RECOVERY-BASED ERROR ESTIMATION AND ADAPTIVE SOLUTION OF ELLIPTIC VARIATIONAL INEQUALITIES OF THE SECOND KIND*

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Abstract. In this paper, we present and analyze gradient recovery type a posteriori error estimates for the finite element approximation of elliptic variational inequalities of the second kind. Both reliability and efficiency of the estimates are addressed. Some numerical results are reported, showing the effectiveness of the error estimates in adaptive solution of elliptic variational inequalities of the second kind.

1. Introduction

A posteriori error analysis has become an important tool for reliability assessment and efficiency improvement of the finite element method in solving both linear and nonlinear problems. An important class of a posteriori error estimates is based on local or global averaging of the gradient, e.g. in the form of Zienkiewicz-Zhu gradient recovery technique, [19, 20, 21]. It is known that in the case of structured grids and higher regularity solutions, such estimators are both efficient and reliable. Some work has been done for unstructured meshes as well, e.g. [16, 18]. A systematic study of various averaging techniques for a posteriori error estimation can be found in [2, 8].

The papers mentioned above are on a posteriori error estimation for solving elliptic boundary value problems of partial differential equations. For numerical solutions of variational inequalities, a few papers can be found for their a posteriori error analysis. E.g., residual type error estimators were obtained for an elliptic obstacle problem in [10] and for an elliptic variational inequality of the second kind in [5], and gradient recovery type error estimates for an elliptic obstacle problem have been shown recently in [3, 17].

In this paper, we derive and study a posteriori error estimates of gradient recovery type for finite element solutions of elliptic variational inequalities of the second kind. The paper is organized as follows. In Section 2 we introduce the model problem. Section 3 contains the finite element method setting. In Section 4 we derive a posteriori error estimates based on both global and local averaging techniques. Section 5 is devoted to an analysis of the efficiency of the estimators. In Section 6 we report some numerical results. We restrict ourselves in this paper to linear elements since they are popularly used in solving variational inequalities due to a lack of higher solution regularity. Also, our discussion focuses on one particular gradient recovery technique. The discussion and results presented here can be extended to other elements and other averaging techniques as studied in [2, 8].

We now list some notations used in the paper. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with Lipschitz boundary $\Gamma = \partial\Omega$. Let Γ_D be a closed subset of Γ with $\text{meas}(\Gamma_D) > 0$ and Γ_C be the remaining part. For any open subset ω of Ω with Lipschitz boundary $\partial\omega$, we denote by $H^m(\omega)$, $L^2(\omega)$ and $L^2(\partial\omega)$ the usual Sobolev

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and Lebesgue spaces with the standard norms $\|\cdot\|_{m;\omega} := \|\cdot\|_{H^m(\omega)}$, $\|\cdot\|_{0;\omega} := \|\cdot\|_{L^2(\omega)}$ and $\|\cdot\|_{0;\partial\omega} := \|\cdot\|_{L^2(\partial\omega)}$. Also, we will make use of the standard seminorm $|\cdot|_{m,\omega} := \|D^m \cdot\|_{0;\omega}$ on $H^m(\omega)$. Throughout this paper we will use the same notation v to denote both $v \in H^1(\Omega)$ and its trace $\gamma v \in L^2(\Gamma)$ on the boundary. We reserve the symbol γ to denote sides of the finite elements.

2. Model problem

The discussion of the a posteriori error estimates in this paper is focused on a model elliptic variational inequality of the second kind. Such a variational inequality is featured by the presence of non-differentiable terms in the formulation. The techniques and results presented in this paper can be extended to general elliptic variational inequalities of the second kind in a straightforward fashion.

Let there be given $f \in L^2(\Omega)$ and a constant $g > 0$. Over the space $V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}$, we define

$$a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + u\,v)\,dx,$$

$$l(v) = \int_\Omega fv\,dx,$$

$$j(v) = \int_{\Gamma_C} g\,|v|\,ds.$$

The model problem is the following elliptic variational inequality of the second kind: Find $u \in V$ such that

$$a(u, v - u) + j(v) - j(u) \geq l(v - u) \quad \forall v \in V. \tag{2.1}$$

This model is a so-called simplified friction problem (cf. [13]) as it can be viewed as a simplified version of a frictional contact problem in linearized elasticity. Existence and uniqueness of a solution for the problem (2.1) follow from a classical result on elliptic variational inequalities (see e.g. [13] or [14]).

In the analysis of a posteriori error estimators later, we will need the following characterization of the solution u of (2.1):

There exists a unique $\lambda \in L^\infty(\Gamma_C)$ such that

$$a(u, v) + \int_{\Gamma_C} g\,\lambda\,v\,ds = l(v) \quad \forall v \in V, \tag{2.2}$$

$$|\lambda| \leq 1, \quad \lambda\,u = |u| \quad \text{on } \Gamma_C. \tag{2.3}$$

A proof of this characterization in the case $\Gamma_C = \Gamma$ can be found in [13]; the proof can be easily extended to the general case, cf. [6]. It follows from the above characterization that the solution u of (2.1) is the weak solution of the boundary value problem

$$-\Delta u + u = f \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} + g\lambda = 0 \quad \text{on } \Gamma_C,$$

$$u = 0 \quad \text{on } \Gamma_D.$$

Here, $\partial/\partial n$ denotes the outward normal derivative.

3. Finite element approximation

For simplicity, we suppose that Ω has a polyhedral boundary Γ. In order to define the finite element method for (2.1) we introduce a family of regular partitions $\mathcal{P}_h$ of $\overline{\Omega}$ consisting of triangles ($d = 2$) or tetrahedrons ($d = 3$). The partitions $\mathcal{P}_h$ are compatible with the decomposition of $\partial\Omega$ into Γ_D and Γ_C. In other words, if an element side lies on the boundary, then it belongs to one of the sets Γ_D or $\overline{\Gamma_C}$. For every element $K \in \mathcal{P}_h$, let h_K be the diameter of K and ρ_K be the diameter of the largest ball inscribed in K. For a side γ of the element K, we denote by h_γ the diameter of γ. We shall assume that the family of partitions $\mathcal{P}_h$, $h > 0$, satisfies the shape regularity assumption, i.e. the ratio h_K/ρ_K is uniformly bounded over the whole family by a constant C. Note that the shape regularity assumption does not require that the elements be of comparable size and thus locally refined meshes are allowed. Define the finite element spaces

$$S_h = \{v_h \in C(\overline{\Omega}) \mid v_h|_K \text{ is affine } \forall K \in \mathcal{P}_h\}$$

and

$$V_h = \{v_h \in S_h \mid v_h = 0 \text{ on } \Gamma_D\}.$$

We will use $\mathcal{E}_h$ for the set of the element sides, $\mathcal{E}_{h,C}$ for the subset of the element sides lying on the $\overline{\Gamma_C}$ part of the boundary, and $\mathcal{E}_{h,0}$ is the subset of the element sides that do not lie on Γ. Let $\mathcal{N}_h$ be the set of all vertices in $\mathcal{P}_h$ and $\mathcal{N}_{h,0} \subset \mathcal{N}_h$ the set of free vertices, i.e. those vertices that do not lie on Γ_D. For a given element $K \in \mathcal{P}_h$, $\mathcal{N}(K)$ and $\mathcal{E}(K)$ denote the sets of the vertices of K and sides of K, respectively. The patch $\widetilde{K}$ associated with any element K from a partition $\mathcal{P}_h$ consists of all elements sharing at least one vertex with K, i.e. $\widetilde{K} = \bigcup\{K' \in \mathcal{P}_h : \overline{K'} \cap \overline{K} \neq \emptyset\}$. Similarly, for any side $\gamma \in \mathcal{E}_h$, the patch $\widetilde{\gamma}$ consists of the elements sharing γ as a common side. Note that in the case when the side γ lies on the boundary Γ, the patch $\widetilde{\gamma}$ consists of only one element. For a given element $K \in \mathcal{P}_h$ or a given side $\gamma \in \mathcal{E}_h$, n_K or n_γ denotes the unit normal vector to the sides of K or to the side γ, respectively. In what follows, for any piecewise continuous function φ and any interior side $\gamma \in \mathcal{E}_{h,0}$, $[\varphi]_\gamma$ denotes the jump of φ across γ in the direction n_γ, i.e.

$$[\varphi]_\gamma(x) = \lim_{t \to 0+} (\varphi(x + tn_\gamma) - \varphi(x - tn_\gamma)), \quad x \in \gamma.$$

The discretization of variational inequality (2.1) is defined by the Galerkin finite element method: Find $u_h \in V_h$ such that

$$a(u_h, v_h - u_h) + j(v_h) - j(u_h) \geq l(v_h - u_h) \quad \forall v_h \in V_h. \tag{3.1}$$

We need the following characterization of the finite element solution (cf. [6]), similar to that of the solution of the continuous problem.

THEOREM 3.1. *The unique solution $u_h \in V_h$ of the discretized problem (3.1) is characterized by the existence of $\lambda_h \in L^\infty(\Gamma_C)$ such that*

$$a(u_h, v_h) + \int_{\Gamma_C} g\,\lambda_h v_h\,ds = l(v_h) \quad \forall v_h \in V_h, \tag{3.2}$$

$$|\lambda_h| \leq 1, \quad \lambda_h u_h = |u_h| \quad \text{a.e. on } \Gamma_C. \tag{3.3}$$

In the derivation of the a posteriori error estimate we use the so-called weighted Clément-type interpolation operator. There are several variants of such operators (see e.g. [2, 7, 8, 9, 17]), which are all versions of the interpolation operator introduced by Clément [11]. The main difference among these interpolants lies in the way the interpolation is performed near the boundary. In this paper we will follow the approach used in [8].

Given a vertex $A \in \mathcal{N}_h$, let $\varphi_A \in S_h$ be the nodal basis function associated with A. For each fixed vertex $A \in \Gamma_D$, choose $\xi(A) \in \mathcal{N}_{h,0}$ to be a vertex of an element containing A. Let $\xi(A) = A$ if $A \in \mathcal{N}_{h,0}$. For each vertex $A \in \mathcal{N}_{h,0}$ define the class $I(A) = \{\tilde{A} \in \mathcal{N}_h \mid \xi(\tilde{A}) = A\}$. In this way, the set of all vertices $\mathcal{N}_h$ is partitioned into $\mathrm{card}(\mathcal{N}_{h,0})$ classes of equivalence. For each $A \in \mathcal{N}_{h,0}$ set

$$\psi_A = \sum_{\xi \in I(A)} \varphi_\xi$$

and notice that $\{\psi_A \mid A \in \mathcal{N}_{h,0}\}$ is a partition of unity. Let $\widetilde{K}_A = \mathrm{supp}(\psi_A)$ and $h_A = \mathrm{diam}(\widetilde{K}_A)$. It is evident that $\widetilde{K}_A$ is connected and that $\psi_A \neq \varphi_A$ implies that $\Gamma_D \cap \widetilde{K}_A$ has a positive surface measure. For a given $v \in L^1(\Omega)$, let

$$v_A = \frac{\int_{\widetilde{K}_A} \psi_A v \, dx}{\int_{\widetilde{K}_A} \varphi_A dx}, \qquad A \in \mathcal{N}_{h,0}.$$

Then define the interpolation operator $\Pi_h : V \to V_h$ as follows:

$$\Pi_h v = \sum_{A \in \mathcal{N}_{h,0}} v_A \varphi_A. \tag{3.4}$$

The next result summarizes some basic estimates for Π_h. Its proof can be found in [8].

THEOREM 3.2. *There exists an h-independent constant $C > 0$ such that for all $v \in V$ and $f \in L^2(\Omega)$ there holds*

$$|v - \Pi_h v|_{1;\Omega}^2 \leq C |v|_{1;\Omega}^2, \tag{3.5}$$

$$\int_\Omega f(v - \Pi_h v) \, dx \leq C |v|_{1;\Omega} \left(\sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{f_A \in \mathbb{R}} \|f - f_A\|_{0;\widetilde{K}_A}^2 \right)^{1/2}, \tag{3.6}$$

$$\sum_{K \in \mathcal{P}_h} \|h_K^{-1}(v - \Pi_h v)\|_{0;K}^2 \leq C |v|_{1;\Omega}^2, \tag{3.7}$$

$$\sum_{\gamma \in \mathcal{E}_h} \|h_\gamma^{-1/2}(v - \Pi_h v)\|_{0;\gamma}^2 \leq C |v|_{1;\Omega}^2. \tag{3.8}$$

4. Gradient recovery-based a posteriori error estimate

The a posteriori error estimate presented in this paper is derived from the following theorem, which is given in [6], in a slightly different problem setting. Its proof makes use of the duality theory in convex analysis.

THEOREM 4.1. *Let $u \in V$ be the unique solution of (2.1), and $w \in V$ an approximation of u. Then for any $\boldsymbol{r}^* = (\boldsymbol{r}_1^*, r_2^*, r_3^*) \in (L^2(\Omega))^d \times L^2(\Omega) \times L^2(\Gamma_C)$, $|r_3^*| \leq g$ a.e.*

on Γ_C, the following error bound is valid:

$$\frac{1}{2}a(u-w, u-w) \leq \int_{\Omega}(|\nabla w + \boldsymbol{r}_1^*|^2 + |w + r_2^*|^2)\, dx$$

$$+ \mathcal{R}(\boldsymbol{r}^*)^2 + \int_{\Gamma_C}(g\,|w| + r_3^* w)\, ds, \qquad (4.1)$$

where the residual $\mathcal{R}(\boldsymbol{r}^*)$ is given by

$$\mathcal{R}(\boldsymbol{r}^*) = \sup_{v \in V}\frac{1}{\|v\|_V}\left\{\int_{\Omega}(\boldsymbol{r}_1^* \cdot \nabla v + (r_2^* + f)\, v)\, dx + \int_{\Gamma_C} r_3^* v\, ds\right\}. \qquad (4.2)$$

In Theorem 4.1, w can be any approximation of the solution u, though we will only use the Galerkin finite element solution u_h in the following. Theorem 4.1 provides a general framework for deriving various a posteriori error estimates with different choices of the auxiliary variable $\boldsymbol{r}^*$. For example, residual-type error estimates were derived in [6] by choosing $\boldsymbol{r}^* = -(\nabla u_h, u_h, g\lambda_h)$, where u_h is the finite element solution and λ_h is provided by Theorem 3.1. In this paper we consider a different choice for auxiliary variable $\boldsymbol{r}^*$, which will lead to a gradient recovery type a posteriori error estimates.

To formulate the error estimator we need a gradient recovery operator. There are many types of gradient recovery operators. In order to have a "good" approximation of the true gradient ∇u, a set of conditions to be satisfied by the recovery operator were identified in [1]. These conditions lead to a more precise characterization of the form of the gradient recovery operator, summarized in Lemma 4.5 in [1]. In particular, the recovered gradient at a vertex A is a linear combination of the values of ∇u_h in a patch surrounding A.

We define the gradient recovery operator $G_h : V_h \to (V_h)^d$ as follows

$$G_h v_h = \sum_{A \in \mathcal{N}_h} G_h v_h(A)\varphi_A, \quad G_h v_h(A) = \frac{1}{|\widetilde{K}_A|}\int_{\widetilde{K}_A}\nabla v_h\, dx. \qquad (4.3)$$

Since linear elements are used,

$$G_h v_h(A) = \sum_{i=1}^{N_A}\alpha_A^i(\nabla v_h)_{K_A^i}, \qquad (4.4)$$

where $(\nabla v_h)_{K_A^i}$ denotes the vector value of the gradient ∇v_h on the element K_A^i, $\widetilde{K}_A = \bigcup_{i=1}^{N_A} K_A^i$, $\alpha_A^i = |K_A^i|/|\widetilde{K}_A|$, $i = 1, \ldots, N_A$.

With the finite element solution u_h, the error estimator based on the gradient recovery can be defined as follows:

$$\eta_G^2 = \sum_{K \in \mathcal{P}_h}\eta_{K,G}^2, \qquad \eta_{K,G}^2 = \|\nabla u_h - G_h u_h\|_{0;K}^2 + \sum_{\gamma \in \mathcal{E}(K) \cap \mathcal{E}_{h,C}} h_\gamma \|G_h u_h \cdot n_\gamma + g\,\lambda_h\|_{0;\gamma}^2.$$

$$(4.5)$$

THEOREM 4.2. *Let u and u_h be the unique solutions of (2.1) and (3.1), respectively. Then the error $u - u_h$ satisfies the a posteriori estimate*

$$\|u - u_h\|_{1;\Omega}^2 \leq C\,\eta_G^2 + C\sum_{A \in \mathcal{N}_{h,0}}\left(h_A^4\|\nabla u_h\|_{0;\widetilde{K}_A}^2 + h_A^2\min_{f_A \in \mathbb{R}}\|f - f_A\|_{0;\widetilde{K}_A}^2\right). \qquad (4.6)$$

Proof. Let u_h be the finite element solution, $\lambda_h \in L^\infty(\Gamma_C)$ be provided by Theorem 3.1 and $G_h u_h$ be the recovered gradient defined in (4.3). Apply Theorem 4.1 with $w = u_h$ and $\boldsymbol{r}^* = -(G_h u_h, u_h, g\lambda_h)$ to obtain

$$\frac{1}{2}a(u - u_h, u - u_h) \leq \int_\Omega |\nabla u_h - G_h u_h|^2 \, dx + \mathcal{R}^2, \tag{4.7}$$

where residual $\mathcal{R}$ is given by

$$\mathcal{R} = \sup_{v \in V} \frac{1}{\|v\|_V} \left\{ \int_\Omega [G_h u_h \cdot \nabla v + (u_h - f)v] \, dx + \int_{\Gamma_C} g\lambda_h v \, ds \right\}. \tag{4.8}$$

Let Π_h be the interpolation operator defined by (3.4). Use characterization (3.2) with $v_h = \Pi_h v$ to get

$$\int_\Omega [\nabla u_h \cdot \nabla \Pi_h v + (u_h - f) \, \Pi_h v] \, dx + \int_{\Gamma_C} g\, \lambda_h \Pi_h v \, ds = 0.$$

Therefore, we can rewrite (4.8) as

$$\mathcal{R} = \sup_{v \in V} \frac{1}{\|v\|_V} \left\{ \int_\Omega [G_h u_h \cdot \nabla(v - \Pi_h v) + (u_h - f)(v - \Pi_h v)] \, dx \right.$$
$$\left. + \int_{\Gamma_C} g\lambda_h(v - \Pi_h v) \, ds + \int_\Omega (G_h u_h - \nabla u_h) \cdot \nabla \Pi_h v \, dx \right\}.$$

By estimate (3.5), we have $\|\nabla \Pi_h v\|_{0;\Omega} \leq C \|v\|_V$ and so

$$\sup_{v \in V} \frac{1}{\|v\|_V} \int_\Omega (G_h u_h - \nabla u_h) \cdot \nabla \Pi_h v \, dx \leq C \|\nabla u_h - G_h u_h\|_{0;\Omega}.$$

Thus,

$$\mathcal{R} \leq C \|\nabla u_h - G_h u_h\|_{0;\Omega} + \mathcal{R}_1 \tag{4.9}$$

where residual $\mathcal{R}_1$ is given by

$$\mathcal{R}_1 = \sup_{v \in V} \frac{1}{\|v\|_V} \left\{ \int_\Omega [G_h u_h \cdot \nabla(v - \Pi_h v) + (u_h - f)(v - \Pi_h v)] \, dx + \int_{\Gamma_C} g\lambda_h(v - \Pi_h v) \, ds \right\}.$$

Integrate by parts over each element $K \in \mathcal{P}_h$ to get

$$\mathcal{R}_1 = \sup_{v \in V} \frac{1}{\|v\|_V} \sum_{K \in \mathcal{P}_h} \left\{ \int_K (-\operatorname{div}(G_h u_h) + u_h - f)(v - \Pi_h v) \, dx \right.$$
$$\left. + \sum_{\gamma \in \mathcal{E}(K)} \int_\gamma G_h u_h \cdot n_\gamma (v - \Pi_h v) \, ds + \sum_{\gamma \in \mathcal{E}(K) \cap \mathcal{E}_{h,C}} \int_\gamma g\, \lambda_h(v - \Pi_h v) \, ds \right\}.$$
$$\tag{4.10}$$

Since $G_h u_h$ is continuous, the integrals on interior sides $\gamma \in \mathcal{E}_{h,0}$ in (4.10) cancel each other. Rewrite

$$-\operatorname{div}(G_h u_h) + u_h - f = \operatorname{div}(\nabla u_h - G_h u_h) + (-\Delta u_h + u_h - f)$$

and rearrange the terms in (4.10) to obtain

$$
\begin{aligned}
\mathcal{R}_1 = \sup_{v \in V} \frac{1}{\|v\|_V} \Bigg\{ &\sum_{K \in \mathcal{P}_h} \int_K \mathrm{div}(\nabla u_h - G_h u_h)(v - \Pi_h v)\, dx \\
&+ \sum_{K \in \mathcal{P}_h} \int_K r_K (v - \Pi_h v)\, dx \\
&+ \sum_{\gamma \in \mathcal{E}_{h,C}} \int_\gamma (G_h u_h \cdot n_\gamma + g\, \lambda_h)(v - \Pi_h v)\, ds \Bigg\} \\
= \sup_{v \in V} \frac{1}{\|v\|_V} &\{ I + II + III \},
\end{aligned}
\tag{4.11}
$$

where $r_K = -\Delta u_h + u_h - f = u_h - f$ denotes the interior residual on element $K \in \mathcal{P}_h$. We will use r to denote the piecewise interior residual, i.e., $r|_K = r_K$ for $K \in \mathcal{P}_h$. To estimate the first summand on the right side of (4.11), we use an elementwise inverse inequality of the form

$$
\|\mathrm{div}(\nabla u_h - G_h u_h)\|_{0;K} \le C h_K^{-1} \|\nabla u_h - G_h u_h\|_{0;K}.
\tag{4.12}
$$

Apply the Cauchy-Schwarz inequality, the inverse inequality (4.12) and the estimate (3.7) to get

$$
\begin{aligned}
I &\le C \sum_{K \in \mathcal{P}_h} \|\mathrm{div}(\nabla u_h - G_h u_h)\|_{0;K} \|v - \Pi_h v\|_{0;K} \\
&\le C \sum_{K \in \mathcal{P}_h} \|\nabla u_h - G_h u_h\|_{0;K} \|h_K^{-1}(v - \Pi_h v)\|_{0;K} \\
&\le C \left(\sum_{K \in \mathcal{P}_h} \|\nabla u_h - G_h u_h\|_{0;K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{P}_h} \|h_K^{-1}(v - \Pi_h v)\|_{0;K}^2 \right)^{1/2} \\
&\le C\, |v|_{1;\Omega} \left(\sum_{K \in \mathcal{P}_h} \|\nabla u_h - G_h u_h\|_{0;K}^2 \right)^{1/2}.
\end{aligned}
\tag{4.13}
$$

For the second summand, we apply the estimate (3.6) to obtain

$$
II \le C|v|_{1;\Omega} \left(\sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{r_A \in \mathbb{R}} \|r - r_A\|_{0;\widetilde{K}_A}^2 \right)^{1/2}.
$$

Now

$$
\begin{aligned}
\sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{r_A \in \mathbb{R}} \|r - r_A\|_{0;\widetilde{K}_A}^2 \le\; &2 \sum_{A \in \mathcal{N}_{h,0}} h_A^2 \|u_h - \overline{u_h}\|_{0;\widetilde{K}_A}^2 \\
&+ 2 \sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{f_A \in \mathbb{R}} \|f - f_A\|_{0;\widetilde{K}_A}^2,
\end{aligned}
$$

where $\overline{u_h}$ denotes the integral mean of u_h over $\widetilde{K}_A$. Use the Poincare inequality and

an inverse inequality of the form (4.12) to get

$$
\sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{r_A \in \mathbb{R}} \|r - r_A\|_{0;\widetilde{K}_A}^2 \leq C \sum_{A \in \mathcal{N}_{h,0}} h_A^4 \|\nabla u_h\|_{0;\widetilde{K}_A}^2
$$
$$
+ 2 \sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{f_A \in \mathbb{R}} \|f - f_A\|_{0;\widetilde{K}_A}^2 .
$$

Therefore,

$$
II \leq C|v|_{1;\Omega} \sum_{A \in \mathcal{N}_{h,0}} \left(h_A^4 \|\nabla u_h\|_{0;\widetilde{K}_A}^2 + h_A^2 \min_{f_A \in \mathbb{R}} \|f - f_A\|_{0;\widetilde{K}_A}^2 \right). \tag{4.14}
$$

Finally, with the aid of the Cauchy-Schwarz inequality and the estimate (3.8), the third summand on the right side of (4.11) can be bounded by

$$
III \leq C\,|v|_{1;\Omega} \left(\sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma \|G_h u_h \cdot n_\gamma + g\,\lambda_h\|_{0;\gamma}^2 \right)^{1/2}. \tag{4.15}
$$

Inserting (4.13), (4.14) and (4.15) into (4.11), and using the Cauchy-Schwarz inequality and (4.9), we deduce that

$$
\mathcal{R} \leq C \left\{ \sum_{K \in \mathcal{P}_h} \|\nabla u_h - G_h u_h\|_{0;K}^2 + \sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma \|G_h u_h \cdot n_\gamma + g\,\lambda_h\|_{0;\gamma}^2 \right.
$$
$$
\left. + \sum_{A \in \mathcal{N}_{h,0}} \left(h_A^4 \|\nabla u_h\|_{0;\widetilde{K}_A}^2 + h_A^2 \min_{f_A \in \mathbb{R}} \|f - f_A\|_{0;\widetilde{K}_A}^2 \right) \right\}^{1/2}. \tag{4.16}
$$

Split the first term on the right side of estimate (4.7) into local contributions from each $K \in \mathcal{P}_h$ and insert (4.16) to conclude the proof. $\qquad\square$

We can write (4.6) as

$$
\|u - u_h\|_{1;\Omega} \leq C\eta_G + R_h \tag{4.17}
$$

where

$$
R_h = C \left[\sum_{A \in \mathcal{N}_{h,0}} h_A^4 \|\nabla u_h\|_{0;\widetilde{K}_A}^2 \right]^{1/2} + C \left[\sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{f_A \in \mathbb{R}} \|f - f_A\|_{0;\widetilde{K}_A}^2 \right]^{1/2}.
$$

We observe that usually the term R_h is of higher order compared to $\|u - u_h\|_{1;\Omega}$ which is of order $O(h)$ in the nondegenerate situations. This observation is argued as follows: First, it is easy to show from the definition of the finite element solutions that there is a constant C such that $\|u_h\|_{1;\Omega} \leq C$ for any h. So the term

$$
\left[\sum_{A \in \mathcal{N}_{h,0}} h_A^4 \|\nabla u_h\|_{0;\widetilde{K}_A}^2 \right]^{1/2}
$$

is bounded by $O(h^2)$. Next, for $f \in L^2(\Omega)$,

$$
\left(\sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{f_A \in \mathbb{R}} \|f - f_A\|^2_{0;\widetilde{K}_A} \right)^{1/2} = o(h);
$$

and if $f \in H^1(\Omega)$, then

$$
\left(\sum_{A \in \mathcal{N}_{h,0}} h_A^2 \min_{f_A \in \mathbb{R}} \|f - f_A\|^2_{0;\widetilde{K}_A} \right)^{1/2} = O(h^2).
$$

Thus, (4.17) illustrates the reliability of the error estimator η_G.

5. Efficiency of the estimator

In what follows, we investigate the efficiency of the error estimator η_G. For this purpose, we will prove that for each element $K \in \mathcal{P}_h$, $\eta_{K,G}$ is dominated by the residual-type estimator $\eta_{K,R}$ derived in [6]:

$$
\eta^2_{K,R} = h_K^2 \|r_K\|^2_{0;K} + \sum_{\gamma \in \mathcal{E}(K)} h_\gamma \|R_\gamma\|^2_{0;\gamma}
$$

with the interior residual $r_K = -\Delta u_h + u_h - f = u_h - f$ in $K \in \mathcal{P}_h$ and side residual

$$
R_\gamma = \begin{cases} [\frac{\partial u_h}{\partial n}]_\gamma & \text{if } \gamma \in \mathcal{E}_{h,0}, \\ \frac{\partial u_h}{\partial n} + g\,\lambda_h & \text{if } \gamma \in \mathcal{E}_{h,C}. \end{cases} \tag{5.1}
$$

For the error estimator $\eta^2_R = \sum_{K \in \mathcal{P}_h} \eta^2_{K,R}$, we have proved in [6] the following estimate

$$
\eta^2_R \leq C \Bigg(\|u - u_h\|^2_{1;\Omega} + \sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma \|\lambda - \lambda_h\|^2_{0;\gamma}
$$

$$
+ \sum_{K \in \mathcal{P}_h} h_K^2 \|r_K - \overline{r_K}\|^2_{0;K} + \sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma \|R_\gamma - \overline{R_\gamma}\|^2_{0;\gamma} \Bigg) \tag{5.2}
$$

with discontinuous piecewise polynomial approximations $\overline{r_K}$, $\overline{R_\gamma}$ of r_K, R_γ.

LEMMA 5.1. *Let $\eta_{K,G}$ be defined by (4.5). Then the following bound holds*

$$
\eta^2_{K,G} \leq C \left(\sum_{\gamma \in \mathcal{E}(K) \cap \mathcal{E}_{h,C}} h_\gamma \|R_\gamma\|^2_{0;\gamma} + \sum_{\gamma' \in \mathcal{E}_{\widetilde{K}}} h_{\gamma'} \|R_{\gamma'}\|^2_{0;\gamma'} \right), \tag{5.3}
$$

where $\mathcal{E}_{\widetilde{K}}$ denotes the set of inner edges of the patch $\widetilde{K}$ corresponding to the element K.

Proof. It follows from the definition of G_h that on each element K,

$$
|\nabla u_h - G_h u_h|^2 = \left| \sum_{A \in \mathcal{N}(K)} \varphi_A \Big(\sum_{i=1}^{N_A} \alpha^i_A ((\nabla u_h)_K - (\nabla u_h)_{K^i_A}) \Big) \right|^2
$$

$$
\leq C \sum_{K' \subset \widetilde{K}} |(\nabla u_h)_K - (\nabla u_h)_{K'}|^2. \tag{5.4}
$$

For any $K' \subset \widetilde{K}$ there is a sequence of inner edges $\gamma_1, \ldots, \gamma_m$, such that $\widetilde{\gamma}_j \cap \widetilde{\gamma}_{j+1} \neq \emptyset$ and $K \subset \widetilde{\gamma}_1$, $K' \subset \widetilde{\gamma}_m$. Hence,

$$|(\nabla u_h)_K - (\nabla u_h)_{K'}| \leq \left| \sum_{j=1}^{m} [\nabla u_h]_{\gamma_j} \right| \leq \sum_{j=1}^{m} |[\nabla u_h]_{\gamma_j}| \leq \sum_{\gamma \in \mathcal{E}_{\widetilde{K}}} |[\nabla u_h]_\gamma|. \tag{5.5}$$

Since u_h is continuous in $\overline{\Omega}$, $[\partial u_h / \partial t]_\gamma = 0$ for all $\gamma \in \mathcal{E}_{h,0}$, where $\partial u_h / \partial t$ is the tangential derivative of u_h. Therefore, $|[\nabla u_h]_\gamma| = |[\partial u_h / \partial n]_\gamma|$ if $\gamma \in \mathcal{E}_{h,0}$. The estimates (5.4) and (5.5) together with the shape regularity of the partition $\mathcal{P}_h$ imply

$$\|\nabla u_h - G_h u_h\|_{0;K}^2 \leq C h_K^2 \sum_{\gamma \in \mathcal{E}_{\widetilde{K}}} \left[\frac{\partial u_h}{\partial n} \right]_\gamma^2 \leq C \sum_{\gamma \in \mathcal{E}_{\widetilde{K}}} h_\gamma \int_\gamma \left[\frac{\partial u_h}{\partial n} \right]_\gamma^2 ds. \tag{5.6}$$

Let $K \in \mathcal{P}_h$ be such that $\mathcal{E}(K) \cap \mathcal{E}_{h,C} = \emptyset$. It follows from (5.6) and definitions of η_K and R_γ that

$$\eta_K^2 \leq C \sum_{\gamma \in \mathcal{E}_{\widetilde{K}}} h_\gamma \|R_\gamma\|_{0;\gamma}^2. \tag{5.7}$$

Consider now the case when the element K has at least one side lying on the boundary Γ_C. Let $\gamma \in \mathcal{E}(K) \cap \mathcal{E}_{h,C}$. Apply the triangle inequality to get

$$\eta_{K,G}^2 = \|\nabla u_h - G_h u_h\|_{0;K}^2 + \sum_{\gamma \in \mathcal{E}(K) \cap \mathcal{E}_{h,C}} h_\gamma \|G_h u_h \cdot n_\gamma + g\lambda_h\|_{0;\gamma}^2$$

$$\leq \|\nabla u_h - G_h u_h\|_{0;K}^2 + \sum_{\gamma \in \mathcal{E}(K) \cap \mathcal{E}_{h,C}} h_\gamma \left(\|(\nabla u_h - G_h u_h) \cdot n_\gamma\|_{0;\gamma} + \|R_\gamma\|_{0;\gamma} \right)^2$$

$$\leq \|\nabla u_h - G_h u_h\|_{0;K}^2 + \sum_{\gamma \in \mathcal{E}(K) \cap \mathcal{E}_{h,C}} 2h_\gamma \left(\|(\nabla u_h - G_h u_h) \cdot n_\gamma\|_{0;\gamma}^2 + \|R_\gamma\|_{0;\gamma}^2 \right).$$

$$\tag{5.8}$$

From an inverse inequality and (5.6) we have

$$h_\gamma \|(\nabla u_h - G_h u_h) \cdot n_\gamma\|_{0;\gamma}^2 \leq h_K \|\nabla u_h - G_h u_h\|_{0;\partial K}^2$$

$$\leq C \|\nabla u_h - G_h u_h\|_{0;K}^2$$

$$\leq C \sum_{\gamma' \in \mathcal{E}_{\widetilde{K}}} h_{\gamma'} \|R_{\gamma'}\|_{0;\gamma'}^2. \tag{5.9}$$

Inserting (5.6) and (5.9) into (5.8) concludes the proof.   □

From Lemma 5.1 and the inequality (5.2) we obtain

$$\eta_G^2 \leq C \left(\|u - u_h\|_{1;\Omega}^2 + \sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma \|\lambda - \lambda_h\|_{0;\gamma}^2 \right.$$

$$\left. + \sum_{K \in \mathcal{P}_h} h_K^2 \|r_K - \overline{r_K}\|_{0;K}^2 + \sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma \|R_\gamma - \overline{R_\gamma}\|_{0;\gamma}^2 \right). \tag{5.10}$$

Since linear elements are used, the term $\sum_{K \in \mathcal{P}_h} h_K^2 \|r_K - \overline{r_K}\|_{0;K}^2$ can be replaced by $\sum_{K \in \mathcal{P}_h} h_K^2 \|f - \overline{f}\|_{0,K}^2$, and the term $\sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma \|R_\gamma - \overline{R_\gamma}\|_{0;\gamma}^2$ can be replaced

by $\sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma g^2 \|\lambda_h - \overline{\lambda_h}\|_{0,\gamma}^2$, for discontinuous piecewise polynomial approximations $\overline{f}$ and $\overline{\lambda_h}$. The term $\sum_{K \in \mathcal{P}_h} h_K^2 \|f - \overline{f}\|_{0,K}^2$ is clearly of higher order than $O(h^2)$. Numerical evidence from the next section shows that $\sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma \|\lambda - \lambda_h\|_{0;\gamma}^2$, and $\sum_{\gamma \in \mathcal{E}_{h,C}} h_\gamma g^2 \|\lambda_h - \overline{\lambda_h}\|_{0,\gamma}^2$ as well, can be expected to be of higher order than $\|u - u_h\|_{1;\Omega}^2$.

6. Numerical experiment

In this section, we show some numerical results in solving the following problem: Find $u \in H^1(\Omega)$ such that

$$\int_\Omega [\nabla u \cdot \nabla (v-u) + u\,(v-u)]\,dx + g \int_\Gamma |v|\,ds - g \int_\Gamma |u|\,ds \geq \int_\Omega f\,(v-u)\,dx \quad \forall\, v \in H^1(\Omega).$$

We choose $\Omega = [0,1] \times [0,1]$, $\Gamma_C = \Gamma$, $g = 1$ and $f = -\Delta w + w$ with

$$w(\boldsymbol{x}) = \arctan(20(r - 1/2)), \quad r^2 = (x_1 - 0.8)^2 + (x_2 + 0.2)^2.$$

Triangular partitioning and linear elements are used in the experiment. The implementation was performed in MATLAB and a seven-point Gauss-Legendre quadrature was used to compute the load vector on each triangle. Numerical integration over a general triangle is reduced that over a reference triangle through a linear transformation of the variables. On the reference triangle

$$\hat{K} = \{(\xi, \eta) \mid \xi \geq 0,\ \eta \geq 0,\ 1 - \xi - \eta \geq 0\},$$

the seven-point Gauss-Legendre quadrature formula is defined by

$$\int_{\hat{K}} F(\xi, \eta)\,d\xi d\eta \approx \sum_{i=1}^{7} \omega_i F(\xi_i, \eta_i),$$

where the nodes (ξ_i, η_i) and weights ω_i are given in Table 6.1. The discretized solution is computed by solving the equivalent minimization problem using an over-relaxation method with a relative error tolerance, in the maximum norm, of 10^{-6} (see [13], [14]).

i	ξ_i	η_i	ω_i
1	$1/3$	$1/3$	$9/80$
2	$(6 + \sqrt{15})/21$	$(6 + \sqrt{15})/21$	$(155 + \sqrt{15})/2400$
3	$(9 - 2\sqrt{15})/21$	$(6 + \sqrt{15})/21$	$(155 + \sqrt{15})/2400$
4	$(6 + \sqrt{15})/21$	$(9 - 2\sqrt{15})/21$	$(155 + \sqrt{15})/2400$
5	$(6 - \sqrt{15})/21$	$(6 - \sqrt{15})/21$	$(155 - \sqrt{15})/2400$
6	$(9 + 2\sqrt{15})/21$	$(6 - \sqrt{15})/21$	$(155 - \sqrt{15})/2400$
7	$(6 - \sqrt{15})/21$	$(9 + 2\sqrt{15})/21$	$(155 - \sqrt{15})/2400$

TABLE 6.1. *Nodes and weights of 7-point Gauss-Legendre quadrature formula over reference triangle.*

In order to show the effectiveness of the adaptive procedure we compare numerical convergence orders of the approximate solutions. We compute these orders by considering families of uniform and adaptively refined partitions. Consider a sequence

of finite element solutions u_h^{un} based on uniform partitions of the domain Ω. Starting with an initial coarse partition $\mathcal{P}_1$, we construct a family of nested meshes by subdividing each element into four congruent elements. The solution from the most refined mesh will be taken as the "exact" solution u, that will be used to compute the errors of the approximate solutions obtained on the other meshes.

Adaptive finite element solutions are obtained by the following algorithm:
1. Start with the initial partition $\mathcal{P}_h$ and corresponding finite element subspace V_h.
2. Compute the finite element solution $u_h^{ad} \in V_h$.
3. For each element $K \in \mathcal{P}_h$ compute the error estimator η_K (residual or gradient recovery type).
4. Let $\eta = \frac{1}{N} \sum_{K \in \mathcal{P}_h} \eta_K$ with N being the number of elements in partition $\mathcal{P}_h$. An element K is marked for refinement if $\eta_K > \mu\,\eta$, where μ is a prescribed threshold. We take $\mu = 0.5$.
5. Perform refinement and obtain a new triangulation $\mathcal{P}_h$.
6. Return to step 2.

In the computation of the error indicator η_K we make use of the multiplier λ_h defined on Γ. In what follows we describe how λ_h can be (approximately) recovered from the solution u_h using characterization

$$a(u_h, v_h) + \int_\Gamma g\,\lambda_h\,v_h\,ds = l(v_h) \quad \forall\,v_h \in V_h. \tag{6.1}$$

Denote by $\{\boldsymbol{x}^i\}_{i=1}^m$ the nodes of the partition $\mathcal{P}_h$ belonging to Γ, and let $\{\varphi_i\}_{i=1}^m$ be the basis functions corresponding to the nodes $\{\boldsymbol{x}^i\}$. We first determine a piecewise linear function

$$\lambda_{h,0} = \sum_{i=1}^m \lambda_{h,0}^i \varphi_i$$

by requiring an analogue of (6.1):

$$a(u_h, v_h) + \int_\Gamma g\,\lambda_{h,0} v_h\,ds = l(v_h) \quad \forall\,v_h \in V_h. \tag{6.2}$$

Denote $\boldsymbol{\lambda}_{h,0} = (\lambda_{h,0}^1, \ldots, \lambda_{h,0}^m)^T$. We then project the components of $\boldsymbol{\lambda}_{h,0}$ onto the interval $[-1, 1]$ to get $\boldsymbol{\lambda}_{h,1} = (\lambda_{h,1}^1, \ldots, \lambda_{h,1}^m)^T$:

$$\lambda_{h,1}^i = \begin{cases} -1 & \text{if } \lambda_{h,0}^i < -1, \\ \lambda_{h,0}^i & \text{if } -1 \le \lambda_{h,0}^i \le 1, \\ 1 & \text{if } \lambda_{h,0}^i > -1, \end{cases}$$

for $i = 1, 2, \ldots, m$. The piecewise linear approximation of the multiplier λ_h on Γ can be computed as

$$\lambda_{h,1} = \sum_{i=1}^m \lambda_{h,1}^i \varphi_i. \tag{6.3}$$

We briefly comment on the method for finding $\boldsymbol{\lambda}_{h,0}$. Let $n = \dim V_h$. Denote by $\boldsymbol{K}$ the standard $(n \times n)$ stiffness matrix and by $\boldsymbol{l} \in \mathbb{R}^n$ the standard load vector. Let
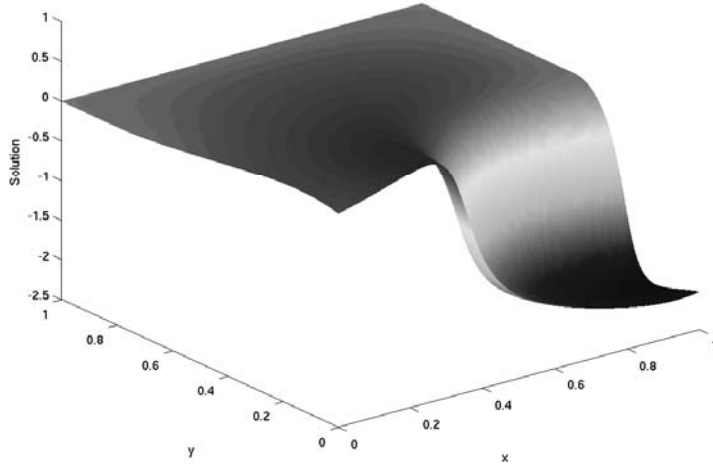
FIG. 6.1. *Solution on a uniform mesh with 66,049 nodes.*

$\boldsymbol{u} \in \mathbb{R}^n$ be the nodal value vector of the finite element solution u_h. Then the algebraic representation of (6.2) becomes

$$(\boldsymbol{K}\boldsymbol{u}, \boldsymbol{v})_{\mathbb{R}^n} + (\boldsymbol{M}\boldsymbol{\lambda}_{h,0}, \boldsymbol{v}_c)_{\mathbb{R}^m} = (\boldsymbol{l}, \boldsymbol{v})_{\mathbb{R}^n} \quad \forall \boldsymbol{v} \in \mathbb{R}^n, \tag{6.4}$$

where $\boldsymbol{v}_c$ denotes the subvector of $\boldsymbol{v}$, containing the nodal values of v_h at the nodes $\{\boldsymbol{x}^i\}_{i=1}^m \subset \Gamma$ and $\boldsymbol{M}$ is a tridiagonal $(m \times m)$ matrix. We can write $\boldsymbol{v} = (\boldsymbol{v}_i^T, \boldsymbol{v}_c^T)^T \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ by assuming that the components of $\boldsymbol{v}_c$ are listed last. We similarly split $\boldsymbol{l}$ to $\boldsymbol{l}_i$ and $\boldsymbol{l}_c$. This decomposition yields a block structure for $\boldsymbol{K}$,

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_{ii} & \boldsymbol{K}_{ic} \\ \boldsymbol{K}_{ci} & \boldsymbol{K}_{cc} \end{pmatrix}.$$

Then (6.4) is equivalent to the following two relations:

$$\boldsymbol{K}_{ii}\boldsymbol{u}_i + \boldsymbol{K}_{ic}\boldsymbol{u}_c = \boldsymbol{l}_i,$$
$$\boldsymbol{K}_{ci}\boldsymbol{u}_i + \boldsymbol{K}_{cc}\boldsymbol{u}_c + \boldsymbol{M}\boldsymbol{\lambda}_{h,0} = \boldsymbol{l}_c.$$

Once the approximate solution u_h is computed, we can obtain from the second relation that

$$\boldsymbol{\lambda}_{h,0} = \boldsymbol{M}^{-1}(\boldsymbol{l}_c - \boldsymbol{K}_{ci}\boldsymbol{u}_i - \boldsymbol{K}_{cc}\boldsymbol{u}_c).$$

We start with a coarse triangulation shown on the left plot in Figure 6.2. Here, the interval $[0, 1]$ is divided into $1/h$ equal parts with $h = 1/4$ which is successively halved. The numerical solution corresponding to $h = 1/256$ is taken as the "exact" solution u. The "exact" solution is shown in Figure 6.1.

We use the regular refinement technique (*red-blue-green* refinement), where the triangle is divided into four triangles by joining the midpoints of edges and adjacent

triangles are refined in order to avoid hanging nodes. For a detailed description of this and other refinement techniques currently used see e.g. [15] and references therein. Also, in order to improve the quality of triangulation, a smoothing procedure is used after each refinement. For each triangle K of the triangulation we compute the triangle quality measure defined by

$$Q(K) = \frac{4\sqrt{3}\,\text{area}(K)}{h_1^2 + h_2^2 + h_3^2},$$

where h_i, $i = 1, 2, 3$, are the side lengths of the triangle K. Note that $Q(K) = 1$ if $h_1 = h_2 = h_3$. A triangle is viewed to be of acceptable quality if $Q > 0.6$, otherwise we modify the mesh by moving the interior nodes toward the center of mass of the polygon formed by the adjacent triangles.

We use u_h^{un} for finite element solutions on uniform meshes, and u_h^{ad} for finite element solutions on adaptive meshes. Since adaptive solutions are involved, numerical solution errors will be plotted against the number of degrees of freedom, rather than the meshsize.

We use an adaptive procedure based on both residual type and gradient recovery type estimates to obtain a sequence of approximate solutions u_h^{ad}. The adaptive finite element mesh after 5 adaptive iterations is shown on the right plot in Figure 6.2. Figures 6.3 and 6.4 contain the error values $\|u - u_h^{un}\|_{1;\Omega}$ and $\|u - u_h^{ad}\|_{1;\Omega}$. We observe a substantial improvement of the efficiency using adaptively refined meshes. The errors $\|u - u_h^{ad}\|_{1;\Omega}$ computed using residual type and gradient recovery type estimators are shown in Figure 6.5. Table 6.2 contains the values of C_I computed for uniform and adaptive solutions:

$$C_I = \frac{\|u - u_h\|_{1;\Omega}}{\eta_I}, \quad \eta_I = (\sum_K \eta_{K,I}^2)^{1/2}, \quad I \in \{R, G\}. \tag{6.5}$$

It is clear from Table 6.2 that for this numerical example, the gradient recovery type error estimator provides a better prediction of the true error than the residual type error estimator, a phenomenon observed in numerous references. Note that implementation of the gradient recovery type error estimator is simpler than that of the residual type error estimator.

To have an idea of the convergence behavior of the discrete Lagrange multipliers, we analyze the errors $\|\lambda - \lambda_h\|_{0;\Gamma}$ corresponding to the sequence of uniform refinements. Here, λ is the Lagrange multiplier corresponding to the parameter $h = 1/256$. Figure 6.6 provides the error values $\|u - u_h^{un}\|_{1;\Omega}$ and $h^{1/2}\|\lambda - \lambda_h\|_{0;\Gamma}$. The numerical convergence order of $h^{1/2}\|\lambda - \lambda_h\|_{0;\Gamma}$ is obviously higher than that of $\|u - u_h^{un}\|_{1;\Omega}$, indicating that the second term in the efficiency bound (5.10) is expected to be of higher order compared to the first term $\|u - u_h\|_{1;\Omega}^2$. Graphs of $\lambda_{h,0}$ and $\lambda_{h,1}$ with $h = 1/256$ are provided in Figures 6.7 and 6.8.

<div align="center">REFERENCES</div>

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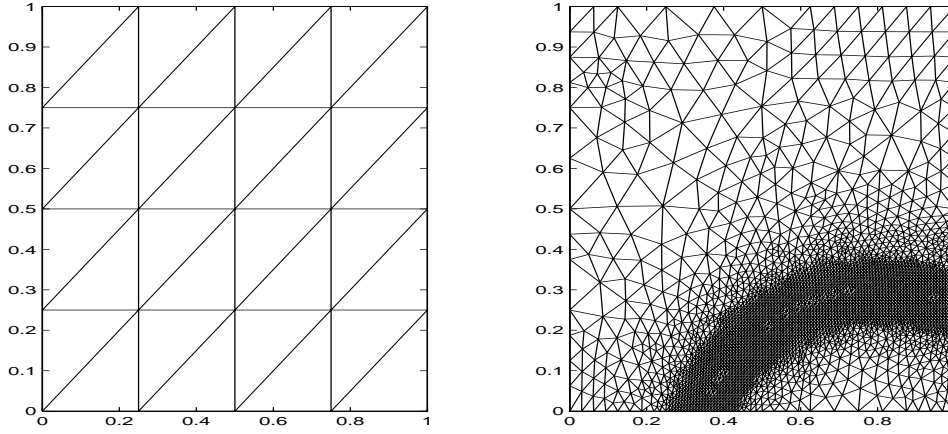
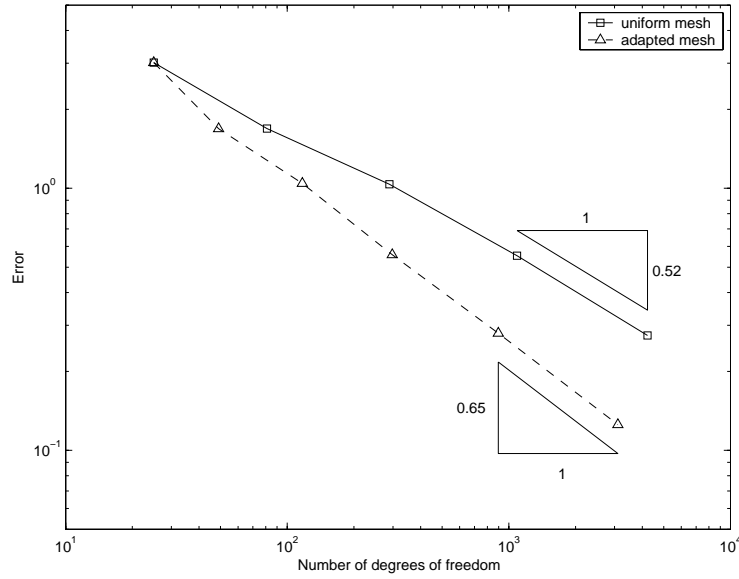FIG. 6.2. *Initial partition and locally refined partition after 5 iterations.*

FIG. 6.3. *Residual type estimator.* $\|u - u_h^{un}\|_{1;\Omega}$ *($\square$) vs.* $\|u - u_h^{ad}\|_{1;\Omega}$ *($\triangle$).*

h	1/4	1/8	1/16	1/32	1/64	1/128
C_R^{un}	1.13e-01	1.23e-01	1.44e-01	1.47e-01	1.44e-01	1.29e-01
C_R^{ad}	1.13e-01	1.23e-01	1.45e-01	1.46e-01	1.39e-01	1.21e-01
C_G^{un}	1.17e+00	8.23e-01	8.56e-01	8.89e-01	9.18e-01	8.48e-01
C_G^{ad}	1.17e+00	8.17e-01	8.52e-01	8.76e-01	8.80e-01	7.97e-01

TABLE 6.2. *Constants C_R and C_G.*
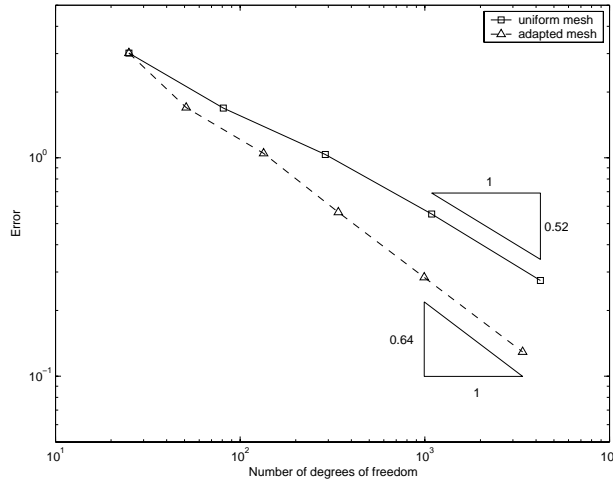
FIG. 6.4. *Gradient recovery type estimator.* $\|u - u_h^{un}\|_{1;\Omega}$ ($\square$) *vs.* $\|u - u_h^{ad}\|_{1;\Omega}$ ($\triangle$).
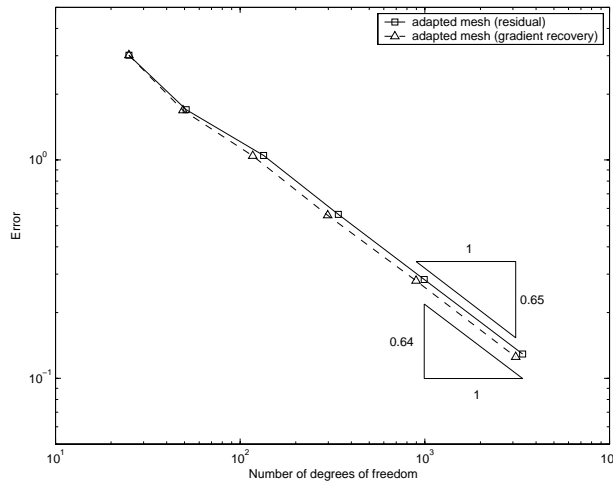

FIG. 6.5. $\|u - u_h^{ad}\|_{1;\Omega}$ *for residual type estimator* ($\square$) *vs.* $\|u - u_h^{ad}\|_{1;\Omega}$ *for gradient recovery type estimator* ($\triangle$).

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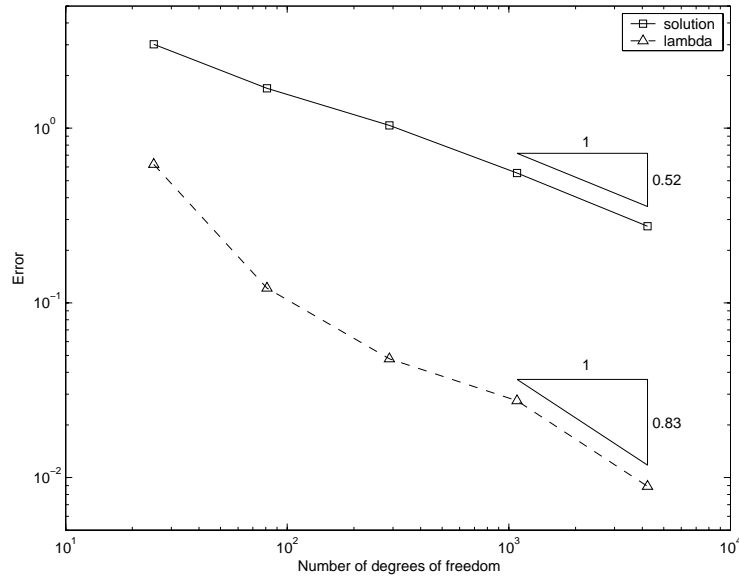
FIG. 6.6. $\|u - u_h^{un}\|_{1;\Omega}$ ($\square$) vs. $h^{1/2}\|\lambda - \lambda_h\|_{0;\Gamma}$ ($\triangle$).
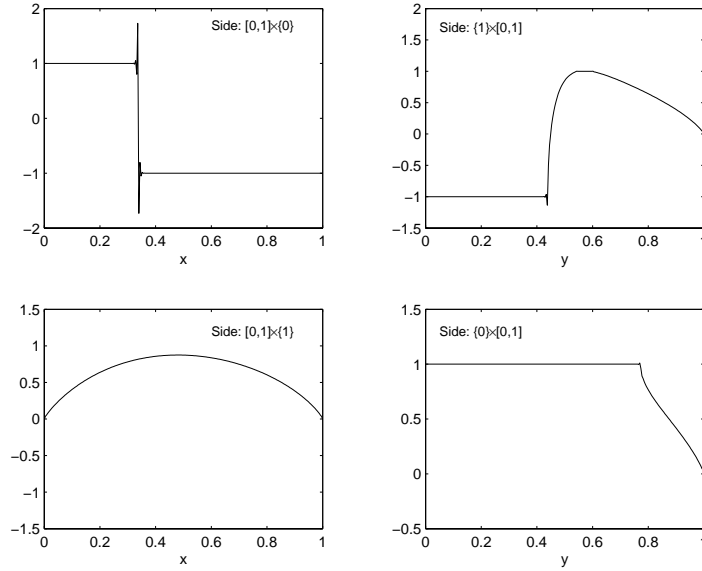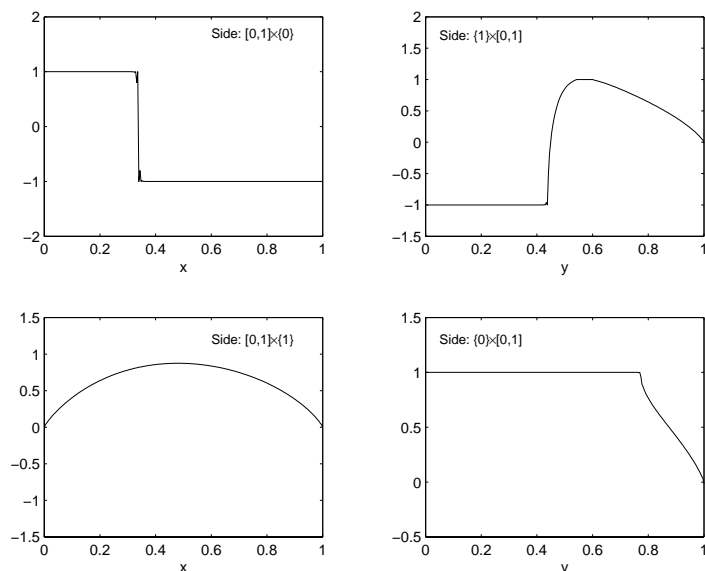

FIG. 6.7. Plots of $\lambda_{h,0}$.

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FIG. 6.8. *Plots of* $\lambda_{h,1}$.

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