, U, \Theta)|_{\xi \le -k_- t} = (v_*, u_*, \theta_*), (V_{3\xi}, U_{3\xi}, \Theta_{3\xi}, V_{3\xi\xi}, U_{3\xi\xi}, \Theta_{3\xi\xi})|_{\xi \le -k_- t} = 0.$

   iv. $\limsup\limits_{t \to +\infty, \xi \in \{\xi \ge 0\}} |(V, U, \Theta)(\xi, t) - (v^r, u^r, \theta^r)(\xi, t)| = 0.$

Let

$$
\begin{pmatrix} V \\ U \\ \Theta \end{pmatrix} (\xi, t) = \begin{pmatrix} V_B(\xi) + V_3(\xi, t) - v_* \\ U_B(\xi) + U_3(\xi, t) - u_* \\ \Theta_B(\xi) + \Theta_3(\xi, t) - \theta_* \end{pmatrix}, \tag{3.35}
$$

and

$$
(\phi, \psi, \theta)(\xi, t) = (v - V, u - U, \theta - \Theta)(\xi, t). \tag{3.36}
$$

Then the system (1.6) is rewritten as

$$
\begin{cases}
\phi_t - k_- \phi_\xi - \psi_\xi = 0, \\[2mm]
\psi_t - k_- \psi_\xi + [p(v, \theta) - p(V, \Theta)]_\xi = \mu \left(\dfrac{u_\xi}{v} \right)_\xi - \mu \left(\dfrac{U_\xi}{V} \right)_\xi + F_\xi, \\[2mm]
\zeta_t - k_- \zeta_\xi + \dfrac{\theta p_\theta}{e_\theta} \psi_\xi + \left\{ \dfrac{\theta p_\theta}{e_\theta}(v, \theta) - \dfrac{\Theta p_\theta}{e_\theta}(V, \Theta) \right\} U_\xi \\[3mm]
= \dfrac{\left\{ \left(\dfrac{\kappa \theta_\xi}{v} \right)_\xi + \dfrac{\mu u_\xi^2}{v} \right\}}{e_\theta(v, \theta)} - \dfrac{\left\{ \left(\dfrac{\kappa \Theta_\xi}{V} \right)_\xi + \dfrac{\mu U_\xi^2}{V} \right\}}{e_\theta(V, \Theta)} + G, \\[3mm]
(\phi, \psi, \zeta)|_{\xi = 0} = (0, 0, 0), \\[2mm]
(\phi, \psi, \zeta)|_{t = 0} = (v - V, u - U, \theta - U)(\xi, 0) =: (\phi_0, \psi_0, \zeta_0)(\xi).
\end{cases} \tag{3.37}
$$

where

$$
F = -[p(V, \Theta) - p(V_B, \Theta_B) - p(V_3, \Theta_3) + p(v_*, \theta_*)] + \mu \left(\dfrac{U_\xi}{V} - \dfrac{U_{B\xi}}{V_B} \right)
$$

$$
=: -F_1 + F_2,
$$

$$
G = - \left[\dfrac{\Theta p_\theta}{e_\theta}(V, \Theta) U_\xi - \dfrac{\Theta_B p_\theta}{e_\theta}(V_B, \Theta_B) U_{B\xi} - \dfrac{\Theta_3 p_\theta}{e_\theta}(V_3, \Theta_3) U_{3\xi} \right] \tag{3.38}
$$

$$
+ \left\{ \dfrac{\left[\left(\dfrac{\kappa \Theta_\xi}{V} \right)_\xi + \dfrac{\mu U_\xi^2}{V} \right]}{e_\theta(V, \Theta)} - \dfrac{\left[\left(\dfrac{\kappa \Theta_{B\xi}}{V_B} \right)_\xi + \dfrac{\mu U_{B\xi}^2}{V_B} \right]}{e_\theta(V_B, \Theta_B)} \right\} =: -G_1 + G_2.
$$

The equation (3.37) is almost the same as (3.3) except for the terms F_ξ and G. Since

$$
\begin{aligned}
|F_{1\xi}| \leq & C \left(|V_3 - v_*| + |\Theta_3 - \theta_*| \right) \left(|V_{B\xi}| + |\Theta_{B\xi}| \right) \\
& + C \left(|V_B - v_*| + |\Theta_B - \theta_*| \right) \left(|V_{3\xi}| + |\Theta_{3\xi}| \right),
\end{aligned} \tag{3.39}
$$

and $V_{3\xi} = \Theta_{3\xi} = V_3 - v_* = \Theta_3 - \theta_* = 0$ for any $\xi < -k_- t$ duo to Lemma 3.6, we have

$$
\begin{aligned}
\int_0^\infty |F_{1\xi}| d\xi & \leq C \delta_1 \int_{-k_- t}^\infty |V_{B\xi}| + |\Theta_{B\xi}| + |V_B - v_*| + |\Theta_B - \theta_*| d\xi \\
& \leq C \delta \delta_1 e^{-k_- t}.
\end{aligned} \tag{3.40}
$$

Thus

$$
\int_0^t \int_0^{+\infty} |F_{1\xi} \psi| d\xi dt \leq C \delta \delta_1. \tag{3.41}
$$

On the other hand, since

$$|F_{2\xi}| \le C(|V_3 - v_*||U_{B\xi\xi}| + |U_{3\xi\xi}| + |V_{3\xi}U_{3\xi}| + |V_{3\xi}U_{B\xi}| \\ + |U_{3\xi}V_{B\xi}| + |V_3 - v_*||V_{B\xi}|), \tag{3.42}$$

we have

$$\int_0^\infty |F_{2\xi}| d\xi \le C\delta\delta_1 e^{-k_- t} + C\delta_1^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}, \tag{3.43}$$

and then

$$\int_0^t \int_0^\infty |F_{2\xi}\psi| d\xi dt \le C\delta\delta_1 + \nu \int_0^t \|\psi_\xi\|^2 dt + C_\nu \delta_1^{\frac{1}{6}}, \tag{3.44}$$

for any $\nu > 0$. The terms $\int_0^t \int_0^\infty |G_1\psi| d\xi d\tau$ and $\int_0^t \int_0^\infty |G_2\psi| d\xi d\tau$ can be treated in the same way. Using the same argumens as in section 3.1, we obtain

$$\|(\phi,\psi,\zeta)(t)\|^2 + \int_0^t \|\partial_\xi(\psi,\zeta)(\tau)\|^2 d\tau \\ \le C\left(\|(\phi,\psi,\zeta)(0)\|_1^2 + \delta \int_0^t \|\phi_\xi\|^2 d\tau + \delta + \delta_1^{\frac{1}{6}}\right). \tag{3.45}$$

The high order estimates are omitted here because they are similar to those in section 3.1. Therefore we have

THEOREM 3.7. *Suppose that $(\phi_0,\psi_0,\zeta_0) \in H_0^1(R_+)$. Then there exists a positive constant δ_0 such that if*

$$\|(\phi_0,\psi_0,\zeta_0)\|_1 + \delta + \delta_1 \le \delta_0 \tag{3.46}$$

then the initial boundary value problem (3.37) has a unique global solution in time satisfying

$$\begin{cases} (\phi,\psi,\zeta) \in C^0(0,+\infty;H_0^1), \\ \phi_\xi \in L^2(0,+\infty;L^2), \\ (\psi,\zeta)_\xi \in L^2(0,+\infty;H^1), \end{cases} \tag{3.47}$$

and

$$\limsup_{t\to+\infty,\ \xi>0} |(\phi,\psi,\zeta)(\xi,t)| \to 0. \tag{3.48}$$

REMARK 3.1. The case $(v_-,u_-,\theta_-) \in R_1R_3(v_+,u_+,\theta_+) \cap \Omega_{super}$ can be treated in a similar way and then the assertion (III) holds.

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