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GLOBAL PROPERTIES OF THE SOLUTIONS OF THE EINSTEIN-BOLTZMANN SYSTEM WITH COSMOLOGICAL CONSTANT IN THE ROBERTSON-WALKER SPACE-TIME*

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Abstract. We study the global properties of the solutions for the initial value problem for the Einstein-Boltzmann system with positive cosmological constant and arbitrarily large initial data, in the spatially homogeneous case, in a Robertson-Walker space-time.

Key words. Asymptotic behavior, geodesic completness, Einstein-Boltzmann, cosmological constant.

AMS subject classifications. 83xx.

1. Introduction

In the mathematical study of General Relativity, after establishing global existence of the solutions for the Einstein equations coupled to various field equations one of the main problems is the properties of these solutions. On the other hand, it is necessary to know if the space time obtained is future complete.

In the case of collisionless matter, the particle distributions are governed by the Einstein-Vlasov system in the pure gravitational case, and by this system coupled to other field equations, if other fields are involved. Due to its importance in kinetic theory, several authors have studied and proved the local and global in time theorem for the Einstein-Vlasov system. Several authors also studied in the case of the Einstein-Vlasov equation, and the asymptotic behavior of its solutions. They have also investigated future geodesic completness. In [4], Lee studied asymptotic behavior of the Einstein-Vlasov system with positive cosmological constant. She dealt with a class of space time possessing a compact Cauchy hypersurface. This allows her to study the asymptotic behavior of the Einstein-Vlasov system with a positive cosmological constant.

In the present paper, we consider the collisional evolution of a kind of uncharged massive particles under the only influence of their own gravitational field which is a function of the position of the particles. The phenomena are governed by the coupled Einstein-Boltzmann system. The Boltzmann equation generalizes the Vlasov equation, in the sense that it takes into account the interaction between the particles. The interaction is defined by a non-linear operator Q called the "collision operator". In the binary and elastic scheme, due to Lichnerowicz and Chernikov [6] we adopt, at a given position, only two particles (or two kinds of particles) collide in an instantaneous shock, without destroying each other. The collision only affects their momenta, which are not the same, before and after the shock. Only the sum of the two momenta being preserved.

The Einstein-Boltzmann system is coupled in the sense that the distribution function f, which is subject to the Boltzmann equation, generates the source $T_{\alpha\beta}$ of the Einstein equations, whereas the metric g, which is subject to the Einstein equations, is a factor in the collision operator.

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The geometric frame we are looking for is a spatially homogeneous Frieman-Lemaitre-Robertson-Walker space time, we will call a "Robertson-Walker space time". In Cosmology, it is the basic model for the study of the spanding universe. The metric tensor has only one unknown component which we denote by a, which is a strictly positive function called the cosmological expansion factor. The spatial homogeneity means that a depends only on the time t, and the distribution function f depends only on the time t and the 4-momentum p of the particles.

In this paper, we consider the Einstein equation with cosmological constant Λ. Our motivation is from a physical point of view. Recent measurements show that the case $\Lambda>0$ is physically very interesting in the sense that it models the astrophysical observation that the expansion of universe is accelerating. In mathematical terms, this means that the mean curvature of the space time tends to this constant at late times; see [11]. In this paper we study the asymptotic behavior of the solution of this system. Global existence of the solution is proved in [9]. We also study the geodesic completness of the solutions obtained.

The paper is organized as follows: In section 2, we recall the essential results of the existence theorem of the Einstein-Boltzmann system with positive cosmological constant in the Robertson-Walker space time. In section 3, we study the asymptotic behavior. Section 4 is devoted to the geodesic completness.

2. Preliminary results

2.1. Notations and function spaces. Greek indicies range from 0 to 3 and Latin indicies range from 1 to 3. We adopt the Einstein summation convention $a_\alpha b^\beta = \Sigma a_\alpha b^\beta$. We consider as background space time a Robertson-Walker space time where for $x=(x^\alpha)=(t,x^i)\in\mathbb{R}^4$, t denotes the time and $\bar{x}=(x^i)$ the space. g stands for the metric tensor with signature $(-,+,+,+)$ which can be written as:

$$g=-dt^2+a^2(t)[(dx^1)^2+(dx^2)^2+(dx^3)^2], \tag{2.1}$$

in which a is a strictly positive function of t, called cosmological expansion factor.

We consider the collisional evolution of a kind of uncharged massive relativistic particles in the time oriented space time $(\mathbb{R}^4,g)$. The particles are statistically described by their distribution function, denoted by f, which is a non-negative real valued function of both the position (x^α) and the 4-momentum (p^α) of the particles, and which defines the coordinates of the tangent bundle $T(\mathbb{R}^4)$. We then have:

$$f:T(\mathbb{R}^4)\simeq\mathbb{R}^4\times\mathbb{R}^4\to\mathbb{R}_+, \qquad (x^\alpha,p^\alpha)\mapsto f(x^\alpha,p^\alpha)\in\mathbb{R}_+. \tag{2.2}$$

For $\bar{p}=(p^i)$, $\bar{q}=(q^i)\in\mathbb{R}^3$, we set

$$\bar{p}.\bar{q}=\sum_{i=1}^3 p^i q^i \qquad \text{and} \qquad |\bar{p}|=[\sum_{i=1}^3 (p^i)^2]^{\frac{1}{2}}. \tag{2.3}$$

We suppose the rest mass $m=1$. The relativistic particles are then required to move in the future sheet of the mass-shell whose equation is $g(p,p)=-1$. From this, we deduce using (2.1) and (2.3)

$$p^0=\sqrt{1+a^2|\bar{p}|^2}. \tag{2.4}$$

The choice of $p^0>0$ symbolizes the fact that the particles eject towards the future.

From (2.4), f depends only on t and $\bar{p}=(p^i)$. The appropriate frame we will refer to will be the subspace of $L^1(\mathbb{R}^3)$, denoted $L^1_2(\mathbb{R}^3)$ and defined by:

$$L^1_2(\mathbb{R}^3)=\{f\in L^1(\mathbb{R}^3);\|f\|\equiv\int_{\mathbb{R}^3}\sqrt{1+|\bar{p}|^2}|f(\bar{p})|d\bar{p}<+\infty\} \tag{2.5}$$

Endowed with $\|.\|$; which is a norm, $L^1_2(\mathbb{R}^3)$ is a Banach space.

Let r be a strictly positive real number and I a real interval. We set

$$\begin{cases} X_r &=\{f\in L^1_2(\mathbb{R}^3), f\geq 0 \quad a.e, \|f\|\leq r\} \\ C[I;L^1_2(\mathbb{R}^3)] &=\{f:I\to L^1_2(\mathbb{R}^3), f \text{ continuous and bounded}\} \end{cases}. \tag{2.6}$$

Endowed with the metric induced by $\|.\|$, X_r is a complete and connected metric subspace of $(L^1_2(\mathbb{R}^3),\|.\|)$

One observes that $(C[I;L^1_2(\mathbb{R}^3)],|||.|||)$, where $|||f|||=\underset{t\in I}{Sup}\|f(t)\|$ is a Banach space.

We set:

$$C(I;X_r)=\{f\in C[I;L^1_2(\mathbb{R}^3)], f(t)\in X_r \quad \forall t\in I\}. \tag{2.7}$$

Endowed with the metric induced by the norm $|||.|||$, $C[I;X_r]$ is a complete metric subspace of $[C[I;L^1_2(\mathbb{R}^3)],|||.|||]$

2.2. The Einstein-Boltzmann system in $(\mathbb{R}^4,g)$**.**

The Boltzmann equation. The Boltzmann equation on the curved space time $(\mathbb{R}^4,g)$ can be written as

$$p^\alpha\frac{\partial f}{\partial x^\alpha}-\Gamma^i_{\mu\nu}p^\mu p^\nu\frac{\partial f}{\partial p^i}=Q(f,f), \tag{2.8}$$

in which $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols of g defined by:

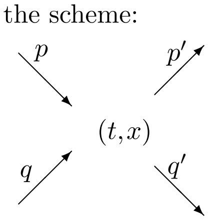
$$\Gamma^\lambda_{\alpha\beta}=\frac{1}{2}g^{\lambda\mu}[\partial_\alpha g_{\mu\beta}+\partial_\beta g_{\alpha\mu}-\partial_\mu g_{\alpha\beta}], \tag{2.9}$$

in which the metric g is defined by (2.1) and $g^{\lambda\mu}$ denotes the inverse of $g_{\lambda\mu}$. Setting $\dot{a}=\frac{da}{dt}$, a direct computation, using (2.1) and (2.9) gives:

$$\Gamma^0_{ii}=\dot{a}a;\quad \Gamma^i_{i0}=\Gamma^i_{0i}=\frac{\dot{a}}{a};\quad \Gamma^0_{\alpha\beta}=0\quad for\quad \alpha\neq\beta;\quad \Gamma^k_{ij}=\Gamma^0_{00}=0. \tag{2.10}$$

Q is a non-linear operator called the collision operator and will be specified subsequently.

In the instantaneous, binary and elastic scheme due to Lichnerowicz and Chernikov, we consider that at a given point (t,x), only two particles collide instantaneously without destroying each other. The collision affects only the momenta of the two particles that change after the collision; only the sum of the two momenta is preserved, following the scheme:



$$\begin{cases} p^0+q^0 = p'^0+q'^0 \\ \bar{p}+\bar{q} = \bar{p}'+\bar{q}' \end{cases} \tag{2.11}$$

The second relation of (2.11) is interpreted by setting, following Glassey in [2]:

$$\begin{cases} \bar{p}' = \bar{p}+b(\bar{p},\bar{q},\omega)\omega \\ \bar{q}' = \bar{q}-b(\bar{p},\bar{q},\omega)\omega \qquad \omega\in S^2, \end{cases} \tag{2.12}$$

in which $b(\bar{p},\bar{q},\omega)$ is a real-valued function. We directly prove that

$$b(\bar{p},\bar{q},\omega)=\frac{2p^oq^oea^2\omega.(\hat{\bar{q}}-\hat{\bar{p}})}{e^2-a^4(\omega.(\bar{p}+\bar{q}))^2}, \tag{2.13}$$

in which $\hat{\bar{p}}=\frac{\bar{p}}{p^0}$, $\hat{\bar{q}}=\frac{\bar{q}}{q^0}$. $e=\sqrt{1+a^2\,|\,\bar{p}\,|^2}+\sqrt{1+a^2\,|\,\bar{q}\,|^2}$ is given by the first relation of (2.11). Another direct computation shows, using the classical properties of the determinants, that the Jacobian of the change of variables $(\bar{p},\bar{q})\to(\bar{p}',\bar{q}')$ in $\mathbb{R}^3\times\mathbb{R}^3$, defined by (2.12) is given by:

$$\frac{\partial(\bar{p}',\bar{q}')}{\partial(\bar{p},\bar{q})}=-\frac{p'^oq'^o}{p^oq^o}. \tag{2.14}$$

The collision operator Q is then defined, using functions f, g on $\mathbb{R}^3$ by

$$Q(f,g)=Q^+(f,g)-Q^-(f,g) \tag{2.15}$$

where

$$Q^+(f,g)(\bar{p})=\int_{\mathbb{R}^3}\frac{a^3d\bar{q}}{q^0}\int_{S^2}f(\bar{p}')g(\bar{q}')A(a,\bar{p},\bar{q},\bar{p}',\bar{q}')d\omega, \tag{2.16}$$

$$Q^-(f,g)(\bar{p})=\int_{\mathbb{R}^3}.\frac{a^3d\bar{q}}{q^0}\int_{S^2}f(\bar{p})g(\bar{q})A(a,\bar{p},\bar{q},\bar{p}',\bar{q}')d\omega. \tag{2.17}$$

We now introduce step by step the elements which define Q, specifying properties and hypotheses:

1) S^2 is the unit sphere of $\mathbb{R}^3$ whose volume element is denoted dw.

2) A is a non-negative real-valued regular function of all its arguments, called the **collision kernel** or the **cross-section** of the collisions, on which we require the following boundedness, symmetry and Lipschitz continuity assumptions:

$$0\le A(a,\bar{p},\bar{q},\bar{p}',\bar{q}')\le C_1 \tag{2.18}$$

$$A(a,\bar{p},\bar{q},\bar{p}',\bar{q}')=A(a,\bar{q},\bar{p},\bar{q}',\bar{p}') \tag{2.19}$$

$$A(a,\bar{p},\bar{q},\bar{p}',\bar{q}')=A(a,\bar{p}',\bar{q}',\bar{p},\bar{q}) \tag{2.20}$$

$$|A(a_1,\bar{p},\bar{q},\bar{p}',\bar{q}')-A(a_2,\bar{p},\bar{q},\bar{p}',\bar{q}')|\le\gamma|a_1-a_2|, \tag{2.21}$$

where C_1 and γ are strictly positive constants.

The Einstein equations. We consider the Einstein equations with a **cosmological constant** Λ which can be written as

$$R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}+\Lambda g_{\alpha\beta}=8\pi T_{\alpha\beta} \tag{2.22}$$

in which $R_{\alpha\beta}$ is the Ricci tensor of g, $R=g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature, $T_{\alpha\beta}$ is the stress-matter tensor that represents the matter contents, generated by the distribution function f of the particles by

$$T^{\alpha\beta}(t)=\int_{\mathbb{R}^3}\frac{p^\alpha p^\beta f(t,\bar{p})|g|^{\frac{1}{2}}}{p^0}dp^1dp^2dp^3, \tag{2.23}$$

and $|g|=a^6$ is the determinant of g. By a direct computation, we have:

$$R_{00}=-3\frac{\ddot{a}}{a} \quad \textit{and} \quad R_{11}=a\ddot{a}+2(\dot{a})^2 \tag{2.24}$$

$$R=R^\alpha_\alpha=g^{\alpha\beta}R_{\alpha\beta}=6\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right]. \tag{2.25}$$

The Einstein-Boltzmann system in (a,f), can then be written as

$$\frac{\partial f}{\partial t}-2\frac{\dot{a}}{a}\sum_{i=1}^{3}p^i\frac{\partial f}{\partial p^i}=\frac{1}{p^0}Q(f,f) \tag{2.26}$$

$$(\frac{\dot{a}}{a})^2=\frac{8\pi}{3}T_{00}+\frac{\Lambda}{3} \tag{2.27}$$

$$\frac{\ddot{a}}{a}=\frac{4\pi}{3}(T^{00}+3a^2T_{11})+\frac{\Lambda}{3}. \tag{2.28}$$

(2.27) is called the Hamiltonian constraint and (2.28) is called the evolution equation.

2.3. Existence theorem for the Einstein-Boltzmann system.

2.3.1. Existence theorem for the Boltzmann equation. We consider the Boltzmann equation on $[t_0,t_0+T]$ with $t_0\in\mathbb{R}_+$, $T\in\mathbb{R}^*_+$, and a is assumed to be given and defined on $[t_0,t_0+T]$.

(2.26) is a first order PDE. Its resolution is equivalent to the resolution of the associated characteristic system, which can be written as, taking t as a parameter:

$$\frac{dp^i}{dt}=-2\,\frac{\dot{a}}{a}p^i; \qquad \frac{df}{dt}=\frac{1}{p^0}Q(f,f) \tag{2.29}$$

We solve the initial value problem on $I=[t_0,t_0+T]$ with initial data:

$$p^i(t_0)=y^i; \qquad f(t_0)=f_{t_0} \tag{2.30}$$

The equation in $\bar{p}=(p^i)$ is directly solved to give, setting $y=(y^i)\in\mathbb{R}^3$;

$$\bar{p}(t_0+t,y)=\frac{a^2(t_0)}{a^2(t_0+t)}y, \quad t\in[0,T]. \tag{2.31}$$

The initial value problem for f is equivalent to the following integral equation in f, in which $\bar{p}$ stands this time for any independent variable in $\mathbb{R}^3$:

$$f(t_0+t,\bar{p})=f_{t_0}(\bar{p})+\int_{t_0}^{t_0+t}\frac{1}{p^0}Q(f,f)(s,\bar{p})ds \quad t\in[0,T]. \tag{2.32}$$

Solving the Boltzmann Equation (2.26) is equivalent to solving the integral equation (2.32). We prove:

THEOREM 2.1. *Let a be a strictly positive continuous function such that* $a(t)\geq\frac{3}{2}$ *whenever a is defined. Let* $f_{t_0}\in L^1_2(\mathbb{R}^3)$, $f_{t_0}\geq 0$, *a.e,* $r\in\mathbb{R}^*_+$ *such that* $r>\|f_{t_0}\|$. *Then, the initial value problem for the Boltzmann equation on* $[t_0,t_0+T]$, *with initial data* f_{t_0}, *has a unique solution* $f\in C[[t_0,t_0+T];X_r]$.
Moreover, f satisfies the estimate

$$\underset{t\in[t_0,t_0+T]}{sup}\|f(t)\|\leq\|f_{t_0}\| \tag{2.33}$$

Proof: See [9].

2.3.2. Existence theorem for the Einstein equation. In this paragraph, we suppose that f is fixed in $C[[0,T];X_r]$, with $f(0)=f_0\in L^1_2(\mathbb{R}^3)$, $f_0\geq 0$ a.e. and $r>\|f_0\|$, and the Einstein equations are (2.27)–(2.28) in the unknown a.

The relations $R_{0i}=0$, $R_{ij}=0$ if $i\neq j$, $R_{11}=R_{22}=R_{33}$, imply for the Einstein tensor $S_{\alpha\beta}=R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R$ that

$$T_{11}=T_{22}=T_{33}, \quad T_{0i}=0, \quad T_{ij}=0 \quad for \quad i\neq j/ \tag{2.34}$$

However the stress-matter tensor $T_{\alpha\beta}$ is defined by (2.23) in terms of the distribution function f.

PROPOSITION 2.2. *Let* f_{t_0} *and* $r>0$ *be defined as in Theorem 3.1. Assume that, in addition,* f_{t_0} *is invariant by* S_{O_3} *and the collision kernel A satisfies*

$$A(a(t),M\bar{p},M\bar{q},M\bar{p}',M\bar{q}')=A(a(t),\bar{p},\bar{q},\bar{p}',\bar{q}'), \qquad \forall M\in S_{O_3}. \tag{2.35}$$

Then

1) The solution f of the integral equation (2.32) satisfies:

$$f(t_0+t,M\bar{p})=f(t_0+t,\bar{p}), \quad \forall t\in[0,T], \quad \forall\bar{p}\in\mathbb{R}^3, \quad \forall M\in S_{O_3}, \tag{2.36}$$

2) The stress-matter tensor $T_{\alpha\beta}$ *satisfies the conditions (2.34).*

Proof: See [9].

In all that follows, we assume that f_{t_0} is invariant by S_{O_3} and that the collision kernel A satisfies assumption (2.35). It is proved (see for instance [6] p. 29) that the Hamiltonian constraint (2.27) is satisfied in the whole existence domain of the solution a of (2.28) on [0, T], once it is satisfied for $t=0$. So, it will be the case if the initial data a_0, $\dot{a}_0$, f_0 satisfy, using expression (2.23) of T^{00}, the initial constraint:

$$\left(\frac{\dot{a}_0}{a_0}\right)^2=\frac{8\pi a_0^3}{3}\int_{\mathbb{R}^3}\sqrt{1+a_0^2|\bar{p}|^2}f_0(\bar{p})d\bar{p}+\frac{\Lambda}{3} \tag{2.37}$$

where $a_0>0$, $\dot{a}_0\in\mathbb{R}$, and f_0 are the initial data, i.e.,

$$a(0)=a_0; \quad \dot{a}(0)=\dot{a}_0; \quad f(0,\bar{p})=f_0(\bar{p}). \tag{2.38}$$

We will choose, taking into account the hypothesis on $a(t)$ in Theorem 2.1.

$$a_0 \geq \frac{3}{2}; \quad f_0 \in L^1_2(\mathbb{R}^3); \quad f_0 \geq 0\, a.e \quad \dot{a}_0 > 0. \tag{2.39}$$

Set $\theta = 3\frac{\dot{a}}{a}$, the evolution equation (2.28) gives

$$\dot{\theta} = -\frac{\theta^2}{3} - 4\pi(T^{00} + 3a^2T^{11}) + \Lambda. \tag{2.40}$$

(2.40) is called the Raychaudhuri equation in θ.

Taking into account the continuity of $t \mapsto \frac{\dot{a}(t)}{a(t)}$ and using the Hamiltonian constraint (2.27), we have $\frac{\dot{a}(t)}{a(t)} \geq \sqrt{\frac{\Lambda}{3}}$. It is then easy to prove that

$$a(t) \geq a(t_0); \quad \sqrt{\frac{\Lambda}{3}} \leq \frac{\dot{a}(t)}{a(t)} \leq \frac{\dot{a}(t_0)}{a(t_0)}, \quad t \geq t_0. \tag{2.41}$$

In order to use standard results, we make the change of variable

$$e = \frac{1}{a}. \tag{2.42}$$

Let us set $\rho = T^{00}$ and $P = a^2T^{11}$, where ρ stands for the density and P for the pressure.

The Einstein evolution equation (2.28) is equivalent to the following first order system in (e,θ):

$$\dot{e} = -\frac{\theta}{3} \times e \tag{2.43}$$

$$\dot{\theta} = -\frac{\theta^2}{3} - 4\pi(\rho + 3P) + \Lambda \tag{2.44}$$

with $\rho = \rho(e,f)$ and $P = P(e,f)$.

Proposition 2.3. *Let $T > 0$ and $f \in C[[0,T]; X_r]$ be given. Then the initial value problem for system (2.43)–(2.44) with initial data (e_0,θ_0) satisfying the initial constraint (2.37) the relations $0 < e_0 \leq \frac{2}{3}$, $0 < \theta_0 \leq d_0$, and where $d_0 = 3\sqrt{\frac{\Lambda}{3} + \frac{16\pi}{3} r a_0^4}$, has an unique solution on $[0,T]$.*

Proof: See [9].

2.3.3. Global existence for the coupled system. Recall that the coupled Einstein-Boltzmann equation in (a,f) is equivalent to the following system in (f,e,θ):

$$\frac{df}{dt} = \frac{1}{p^0} Q(f,f) \tag{2.45}$$

$$\dot{e} = -\frac{\theta}{3} e \tag{2.46}$$

$$\dot{\theta} = -\frac{\theta^2}{3} - 4\pi(\rho + 3P). \tag{2.47}$$

PROPOSITION 2.4 (local existence theorem). *There exists an interval* $[0,l]$, $l>0$ *such that, the initial value problem for the system (2.45)–(2.47) with initial data* $(f_0, e_0, \theta_0) \in L^1_2(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$ *has an unique solution* (f, e, θ) *on* $[0,l]$.

Proof: See [9].

PROPOSITION 2.5. *1) There exists a strictly positive real number* $\delta > 0$ *depending only on the absolute constants* a_0, Λ , r *and* T *such that the initial value problem for system (2.45)–(2.47) with initial data satisfying the initial constraint has a solution* $(f, e = \frac{1}{a}, \theta) \in C[[t_0, t_0+\delta]; X_r] \times E^\delta_{t_0} \times F^\delta_{t_0}$ *where*

$$E^\delta_{t_0} = \left\{ e \in C[t_0, t_0+\delta], \frac{1}{C_2} e^{-C_3(t_0+t+1)^2} \leq e(t_0+t) \leq \frac{2}{3} \quad , \forall t \in [0, \delta[\right\},$$

$$F^\delta_{t_0} = \left\{ \theta \in C[t_0, t_0+\delta], \theta, \sqrt{3\Lambda} \leq \theta(t_0+t) \leq D_0 \quad \forall t \in [0, \delta[\right\},$$

in which C_2, C_3, *and* D_0 *are absolute constants. In fact, setting* $\gamma_1 = \gamma_1(a_0, r, T) = \frac{\Lambda}{3} + \sqrt{\frac{\Lambda}{3} + 3ra_0^4}(T+1)$, *we have*

$$C_2 = a_0 e^{\gamma_1}; \quad C_3 = \gamma_1 + \frac{\Lambda}{3}(T+1); \quad D_0 = 3\gamma_1 + \Lambda(T+1). \tag{2.48}$$

2) The problem has a global solution $(f, e = \frac{1}{a}, \theta)$ *over* $[0, +\infty[$.

Proof: See [9].

3. Asymptotic behavior

We are going to study the asymptotic behavior of the solution of the Einstein-Boltzmann system with positive cosmological constant in the Robertson-Walker space time. We will use the following lemma:

LEMMA 3.1. *Let* (a, f) *be the solution of the Einstein-Boltzmann system (2.26)–(2.28). Then* $(\frac{\dot{a}}{a})^2$ *goes to* $\frac{\Lambda}{3}$ *as* t *goes to infinity and the mean curvature goes to a strictly positive limit.*

Proof. Let us compute the derivative of $\frac{\dot{a}}{a}$.

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2.$$

Using (2.27), we have

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = -\frac{4\pi}{3}(T^{00} + 3a^2 T^{11}) + \frac{\Lambda}{3} - \left(\frac{\dot{a}}{a}\right)^2.$$

By direct computation, this implies

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) \leq \frac{\Lambda}{3} - \left(\frac{\dot{a}}{a}\right)^2. \tag{3.1}$$

Due to the Hamiltonian constraint, we have $\frac{\Lambda}{3}-\left(\frac{\dot a}{a}\right)^2<0$ for a non trivial solution of the system (2.26)–(2.28), and then

$$\frac{d}{dt}\left(\frac{\dot a}{a}\right)\leq\frac{\Lambda}{3}-\left(\frac{\dot a}{a}\right)^2<0.$$

This implies that

$$\frac{\frac{d}{dt}\left(\frac{\dot a}{a}\right)}{\left(\frac{\dot a}{a}\right)^2-\frac{\Lambda}{3}}\leq -1. \tag{3.2}$$

We observe that

$$\frac{1}{\left(\frac{\dot a}{a}\right)^2-\frac{\Lambda}{3}}=\frac{1}{\left(\frac{\dot a}{a}-\sqrt{\frac{\Lambda}{3}}\right)\left(\frac{\dot a}{a}+\sqrt{\frac{\Lambda}{3}}\right)}=\frac{1}{2\sqrt{\frac{\Lambda}{3}}}\left[\frac{1}{\frac{\dot a}{a}-\sqrt{\frac{\Lambda}{3}}}-\frac{1}{\frac{\dot a}{a}+\sqrt{\frac{\Lambda}{3}}}\right].$$

It then follows that

$$\frac{\frac{d}{dt}\left(\frac{\dot a}{a}\right)}{\left(\frac{\dot a}{a}\right)^2-\frac{\Lambda}{3}}=\frac{1}{2\sqrt{\frac{\Lambda}{3}}}\left[\frac{\frac{d}{dt}\left(\frac{\dot a}{a}\right)}{\frac{\dot a}{a}-\sqrt{\frac{\Lambda}{3}}}-\frac{\frac{d}{dt}\left(\frac{\dot a}{a}\right)}{\frac{\dot a}{a}+\sqrt{\frac{\Lambda}{3}}}\right]\leq -1. \tag{3.3}$$

Integrating relation (3.3) over $[0,t]$, we obtain

$$\text{Log}\left[\frac{\frac{\dot a}{a}+\sqrt{\frac{\Lambda}{3}}}{\frac{\dot a}{a}-\sqrt{\frac{\Lambda}{3}}}\times\frac{1}{C_4}\right]\leq -2\sqrt{\frac{\Lambda}{3}}t,$$

where C_4 is a constant.

This allows us to obtain the following relation

$$\frac{\frac{\dot a}{a}-\sqrt{\frac{\Lambda}{3}}}{\frac{\dot a}{a}+\sqrt{\frac{\Lambda}{3}}}\leq C_4 exp\left(-2\sqrt{\frac{\Lambda}{3}}t\right) \tag{3.4}$$

From (2.27) we obtain

$$\sqrt{\frac{\Lambda}{3}}\leq\frac{\dot a}{a}\leq\frac{\dot a(t_0)}{a(t_0)}\quad t\geq t_0 \tag{3.5}$$

This implies

$$0\leq\sqrt{\frac{\Lambda}{3}}+\frac{\dot a}{a}\leq\sqrt{\frac{\Lambda}{3}}+\frac{\dot a(t_0)}{a(t_0)} \tag{3.6}$$

Using (3.4) and the fact that $\sqrt{\frac{\Lambda}{3}}\leq\frac{\dot a}{a}$, we have

$$0\leq\frac{\dot a}{a}-\sqrt{\frac{\Lambda}{3}}\leq C_5 exp\left(-2\sqrt{\frac{\Lambda}{3}}t\right)\quad\text{for}\quad t\geq t_0 \tag{3.7}$$

where $C_5 = \sqrt{\frac{\Lambda}{3}} + \frac{a(\dot{t_0})}{a(t_0)}$ is a constant.

From (3.7) we conclude that $\frac{\dot{a}}{a} - \sqrt{\frac{\Lambda}{3}}$ goes to zero as t goes to infinity. Finally $\left(\frac{\dot{a}}{a}\right)^2$ goes to $\frac{\Lambda}{3}$ as t goes to infinity. □

THEOREM 3.2. *At late times in the future, the solution of Einstein-Boltzmann system with positive cosmological constant for the Robertson-Walker space time is asymptotically dust-like.*

Proof.

- Using Lemma 3.1 and the Hamiltonian constraint $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}T^{00} + \frac{\Lambda}{3}$, it is clear that the density given by $\rho = T^{00}$ goes to zero as t goes to infinity.
- recall that both the density $\rho = T^{00}$ and the pressure $P = a^2 T^{11}$ are defined by the relation (2.23). It is then obvious by direct estimation that $P \leq \rho$. Since $P \geq 0$, this allows us to conclude that the pressure goes to zero as t goes to infinity.
- To complete the proof of this theorem, we are going to prove that the expression $\frac{P}{\rho}$ goes to zero as t goes to infinity.

Using the characteristic system for the Boltzmann equation, we have

$$\frac{dp^i}{dt} = -2\frac{\dot{a}}{a}p^i$$

The explicit solutions of the above system are

$$p^i = \frac{a^2(0)p^i(0)}{a^2(t)}. \tag{3.8}$$

Setting $C^i = a^2(0)p^i(0)$, we have

$$p^i = \frac{C^i}{a^2(t)} \quad \text{which implies} \quad a^2 p^i = C^i. \tag{3.9}$$

Recall that

$$\begin{cases} \rho &= a^3(t)\int_{\mathbb{R}^3} \sqrt{1 + a^3(t)|\bar{p}|^2} f(t,\bar{p})d\bar{p} \\ P &= a^5(t)\int_{\mathbb{R}^3} \frac{(p^1)^2}{\sqrt{1+a^3(t)|\bar{p}|^2}} f(t,\bar{p})d\bar{p} \end{cases}. \tag{3.10}$$

We then have

$$\frac{P}{\rho} = \frac{1}{a^2} \frac{\int_{\mathbb{R}^3} \frac{(a^2p^1)^2}{\sqrt{1+a^3(t)|\bar{p}|^2}} f(t,\bar{p})d\bar{p}}{\int_{\mathbb{R}^3} \sqrt{1+a^3(t)|\bar{p}|^2} f(t,\bar{p})d\bar{p}}$$

Using the relation (3.9), we obtain

$$\frac{P}{\rho} = \frac{(C^1)^2}{a^2} \frac{\int_{\mathbb{R}^3} \frac{1}{\sqrt{1+a^3(t)|\bar{p}|^2}} f(t,\bar{p})d\bar{p}}{\int_{\mathbb{R}^3} \sqrt{1+a^3(t)|\bar{p}|^2} f(t,\bar{p})d\bar{p}}$$

$$\frac{P}{\rho} \leq \frac{(C^1)^2}{a^2} \frac{\int_{\mathbb{R}^3} f(t,\bar{p})d\bar{p}}{\int_{\mathbb{R}^3} \sqrt{1+a^3(t)|\bar{p}|^2} f(t,\bar{p})d\bar{p}}, \tag{3.11}$$

and then

$$\frac{P}{\rho} \leq \frac{(C^1)^2}{a^2}. \tag{3.12}$$

Integrating the first inequality of (3.5) over the interval $[t_0,t]$, we have

$$a(t) \geq a(t_0) exp\left(\sqrt{\frac{\Lambda}{3}}(t-t_0)\right), \tag{3.13}$$

which implies that

$$\frac{P}{\rho} \leq \frac{(C^1)^2}{a_0^2} exp\left(2\sqrt{\frac{\Lambda}{3}}(t_0-t)\right). \tag{3.14}$$

By (3.14), we conclude that $\frac{P}{\rho}$ goes to zero as t goes to infinity. □

4. Geodesic completness

THEOREM 4.1. *Let $a_0 = a(0)$, $\dot{a}_0 = \dot{a}(0)$, and $f_0 \in L_2^1(\mathbb{R}^3)$ be the initial data for the Einstein-Boltzmann system on the Robertson-Walker space time. Then the space time proved in the existence theorem is future complete.*

Proof. Recall that the metric we use is the Robertson-Walker metric.

The geodesic equation for such a metric implies that along the geodesics the variables t, p^0, and p^i satisfy the following system of differential equations:

$$\begin{cases} \frac{dt}{d\tau} &= p^0 \\ \frac{dp^0}{d\tau} &= \dot{a}a[(p^1)^2+(p^2)^2+(p^3)^2] \\ \frac{dp^i}{d\tau} &= -2\frac{\dot{a}}{a}p^0p^i, \end{cases} \tag{4.1}$$

where τ is an affine parameter.

The particles are for the rest mass $m=1$ and are supposed to move forward in time. Recall that $p^0 = \sqrt{1+a^2|\bar{p}|^2}$.

The geodesic completness is decided by looking the relation between the time t and the affine parameter τ along any future directed causal geodesic. From (4.1), we have $\frac{dt}{d\tau} = \sqrt{1+a^2|\bar{p}|^2}$, which implies that $\frac{d\tau}{dt} = \frac{1}{\sqrt{1+a^2|\bar{p}|^2}}$.

To control the relation between t and τ, we need to control the quantity $a^2(t)|\bar{p}|^2$ as a function of coordinate time t.

From (3.9), we have $a^2p^i = C^i$. Using (2.41), we have $a(t) \geq a_0$ $\forall t \in [0,+\infty[$. Direct computation gives

$$\frac{d\tau}{dt} \geq \chi \tag{4.2}$$

where $\chi = [1+\frac{1}{a_0^2}(C^1)^2+(C^2)^2+(C^2)^2]^{-\frac{1}{2}}$ is a strictly positive constant. Therefore, τ is recovered by integrating (4.2). The integral of the right hand side diverges as t goes to infinity. Therefore as t goes to infinity so does τ. □

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