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ORBITAL STABILITY OF NUMERICAL PERIODIC NONLINEAR SCHRÖDINGER EQUATION*

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Abstract. This work is devoted to the study of the system that arises by discretization of the periodic nonlinear Schrödinger equation in dimension one. We study the existence of the discrete ground states for this system and their stability property when the potential parameter σ is small enough: i.e., if the initial data are close to the ground state, the solution of the system will remain near to the orbit of the discrete ground state forever. This stability property is an appropriate tool for proving the convergence of the numerical method.

Key words. numerical periodic nonlinear Schrödinger equation, ground states, orbital stability.

AMS subject classifications. 34L15; 35J25; 35P30

1. Introduction

We consider the discrete system associated to the periodic nonlinear Schrödinger equation (PNLS)

$$\begin{cases} \partial_t \phi = i\partial_x^2 \phi + i|\phi|^{2\sigma}\phi & (x,t)\in(0,1)\times\mathbb{R}_{>0} \\ \phi(0,t)=\phi(1,t)=0 & t>0 \\ \partial_x\phi(0,t)=\partial_x\phi(1,t) & t>0 \\ \phi(x,0)=\phi_0(x) & 0\le x\le 1 \end{cases} \tag{PNLS}$$

with $\sigma>0$ and ϕ a function in a suitable Sobolev space.

Problem (PNLS) arises in the propagation of electromagnetic waves in a nonlinear medium, such as a laser beam in an optical fiber. The existence of solutions was studied by Bourgain [2] and Kavian [6] who proved the well-posedness, local and global existence in time variable and, for $\sigma\ge 2$, the existence of blowing-up solutions.

The existence of ground states for this problem in $\mathbb{R}^n$ was studied by W.A. Strauss in [8] and M. Weinstein in [11], who characterized the ground state solution of this equation with a potential nonlinearity, as a minimum of the Gagliardo-Nirenberg functional. Orbital stability of the ground state solution in this case was analyzed by M. Weinstein in [12]. The author uses a variational formulation and that this equation has phase and translation symmetries, in order to construct a Lyapunov function. In this way, he obtain an orbital stability result.

Ground states, existence for the periodic problem (PNLS), and their orbital stability were proved in [1] through perturbation theory, because in the periodic problem it is not possible to apply a rescaling argument as was used in the previous works cited above. From [1] we know that (PNLS) only has phase symmetry; then Kato's perturbation theory ([5]) was used in order to obtain a lower bound for the difference between the value of the Lyapunov function on the ground state profile and at another point of the flux, close to the ground state solution. Thus, in the present article we prove the orbital stability for parameter σ small enough.

In order to give a numerical method for the calculus of the solution of (PNLS) we introduce a discretization in the spatial variable x. For $n\in\mathbb{N}$ let us consider the

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uniform partition of $[0,1]$ in $4n$ subintervals of length $h=\frac{1}{4n}$,

$$x_j = jh, \qquad \text{where } 0 \le j \le 4n. \tag{1.1}$$

If $\phi_j=\phi(x_j)$ and applying a finite difference scheme in Equation (PNLS), we have the following semidiscrete system:

$$\phi_j' = \frac{i}{h^2}(\phi_{j+1}-2\phi_j+\phi_{j-1}) + i|\phi_j|^{2\sigma}\phi_j \qquad \text{with } 0\le j\le 4n.$$

Due to the Diritchlet condition for (PNLS), we know that $\phi_0=\phi_{4n}=0$, so we have for each $n\in\mathbb{N}$ the system

$$\phi_h' = -iA_h\phi_h + iD(\phi_h)\phi_h, \tag{SDPNLS}$$

where $\phi_h=(\phi_1,\phi_1,\dots,\phi_{4n-1})\in\mathbb{C}^{4n-1}$, and the matrices A_h and $D(\phi_h)$ are given by

$$(A_h)_{ij} = \begin{cases} 2/h^2 & \text{if } i=j \\ -1/h^2 & \text{if } |i-j|=1 \\ 0 & \text{in another case} \end{cases} \qquad \text{and} \qquad (D(\phi_h))_{ij} = \begin{cases} |\phi_i|^{2\sigma} & \text{if } i=j \\ 0 & \text{if } i\ne j \end{cases}.$$

In this work we obtain for the system (SDPNLS) analogous results to those already known for the continuous case (PNLS), i.e. for each $n\in\mathbb{N}$ we prove the existence of a ground state solution R_h and its orbital stability, and in a second step, we use this result in order to prove the convergence of the method when n tends to infinity. In this sense, the main point is to prove the convergence of R_h to the ground state solution $R(x)$ of (PNLS) when $h\to 0$.

We know (see [1] and [11]) that a ground state solution profile is a real function. Since the stationary equation is a nonlinear equation, we use shooting methods in order to obtain a ground state profile. An existence result, an algorithmic method of calculus and an estimate of its error are obtained. Furthermore, by an estimate of the difference between R_h and the evaluation of the stationary solution $R(x)$ at the nodes x_i, we prove the orbital stability for solutions of (SDPNLS) and the convergence of the method.

The paper is organized as follows. In Section 2 we define the inner product, the norm and the space in which we work. Section 3 provides a detailed exposition of the existence of the ground state solutions and the convergence of these discrete ground states to the continuous ground state. Section 4 presents the concepts of orbit, distance and Lyapunov function, and we show that stability relies on a suitable lower bound on the second variation of the Lyapunov function. Section 5 presents the analysis of a constrained variational problem in order to find this suitable bound, as it was carried out in [12] for the general case or in [1] for our problem, but adapted to the discrete problem. Section 6 presents the main results of this work: a stability theorem and a convergence theorem.

2. Some definitions and notations

We will denote by $E_h(f)$ the evaluation of the function $f(x)$ at nodes $x_0,x_1,\dots,x_{4n-1}$.

Let $\mathcal{W}$ denote the space of functions $\phi(x)\in H_0^1[0,1]$ which are odd with respect to the midpoint $x=1/2$. From [1] we know that $\mathcal{W}$ is invariant under the flux of Equation (PNLS). Let $\mathcal{V}$ denote the subset of $\mathcal{W}$ of polygonal function with domain in $[0,1]$, and let $\mathcal{V}_h\subset\mathcal{V}$ denote the polygonal functions whose vertices are at nodes

(1.1). If $u_h=(u_1,u_2,\dots,u_{4n-1})\in\mathbb{C}^{4n-1}$ we can associate with it a polygonal function whose value at $x=0$ and $x=1$ is zero and whose value at node x_j is u_j. Moreover, we define the sets $\mathbb{S}_h\subset\mathbb{C}^{4n-1}$ using the property of skew symmetry:

$$\mathbb{S}_h=\left\{u\in\mathbb{C}^{4n-1}:u_{2n+j}=-u_{2n-j}\quad\text{for } 0\le j\le 2n-1\right\},$$

notice that this definition implies that $u_{2n}=0$. This space is invariant under the flux of the discrete equation (SDPNLS), and our discrete ground state $R_h\in\mathbb{S}_h\cap\mathbb{R}^{4n-1}$.

We define functions $G_h:\mathbb{S}_h\to\mathcal{V}_h$ and $E_h:\mathcal{V}_h\to\mathbb{S}_h$, we call them the polygonal and the evaluation function, and they give a continuous polygonal respectively, and a vector, respectively.

We will consider the following inner product and norms in $\mathbb{S}_h$. If u_h and $v_h\in\mathbb{S}_h$, then

$$\langle u_h,v_h\rangle_h=\sum_{j=1}^{4n-1}hu_jv_j^*,$$

$$|u_h|_{2,h}^2=\sum_{j=1}^{4n-1}h|u_j|^2,$$

$$\|u_h\|_{1,h}^2=\langle (I+A_h)u_h,u_h\rangle_h.$$

3. Ground states

3.1. Existence of the numerical ground states. A ground state solution of Equation (PNLS) is a solution in the form

$$\phi(x,t)=R(x)e^{iEt},$$

where profile $R(x)$ is a solution of the stationary problem

$$\begin{cases} R''-ER+R^{2\sigma+1}=0 \ \text{ if } x\in(0,1)\\ R(0)=R(1)=0\\ R'(0)=R'(1), \end{cases}. \tag{3.1}$$

We consider a finite difference scheme for this equation, associated with a regular partition on the interval $[0,1]$,, then we have the system

$$\frac{r_{j-1}-2r_j+r_{j+1}}{h^2}-Er_j+r_j^{2\sigma+1}=0,\quad\text{where } 1\le j\le 4n-1.$$

This system can be written as a recursive sequence

$$r_{j+1}=2r_j-r_{j-1}+h^2r_j\left(E-r_j^{2\sigma}\right) \tag{3.2}$$

or as a slope recursive sequence

$$\frac{r_{j+1}-r_j}{h}=hr_j\left(E-r_j^{2\sigma}\right)+\frac{r_j-r_{j-1}}{h}. \tag{3.3}$$

We use Equation (3.2) in order to prove the existence and the symmetry properties of the numerical ground state. We propose a shooting method which starts at zero without restriction on the right side. But, in order to obtain the existence of ground states with their expected properties, we need to prove some technical results.

3.1.1. Maximum of the ground states profiles. Our goal in this section is to prove that a sequence like (3.2) that starts with values $r_0=0$ and $r_1>0$, increases from zero until it reaches a first maximum value.

LEMMA 3.1. *For each E,h and $\sigma>0$, if $(r_j)_{j\geq 0}$ satisfies Equation (3.2), then the slope sequence $\left(\frac{r_{j+1}-r_j}{h}\right)_{j\geq 0}$ cannot be an increasing sequence for all j.*

Proof. We take $\frac{r_1}{h}>0$. Let us suppose that $\left(\frac{r_{j+1}-r_j}{h}\right)_{j\geq 1}$ is an increasing sequence; following Equation (3.3), the sequence r_j has a superlinear behavior, so $r_j \underset{n\to+\infty}{\to} +\infty$. Therefore there exists an index value j_0 such that $r_{j_0}^{2\sigma}>E$, so $\frac{r_{j_0+1}-r_{j_0}}{h}<\frac{r_{j_0}-r_{j_0-1}}{h}$, but this is a contradiction. □

We need to know if this slope sequence becomes negative after a suitable number of steps. As a first result, we prove a necessary condition for the slope sequence to remain positive.

LEMMA 3.2. *If for all j we have that $\frac{r_{j+1}-r_j}{h}>0$, and there exists j_0 such that $\frac{r_{j_0+1}-r_{j_0}}{h}<\frac{r_{j_0}-r_{j_0-1}}{h}$, then the slope sequence remains decreasing for $j>j_0$.*

Proof. By hypothesis we have that $0<\frac{r_{j_0+1}-r_{j_0}}{h}<\frac{r_{j_0}-r_{j_0-1}}{h}$. We only need to prove that $\frac{r_{j_0+2}-r_{j_0+1}}{h}<\frac{r_{j_0+1}-r_{j_0}}{h}$, then we can proceed by induction.

It is easy to check that $E-r_{j_0}^{2\sigma}<0$, and by hypothesis $\frac{r_{j_0+1}-r_{j_0}}{h}>0$, so $E-r_{j_0+1}^{2\sigma}<E-r_{j_0}^{2\sigma}<0$ and

$$\frac{r_{j_0+2}-r_{j_0+1}}{h}=hr_{j_0+1}\left(E-r_{j_0+1}^{2\sigma}\right)+\frac{r_{j_0+1}-r_{j_0}}{h}<\frac{r_{j_0+1}-r_{j_0}}{h}$$

and the lemma follows. □

In in the following lemma we now prove that is impossible that the slope sequence remains positive forever.

LEMMA 3.3. *There exists an index j_0 such that the slope sequence $\left(\frac{r_{j+1}-r_j}{h}\right)_{j\geq 0}$ becomes non-positive for $j>j_0$.*

Proof. Let us suppose that the hypotheses of Lemma 3.2 are true. Actually, the second one is true by Lemma 3.1, so we only assume that $\frac{r_{j+1}-r_j}{h}>0$ for all j. Thus, from this assumption and Lemma 3.2 we know that $\left(\frac{r_{j+1}-r_j}{h}\right)_{j\geq 0}$ is a positive decreasing sequence, so it has a limit $C\geq 0$.

If $C>0$, then the sequence r_j is superlinear, so $r_j\to+\infty$; therefore, taking the limit in (3.3), then we have a contradiction.

If $C=0$, then we again have two possibilities. If $r_j\to+\infty$, we can conclude in the same way as before. Else, $r_j\to K>0$, taking the limit in (3.3), we obtain that $E=K^{2\sigma}$, but due to the fact that $E-r_{j_0}^{2\sigma}<0$, we conclude that $r_{j_0}>E^{1/2\sigma}=K$, a contradiction. □

3.1.2. Monotonicity at each node. As a first step in the proof of the existence of the ground states we need to prove that the first values of the recurrence (3.2) are increasing functions with respect to r_1.

Fix E, σ, $h=\frac{1}{4N}$. By Equation (3.2) we can say that $r_2,r_3,\ldots,r_N,\ldots$ depend on r_1. In this section we give necessary conditions that the differentiable functions $r_2(r_1)\ldots r_N(r_1)$ are monotonically increasing for r_1 in some interval $I=(0,a)$.

As a first step we prove that each function $r_j(r_1)$ reaches its first maximum at a value $r_1^{*j-1}>0$ with the property that these points are ordered in a decreasing sequence.

LEMMA 3.4. *Function $r_j(r_1)$ has a first positive maximum called r_1^{*j-1}, and if we consider these points all together for $j\geq 1$, they satisfy $r_1^{*1}>r_1^{*2}>\ldots>r_1^{*j}>\ldots>0$. Moreover, $r_1,r_2,\ldots,r_j$ are increasing functions for $r_1\in I_j=\left(0,r_1^{*j}\right)$.*

Proof. The proof is by induction on n. For $n=2$, from (3.2) we have that

$$r_2=h^2r_1\left(E-r_1^{2\sigma}\right)+2r_1,$$

differentiating this expression with respect to r_1, we obtain

$$r_2'=\left(h^2E+2\right)-(2\sigma+1)h^2r_1^{2\sigma}.$$

The equation $r_2'(r_1)=0$ has a solution $r_1^{*1}=\left(\frac{2+Eh^2}{h^2(2\sigma+1)}\right)^{1/2\sigma}$ such that $r_2'(r_1)>0$ for $r_1\in I_1=\left(0,r_1^{*1}\right)$. Notice that $r_2(r_1)>r_1$ for $r_1\in I_1$.

Nevertheless, we need to study the case $n=3$ before proving the inductive step, because for $n=2$ we did not use a general argument.

For $n=3$ we have that

$$r_3=h^2r_2\left(E-r_2^{2\sigma}\right)+2r_2-r_1,$$

and differentiating with respect to r_1,

$$r_3'=\left[h^2E+2-(2\sigma+1)h^2r_2^{2\sigma}\right]r_2'-1.$$

In order to prove existence of $r_1^{*2}\in\left(0,r_1^{*1}\right)$ such that $r_3'\left(r_1^{*2}\right)=0$ and $r_3'(r_1)>0$ for $r_1\in I_2=\left(0,r_1^{*2}\right)$, we claim that r_3' changes sign in I_1; indeed, using that $r_2'\left(r_1^{*1}\right)=0$, it follows that $r_3'\left(r_1^{*1}\right)=-1$, and since $r_2(0)=0$ and $r_2'(0)=Eh^2+2$, we have that

$$r_3'(0)=\left(h^2E+2\right)^2-1>0,$$

therefore, the conclusion for $n=3$ follows.

We now we assume that this assertion is true for j and seek to prove it for $j+1$. Thus, we want to prove that there exists $r_1^{*j}\in\left(0,r_1^{*j-1}\right)$ such that $r_{j+1}'\left(r_1^{*j}\right)=0$ and $r_{j+1}'(r_1)>0$ for $r_1\in I_j=\left(0,r_1^{*j}\right)$. We have that

$$r_{j+1}=h^2r_j\left(E-r_j^{2\sigma}\right)+2r_j-r_{j-1} \tag{3.4}$$

and

$$r_{j+1}'=\left[h^2E+2-(2\sigma+1)h^2r_j^{2\sigma}\right]r_j'-r_{j-1}'. \tag{3.5}$$

The fact that $r_{j+1}'\left(r_1^{*j-1}\right)<0$ follows from the facts that from $r_j'\left(r_1^{*j-1}\right)=0$ and $r_{j-1}'\left(r_1^{*j-1}\right)>0$. In order to prove that $r_{j+1}'(0)>0$, we can observe that evaluation of (3.3) at $x=0$ gives the following linear recurrence:

$$r_{j+1}'(0)-r_j'(0)=h^2Er_j'(0)+\left(r_j'(0)-r_{j-1}'(0)\right), \tag{3.6}$$

with initial data $r_1'(0)=1$ and $r_2'(0)=h^2E+2$. By the inductive hypothesis we know that $r_j'(0)>0$, and repeated application of (3.6) enables us to assert that $r_{j+1}'(0)-r_j'(0)>0$, so $r_{j+1}'(0)>0$. Thus, the proof follows. □

Our goal is to see that if h is small enough, x_j is far enough to the right side of zero, and if two polygonal functions, corresponding with two values r_1 close to r_1^{*j}, pass each other in the interval $[x_j,x_{j+1}]$ then they are descending in this interval. This is equivalent to prove that under these conditions $(r_{j+1}-r_j)\left(r_1^{*j}\right)<0$.

Lemma 3.5. *If h is small enough and x_j is far enough to the right side of zero, then $(r_{j+1}-r_j)\left(r_1^{*j}\right)<0$.*

Proof. By definition $r_{j+1}'\left(r_1^{*j}\right)=0$, so evaluating in (3.5) we have

$$0=\left[h^2E+2-(2\sigma+1)h^2r_j^{2\sigma}\left(r_1^{*j}\right)\right]r_j'\left(r_1^{*j}\right)-r_{j-1}'\left(r_1^{*j}\right),$$

therefore

$$r_j^{2\sigma}\left(r_1^{*j}\right)=\frac{\left(h^2E+2\right)r_j'\left(r_1^{*j}\right)-r_{j-1}'\left(r_1^{*j}\right)}{(2\sigma+1)h^2r_j'\left(r_1^{*j}\right)}.$$

Plugging this expression into (3.4), we obtain

$$\begin{aligned}[r_{j+1}-r_j]\left(r_1^{*j}\right)&=\left[h^2r_j\left(E-r_j^{2\sigma}\right)+(r_j-r_{j-1})\right]\left(r_1^{*j}\right)\\&=\left[\frac{r_j}{2\sigma+1}\left(2\sigma h^2E-2+\frac{r_{j-1}'}{r_j'}\right)+(r_j-r_{j-1})\right]\left(r_1^{*j}\right).\end{aligned}$$

If h is small and x_j is far enough to the right side of zero, we have that $0<(r_j-r_{j-1})\left(r_1^{*j}\right)\ll 1$ and $0<\left|r_j'-r_{j-1}'\right|\left(r_1^{*j}\right)\ll 1$. Therefore, $[r_{j+1}-r_j]\left(r_1^{*j}\right)<0$. □

3.1.3. Existence and uniqueness of the ground state. We can now formulate our main results of this section: the existence of the numerical ground states with their symmetry properties and their uniqueness, for each $E>0,\sigma>0$ and $h=\frac{1}{4n}$, in the sense that they have the lowest frequency.

Proposition 3.6. *Suppose $n\geq 2$; then there exists a unique solution $R_h=\left(r_j^h\right)_{j=1}^{4n-1}$ of the system (3.2) such that:*

1. *$r_j^h>0$ for $1\leq j\leq 2n-1$.*
2. *$r_{n-j}^h=r_{n+j}$ for $0\leq j\leq n$.*
3. *$r_{2n+j}^h=-r_{2n-j}^h$ for $0\leq j\leq 2n$.*

Proof. Let $n\in\mathbb{N}$ and $h=\frac{1}{4n}$, so $x_n=1/4$. From Lem. 3.4 we know that if $r_1\in I(h)=\left[r_1^{*n},r_1^{*n-1}\right]$ the corresponding sequence increases until node x_n and then starts to descend, i.e., the polygonal function has its first maximum at $x_n=1/4$.

Notice that, if $r_1<r_1^{*n}$ then $r_{n+1}>r_n$. Analogously if $r_1>r_1^{*n-1}$ then $r_{n-1}>r_n$.

We want to prove that there exists $r_1^h\in I(h)$ such that the recursion solution has the expected symmetries properties, i.e., it reaches its maximum value r_n^h at $x_n=\frac{1}{4}$

and $r^h_{n-1}=r^h_{n+1}$. As a consequence, $r^h_{2n}=0$, and we also have odd symmetry for $r^h_{2n+1},\ldots,r^h_{4n}$. This solution is called the numerical soliton R_h.

Throughout the proof $r^{\min}$ and $r^{\max}$ denote the vector solutions corresponding to initial values $r_1^{\min}=r_1^{*n}$ and $r_1^{\max}=r_1^{*n-1}$, respectively. By definition we know that $r^{\min}_{n-1}<r^{\min}_n=r^{\min}_{n+1}$ and $r^{\max}_{n-1}=r^{\max}_n>r^{\max}_{n+1}$. Moreover, due to the fact that $r_n(r_1)$ is an increasing function in $I(h)$, we have that

$$r^{\min}_{n-1}<r^{\min}_n<r^{\max}_{n-1}. \tag{3.7}$$

Therefore, following (3.7) we have

$$r^{\min}_{n-1}<r^{\min}_{n+1}=r^{\min}_n<r^{\max}_{n-1}.$$

Also, from the definition of $r_1^{\max}$, it follows that

$$r^{\max}_{n+1}<r^{\max}_n=r^{\max}_{n-1},$$

so the continuous functions $r_{n+1}(r_1)$ and $r_{n-1}(r_1)$ must intersect each other, at least once, in $r_1^h\in I$.

Let us denote by R_h the vector solution of the recursion corresponding to this value r_1^h. We call it the vector soliton or the discrete numerical soliton. By construction R_h has the symmetry properties required.

For the proof of the uniqueness of R_h, we use its symmetry as follows. Since the function $r_{2n}(r_1)$ is decreasing, if we had two different values of $r_1\in I$ such that their corresponding sequences satisfied $r_{n-1}=r_{n+1}$, then both sequences would have $r_{2n}=0$, a contradiction. □

3.2. Convergence of the numerical solitons. In the previous section we proved the existence of $r_1^h\in I(h)$ corresponding to the initial value of the recursion such that the sequence solution is R_h, but we have not given an estimate for it yet.

In this section, we estimate this value r_1^h with an error with quadratic order in h. In a second step we will use this estimate to obtain an order of approximation between the evaluation of the ground state profile $E_h(R(x))$ and the discrete numerical soliton R_h.

3.2.1. First approximation step. We know (see [1]) that the ground state profile $R(x)$ reaches its maximum value at $x=1/4$. We denote by $A=R(1/4)$ this maximum value.

Lemma 3.7. *Let $0<h<1/4$,; then*

$$R(h)=hA\sqrt{\frac{A^{2\sigma}}{\sigma+1}-E}+O\left(h^3\right). \tag{3.8}$$

Proof. From the soliton Equation (3.1)

$$R''=ER-R^{2\sigma+1};$$

multiplying each member by R' and integrating, we have that

$$R'^2=C+ER^2-\frac{1}{\sigma+1}R^{2\sigma+2}.$$

We know that $R'(1/4)=0$,; then, using that $R(0)=0$, we obtain that

$$R'(0)=\sqrt{C}=A\sqrt{\frac{A^{2\sigma}}{\sigma+1}-E},$$

where we have taken the positive branch of the square root because we want $R(x)$ to be increasing in $(0,1/4)$. From the same equation we have that $R''(0)=0$ and $R'''(0)=ER'(0)=EA\sqrt{\frac{A^{2\sigma}}{\sigma+1}-E}$. Thus, by Taylor's expansion of $R(x)$ at $x=h$, equality (3.8) follows. ∎

Equality (3.8) suggests that we start the recursion process (3.2) with the value u_1 defined by the first term of this expansion.

On the other hand, we may consider the following family of initial value problems:

$$\begin{cases} -U''+\left(E-U^{2\sigma}\right)U=0 \\ U(0)=0 \\ U'(0)=\beta. \end{cases} \tag{3.9}$$

For each $\beta>0$ there exists only one solution $U_\beta(x)$. From Equation (3.9) we have that

$$U_\beta(h)=\beta h+O\left(h^3\right),$$

so

$$\frac{U_\beta(h)}{h}=\beta+\left(h^2\right).$$

We will denote by u^β the output sequence (3.2) that starts with the value $u_1=h\beta$. In the following proposition we prove that, for all β, the evaluation $E_h(U_\beta(x))$ and the vector u^β differ by a quadratic order in h.

Proposition 3.8. *Let $\beta>0$ be fixed; then $U_\beta(x)$ and u^β satisfy, for $1\le j\le 4n-1$,*

$$\left|e_j^\beta\right|=\left|U_\beta(jh)-u_j^\beta\right|=O\left(h^2\right). \tag{3.10}$$

Proof. For simplicity of notation, in Equation (3.10) we write e_j instead of e_j^β. The order of accuracy of the scheme at each node x_j is $T_j=O\left(h^2\right)$, and, due to the fact that u^β satisfies the recurrence equation (3.2), we have that the local errors satisfy the following new recurrence equation,

$$e_{j+1}=-e_{j-1}+2e_j+h^2[E-C]e_j+h^2T_j, \tag{3.11}$$

where C is a constant. Actually, C depends on β and h because it is an upper bound of $U_\beta^{2\sigma}(x)$ and of $\left(u_j^\beta\right)^{2\sigma}$, but any change in h has a corresponding change in β such that $u_1^\beta=h\beta$.

The linear part of the recurrence equation

$$e_{j+1}=-e_{j-1}+2e_j \tag{3.12}$$

has only one (double) eigenvalue $\lambda=1$. Thus, a basis for the vector space of solution sequences of (3.12) is given by $(1,1,1,\ldots)$ and $(1,2,3,\ldots)$; therefore any solution $\{e_j\}$ increases linearly in j. This is enough for our purposes, because, if we call

$$q_j = h^2\left[E-(2\sigma+1)C\right]e_j + h^2 T_j,$$

and since $T_j = O\left(h^2\right)$, we have that

$$|q_j| = O\left(h^4\right) + O\left(h^2\right)|e_j| \tag{3.13}$$

and, if let $B=\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, $E_j=\begin{pmatrix} e_j \\ e_{j+1} \end{pmatrix}$ and $Q_j=\begin{pmatrix} 0 \\ q_j \end{pmatrix}$, the recurrence equation (3.11) becomes

$$E_j = BE_{j-1} + Q_j. \tag{3.14}$$

Iterating (3.14), we have

$$E_j = B^j E_0 + \sum_{k=0}^{j-1} B^k Q_{j-k}. \tag{3.15}$$

The linear increase rate of (3.12) implies that

$$\left\|B^k\right\| \le Ck. \tag{3.16}$$

In order to estimate $\|E_j\|$ we use an inductive argument. Using the definition of u_1^β, we have that the error at the first node x_1 satisfies

$$|e_1| = \left|U_\beta(h) - u_1^\beta\right| = |U_\beta(h) - h\beta| = O\left(h^3\right). \tag{3.17}$$

Thus, since $e_0=0$, we have that

$$\|E_0\| = O\left(h^3\right).$$

For estimating $\|E_1\|$ we need to estimate $|e_2|$. Using (3.13), (3.16) and (3.17), for $j=1$ we have that

$$|e_2| \le \|E_1\| = \|BE_0 + Q_1\| \le \|BE_0\| + |q_1| = O\left(h^3\right) + O\left(h^2\right)|e_1| = O\left(h^3\right).$$

By the inductive hypothesis we can assume that $|e_k| = O\left(h^2\right)$, for all $0\le k\le j$,. Then by (3.15) we have that

$$|e_{j+1}| \le \|E_j\| \le \left\|B^j E_0\right\| + \sum_{k=0}^{j-1}\left\|B^k\right\|\|Q_{j-k}\|.$$

For the first term, we have the following estimate:

$$\left\|B^j E_0\right\| \le \left\|B^j\right\|\|E_0\| \le Cj\|E_0\| = CjO\left(h^3\right) = O\left(h^2\right),$$

and for the second term, by (3.13) and (3.16) we have that

$$\sum_{k=0}^{j-1}\left\|B^k\right\|\|Q_{j-k}\| \le \sum_{k=0}^{j-1} Ck\left(O\left(h^4\right) + O\left(h^2\right)|e_{j-k}|\right)$$

by inductive hypothesis $|e_k| = O\left(h^2\right)$; thus for all $0\le k\le j$, we have that

$$\sum_{k=0}^{j-1}\left\|B^k\right\|\|Q_{j-k}\| \le O\left(h^4\right)\sum_{k=0}^{j-1} k = O\left(h^2\right).$$

Therefore we conclude that, for all j, (3.10) is satisfied. ∎

3.2.2. Second approximation step. By Proposition 3.6 we know that, for each $h>0$, there exists a unique $r_1^h \in I(h)$ that gives the discrete soliton R_h. Let $\beta_h^*>0$ be given by $r_1^h = h\beta_h^*$,; then sequence $u^{\beta_h^*}$ is the numerical soliton R_h.

Let $U_{\beta_h^*}(x)$ the solution of (3.9) with initial data $U'_{\beta_h^*}(0)=\beta_h^*$.

We denote by $\beta_0 = R'(0) = A\sqrt{\frac{A^{2\sigma}}{\sigma+1}-E}$ the exact slope of the continuous soliton at $x=0$,; then we have that $U_{\beta_0}(x)=R(x)$. Fixing $h>0$ we have associated to β_0 the recurrence solution u^{β_0}.

Remark 3.1. *By definition of $\beta_h^*, u^{\beta_h^*}$ and β_0, using the estimate (3.10), we have that*

$$\left|U_{\beta_h^*}(1/2)\right| = \left|U_{\beta_h^*}(1/2) - u_{2n}^{\beta_h^*}\right| = O\left(h^2\right), \tag{3.18}$$

$$\left|u_{2n}^{\beta_0}\right| = \left|u_{2n}^{\beta_0} - R(1/2)\right| = O\left(h^2\right). \tag{3.19}$$

Our goal is to prove the convergence of the numerical soliton R_h to the continuous soliton $R(x)$. A first step is to prove the convergence of β_h^* to β_0. A second step will be to calculate the order of this convergence.

Lemma 3.9. *Let β_h^* and β_0 defined as above, then*

$$\beta_h^* \underset{h\to 0^+}{\to} \beta_0. \tag{3.20}$$

Proof. We know that (see equality (3.18)) for $h>0$,

$$\left|U_{\beta_h^*}(1/2)\right| = O\left(h^2\right),$$

so $U_{\beta_h^*}(1/2) \underset{h\to 0}{\to} 0$. If $\beta_h^* \nrightarrow \beta_0$, we would have two different positive solutions of (PNLS) such that either are valued zero at boundaries $x=0$ and $x=\frac{1}{2}$, but this is impossible (see [1]). ∎

Order of convergence

We now want to calculate the order of the convergence that was given in (3.20). For that, we will need some preliminary results.

We can rewrite the second order equation (3.9) as a first order system by introducing the variable $V=U'$:

$$\begin{cases} U' = V \\ V' = \left(E - U^{2\sigma}\right)U, \end{cases} \qquad \text{with initial data } (U(0),V(0)) = (0,\beta), \tag{3.21}$$

and we write (U_β, V_β) for the unique solution of this system, for each β. System (3.21) has properties of continuity and differentiability with respect to the initial parameters, for example, with respect to β (see [4] or [9]).

Lemma 3.10. *The solution $(U_\beta,V_\beta)(x)$ of the system (3.21) is continuous with respect to β. If K is a Lipschitz constant for this operator in* $[0,1]$, *then*

$$\left\|(U_{\beta_2}(x),V_{\beta_2}(x)) - (U_{\beta_1}(x),V_{\beta_1}(x))\right\| \le e^{\frac{K}{2}}\left|\beta_2-\beta_1\right|.$$

Proof. See [9] chapter II. ∎

In our case we have that

$$O\left(h^2\right) = \left|U_{\beta_h^*}(1/2) - U_{\beta_0}(1/2)\right| \le e^{\frac{K}{2}} \left|\beta_h^* - \beta_0\right|; \tag{3.22}$$

thus, two is the best possible order of convergence in (3.20). In the rest of this section we will prove that the order of convergence is a number greater than one. In fact, we will prove that there exists $1 \le \gamma < 2$ such that $|\beta_h^* - \beta_0| = o(h^\gamma)$.

LEMMA 3.11. *The solution $(U_\beta, V_\beta)(x)$ of Equation (3.21) is differentiable with respect to β, and if β_0 is as above, then*

$$\partial_\beta V_{\beta_0}(1/2) = -1. \tag{3.23}$$

Proof. For the first assertion, see [9] chapter II. For the second assertion, we know that the solution of the problem (3.21) with initial data $(0,\beta_0)$ satisfies $(U_{\beta_0}(0), V_{\beta_0}(0)) = (0,\beta_0)$ and $(U_{\beta_0}(1/2), V_{\beta_0}(1/2)) = (0,-\beta_0)$. Now, multiplying the second equation of the system (3.21) by $V_\beta(x)$ and integrating in x, we have that

$$V_\beta^2(x) - EU_\beta^2(x) + \frac{1}{\sigma+1} U_\beta^{2\sigma+2}(x) = \beta^2;$$

differentiating this equation with respect to β, evaluating at $x = \frac{1}{2}$, and taking the limit when $\beta \to \beta_0$, we obtain that

$$-\beta_0 \left(\partial_\beta V_{\beta_0}\right)(1/2) = \beta_0$$

.Since $\beta_0 \neq 0$, equality (3.23) follows. □

We consider the tangent straight line of the function $U_{\beta_h^*}(x)$ at $x = \frac{1}{2}$ and let x_1^h be the value of the intersection between this straight line and the x-axis. Also, we denote by x_0^h the first positive root of $U_{\beta_h^*}(x)$. Notice that, in general, $x_0^h \neq x_1^h$ but we can assert that both are on the same side of $x = \frac{1}{2}$.

LEMMA 3.12. *Let x_0^h and x_1^h be as above; then*

$$x_0^h - \frac{1}{2} = O\left(h^2\right) \quad \text{and} \quad x_1^h - \frac{1}{2} = O\left(h^2\right). \tag{3.24}$$

Proof. We know (see [1]) that $U_{\beta_h^*}(x)$ has only one inflection point in its increasing part (between 0 and its symmetrical axis), therefore there is only one inflection point in the decreasing part (between its symmetrical axis and x_0^h). $U_{\beta_h^*}(x)$ is convex in a neighborhood on the left side of x_0^h, but we do not know the exact position of the inflection point, because it depends on σ and E.

We will only consider the cases x_0^h and x_1^h greater than $\frac{1}{2}$.

1. If $U_{\beta_h^*}(x)$ is convex at $x = \frac{1}{2}$, then the inflection point is on the left side of $x = \frac{1}{2}$, thus it is convex in the interval $\left(\frac{1}{2}, x_0^h\right)$ (see Figure 3.1). Therefore $\frac{1}{2} < x_1^h < x_0^h$ and

$$U'_{\beta_h^*}(1/2) < U'_{\beta_h^*}\left(x_0^h\right) = -\beta_h^* < 0.$$

The value at $x = \frac{1}{2}$ of the tangent straight line of $U_{\beta_h^*}(x)$ at x_0^h is $y_h < U_{\beta_h^*}(1/2)$,; therefore $y_h = O\left(h^2\right)$. Also $U'_{\beta_h^*}\left(x_0^h\right) = \frac{y_h}{x_0^h - \frac{1}{2}} = -\beta_h^*$, and, since $\beta_h^* \to \beta_0 \neq 0$, we have that $x_0^h - \frac{1}{2} = O\left(h^2\right)$, so $x_1^h - \frac{1}{2} = O\left(h^2\right)$.

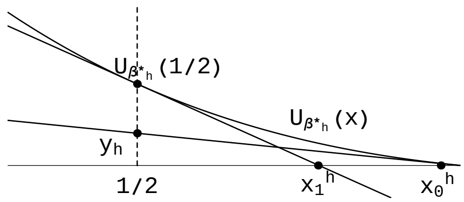

FIGURE 3.1.

2. If $U_{\beta_h^*}(x)$ is concave at $x=\frac{1}{2}$ we have two possibilities:
If $\frac{1}{2}<x_1^h<x_0^h$, then the inflection point is in the interval $\left(\frac{1}{2},x_1^h\right)$ (see Figure 3.2) and

$$U'_{\beta_h^*}(1/2)<U'_{\beta_h^*}\left(x_0^h\right)<0$$

then we can now proceed analogously to the previous case.

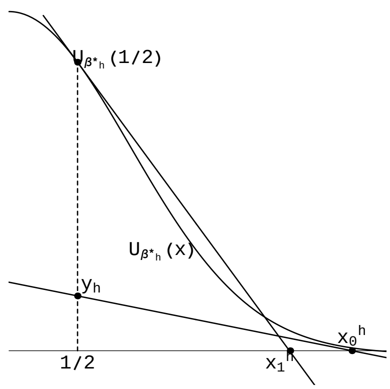

FIGURE 3.2.

If $\frac{1}{2}<x_0^h<x_1^h$ the inflection point of $U_{\beta_h^*}(x)$ is in $\left(\frac{1}{2},x_0^h\right)$ and we have two new possibilities. The first one is (see Figure 3.3)

$$U'_{\beta_h^*}(1/2)\leq U'_{\beta_h^*}\left(x_0^h\right)<0;$$

therefore we can apply a similar argument as in the above cases.

The second one is (see Figure 3.4)

$$U'_{\beta_h^*}\left(x_0^h\right)<U'_{\beta_h^*}(1/2)<0$$

moreover since by definition we have that

$$U'_{\beta_h^*}(1/2)=\frac{U_{\beta_h^*}(1/2)}{\frac{1}{2}-x_1^h}<0.$$

But we know that $U_{\beta_0}(1/2)=0$ and $U'_{\beta_0}(1/2)=-\beta_0$. By (3.23) we obtain

$$-1+O(\beta_h^*-\beta_0)=\frac{U'_{\beta_h^*}(1/2)-U'_{\beta_0}(1/2)}{\beta_h^*-\beta_0}=\frac{\frac{U_{\beta_h^*}(1/2)}{\frac{1}{2}-x_1^h}+\beta_0}{\beta_h^*-\beta_0}. \tag{3.25}$$

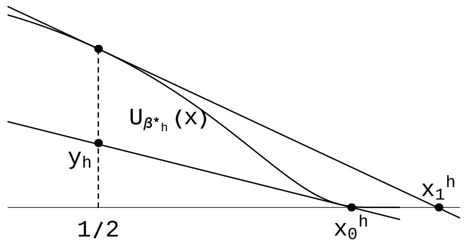

FIGURE 3.3.

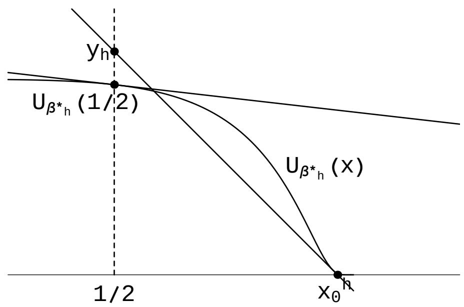

FIGURE 3.4.

Therefore

$$\frac{U_{\beta_h^*}(1/2)}{\frac{1}{2}-x_1^h}=-\beta_h^*+O\left(\beta_h^*-\beta_0\right)^2,$$

so $x_1^h-\frac{1}{2}=O\left(h^2\right)$.
Hence, this lemma follows. ∎

PROPOSITION 3.13. *There exists* $1\leq\gamma<2$ *such that*

$$|\beta_h^*-\beta_0|=o\left(h^\gamma\right). \tag{3.26}$$

Proof. If there exists γ as in (3.26), by (3.22) it will be less than two. For the proof that $\gamma\geq 1$, it suffices to prove that $\beta_h^*-\beta_0=o(h)$.

It is immediate that $U'_{\beta_h^*}(1/2)\rightarrow-\beta_0$ and $U'_{\beta_h^*}\left(x_0^h\right)=-\beta_h^*$. We define

$$g(h)=U'_{\beta_h^*}\left(x_0^h\right)-U'_{\beta_h^*}(1/2)=-\beta_h^*-U'_{\beta_h^*}(1/2).$$

Since $U''_{\beta_h^*}(0)=0$ and by (3.24) we have that

$$0=U''_{\beta_h^*}\left(x_0^h\right)\simeq\frac{U'_{\beta_h^*}(1/2)-U'_{\beta_h^*}\left(x_0^h\right)}{O\left(h^2\right)}=\frac{-g(h)}{O\left(h^2\right)};$$

thus $g(h)=o(h^2)$. Using the definition of $g(h)$ and (3.25) we calculate the order of $\beta_h^*-\beta_0$:

$$\begin{aligned}\beta_0-\beta_h^* &= \beta_0+U'_{\beta_h^*}(1/2)+g(h)\\ &= \left(U'_{\beta_h^*}(1/2)-U'_{\beta_0}(1/2)\right)+g(h)\\ &= -(\beta_h^*-\beta_0)+O(\beta_h^*-\beta_0)^2+g(h),\end{aligned}$$

so

$$g(h)=O(\beta_h^*-\beta_0)^2.$$

Therefore $\beta_h^*-\beta_0=o(h)$. Thus, the proposition follows. □

Approximation from the discrete ground states to the continuous one

We are now, are in a position to show the main result for this section.

Proposition 3.14. *In the above conditions we have that the difference between the evaluation of the continuous soliton E_hR and the discrete soliton R_h has a first order of approximation.*

$$|R_h-E_hR|_{2,h}=O(h)\underset{h\to 0}{\longrightarrow}0. \tag{3.27}$$

Proof. By definition

$$|R_h-E_hR|_{2,h}^2=\sum_{j=1}^{2N-1}h\left|u_j^{\beta_h^*}-R(jh)\right|^2.$$

As in the proof of Proposition (3.8), if

$$\left|e_1^h\right|=\left|u_1^{\beta_h^*}-R(h)\right|=O\left(h^2\right), \tag{3.28}$$

then we have for each $1<j\leq 2n-1$ that

$$\left|e_j^h\right|=\left|u_j^{\beta_h^*}-R(jh)\right|=O(h);$$

hence (3.27) follows. Therefore, we only need to show equality (3.28).

$$\left|u_1^{\beta_h^*}-R(h)\right|\leq\left|u_1^{\beta_h^*}-U_{\beta_h^*}(h)\right|+\left|U_{\beta_h^*}(h)-R(h)\right|.$$

The first term satisfies $\left|u_1^{\beta_h^*}-U_{\beta_h^*}(h)\right|=O\left(h^2\right)$ by (3.10), and the second one also satisfies $\left|U_{\beta_h^*}(h)-R(h)\right|=O\left(h^2\right)$, because

$$\begin{aligned}\left|U_{\beta_h^*}(h)-R(h)\right| &= \left|\beta_h^*h+O\left(h^3\right)-\beta_0h-O\left(h^3\right)\right|\\ &= h\left|\beta_h^*-\beta_0\right|+O\left(h^3\right).\end{aligned}$$

Using (3.26), limit (3.27) follows. □

4. Orbit, distance and Lyapunov function

We give the concepts of orbit and of distance from a point to the orbit of another point for the discrete case, in the same way as was given in [1] and [12]. The system (SDPNLS) has phase symmetry; thus by orbital stability we mean stability modulo this symmetry.

DEFINITION 4.1. *Suppose $v_h \in \mathbb{S}_h$. We define its orbit as the following set:*

$$\mathcal{G}_{v_h} = \left\{ v_h e^{i\gamma} / \gamma \in (0, 2\pi] \right\}. \tag{4.1}$$

DEFINITION 4.2. *Let $w_h, v_h \in \mathbb{S}_h$. The distance from w_h to the orbit $\mathcal{G}_{v_h}$ is defined by*

$$\rho_E^2 (w_h, \mathcal{G}_{v_h}) = \inf_{\gamma \in (0,2\pi]} \left\{ \left\langle A_h \left(v_h e^{i\gamma} - w_h \right), v_h e^{i\gamma} - w_h \right\rangle_h + E \left\langle v_h e^{i\gamma} - w_h, v_h e^{i\gamma} - w_h \right\rangle_h \right\}. \tag{4.2}$$

Hamiltonian and norm two are two quantities that play a central role in the analysis on the continuous case (see [1] and [12]); thus we can consider for the system (SDPNLS) two similar concepts given by translating them from the continuous case. These quantities are conserved by (SDPNLS), as we expected, and they are given by

$$\mathcal{H}_h[\phi_h] = \left\langle \left(A_h - \frac{1}{\sigma+1} D(R_h) \right) \phi_h, \phi_h \right\rangle_h,$$
$$\mathcal{N}_h[\phi_h] = |\phi_h|_{2,h}^2.$$

In the same way as [1] and [12], we define a Lyapunov functional with these quantities:

$$\mathcal{E}_h[\phi_h] = \mathcal{H}_h[\phi_h] + E \mathcal{N}_h[\phi_h]. \tag{4.3}$$

We will estimate $\mathcal{E}_h$ in terms of ρ_E. We can assume that a solution of (SDPNLS) is a perturbation of the ground state R_h:

$$\phi_h(t) e^{i\gamma} = R_h + w_h(t), \quad \text{where } w_h(t) = u_h(t) + i v_h(t). \tag{4.4}$$

Using the conservation of $\mathcal{E}_h$, we can write the difference $\Delta\mathcal{E}_h$ in the following form:

$$\begin{aligned} \Delta\mathcal{E}_h &= \mathcal{E}_h[\phi_0] - \mathcal{E}_h[R_h] \\ &= \mathcal{E}_h\left[\phi_h(t) e^{i\gamma}\right] - \mathcal{E}_h[R_h] \\ &= \mathcal{E}_h[R_h + w_h(t)] - \mathcal{E}_h[R_h]. \end{aligned}$$

We obtain a lower bound for $\Delta\mathcal{E}_h$ through a Taylor expansion in R_h:

$$\Delta\mathcal{E}_h \geq \left\langle L_+^h u_h, u_h \right\rangle + \left\langle L_-^h v_h, v_h \right\rangle - C_1 \|w_h\|_{1,h}^3 - C_2 \|w_h\|_{1,h}^6, \tag{4.5}$$

where

$$L_-^h = A_h + E I_h - D(R_h),$$
$$L_+^h = A_h + E I_h - (2\sigma+1) D(R_h).$$

Notice that functions L_-^h and $L_+^h : \mathbb{S}_h \to \mathbb{S}_h$ are the real and imaginary parts of the linearized (SDPNLS) operator near the ground state R_h.

If we write the solution in the form (4.4) there exists $\gamma_h \in (0,2\pi]$ that achieves the distance (4.2) for R_h:

$$\rho_E^2(R_h,\mathcal{G}_{v_h}) = \inf\left\{\left\langle A_h\left(v_h\ e^{i\gamma}-R_h\right),v_h\ e^{i\gamma}-R_h\right\rangle_h + E\left\langle v_h\ e^{i\gamma}-R_h,v\ e^{i\gamma}-R_h\right\rangle_h\right\}. \tag{4.6}$$

the minimum is taken over all phases $\gamma \in [0,2\pi)$ and it provides an orthogonal condition

$$\langle D(R_h)R_h,v_h\rangle_h = 0, \tag{4.7}$$

where v_h is the imaginary part of the perturbation of the solution in (4.4).

We want to obtain lower bounds for the quadratic forms $\left\langle L_-^h u_h,u_h\right\rangle_h$ and $\left\langle L_+^h v_h,v_h\right\rangle_h$ as follows:

$$\left\langle L_+^h u_h,u_h\right\rangle_h + \left\langle L_-^h v_h,v_h\right\rangle_h \geq C_3\|w_h\|_{1,h}^2 - C_4\|w_h\|_{1,h}^3 - C_4\|w_h\|_{1,h}^4, \tag{4.8}$$

because plugging (4.8) into (4.5) we get a suitable lower bound of $\Delta\mathcal{E}$ and then, applying the techniques introduced in [12], we can obtain an orbital stability result. Thus, it is sufficient to calculate a bound like (4.8).

5. Constrained variational problem for the quadratic forms

In order to obtain a bound like (4.8) we need to analyze the spectrum of L_-^h and L_+^h. The technique that we use is adapted to the discrete problem and differs considerably from the technique used in [1] for the continuous periodic case.

In the remainder of this work we assume, without loss of generality by the property of symmetry, that $\mathbb{S}_h \subseteq \mathbb{C}^{2n-1}$ and that we can take the corresponding inner product and norm in the interval $[0,1/2]$.

5.1. Lower bound for L_-^h. Using the tridiagonal form of the matrix $L_-^h(u_h)$, we prove that its eigenvalues are singles.

LEMMA 5.1. *Eigenvalues of the matrix $L_-^h(u_h)$ are singles, and eigenvectors of two different eigenvalues are orthogonal.*

Proof. Suppose that u_h and $v_h \in \mathbb{S}_h$ are nonzero eigenvectors of eigenvalue λ. If $u_{2n-1} \neq 0$ and $v_{2n-1} \neq 0$ then there exist $\alpha \neq 0$and $\beta \neq 0$ such that

$$\alpha u_{2n-1} + \beta v_{2n-1} = 0.$$

Since

$$(A_h + EI_h - D(R_h) - \lambda I)(\alpha u_h + \beta v_h) = 0,$$

using the tridiagonal form of this matrix we have that

$$\alpha u_h + \beta v_h = 0.$$

Thus u_h and v_h are linearly dependent, so the eigenvalues are single. If $u_{2n-1} = 0$ or $v_{2n-1} = 0$ we have two possibilities. If $u_{2n-2} \neq 0$ and $v_{2n-2} \neq 0$, we can proceed analogously to the above case with a similar argument. If $u_{2n-1} \neq 0$ and $v_{2n-1} = 0$, we define $\widetilde{v}_h = u_h + v_h$; this is an eigenvector of the same eigenvalue λ and $\widetilde{v}_{2n-1} \neq 0$; thus it is covered in a previous case.

For the orthogonality we can use a standard argument (see for instance [10]). $\square$

Since $L_-^h R_h = 0$, we have that R_h is an eigenvector of eigenvalue $\lambda = 0$. Actually, we will prove that this is the first eigenvalue of L_-^h.

Lemma 5.2. *The first eigenvalue of L_-^h in $\mathbb{S}_h$ is $\lambda=0$ with eigenvector R_h. Moreover, an eigenvector of the first eigenvalue has all its coefficients with the same sign, and they never vanish at the inner nodes.*

Proof. By definition R_h is an eigenvector of the eigenvalue zero of the operator L_-^h. Our goal in this lemma is to prove that zero is the first eigenvalue of this operator. For this, we can consider the Rayleigh quotient for L_-^h:

$$Ra(v_h,h)=\frac{\langle A_h v_h+Ev_h-D(R_h)v_h,v_h\rangle_h}{|v_h|_{2,h}^2}. \tag{5.1}$$

If λ is the first eigenvalue for L_-^h in $\mathbb{S}_h$, we know that

$$\lambda=\min_{v_h\in\mathbb{S}_h} Ra(v_h,h), \tag{5.2}$$

and this minimum is achieved in an eigenvector $\widetilde{v}_h$. But it is easy to check that the vector $|\widetilde{v}_h|$ is a minimum of (5.2); thus $|\widetilde{v}_h|$ is also an eigenvector of the eigenvalue λ. Therefore, by Lemma 5.1, we can conclude that $\widetilde{v}_h=|\widetilde{v}_h|$ or $\widetilde{v}_h=-|\widetilde{v}_h|$, so $\widetilde{v}_h$ does not change the sign in $\mathbb{S}_h$. Moreover, due to the tridiagonal form of the matrix $A_h+EI_h-D(R_h)$, all the coefficients of $\widetilde{v}_h$ are nonzero

We shall have established this lemma if we show that $\lambda=0$. We suppose that $\lambda\neq 0$; then by Lemma 5.1, and since $\widetilde{v}_h$ and R_h are orthogonal, but both do not change sign, so we have a contradiction. Hence, zero is the first eigenvalue of L_-^h in $\mathbb{S}_h$ with eigenvector R_h. Therefore, L_-^h is a non negative operator in $\mathbb{S}_h$. □

Let $\mathcal{L}_h$ denote the subspace of $\mathbb{S}_h$ that satisfies condition (4.7):

$$\mathcal{L}_h=\{v\in\mathbb{S}_h/\langle D(R_h)R_h,v\rangle_h=0\}=(D(R_h)R_h)^{\perp}\cap\mathbb{S}_h.$$

The first eigenvalue λ_1^h of L_-^h in the subspace $\mathcal{L}_h$ is given by the minimum of the Rayleigh quotient

$$\lambda_1^h=\inf_{v_h\in\mathcal{L}_h}\frac{\left\langle L_-^h v_h,v_h\right\rangle_h}{|v_h|_{2,h}^2}.$$

Since $R_h\notin\mathcal{L}_h$ we have that $\lambda_1^h>0$. We will prove that there exists a constant $C>0$ such that $\lambda_1^h\geq C$.

Proposition 5.3. *If λ_1^h is the first eigenvalue of L_-^h in $\mathcal{L}_h$, there exists a constant $C>0$ such that $\lambda_1^h\geq C$, for $h>0$ small enough.*

Proof. We consider the usual basis of finite element functions $\{\varphi_i\}_{1\leq i\leq k-1}$ in $\mathcal{V}_h\subset H_0^1[0,1/2]$. If $w\in\mathcal{V}_h$ we have that $w(x)=\sum_{i=1}^{k-1}w(x_i)\varphi_i(x)$ (see [3]). By the Rayleigh-Ritz principle (see [7]), the first eigenvalue of L_-^h in the finite element space is

$$\lambda_{e1}^h=\min_{\langle w\rangle}\max_{\alpha\in\mathbb{R}}\mathcal{R}(\alpha w)=\min_{\substack{w\in\mathcal{V}_h\\ \|w\|_2=1}}\mathcal{R}(w),$$

where $\mathcal{R}$ is the Rayleigh quotient for the continuous operator L_- (see [1]) evaluated at the finite elements, i.e., if $w=\sum q_i\varphi_i$,

$$\mathcal{R}(w)=\frac{\sum\limits_i\sum\limits_j q_iq_j\int\left(\varphi_i'\varphi_j'+E\varphi_i\varphi_j-R^{2\sigma}\varphi_i\varphi_j\right)dx}{\sum\limits_i\sum\limits_j q_iq_j\int\varphi_i\varphi_j dx}. \tag{5.3}$$

Let λ_1 denote the first eigenvalue of the continuous operator in the subspace $\left(R^{2\sigma+1}\right)^{\perp}$ (see [1]). We know that (see [7])

$$\lambda_{e1}^h \geq \lambda_1 > 0.$$

Our goal is to prove that there exists a constant $C>0$ such that $\lambda_1^h \geq C$ for h small enough. Thus, it is sufficient to show that λ_{e1}^h and λ_1^h are close if h is small. In this case we can take, for instance, $C=\frac{\lambda_1}{2}$.

Let $u_h \in \mathbb{S}_h$, thus (see Section 2) we have $G_h(u_h) \in \mathcal{V}_h$ such that $G_h(u_h)(x_i)=u_i$, so $G_h(u_h)(x)=\sum_{i=1}^{n-1} u_i\varphi_i(x)$. Evaluating (5.3) at $G_h(u_h)$ we have that

$$\mathcal{R}(G_h(u_h)) = \frac{\langle A_h u_h + E u_h - D(E_h R) u_h, u_h\rangle_h + O(h)}{|u_h|_h^2 - \frac{h^2}{6}|u_h'|_h^2}.$$

Since (3.27) and (5.1), we obtain

$$Ra(u_h,h) - \mathcal{R}(G_h(u_h)) \underset{h\to 0^+}{\rightarrow} 0.$$

Therefore we conclude that $\left|\lambda_1^h - \lambda_{e1}^h\right|$ is small for h small enough. Hence, the proposition follows. $\square$

Remark 5.1. *By Proposition (5.3) we have that*

$$\left\langle L_-^h v_h, v_h\right\rangle_h \geq C|v_h|_{2,h}^2 \quad \forall v_h \in \mathcal{L}_h,$$

and from this bound we conclude,

$$\left\langle L_-^h v_h, v_h\right\rangle_h \geq C\|v_h\|_{1,h}^2 \quad \forall v_h \in \mathcal{L}. \tag{5.4}$$

5.2. Lower bound for L_+^h. From a simple computation we can see that $\left\langle L_+^h R_h\ , R_h\right\rangle_h < 0$. In this section we will prove that L_+^h defines a positive quadratic form in $R_h^{\perp}$, the subspace of $\mathbb{S}_h$ orthogonal to R_h.

With the notation of Proposition (5.3), we know that (see [7])

$$\lambda_1 \leq \lambda_{e1}^h \leq \lambda_1 + 2\delta h^2 \lambda_1^2. \tag{5.5}$$

Now we can proceed analogously to the previous section. Using inequality (5.5) we obtain a lower positive bound for $\left\langle L_+^h v, v\right\rangle_h$ at $R_h^{\perp}$,

$$\left\langle L_+^h u_h, u_h\right\rangle_h > C|u_h|_{2,h}^2, \qquad u_h \in R_h^{\perp}.$$

We require of an additional condition for the solution $\phi_h(x,t)$,

$$\langle \phi_h(t), \phi_h(t)\rangle_h = \langle R_h, R_h\rangle_h \quad t\in\mathbb{R}. \tag{5.6}$$

By (4.4) and (5.6) we have that

$$\langle u_h, R_h\rangle_h = -\frac{1}{2}\left[\langle u_h, u_h\rangle_h + \langle v_h, v_h\rangle_h\right]. \tag{5.7}$$

Now, in the same way as [12], we can prove the following results.

Proposition 5.4. *Let $\sigma<2$. If $\phi_h(t)$ satisfies (4.4) and (5.6) with $u_h \in \mathbb{S}_h$, then there are constants D, D_1 and $D_2>0$, such that*

$$\left\langle L_+^h u_h, u_h\right\rangle_h \geq D\|u_h\|_{1,h}^2 - D_1\langle A_h w_h, w_h\rangle_h |w|_{2,h}^2 - D_2 |w|_{2,h}^4. \tag{5.8}$$

Proof. Without loss of generality we can assume that $\langle R_h, R_h\rangle_h = 1$. We write

$$u_h = (u_h)_\parallel + (u_h)_\perp$$

with

$$\begin{aligned}(u_h)_\parallel &= \langle u_h, R_h\rangle_h R_h \\ &= -\frac{1}{2}\left[\langle u_h, u_h\rangle_h + \langle v_h, v_h\rangle_h\right] R_h\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}(u_h)_\perp &= u_h - (u_h)_\parallel \\ &= u_h - \frac{1}{2}\left[\langle u_h, u_h\rangle_h + \langle v_h, v_h\rangle_h\right] R_h.\end{aligned}$$

We have that

$$\begin{aligned}\left\langle L_+^h u_h, u_h\right\rangle_h &= \left\langle L_+^h (u_h)_\parallel, (u_h)_\parallel\right\rangle_h \\ &\quad + 2\left\langle L_+^h (u_h)_\parallel, (u_h)_\perp\right\rangle_h + \left\langle L_+^h (u_h)_\perp, (u_h)_\perp\right\rangle_h,\end{aligned}$$

and there exists $d>0$ verifying

$$\begin{aligned}\left\langle L_+^h (u_h)_\perp, (u_h)_\perp\right\rangle_h &\geq d\langle (u_h)_\perp, (u_h)_\perp\rangle_h \\ &= d\left[\langle u_h, u_h\rangle_h + \frac{1}{4}\left[\langle u_h, u_h\rangle_h + \langle v_h, v_h\rangle_h\right]^2\right].\end{aligned} \tag{5.10}$$

By (5.7), (5.9) we have that

$$\left\langle L_+^h (u_h)_\parallel, (u_h)_\parallel\right\rangle_h = \frac{1}{4}\left\langle L_+^h R_h, R_h\right\rangle_h \left[\langle u_h, u_h\rangle_h + \langle v_h, v_h\rangle_h\right]^2. \tag{5.11}$$

Finally, from (5.7) we conclude that

$$\begin{aligned}\left\langle L_+^h (u_h)_\perp, (u_h)_\parallel\right\rangle_h &= \langle u_h, R_h\rangle_h \left\langle L_+^h (u_h)_\perp, R_h\right\rangle_h \\ &\geq -d'\langle A_h w_h, w_h\rangle_h |w|_{2,h}^2 - d''|w|_{2,h}^4.\end{aligned} \tag{5.12}$$

Now, the lower bound (5.8) follows from (5.10), (5.11) and (5.12). □

6. Main results

We are now in position to prove a stability theorem by the discrete operator. Once this stability result is proved, we will show the convergence of the numerical solutions to the solution of the continuous problem (PNLS). In order to prove the stability theorem we will proceed analogously to [12].

THEOREM 6.1. *Let σ be small enough and let $\phi_h(t)$ be the solution of (SDPNLS) with initial data $\phi_0 \in \mathbb{S}_h$. Then there exists a ground state R_h that is orbital stable, i.e., for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$, such that if $|\phi_0|_{2,h}^2 = |R_h|_{2,h}^2$ and $\rho_E(\phi_0, \mathcal{G}_{R_h}) < \delta(\varepsilon)$ then*

$$\rho_E(\phi_h(t), \mathcal{G}_{R_h}) < \varepsilon \quad \textit{for all } t > 0.$$

We will say that the soliton orbit is stable.

Proof. Suppose that the minimum of (4.6) is achieved in γ_h. From (5.4) and (5.8), substituting in (4.5), we have

$$\begin{aligned} \Delta\mathcal{E}_h &\geq \langle L_+^h u_h, u_h \rangle + \langle L_-^h v_h, v_h \rangle - C_1 \|w_h\|_{1,h}^2 - C_2 \|w_h\|_{1,h}^6 \\ &\geq C_3 \|w_h\|_{1,h}^2 - C_4 \|w_h\|_{1,h}^2 - C_5 \|w_h\|_{1,h}^6 . \end{aligned}$$

Hence

$$\Delta\mathcal{E}_h \geq g(\rho(\phi_h(t), \mathcal{G}_{R_h})), \tag{6.1}$$

where

$$g(t) = ct^2\left(1 - at^2 - bt^4\right) \quad \text{with } a,b,c > 0.$$

Notice that $g(0) = 0$, and $g(t) > 0$ for $0 < t \ll 1$. We can obtain the stability from (6.1): let $\varepsilon > 0$ be small enough; then the continuity of $\mathcal{E}_h$ in $\mathbb{S}_h$ implies that there exists $\delta(\varepsilon) > 0$ such that $\Delta\mathcal{E}_h < g(\varepsilon)$ for $\rho_E(\phi_0, \mathcal{G}_{R_h}) < \delta(\varepsilon)$. Since $\Delta\mathcal{E}_h$ does not depend on t, from (6.1) we conclude that $g(\rho_E(\phi_h(t), \mathcal{G}_{R_h})) < g(\varepsilon)$ for $t > 0$.

Hence, since $\rho_E(\phi_h(t), \mathcal{G}_{R_h})$ is a continuous function of t,

$$\rho_E(\phi_h(t), \mathcal{G}_{R_h}) < \varepsilon \quad \text{for } t > 0, \tag{6.2}$$

and the soliton R_h is orbital stable. □

THEOREM 6.2. *Let $E_h : H_0^1[0,1/2] \to \mathbb{S}_h$ be the evaluation operator at nodes $\{x_j = jh\}_j$ and $\phi(x,t)$, be the solution of (PNLS) with initial data $\phi_0(x)$; then for all $t > 0$*

$$\left| e^{i\gamma_h} \phi_h(t) - e^{i\gamma} E_h(\phi(x,t)) \right|_{2,h} \underset{h \to 0}{\to} 0$$

where γ_h and γ are the values of the phases where the distance is achieved in the discrete and continuous cases respectively.

Proof. From (3.27) and (6.2), and by the continuity of the evaluation operator E_h and the orbital stability in the continuous case (see [1]), we have the following inequalities,

$$\begin{aligned} \left| e^{i\gamma_h} \phi_h(t) - e^{i\gamma} E_h(\phi(t)) \right|_h &\leq \left| e^{i\gamma_h} \phi_h(t) - R_h \right|_{2,h} + |E_h(R) - R_h|_{2,h} + \\ &\quad + \left| E_h(R) - e^{i\gamma} E_h(\phi(t)) \right|_{2,h} \\ &\leq \rho_E(\phi_h(t), \mathcal{G}_{R_h}) + |E_h(R) - R_h|_{2,h} + C\rho_E(\phi(t), \mathcal{G}_R) \underset{h \to 0}{\to} 0, \end{aligned}$$

and the proof is complete. □

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