RAYLEIGH-BÉNARD CONVECTION: DYNAMICS AND STRUCTURE IN THE PHYSICAL SPACE[*]

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Abstract. The main objective of this article is part of a research program to link the dynamics of fluid flows with the structure of these fluid flows in physical space and the transitions of this structure. To demonstrate the main ideas, we study the two-dimensional Rayleigh-Bénard convection, which serves as a prototype problem. The analysis is based on two recently developed nonlinear theories: geometric theory for incompressible flows [T. Ma and S. Wang, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 119, 2005] and bifurcation and stability theory for nonlinear dynamical systems (both finite and infinite dimensional) [T. Ma and S. Wang, World Scientific, 2005]. We have shown in [T. Ma and S. Wang, *Commun. Math. Sci., 2(2)*, 159–183, 2004] that the Rayleigh-Bénard problem bifurcates from the basic state to an attractor $\mathcal{A}_R$ when the Rayleigh number R crosses the first critical Rayleigh number R_c for all physically sound boundary conditions, regardless of the multiplicity of the eigenvalue R_c for the linear problem. In this article, in addition to a classification of the bifurcated attractor $\mathcal{A}_R$, the structure of the solutions in physical space and the transitions of this structure are classified, leading to the existence and stability of two different flows structures: pure rolls and rolls separated by a cross the channel flow. It appears that the structure with rolls separated by a cross-channel flow has not been carefully examined although it has been observed in other physical contexts such as the Branstator-Kushnir waves in atmospheric dynamics [G.W. Branstator, *J. Atmos. Sci.*, 44, 2310–2323, 1987] and [K. Kushnir, *J. Atmos. Sci.*, 44, 2727–2742, 1987].

1. Introduction

The Rayleigh-Bénard convection problem was originated in the famous experiments conducted by H. Bénard in 1900. Bénard investigated a fluid with a free surface heated from below in a dish and noticed a rather regular pattern of hexagonal convection cells. Based on the pioneering studies by Lord Rayleigh [13], the convection occurs only when a non-dimensional parameter called the Rayleigh number,

$$R = \frac{g\alpha\beta}{\kappa\nu} h^4, \tag{1.1}$$

exceeds a certain critical value, where g is the acceleration due to gravity, α the coefficient of thermal expansion of the fluid, $\beta = |dT/dz| = (\bar{T}_0 - \bar{T}_1)/h$ the vertical temperature gradient with $\bar{T}_0$ the temperature on the lower surface and $\bar{T}_1$ on the upper surface, h the depth of the layer of the fluid, κ the thermal diffusivity and ν the kinematic viscosity. There have been intensive studies of this problem; see among others Chandrasekhar [2] and Drazin and Reid [3] for linear theories, and Kirchgässner [6], Rabinowitz [12], and Yudovich [16, 17], and the references therein for nonlinear theories.

Recently, the authors have developed a bifurcation theory [9] for nonlinear partial differential equations, which has been used to develop a nonlinear analysis for the

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Rayleigh-Bénard convections [8]. This bifurcation theory is centered on a new notion of bifurcation, called attractor bifurcation, for nonlinear evolution equations. The main ingredients of the theory include a) the attractor bifurcation theory, b) steady state bifurcation for a class of nonlinear problems with even order non-degenerate nonlinearities, regardless of the multiplicity of the eigenvalues, and c) new strategies for the Lyapunov-Schmidt reduction and the center manifold reduction procedures.

In particular, based on this bifurcation theory, for the Rayleigh-Bénard convection problem, we have shown [8, 9] that the problem bifurcates from the trivial solution to an attractor $\mathcal{A}_R$ when the Rayleigh number R crosses the first critical Rayleigh number R_c, for all physically sound boundary conditions and regardless of the multiplicity of the eigenvalue R_c for the linear problem.

The main objectives of this article are 1) to classify the solutions in the bifurcated attractor $\mathcal{A}_R$, and 2) to study the structure (and the transitions of this structure) of the solutions of the Bénard problem in physical space. The first objective is an important part of the above-mentioned new bifurcation and stability theory. The second objective is part of a program recently initiated by the authors to develop a geometric theory of two-dimensional incompressible fluid flow in physical space; see [10]. This program of study consists of research in two directions: 1) the study of the structure (and its transitions/evolutions) of divergence-free vector fields (kinematics), and 2) the study of the structure (and its transitions) of velocity fields for 2-D incompressible fluid flows governed by the Navier-Stokes equations or the Euler equations. The study in this article is in the second direction, linking kinematics to dynamics.

To demonstrate ideas, in this article we only consider the two-dimensional Bénard convection problem. The three-dimensional case is technically more involved, and shall be reported elsewhere. From the physical point of view, two-dimensional Boussinesq equations can be considered as idealized models for many physical phenomena, including 1) the Walker circulation over the tropics [14], which has the same topological structure as the cells given in Fig. 4.3 in Thm. 4.2, and 2) the Branstator-Kushnir waves of atmospheric dynamics [1, 7], which have similar topological structure as given in Fig. 4.2 in Thm. 4.1.

We end this introduction with a few remarks. First, the main idea of the study is 1) to explicitly reduce the bifurcation problem to a bifurcation problem on the center manifold, and 2) to solve this reduced problem using an S^1 attractor bifurcation theorem and structural stability theorems for 2D incompressible flows. The types of solutions in this S^1 attractor depend on the boundary conditions. With the periodic boundary condition (2.7) in the x_1 direction in this article, the bifurcated attractor consists of only steady states. When the boundary conditions for the velocity field are free-slip boundary conditions and $\Omega = (0, L)^2 \times (0, 1)$ with $0 < L^2 < (2 - \sqrt[3]{2})/(\sqrt[3]{2} - 1)$, using the same method established in this article, we can prove that the bifurcated attractor is still an S^1 attractor, consisting of exactly eight singular steady states (with four saddles and four minimal attractors) and eight heteroclinic orbits connecting these steady states. The bifurcated attractor and its detailed classification provide a global dynamic transitions in both the physical and phase spaces.

Second, the method and ideas presented in this article are crucial to obtain these results, which cannot be obtained using only the classical bifurcation theories. For the case studied in this article, the classical bifurcation theory with symmetry arguments implies that the bifurcation attractor in the main theorems, theorems 4.1 and 4.2, *contains* a circle of steady states. We need, however, the new bifurcation theory to prove in particular that the bifurcated attractors are *exactly* of type S^1. Furthermore,

for general boundary conditions such as the free-slip boundary conditions mentioned above, no symmetry can be used, and the classical amplitude equation methods fails to derive the dynamics.

Third, the newly developed geometric theory for incompressible flows is crucial for the structure (and its stability) of the solutions in the physical spaces obtained in the main theorems.

Fourth, it appears that the structure with rolls separated by a cross-channel flow has not been carefully examined, although it has been observed in other physical contexts such as the Branstator-Kushnir waves in atmospheric dynamics [1, 7].

Finally, as previously mentioned, this article is part of a research program initiated in the mid-1990s to make connections between the dynamics and the structure in physical spaces.

This article is organized as follows. First, the functional setting and the attractor bifurcation theorem for the Bénard convection problem obtained in [8] are introduced in Section 2. Section 3 recapitulates 1) the approximation of the center manifold function, 2) an S^1-attractor bifurcation theorem, and 3) structural stability theorems for incompressible flows. The main theorems of this article are stated in Section 4, and proved in Section 5.

2. Bénard problem

2.1. Boussinesq equations.   The Bénard problem can be modeled by the Boussinesq equations. In this paper, we consider the Bénard problem in a two-dimensional (2D) domain $\mathbb{R}^1 \times (0,h) \subset \mathbb{R}^2$ ($h > 0$). The Boussinesq equations, which govern the motion and states of the fluid flow, are as follows; see among others Rayleigh [13], Drazin and Reid [3] and Chandrasekhar [2]:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \rho_0^{-1} \nabla p = -gk[1 - \alpha(T - \bar{T}_0)], \tag{2.1}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \kappa \Delta T = 0, \tag{2.2}$$

$$\operatorname{div} u = 0, \tag{2.3}$$

where ν, κ, α, g are constants defined as in (1.1), $u = (u_1, u_2)$ the velocity field, p the pressure function, T the temperature function, $\bar{T}_0$ and $\bar{T}_1$ constants representing the lower and upper surface temperatures at $x_2 = 0$ and $x_2 = h$ respectively, and $k = (0,1)$ the unit vector in the x_3 direction.

To make the equations non-dimensional, let

$$
\begin{aligned}
&x = hx', \\
&t = h^2 t'/\kappa, \\
&u = \kappa u'/h, \\
&T = \beta h(T'/\sqrt{R}) + \bar{T}_0 - \beta h x'_2, \\
&p = \rho_0 \kappa^2 p'/h^2 + p_0 - g\rho_0(hx'_2 + \alpha\beta h^2 (x'_2)^2/2), \\
&P_r = \nu/\kappa.
\end{aligned}
$$

Here the Rayleigh number R is defined by (1.1), and $P_r = \nu/\kappa$ is the Prandtl number.

Omitting the primes, the equations (2.1)-(2.3) can be rewritten as follows

$$\frac{1}{P_r}\left[\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\nabla p\right]-\Delta u-\sqrt{R}Tk=0, \tag{2.4}$$

$$\frac{\partial T}{\partial t}+(u\cdot\nabla)T-\sqrt{R}u_3-\Delta T=0, \tag{2.5}$$

$$\mathrm{div}\,u=0. \tag{2.6}$$

The non-dimensional domain is $\Omega=\mathbb{R}^1\times(0,1)\subset\mathbb{R}^2$. We consider periodic boundary condition in the x_1 direction:

$$(u,T)(x_1,x_2,t)=(u,T)(x_1+kL,x_2,t)\quad\forall k\in\mathbb{Z}. \tag{2.7}$$

At the top and bottom boundary $(x_2=0,1)$, different combinations of top and bottom boundary conditions are normally used in different physical settings such as *rigid-rigid*, *rigid-free*, *free-rigid*, and *free-free*. For instance, we have

DIRICHLET BOUNDARY CONDITION (RIGID-RIGID):

$$T=0,\quad u=0\quad\text{at }x_2=0,1. \tag{2.8}$$

FREE-FREE BOUNDARY CONDITION:

$$T=0,\quad u_2=0\quad\frac{\partial u_1}{\partial x_2}=0\quad\text{at }x_2=0,1. \tag{2.9}$$

FREE-RIGID BOUNDARY CONDITION:

$$\begin{cases}T=0,\quad u=0 & \text{at }x_2=0,\\[2mm]T=0,\quad u_2=0,\quad\dfrac{\partial u_1}{\partial x_2}=0 & \text{at }x_2=1.\end{cases} \tag{2.10}$$

The initial value conditions are given by

$$(u,T)=(u_0,T_0)\qquad\text{at }t=0. \tag{2.11}$$

2.2. Functional setting. For simplicity, we proceed in this article with the set of boundary conditions given by (2.7) and (2.8), and similar results hold true as well for other combinations of boundary conditions.

Let

$$H=\{(u,T)\in L^2(\Omega)^3\mid\mathrm{div}\,u=0,\ u_2|_{x_2=0,1}=0,\ u_1\text{ is periodic in the }x_1\text{ direction}\},$$
$$V=\{(u,T)\in H_0^1(\Omega)^3\mid\mathrm{div}\,u=0,\ (u,T)\text{ is periodic in the }x_1\text{ direction}\},$$
$$H_1=V\cap H^2(\Omega)^3.$$

Let $G:H_1\to H$, and $L_\lambda=-A+B_\lambda:H_1\to H$ be defined by

$$G(\psi)=(-P[(u\cdot\nabla)u],-(u\cdot\nabla)T),$$
$$A\psi=(-P(\Delta u),-\Delta T),$$
$$B_\lambda\psi=\lambda(P(Tk),u_2),$$

for any $\psi=(u,T)\in H_1$. Here $\lambda=\sqrt{R}$, and P the Leray projection to L^2 fields.

Then the Boussinesq equations (2.4)–(2.8) can be rewritten in the following operator form

$$\frac{d\psi}{dt}=L_\lambda\psi+G(\psi),\qquad\psi=(u,T). \tag{2.12}$$

2.3. Attractor bifurcation of the Bénard problem.   Let $\{S_\lambda(t)\}_{t\geq 0}$ be an operator semi-group generated by the equation (2.12). Then the solution of (2.12) can be expressed as

$$\psi(t,\psi_0) = S_\lambda(t)\psi_0, \qquad t \geq 0.$$

DEFINITION 2.1. *A set $\Sigma \subset H$ is called an invariant set of (2.12) if $S(t)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (2.12) is called an attractor if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that for any $\psi_0 \in U$ we have*

$$\lim_{t\to\infty} \mathrm{dist}_H(\psi(t,\psi_0),\Sigma) = 0.$$

DEFINITION 2.2.
1. *We say that the equation (2.12) bifurcates from $(\psi,\lambda) = (0,\lambda_0)$ to invariant sets Ω_λ, if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of (2.12) such that $0 \notin \Omega_{\lambda_n}$ and*

$$\lim_{n\to\infty} \lambda_n = \lambda_0,$$
$$\lim_{n\to\infty} \max_{x\in\Omega_{\lambda_n}} |x| = 0.$$

2. *If the invariant sets Ω_λ are attractors of (2.12), then the bifurcation is called an attractor bifurcation.*

We are now in position to state the attractor bifurcation theorem for the Bénard problem (2.4)-(2.8). The linearized equations of (2.4)-(2.6) are given by

$$\begin{cases} -\Delta u + \nabla p - \sqrt{R}Tk = 0, \\ \qquad -\Delta T - \sqrt{R}u_2 = 0, \\ \qquad\qquad \mathrm{div}\, u = 0, \end{cases} \tag{2.13}$$

where R is the Rayleigh number. These equations are supplemented with the same boundary conditions (2.7) and (2.8) as the nonlinear Boussinesq system. This eigenvalue problem for the Rayleigh number R is symmetric. Hence, we know that all eigenvalues R_k with multiplicities m_k of (2.13) with (2.7) and (2.8) are real numbers, and

$$0 < R_1 < \cdots < R_k < R_{k+1} < \cdots.$$

The first eigenvalue R_1 is a function of the period L. The critical Rayleigh number R_c is given by

$$R_c = \min_{L>0} R_1(L). \tag{2.14}$$

Let the multiplicity of R_c be $m_1 = m$ (m even), and the first eigenspace be denoted by E_0. Then we have the following attractor bifurcation theorem.

THEOREM 2.3. [8, 9] *For the Bénard problem (2.4-2.8), the following assertions hold true.*
1. *When $R \leq R_c$, the steady state $(u,T) = 0$ is globally asymptotically stable in H.*

2. *The equations bifurcate from* $((u,T),R) = (0,R_c)$ *to attractors* Σ_R *for* $R >$ R_c, *with* $m-1 \leq \dim \Sigma_R \leq m$, *and* Σ_R *is an* $(m-1)$ *dimensional homological sphere, i.e.,* Σ_R *has the same homology as* S^{m-1}.

3. *For any* $(u,T) \in \Sigma_R$, *the velocity field* u *can be expressed as*

$$u = \sum_{k=1}^{m} \alpha_k e_k + o\left(\sum_{k=1}^{m} \alpha_k e_k\right),$$

   where e_k *are the velocity fields of the first eigenvectors in* E_0.

4. *For any open bounded neighborhood* $U \subset H$ *of* $(u,T) = 0$, *the attractor* Σ_R *attracts* $U \setminus \Gamma$ *in* H, *where* Γ *is the stable manifold of* $(u,T) = 0$ *with codimension* m *in* H.

REMARK 2.4. Results similar to this attractor bifurcation theorem hold true as well for 3D Bénard problems; see [8, 9].

3. Preliminaries

3.1. Center manifold functions.   To study the structure of the bifurcated attractors of (2.4-2.8), it is necessary to consider the reduction of nonlinear evolution equations to center manifolds. To this end, we introduce in this section a method to derive a first order approximation of the central manifold functions. This method was introduced and used in [9].

Let H and H_1 be two Hilbert spaces, and let $H_1 \hookrightarrow H$ be a dense and compact inclusion. We consider the nonlinear evolution equation,

$$\begin{cases} \dfrac{du}{dt} = L_\lambda u + G(u,\lambda), \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where $u : [0,\infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parameterized linear completely continuous fields depending continuously on $\lambda \in \mathbb{R}^1$, which satisfy

$$\begin{cases} L_\lambda = -A + B_\lambda & \text{is a sectorial operator,} \\ A : H_1 \to H & \text{is a linear homeomorphism, and} \\ B_\lambda : H_1 \to H & \text{are parameterized linear compact operators.} \end{cases} \tag{3.2}$$

It is easy to see [5, 11] that L_λ generates an analytic semigroup $\{e^{-tL_\lambda}\}_{t \geq 0}$. Then we can define fractional power operators L_λ^α for any $0 \leq \alpha \leq 1$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ if $\alpha_1 > \alpha_2$, and $H_0 = H$.

Furthermore, we assume that the nonlinear term $G(\cdot,\lambda) : H_\alpha \to H$, for some $0 \leq \alpha < 1$, is a family of parameterized C^r bounded operators $(r \geq 1)$ continuously depending on the parameter $\lambda \in \mathbb{R}^1$, such that

$$G(u,\lambda) = o(\|u\|_{H_\alpha}), \quad \forall \lambda \in \mathbb{R}^1. \tag{3.3}$$

In this paper, we are interested in the case where $L_\lambda = -A + B_\lambda$ are sectorial operators such that there exist an eigenvalue sequence $\{\rho_k\} \subset \mathbb{C}^1$ and an eigenvector sequence $\{e_k, h_k\} \subset H_1$ of A:

$$\begin{cases} A z_k = \rho_k z_k, & z_k = e_k + i h_k, \\ \text{Re}\,\rho_k \to \infty \ (k \to \infty), \\ |\text{Im}\,\rho_k / (a \text{Re}\,\rho_k)| \leq c, \end{cases} \tag{3.4}$$

for some $a, c > 0$, such that $\{e_k, h_k\}$ is a basis of H.

Condition (3.4) implies that A is a sectorial operator. For the operator $B_\lambda : H_1 \to H$, we also assume that there is a constant $0 < \theta < 1$ such that

$$B_\lambda : H_\theta \longrightarrow H \text{ is bounded, } \forall \lambda \in \mathbb{R}^1. \tag{3.5}$$

Under conditions (3.4) and (3.5), the operator $L_\lambda = -A + B_\lambda$ is a sectorial operator.

Let H_1 and H be decomposed into

$$\begin{cases} H_1 = E_1^\lambda \oplus E_2^\lambda, \\ H = \widetilde{E}_1^\lambda \oplus \widetilde{E}_2^\lambda, \end{cases} \tag{3.6}$$

for λ near $\lambda_0 \in \mathbb{R}^1$, where E_1^λ, E_2^λ are invariant subspaces of L_λ, such that

$$\dim E_1^\lambda < \infty,$$
$$\widetilde{E}_1^\lambda = E_1^\lambda,$$
$$\widetilde{E}_2^\lambda = \text{closure of } E_2^\lambda \text{ in } H.$$

In addition, L_λ can be decomposed into $L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda$ such that for any λ near λ_0,

$$\begin{cases} \mathcal{L}_1^\lambda = L_\lambda|_{E_1^\lambda} : E_1^\lambda \longrightarrow \widetilde{E}_1^\lambda, \\ \mathcal{L}_2^\lambda = L_\lambda|_{E_2^\lambda} : E_2^\lambda \longrightarrow \widetilde{E}_2^\lambda, \end{cases} \tag{3.7}$$

where all eigenvalues of $\mathcal{L}_2^\lambda$ possess negative real parts, and the eigenvalues of $\mathcal{L}_1^\lambda$ possess nonnegative real parts at $\lambda = \lambda_0$.

Thus, for λ near λ_0, equation (3.1) can be written as

$$\begin{cases} \dfrac{dx}{dt} = \mathcal{L}_1^\lambda x + G_1(x, y, \lambda), \\ \dfrac{dy}{dt} = \mathcal{L}_2^\lambda y + G_2(x, y, \lambda), \end{cases} \tag{3.8}$$

where $u = x + y \in H_1$, $x \in E_1^\lambda$, $y \in E_2^\lambda$, $G_i(x, y, \lambda) = P_i G(u, \lambda)$, and $P_i : H \to \widetilde{E}_i^\lambda$ are canonical projections. Furthermore, let

$$E_2^\lambda(\alpha) = \text{ closure of } E_2^\lambda \text{ in } H_\alpha,$$

with $\alpha < 1$ given by (3.3).

The following center manifold theorem is classical; see [5, 15].

THEOREM 3.1. *Assume (3.3)–(3.7) hold true. Then there exists a neighborhood of λ_0 given by $|\lambda - \lambda_0| < \delta$ for some $\delta > 0$, a neighborhood $B_\lambda \in E_1^\lambda$ of $x = 0$, and a C^1 function $\Phi(\cdot, \lambda) : B_\lambda \to E_2^\lambda(\theta)$ depending continuously on λ, such that*

1. $\Phi(0, \lambda) = 0$, $\Phi_x'(0, \lambda) = 0$;

2. the sets

$$M_\lambda = \left\{ (x, y) \in H \mid x \in B_\lambda, y = \Phi(x, \lambda) \in E_2^\lambda(\theta) \right\},$$

called the center manifolds, are locally invariant for (3.1); i.e. for each $u_0 \in M_\lambda$

$$u_\lambda(t, u_0) \in M_\lambda, \qquad \forall\, 0 \leq t < t(u_0)$$

for some $t(u_0) > 0$, where $u_\lambda(t, u_0)$ is the solution of (3.1);

3. *if $(x_\lambda(t), y_\lambda(t))$ is a solution of (3.8), then there is a $\beta_\lambda > 0$ and $k_\lambda > 0$ with k_λ depending on $(x_\lambda(0), y_\lambda(0))$ such that*

$$\|y_\lambda(t) - \Phi(x_\lambda(t), \lambda)\|_H \le k_\lambda e^{-\beta_\lambda t}.$$

Also, it is classical that the bifurcation problem of (3.1) reduces to the bifurcation problem for the following finite dimensional system:

$$\frac{dx}{dt} = \mathcal{L}_1^\lambda x + g_1(x, \Phi_\lambda(x), \lambda), \tag{3.9}$$

for $x \in B_\lambda \subset E_1^\lambda$.

Now we give a formula to calculate the center manifold functions. Let the nonlinear operator G be given by

$$G(u, \lambda) = G_k(u, \lambda) + o(|u|^k), \tag{3.10}$$

for $k \ge 2$, where $G_k(u, \lambda)$ is a k-multilinear operator:

$$G_k : H_1 \times \cdots \times H_1 \to H,$$
$$G_k(u, \lambda) = G_k(u, \ldots, u, \lambda).$$

The following theorem was proved in [9].

THEOREM 3.2.   *Under the conditions (3.3)-(3.7) and (3.10), the center manifold function $\Phi(x, \lambda)$ can be expressed as*

$$\Phi(x, \lambda) = (-\mathcal{L}_2^\lambda)^{-1} P_2 G_k(x, \lambda) + O(|\mathrm{Re}\beta(\lambda)| \cdot \|x\|^k) + o(\|x\|^k), \tag{3.11}$$

where $\mathcal{L}_2^\lambda$ is given by (3.7), $P_2 : H \to \widetilde{E}_2$ is the canonical projection, $x \in E_1^\lambda$, and $\beta(\lambda) = (\beta_1(\lambda), \ldots, \beta_m(\lambda))$ is the tuple of eigenvalues of $\mathcal{L}_1^\lambda$.

REMARK 3.3.   Consider the case where $L_\lambda : H_1 \to H$ is symmetric. Then the eigenvalues are real, and the eigenvectors form an orthogonal basis of H. Therefore, we have

$$u = x + y \in E_1^\lambda \oplus E_2^\lambda,$$
$$x = \sum_{i=1}^m x_i e_i \in E_1^\lambda,$$
$$y = \sum_{i=m+1}^\infty x_i e_i \in E_2^\lambda.$$

Then near $\lambda = \lambda_0$, the formula (3.11) can be expressed as follows.

$$\Phi(x, \lambda) = \sum_{j=m+1}^\infty \Phi_j(x, \lambda) e_j + O(|\mathrm{Re}\beta(\lambda)| \cdot \|x\|^k) + o(\|x\|^k), \tag{3.12}$$

where

$$\Phi_j(x, \lambda) = -\frac{1}{\beta_j(\lambda)} \sum_{1 \le j_1, \ldots, j_k \le m} a^j_{j_1 \cdots j_k} x_{j_1} \cdots x_{j_k},$$
$$a^j_{j_1 \cdots j_k} = (G_k(e_{j_1}, \ldots, e_{j_k}, \lambda), e_j)_H.$$

In many applications, the coefficients $a^j_{j_1 \cdots j_k}$ can be computed, and the first m eigenvalues $\beta_1(\lambda), \ldots, \beta_m(\lambda)$ satisfy

$$|\mathrm{Re}\,\beta(\lambda_0)| = \sqrt{\sum_{j=1}^{m} (\mathrm{Re}\,\beta_j(\lambda_0))^2} = 0.$$

Hence (3.12) gives an explicit formula for the first approximation of the center manifold functions.

3.2. S^1-attractor bifurcation.   In this section, we study the structure of the bifurcated attractor of (3.9) when $m = 2$. Namely, we consider the following two-dimensional system:

$$\frac{dx}{dt} = \beta(\lambda)x - g(x, \lambda), \quad x \in \mathbb{R}^2. \tag{3.13}$$

Here $\beta(\lambda)$ is a continuous function of λ satisfying

$$\beta(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0, \end{cases} \tag{3.14}$$

and

$$\begin{cases} g(x, \lambda) = g_k(x, \lambda) + o(|x|^k), \\ g_k(\cdot, \lambda) \text{ is a k-multilinear field,} \\ C_1 |x|^{k+1} \leq (g_k(x, \lambda), x), \end{cases} \tag{3.15}$$

for some integer $k = 2m + 1 \geq 3$, and some constant $C_1 > 0$.

The following theorem was proved in [9], which shows that under conditions (3.14) and (3.15), the system (3.13) bifurcates to an S^1-attractor.

THEOREM 3.4. *Let the condition (3.14) and (3.15) hold true. Then the system (3.13) bifurcates from $(x, \lambda) = (0, \lambda_0)$ to an attractor Σ_λ, which is homeomorphic to S^1, for $\lambda_0 < \lambda < \lambda_0 + \varepsilon$ for some $\varepsilon > 0$. Moreover, one and only one of the following is true.*

1. *Σ_λ is a periodic orbit,*

2. *Σ_λ consists of an infinite number of singular points, or*

3. *Σ_λ contains at most $2(k+1) = 4(m+1)$ singular points, consisting of $2N$ saddle points, $2N$ stable node points, and n ($\leq 4(m+1) - 4N$) singular points with index zero, as shown in Fig. 3.1.*

3.3. Structural stability theorems.   In this subsection, we recall some results on structural stability for 2D divergence-free vector fields developed in [10], which are crucial to study the asymptotic structure in the physical space of the bifurcated solutions of the Bénard problem.

Let $C^r(\Omega, \mathbb{R}^2)$ be the space of all C^r ($r \geq 1$) vector fields on $\Omega = \mathbb{R}^1 \times (0, 1)$ which are periodic in the x_1 direction with period L, and let $D^r(\Omega, \mathbb{R}^2)$ be the space of all C^r divergence-free vector fields on $\Omega = \mathbb{R}^1 \times (0, 1)$ which are periodic in the x_1 direction with period L and satisfy the no-normal-flow condition in the the x_2 direction:

$$D^r(\Omega, \mathbb{R}^2) = \left\{ v \in C^r(\Omega, \mathbb{R}^2) \,|\, v_2 = 0 \text{ at } x_2 = 0, 1 \right\}.$$
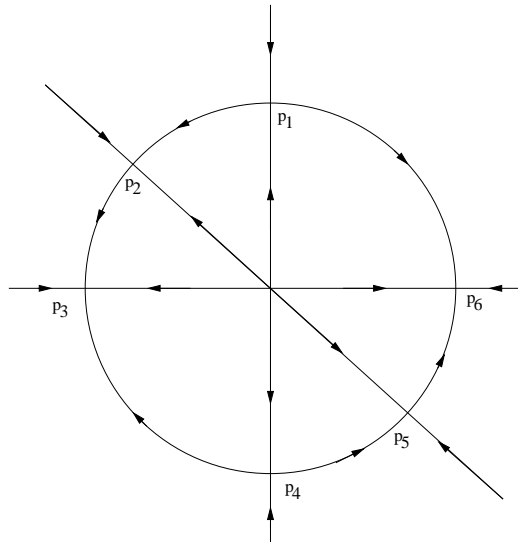
FIG. 3.1. Ω_λ has $4N+n$ ($N=1$ and $n=2$ shown here) singular points, where p_1 and p_4 are saddles, p_3 and p_6 are nodes, and p_2 and p_5 are singular points with index zero.

Furthermore, let

$$B_0^r(\Omega,\mathbb{R}^2) = \left\{ v \in D^r(\Omega,\mathbb{R}^2) \mid v=0 \text{ at } x_2=0,1 \right\},$$

$$B_1^r(\Omega,\mathbb{R}^2) = \left\{ v \in D^r(\Omega,\mathbb{R}^2) \ \middle| \ \begin{array}{c} v=0 \text{ at } x_2=0, \\ v_2 = \dfrac{\partial v_1}{\partial x_2} = 0 \text{ at } x_2=1 \end{array} \right\}.$$

DEFINITION 3.5. *Two vector fields $u,v \in C^r(\Omega,\mathbb{R}^2)$ are called topologically equivalent if there exists a homeomorphism of $\varphi:\Omega \to \Omega$ which takes the orbits of u to orbits of v and preserves their orientation.*

DEFINITION 3.6. *Let $X = D^r(\Omega,\mathbb{R}^2)$ or $X = B_0^r(\Omega,\mathbb{R}^2)$. A vector field $v \in X$ is called structurally stable in X if there exists a neighborhood $U \subset X$ of v such that for any $u \in U$, u and v are topologically equivalent.*

Let $v \in D^r(\Omega,\mathbb{R}^2)$. We recall next some basic facts and definitions on divergence-free vector fields.

1. A point $p \in \Omega$ is called a singular point of v if $v(p)=0$; a singular point p of v is called non-degenerate if the Jacobian matrix $Dv(p)$ is invertible; v is called regular if all singular points of v are non-degenerate.

2. An interior non-degenerate singular point of v can be either a center or a saddle, and a non-degenerate boundary singularity must be a saddle.

3. Saddles of v must be connected to saddles. An interior saddle $p \in \Omega$ is called self-connected if p is connected only to itself, i.e., p occurs in a graph whose topological form is that of the number 8.

Let $v \in B_0^r(\Omega,\mathbb{R}^2)$; then we know that each point on $x_2=0,1$ is a singular point of v in the usual sense. To study the structure of v, we need to classify the boundary points as follows.

DEFINITION 3.7. *Let $u \in B_0^r(TM)$ $(r \geq 2)$.*

1. *A point $p \in \partial M$ is called a ∂-regular point of u if $\frac{\partial u_\tau(p)}{\partial n} \neq 0$; otherwise, $p \in \partial M$ is called a ∂-singular point of u.*

2. *A ∂-singular point $p \in \partial M$ of u is called non-degenerate if*

$$\det \begin{pmatrix} \dfrac{\partial^2 u_\tau(p)}{\partial \tau \partial n} & \dfrac{\partial^2 u_\tau(p)}{\partial n^2} \\[3mm] \dfrac{\partial^2 u_n(p)}{\partial \tau \partial n} & \dfrac{\partial^2 u_n(p)}{\partial n^2} \end{pmatrix} \neq 0. \tag{3.16}$$

   A non-degenerate ∂-singular point of u is also called a ∂-saddle point of u.

3. *$u \in B_0^r(TM)$ $(r \geq 2)$ is called D-regular if a) u is regular in $\overset{\circ}{M}$, and b) all ∂-singular points of u on ∂M are non-degenerate.*

The following theorem provides necessary and sufficient conditions for structural stability of a divergence-free vector field.

THEOREM 3.8. [10] *Let $u \in B_0^r(TM)$ $(r \geq 2)$. Then u is structurally stable in $B_0^r(TM)$ if and only if*

   1) *u is D-regular,*

   2) *all interior saddle points of u are self-connection, and*

   3) *each ∂-saddle point of u on ∂M is connected to a ∂-saddle point on the same connected component of ∂M.*

Moreover, the set of all structurally stable vector fields is open and dense in $B_0^r(TM)$.

REMARK 3.9. For vector fields with free-rigid boundary conditions, the conditions for structural stability differ slightly. More precisely, $u \in B_1^r(TM)$ $(r \geq 2)$ is structurally stable in $B_1^r(TM)$ if and only if

   1) all singular points of u in Ω and on $x_2 = 1$ are regular, and all ∂-singular points on $x_2 = 0$ are ∂-regular,

   2) all interior saddle points of u are self-connected, and

   3) each saddle of u on $x_2 = 1$ is connected to saddles on $x_2 = 1$, and each ∂-saddle point of u on $x_2 = 0$ is connected to a ∂-saddle point on $x_2 = 0$.

REMARK 3.10. For vector fields satisfying free-free boundary conditions, we set

$$B_2^r(\Omega, \mathbb{R}^2) = \left\{ v \in D^r(\Omega, \mathbb{R}^2) \ \Big| \ v_2 = \frac{\partial v_1}{\partial x_2} = 0 \text{ at } x_2 = 0, 1 \right\},$$

$$B_3^r(\Omega, \mathbb{R}^2) = \left\{ v \in D_2^r(\Omega, \mathbb{R}^2) \ \ \Big| \ \int_\Omega u \, dx = 0 \right\}.$$

Then $u \in B_2^r(\Omega, \mathbb{R}^2)$ (respectively $u \in B_3^r(\Omega, \mathbb{R}^2)$) is structurally stable in $B_2^r(\Omega, \mathbb{R}^2)$ (respectively in $B_2^r(\Omega, \mathbb{R}^2)$) if and only if

   1) u is regular,

   2) all interior saddle points of u are self-connected, and

   3) each boundary saddle of u is connected to boundary saddles on the same connected component of $\partial \Omega$ (respectively each boundary saddle of u is connected to boundary saddles not necessarily on the same connected component).

The difference between these two cases is due to the zero-average condition in the definition in $B_3^r(\Omega,\mathbb{R}^2)$, which implies that $B_3^r(\Omega,\mathbb{R}^2)$ does not contain the harmonic field $v_0 = (\alpha, 0)$ for any constant $\alpha \neq 0$. Hence, an orbit connecting two saddles on different components of the boundary cannot be broken with a perturbation in $B_3^r(\Omega,\mathbb{R}^2)$ into orbits connecting only saddles on the same connected component of the boundary.

4. Structure of bifurcated solutions for the Bénard problem

In this section, we study the topological structure of the bifurcated attractor and the asymptotic structure of solutions for the Bénard problem. It is known that for each type of boundary conditions, there is a minimal period L_c satisfying (2.14). Hereafter, we always take L_c to be the period of (2.7).

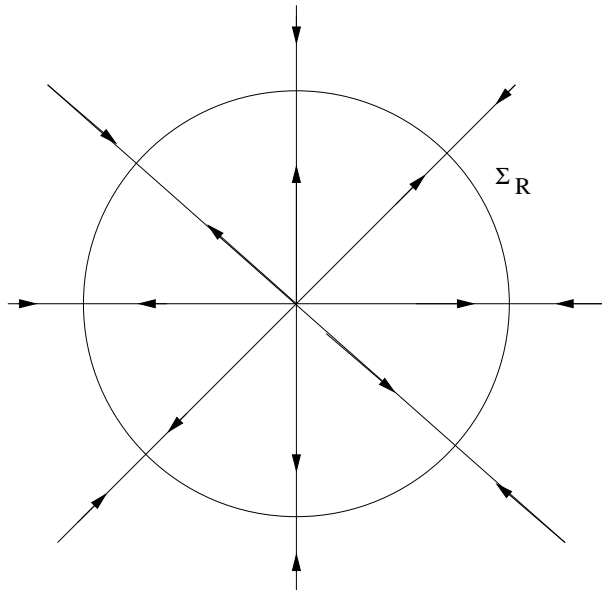The main theorem in this article is as follows.

FIG. 4.1. *All points on the bifurcated attractor $\Sigma_R = S^1$ are steady state solutions.*

THEOREM 4.1. *For the Bénard problem (2.4)-(2.8), the following assertions hold true.*

1. *For $R > R_c$, the equations bifurcate from the trivial solution $((u,T),R) = (0,R_c)$ to an attractor Σ_R, homeomorphic to S^1, which consists of steady state solutions as shown in Fig. 4.1, where R_c is the critical Rayleigh number.*

2. *For any $\psi_0 = (u_0, T_0) \in H \setminus (\Gamma \cup E)$, there exists a time $t_0 \geq 0$ such that for any $t \geq t_0$, the vector field $u(t, \psi_0)$ is topologically equivalent to the structure as shown in either Fig. 4.2(a) or Fig. 4.2(b), where $\psi = (u(t,\psi_0), T(t,\psi_0))$ is the solution of (2.4)-(2.8) with initial data ψ_0, Γ is the stable manifold of the trivial solution $(u,T) = 0$ with codimension 2, and*

$$E = \left\{ (u,T) \in H \ \Big| \ \int_0^1 u_1 \, dx_2 = 0 \right\}.$$
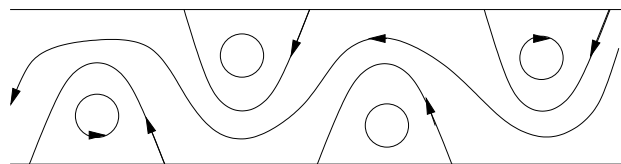
(a)
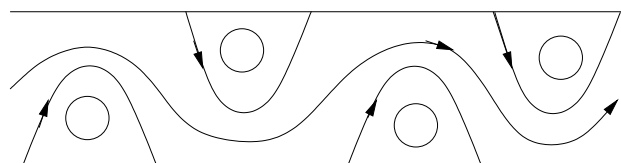
(b)

FIG. 4.2. *Here the horizontal axis is the x_1-axis, and the vertical axis is the x_2-axis. With the Dirichlet boundary conditions on $x_2 = 0, 1$, the flow is not moving on both the top $x_2 = 1$ and the bottom $x_2 = 0$ boundaries.*

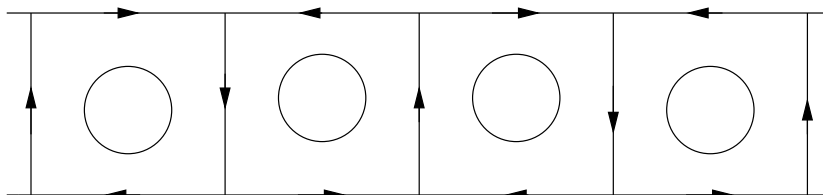
FIG. 4.3. *Here the horizontal axis is the x_1-axis, and the vertical axis is the x_2-axis. With the free slip boundary conditions on $x_2 = 0, 1$, the flow does move on both the top $x_2 = 1$ and the bottom $x_2 = 0$ boundaries.*

The zonally moving meandering flow shown in Fig. 4.2 appears often in many physical problems such as the Bransdator-Kushnir waves in atmospheric circulation [1, 7].

Thm. 4.1 is also valid for the Bénard problem (2.4)-(2.7) with (2.10). However the case with the free-free boundary condition is different. More precisely, for the free-free boundary condition, it is easy to see that for any constant α, the harmonic field $\psi_0 = ((\alpha, 0), 0)$ is a solution of (2.4)-(2.6). Therefore, we have to consider the problem (2.4)-(2.7) with (2.9) in the following function spaces:

$$\widetilde{H} = \left\{ (u, T) \in L^2(\Omega)^3 \quad | \quad \operatorname{div} u = 0, \int_\Omega u \, dx = 0 \right\},$$

$$\widetilde{H}_1 = \{(u, T) \in \widetilde{H} \cap H^2(\Omega)^3 \text{ satisfying (2.7) and (2.9)}\}.$$

Then we have the following theorem.

THEOREM 4.2. *For the Bénard problem (2.4)-(2.7) with boundary condition (2.9), the following assertions hold true.*

1. For $R > R_c$, the equations bifurcate from the trivial solution $((u,T),R) = (0,R_c)$ to an attractor Σ_R, homeomorphic to S^1, which consists of steady state solutions, where $R_c = 27\pi^4/4$ is the critical Rayleigh number.

2. For any $\psi_0 = (u_0,T_0) \in \widetilde{H} \setminus \Gamma$, there exists a time $t_0 \geq 0$ such that for any $t \geq t_0$, the vector field $u(t,\psi_0)$ is topologically equivalent to the structure as shown in Fig. 4.3, where $\psi = (u(t,\psi_0),T(t,\psi_0))$ is the solution of (2.4)-(2.7) with (2.9), and Γ is the stable manifold of the trivial solution $(u,T) = 0$ with codimension 2 in $\widetilde{H}$.

5. Proof of main theorems

5.1. Eigenvectors of the linear Boussinesq equations.   We shall only prove Thm. 4.1. The proof of Thm. 4.2 is essentially the same, and we omit the details. We proceed by first considering the eigenvalues and eigenvectors of the linearized equations of (2.4)–(2.6):

$$\begin{cases} \triangle u - \nabla p + \sqrt{R}Tk = \beta(R)u, \\ \triangle T + \sqrt{R}u_2 = \beta(R)T, \\ \operatorname{div} u = 0, \end{cases} \qquad (5.1)$$

supplemented with the boundary conditions (2.7) and (2.8).

For $\psi = (u_1,u_2,T) \in H_1$, we take the separation of variables as follows:

$$\psi = \left(-\sin\frac{2k\pi x_1}{L}h'(x_2), \frac{2k\pi}{L}\cos\frac{2k\pi x_1}{L}h(x_2), \cos\frac{2k\pi x_1}{L}\theta(x_2) \right),$$

$$\widetilde{\psi} = \left(\cos\frac{2k\pi x_1}{L}h'(x_2), \frac{2k\pi}{L}\sin\frac{2k\pi x_1}{L}h(x_2), \sin\frac{2k\pi x_1}{L}\theta(x_2) \right).$$

Then it follows from (5.1) that (h,θ) satisfies the following differential equations

$$\begin{cases} \left(\dfrac{d^2}{dx_2^2} - a_k^2 \right)^2 h - \sqrt{R}a_k\theta = \beta(R)\left(\dfrac{d^2}{dx_2^2} - a_k^2 \right)h, \\ -\left(\dfrac{d^2}{dx_2^2} - a_k^2 \right)\theta - \sqrt{R}a_kh = -\beta(R)\theta, \end{cases} \qquad (5.2)$$

supplemented with the following boundary conditions

$$\theta = 0, \quad h = h' = 0 \quad \text{at } x_2 = 0,1, \qquad (5.3)$$

where $a_k = 2k\pi/L$ and $L = L_c$ satisfies (2.14).

The eigenvalue problem (5.2) with (5.3) is symmetric, and has a complete eigenvalue and eigenvector sequences for given k and R:

$$\begin{cases} \beta_{k1}(R) > \beta_{k2}(R) > \cdots, \\ \lim_{j \to \infty} \beta_{kj}(R) = -\infty, \\ h_{kj} \in H^4(0,1) \cap H_0^2(0,1) \quad j = 1,2,\ldots, \\ \theta_{kj} \in H^2(0,1) \cap H_0^1(0,1) \quad j = 1,2,\ldots. \end{cases} \qquad (5.4)$$

Moreover,

$$\{(h_{kj},\theta_{kj}) \mid j = 1,2,\ldots\}$$

constitutes an orthogonal basis of $L^2(0,1) \times L^2(0,1)$.

Thus, we obtain the following complete set of eigenvectors for (5.1) with boundary conditions (2.7) and (2.8):

$$\psi_{kj} = \left(-\sin\frac{2k\pi x_1}{L} h'_{kj}(x_2), \frac{2k\pi}{L}\cos\frac{2k\pi x_1}{L} h_{kj}(x_2), \cos\frac{2k\pi x_1}{L}\theta_{kj}(x_2) \right), \qquad (5.5)$$

$$\widetilde{\psi}_{kj} = \left(\cos\frac{2k\pi x_1}{L} h'_{kj}(x_2), \frac{2k\pi}{L}\sin\frac{2k\pi x_1}{L} h_{kj}(x_2), \sin\frac{2k\pi x_1}{L}\theta_{kj}(x_2) \right), \qquad (5.6)$$

where $0 \leq k < \infty$, $1 \leq j < \infty$. When $k=0$, we derive from (5.1), (5.5) and (5.6) that

$$\begin{cases} \psi_{0j} = (0,0,\sin j\pi x_2), \\ \widetilde{\psi}_{0j} = (h'_{0j}(x_2),0,0). \end{cases} \qquad (5.7)$$

5.2. Singularity cycle. We shall show that the bifurcated attractor Σ_R of (2.4)-(2.8) given in Thm. 2.3 contains a cycle of steady state solutions.

First, we note that the equations (2.4)-(2.8) are invariant under the following translation:

$$\psi(x_1,x_2,t) \to \psi(x_1+\alpha,x_2,t) \quad \forall \alpha \in \mathbb{R}.$$

Hence, if $\psi_0(x)$ is a steady state solution, then $\psi_0(x_1+\alpha,x_2)$ are steady state solutions as well. It is easy to see that the set

$$\mathcal{S} = \{\psi_0(x_1+\alpha,x_2) \mid \alpha \in \mathbb{R}\}$$

is a cycle S^1 in H_1. Therefore, each steady state of (2.4)-(2.8) generates a cycle of steady state solutions.

Let

$$H' = \{(u,T) \in H \mid u_1(-x_1,x_2) = -u_1(x_1,x_2)\},$$
$$H'_1 = H_1 \cap H'.$$

It is easy to check that H' and H'_1 are invariant spaces for the operator $L_\lambda + G$ given by (2.12) in the sense that

$$L_\lambda + G : H'_1 \to H',$$

where $\lambda = \sqrt{R}$. On the other hand, it is clear that the sequence $\{\psi_{kj} \mid k = 0,1,\ldots$ and $j=1,2,\ldots\}$ defined by (5.5) is a basis of H'. Since the first eigenvalue R_c of (2.13) is simple, the first eigenvalue $\beta_1(R_c)$ of L_λ in H'_1 is also simple, where

$$\beta_1(R_c) = \beta_{11}(R_c) = 0,$$

and $\beta_{11}(R_c)$ is defined by (5.4). Hence by the classical Krasnoselskii bifurcation theorem, we know that the operator $L_\lambda + G$ bifurcates from $(\psi,\lambda) = (0,\sqrt{R_c})$ to a singular point in H'_1. Namely, the Bénard problem (2.4)-(2.8) bifurcates from $(\psi,R) = (0,R_c)$, a steady state solution. Therefore, the bifurcated attractor Σ_R contains at least a cycle of steady state solutions.

5.3. S^1-attractor: $\Sigma_R = S^1$. To prove that $\Sigma_R = S^1$, by Thm. 3.4, it suffices to verify that the equations of (2.4)-(2.8) reduced to the center manifold satisfy conditions (3.14) and (3.15).

For any $\psi = (u, T) \in H$, we have

$$\psi = \sum_{k \geq 0, j \geq 1} \left(x_{kj} \psi_{kj} + y_{kj} \widetilde{\psi}_{kj} \right).$$

Since L is the minimal period satisfying (2.14), ψ_{11} and $\widetilde{\psi}_{11}$ are the first eigenvectors of (5.1). Therefore the reduced equations of (2.4)-(2.8) are given by

$$\begin{cases} \dfrac{dx_{11}}{dt} = \beta_1(R) x_{11} + \dfrac{1}{\|\psi_{11}\|_H^2} (G(\psi, \psi), \psi_{11}), \\[3mm] \dfrac{dy_{11}}{dt} = \beta_1(R) y_{11} + \dfrac{1}{\|\psi_{11}\|_H^2} (G(\psi, \psi), \widetilde{\psi}_{11}). \end{cases} \tag{5.8}$$

Here for $\psi_1 = (u, T_1)$, $\psi_2 = (v, T_2)$, and $\psi_3 = (w, T_3)$,

$$(G(\psi_1, \psi_2), \psi_3) = -\int_\Omega [(u \cdot \nabla v) w + (u \cdot \nabla T_2) T_3] \, dx.$$

Let the center manifold function be denoted by

$$\Phi = \sum_{(k,j) \neq (1,1)} \left(\Phi_{kj}(x_{11}, y_{11}) \psi_{kj} + \widetilde{\Phi}_{kj}(x_{11}, y_{11}) \widetilde{\psi}_{kj} \right). \tag{5.9}$$

Note that for any $\psi_i \in H_1$ ($i = 1, 2, 3$),

$$\begin{aligned} (G(\psi_1, \psi_2), \psi_2) &= 0, \\ (G(\psi_1, \psi_2), \psi_3) &= -(G(\psi_1, \psi_3,), \psi_2). \end{aligned}$$

Then by $\psi = x_{11} \psi_{11} + y_{11} \widetilde{\psi}_{11} + \Phi$, we have

$$\begin{aligned} (G(\psi, \psi), \psi_{11}) =\ & (G(\widetilde{\psi}_{11}, \widetilde{\psi}_{11}), \psi_{11}) y_{11}^2 \\ & - (G(\psi_{11}, \psi_{11}), \widetilde{\psi}_{11}) x_{11} y_{11} \\ & - (G(\psi_{11}, \psi_{11}), \Phi) x_{11} \\ & + (G(\widetilde{\psi}_{11}, \Phi) + G(\Phi, \widetilde{\psi}_{11}), \psi_{11}) y_{11} \\ & + (G(\Phi, \Phi), \psi_{11}), \end{aligned} \tag{5.10}$$

$$\begin{aligned} (G(\psi, \psi), \widetilde{\psi}_{11}) =\ & (G(\psi_{11}, \psi_{11}), \widetilde{\psi}_{11}) x_{11}^2 \\ & - (G(\widetilde{\psi}_{11}, \widetilde{\psi}_{11}), \psi_{11}) x_{11} y_{11} \\ & - (G(\widetilde{\psi}_{11}, \widetilde{\psi}_{11}), \Phi) y_{11} \\ & + (G(\psi_{11}, \Phi) + G(\Phi, \psi_{11}), \widetilde{\psi}_{11}) x_{11} \\ & + (G(\Phi, \Phi), \widetilde{\psi}_{11}). \end{aligned} \tag{5.11}$$

It is easy to check that for $k \neq 0, 2$,

$$\begin{cases} (G(\psi_{11}, \psi_{11}), \psi_{kj}) = 0, \\ (G(\widetilde{\psi}_{11}, \widetilde{\psi}_{11}), \psi_{kj}) = 0, \\ (G(\widetilde{\psi}_{kj}, \widetilde{\psi}_{11}), \psi_{11}) = 0, \\ (G(\widetilde{\psi}_{kj}, \psi_{11}), \widetilde{\psi}_{11}) = 0, \end{cases} \tag{5.12}$$

and for any $k \geq 0$, $j \geq 1$, we have

$$\begin{cases} (G(\psi_{11}, \psi_{11}), \widetilde{\psi}_{kj}) = 0, \\ (G(\widetilde{\psi}_{11}, \widetilde{\psi}_{11}), \psi_{kj}) = 0, \\ (G(\psi_{kj}, \widetilde{\psi}_{11}), \psi_{11}) = 0, \\ (G(\psi_{kj}, \psi_{11}), \widetilde{\psi}_{11}) = 0. \end{cases} \tag{5.13}$$

By (5.9), (5.12) and (5.13), the equalities (5.10) and (5.11) can be rewritten as

$$(G(\psi, \psi), \psi_{11}) = -\sum_{j=1}^{\infty} [(G(\psi_{11}, \psi_{11}), \psi_{0j})\Phi_{0j} + (G(\psi_{11}, \psi_{11}), \psi_{2j})\Phi_{2j}]x_{11}$$

$$-\sum_{j=1}^{\infty} [(G(\widetilde{\psi}_{11}, \psi_{11}), \widetilde{\psi}_{0j}) + (G(\widetilde{\psi}_{0j}, \psi_{11}), \psi_{11})y_{11}\widetilde{\Phi}_{0j}$$

$$-\sum_{j=1}^{\infty} [(G(\widetilde{\psi}_{11}, \psi_{11}), \widetilde{\psi}_{2j}) + (G(\widetilde{\psi}_{2j}, \psi_{11}), \widetilde{\psi}_{11})y_{11}\widetilde{\Phi}_{2j}$$

$$+ (G(\Phi, \Phi), \psi_{11}), \tag{5.14}$$

$$(G(\psi, \psi), \widetilde{\psi}_{11}) = -\sum_{j=1}^{\infty} [(G(\widetilde{\psi}_{11}, \widetilde{\psi}_{11}), \psi_{0j})\Phi_{0j} + (G(\widetilde{\psi}_{11}, \widetilde{\psi}_{11}), \psi_{2j})\Phi_{2j}]y_{11}$$

$$-\sum_{j=1}^{\infty} [(G(\psi_{11}, \widetilde{\psi}_{11}), \widetilde{\psi}_{0j}) + (G(\widetilde{\psi}_{0j}, \widetilde{\psi}_{11}), \widetilde{\psi}_{11})x_{11}\widetilde{\Phi}_{0j}$$

$$-\sum_{j=1}^{\infty} [(G(\psi_{11}, \widetilde{\psi}_{11}), \widetilde{\psi}_{2j}) + (G(\widetilde{\psi}_{2j}, \widetilde{\psi}_{11}), \psi_{11})x_{11}\widetilde{\Phi}_{2j}$$

$$+ (G(\Phi, \Phi), \widetilde{\psi}_{11}). \tag{5.15}$$

Since the center manifold functions contains only higher order terms

$$\Phi(x_{11}, y_{11}) = O(|x_{11}|^2, |y_{11}|^2),$$

we derive that

$$\begin{cases} (G(\Phi, \Phi), \psi_{11}) = o(|x_{11}|^3, |y_{11}|^3), \\ (G(\Phi, \Phi), \widetilde{\psi}_{11}) = o(|x_{11}|^3, |y_{11}|^3). \end{cases} \tag{5.16}$$

Then direct calculation yields that

$$(G(\widetilde{\psi}_{2j}, \psi_{11}), \widetilde{\psi}_{11}) = -(G(\widetilde{\psi}_{2j}, \widetilde{\psi}_{11}), \psi_{11})$$

$$= \frac{\pi}{2} \int_0^1 [-h'_{11}(h'_{2j}h'_{11} + 2h_{2j}h''_{11})$$

$$+ h_{11}(h'_{2j}h_{11} + 2h_{2j}h'_{11})$$

$$+ \theta_{11}(h'_{2j}\theta_{11} + 2h_{2j}\theta'_{11})]dx_2$$

$$= \frac{\pi}{2} \int_0^1 \frac{d}{dx_2}(-h_{2j}(h'_{11})^2 + h_{2j}h_{11}^2 + h_{2j}\theta_{11}^2)dx_2$$

$$= 0. \tag{5.17}$$

It is clear that for any $k \geq 0$ and $j \geq 1$,

$$\|\psi_{kj}\|_H^2 = \|\widetilde{\psi}_{kj}\|_H^2. \tag{5.18}$$

Hence the reduced equations (5.8) can be expressed as follows:

$$
\begin{aligned}
\frac{dx_{11}}{dt} =& \beta_1(R)x_{11} - \frac{1}{\|\psi_{11}\|_H^2} \sum_{j=1}^{\infty} [(G(\psi_{11},\psi_{11}),\psi_{0j})\Phi_{0j}x_{11} \\
& + (G(\psi_{11},\psi_{11}),\psi_{2j})\Phi_{2j}x_{11} + (G(\widetilde{\psi}_{11},\psi_{11}),\widetilde{\psi}_{2j})y_{11}\widetilde{\Phi}_{2j} \\
& + (G(\widetilde{\psi}_{11},\psi_{11}),\widetilde{\psi}_{0j})y_{11}\widetilde{\Phi}_{0j} + (G(\widetilde{\psi}_{0j},\psi_{11}),\widetilde{\psi}_{11})y_{11}\widetilde{\Phi}_{0j}] \\
& + o(|x_{11}|^3, |y_{11}|^3),
\end{aligned} \tag{5.19}
$$

$$
\begin{aligned}
\frac{dy_{11}}{dt} =& \beta_1(R)y_{11} - \frac{1}{\|\psi_{11}\|_H^2} \sum_{j=1}^{\infty} [(G(\widetilde{\psi}_{11},\widetilde{\psi}_{11}),\psi_{0j})\Phi_{0j}y_{11} \\
& + (G(\widetilde{\psi}_{11},\widetilde{\psi}_{11}),\psi_{2j})\Phi_{2j}y_{11} + (G(\psi_{11},\widetilde{\psi}_{11}),\widetilde{\psi}_{2j})x_{11}\widetilde{\Phi}_{2j} \\
& + (G(\psi_{11},\widetilde{\psi}_{11}),\widetilde{\psi}_{0j})x_{11}\widetilde{\Phi}_{0j} + (G(\widetilde{\psi}_{0j},\widetilde{\psi}_{11}),\psi_{11})x_{11}\widetilde{\Phi}_{0j}] \\
& + o(|x_{11}|^3, |y_{11}|^3).
\end{aligned} \tag{5.20}
$$

By Thm. 3.2 and (3.12), we have

$$
\begin{aligned}
\Phi_{0j} =& \frac{-1}{\|\psi_{0j}\|_H^2 \beta_{0j}} [(G(\psi_{11},\psi_{11}),\psi_{0j})x_{11}^2 + (G(\widetilde{\psi}_{11},\widetilde{\psi}_{11}),\psi_{0j})y_{11}^2] \\
& + o(x_{11}^2 + y_{11}^2) + O(\beta_1(R)(x_{11}^2 + y_{11}^2)), \\
\Phi_{2j} =& \frac{-1}{\|\psi_{2j}\|_H^2 \beta_{2j}} [(G(\psi_{11},\psi_{11}),\psi_{2j})x_{11}^2 + (G(\widetilde{\psi}_{11},\widetilde{\psi}_{11}),\psi_{2j})y_{11}^2] \\
& + o(x_{11}^2 + y_{11}^2) + O(\beta_1(R)(x_{11}^2 + y_{11}^2)), \\
\widetilde{\Phi}_{0j} =& \frac{-1}{\|\widetilde{\psi}_{0j}\|_H^2 \beta_{0j}} [(G(\psi_{11},\widetilde{\psi}_{11}) + G(\widetilde{\psi}_{11},\psi_{11}),\widetilde{\psi}_{0j})x_{11}y_{11}] \\
& + o(x_{11}^2 + y_{11}^2) + O(\beta_1(R)(x_{11}^2 + y_{11}^2)), \\
\widetilde{\Phi}_{2j} =& \frac{-1}{\|\widetilde{\psi}_{2j}\|_H^2 \beta_{2j}} [(G(\psi_{11},\widetilde{\psi}_{11}) + G(\widetilde{\psi}_{11},\psi_{11}),\widetilde{\psi}_{2j})x_{11}y_{11}] \\
& + o(x_{11}^2 + y_{11}^2) + O(\beta_1(R)(x_{11}^2 + y_{11}^2)), \\
\Phi_{kj} =& o(x_{11}^2 + y_{11}^2) \qquad \forall k \neq 0, 2, \\
\widetilde{\Phi}_{kj} =& o(x_{11}^2 + y_{11}^2) \qquad \forall k \neq 0, 2,
\end{aligned}
$$

where $\beta_{kj}(R)$ are as in (5.4).

Also by direct computation, we obtain that

$$
\begin{aligned}
& (G(\psi_{11},\psi_{11}),\psi_{0j}) = (G(\widetilde{\psi}_{11},\widetilde{\psi}_{11}),\psi_{0j}), \\
& (G(\psi_{11},\psi_{11}),\psi_{2j}) = -(G(\widetilde{\psi}_{11},\widetilde{\psi}_{11}),\psi_{2j}), \\
& (G(\psi_{11},\widetilde{\psi}_{11}) + G(\widetilde{\psi}_{11},\psi_{11}),\widetilde{\psi}_{0j}) = 0, \\
& (G(\psi_{11},\widetilde{\psi}_{11}),\widetilde{\psi}_{2j}) = (G(\widetilde{\psi}_{11},\psi_{11}),\widetilde{\psi}_{2j}) = (G(\psi_{11},\psi_{11}),\psi_{2j}).
\end{aligned}
$$

Hence we have

$$\Phi_{0j} = \frac{-1}{\|\psi_{0j}\|_H^2 \beta_{0j}} (G(\psi_{11},\psi_{11}),\psi_{0j})(x_{11}^2 + y_{11}^2)$$
$$+ o(x_{11}^2 + y_{11}^2) + O(\beta_1(R)(x_{11}^2 + y_{11}^2)), \tag{5.21}$$

$$\Phi_{2j} = \frac{-1}{\|\psi_{2j}\|_H^2 \beta_{2j}} (G(\psi_{11},\psi_{11}),\psi_{2j})(x_{11}^2 - y_{11}^2)$$
$$+ o(x_{11}^2 + y_{11}^2) + O(\beta_1(R)(x_{11}^2 + y_{11}^2)), \tag{5.22}$$

$$\widetilde{\Phi}_{0j} = o(x_{11}^2 + y_{11}^2) + O(\beta_1(R)(x_{11}^2 + y_{11}^2)), \tag{5.23}$$

$$\widetilde{\Phi}_{2j} = \frac{-2}{\|\psi_{2j}\|_H^2 \beta_{2j}} (G(\psi_{11},\psi_{11}),\psi_{2j})x_{11}y_{11}$$
$$+ o(x_{11}^2 + y_{11}^2) + O(\beta_1(R)(x_{11}^2 + y_{11}^2)). \tag{5.24}$$

Inserting (5.21)-(5.24) into (5.19) and (5.20), we have

$$\frac{dx_{11}}{dt} = \beta_1(R)x_{11} - \alpha x_{11}(x_{11}^2 + y_{11}^2)$$
$$+ o(x_{11}^3 + y_{11}^3) + O(\beta_1(R)(x_{11}^3 + y_{11}^3)), \tag{5.25}$$

$$\frac{dy_{11}}{dt} = \beta_1(R)y_{11} - \alpha y_{11}(x_{11}^2 + y_{11}^2)$$
$$+ o(x_{11}^3 + y_{11}^3) + O(\beta_1(R)(x_{11}^3 + y_{11}^3)), \tag{5.26}$$

where

$$\alpha = \frac{-1}{\|\psi_{11}\|_H^2} \sum_{j=1}^{\infty} \left[\frac{(G(\psi_{11},\psi_{11}),\psi_{0j})^2}{\|\psi_{0j}\|_H^2 \beta_{0j}(R)} + \frac{(G(\psi_{11},\psi_{11}),\psi_{2j})^2}{\|\psi_{2j}\|_H^2 \beta_{2j}(R)} \right].$$

We know that

$$\beta_1(R) = \beta_{11}(R) > \beta_{kj}(R) \quad \forall (k,j) \neq (1,1),$$
$$\beta_{11}(R_c) = 0.$$

Hence near $R = R_c$,

$$\beta_{kj}(R) < 0 \quad \forall (k,j) \neq (1,1).$$

Consequently, $\alpha > 0$ and (5.25) and (5.26) satisfy (3.15).
    Also, we know that [8, 9]

$$\beta_1(R) \begin{cases} < 0 & \text{if } R < R_c, \\ = 0 & \text{if } R = R_c, \\ > 0 & \text{if } R > R_c. \end{cases}$$

Therefore (3.15) holds true.

5.4. Asymptotic structure of solutions.   By [4], we know that for any initial value $\psi_0 = (u_0, T_0) \in H$, there is a time $\tau \geq 0$ such that the solution $\psi = (u(t,\psi_0), T(t,\psi_0))$ is C^∞ for $t > \tau$, and is uniformly bounded in C^r-norm for any given $r \geq 1$. Hence, by Thm. 2.3, we have

$$\lim_{t \to \infty} \min_{\phi \in \Sigma_R} \|\psi(t,\psi_0) - \phi\|_{C^r} = 0. \tag{5.27}$$

We infer then from (5.25) and (5.26) that for any steady state solution $\Phi = (e, T) \in \Sigma_R$ of (2.4)-(2.8), the vector field $e = (e_1, e_2)$ can be expressed as

$$\begin{cases} e_1 = r\cos\dfrac{2\pi}{L}(x_1+\theta)\ h'_{11}(x_2) + v_1(x_{11}, y_{11}, \beta_1), \\[2mm] e_2 = \dfrac{2\pi}{L}r\sin\dfrac{2\pi}{L}(x_1+\theta)\ h_{11}(x_2) + v_2(x_{11}, y_{11}, \beta_1), \end{cases} \tag{5.28}$$

for some $0 \le \theta \le 2\pi$. Here

$$\begin{cases} r = \sqrt{x_{11}^2 + y_{11}^2} = \sqrt{\beta_1(R)} + o(\sqrt{\beta_1(R)}) & \text{if } R > R_c, \\[2mm] v_i(x_{11}, y_{11}, \beta_1) = o(\sqrt{\beta_1(R)}) & \text{for } i = 1, 2. \end{cases} \tag{5.29}$$

On the other hand, it is known that the first eigenfunction $h_{11}(x_2)$ of (5.2) and (5.3) at $R = R_c$ is given by

$$h_{11}(x_2) \simeq \cos\alpha_0\left(x_2 - \frac{1}{2}\right) - 0.06\cosh\alpha_1\left(x_2 - \frac{1}{2}\right)\cos\alpha_2\left(x_2 - \frac{1}{2}\right)$$

$$+ 0.1\sinh\alpha_1\left(x_2 - \frac{1}{2}\right)\sin\alpha_2\left(x_2 - \frac{1}{2}\right), \tag{5.30}$$

where $\alpha_0 \simeq 3.97$, $\alpha_1 \simeq 5.2$ and $\alpha_2 \simeq 2.1$. This function is schematically given by Fig. 5.1; see [2].
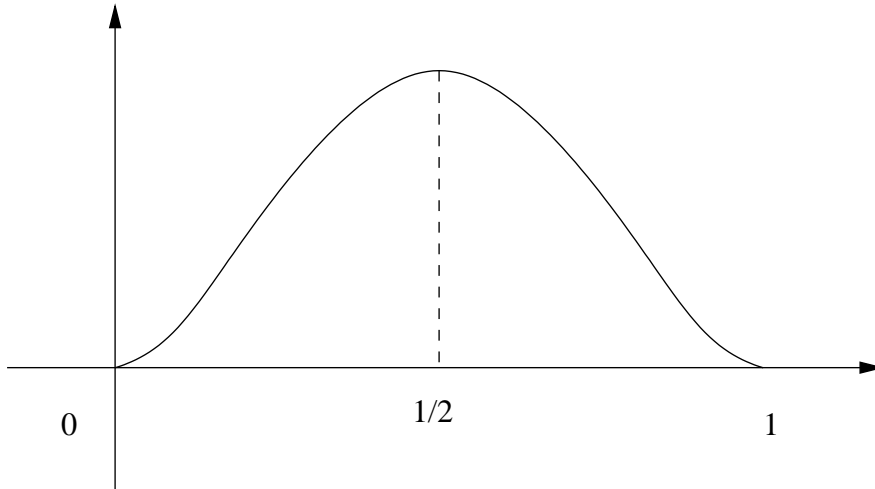

FIG. 5.1.

Now we show that the vector field

$$e_0 = \left(r\cos\frac{2\pi x_1}{L}h'_{11}(x_2),\ \frac{2\pi r}{L}\sin\frac{2\pi x_1}{L}h_{11}(x_2)\right) \tag{5.31}$$

is D-regular in $\Omega = \mathbb{R}^1 \times (0, 1)$.

To this end, by (5.30) we see that

$$h''_{11}(x_2) \ne 0 \qquad \text{at } x_2 = 0, \frac{1}{2}, 1,$$

$$h'_{11}\left(\frac{1}{2}\right) = 0, \qquad h''_{11}\left(\frac{1}{2}\right) \ne 0.$$

Hence

$$\det De_0(x_1, x_2) \neq 0,$$

for any $(x_1, x_2) = (kL/2, 1/2)$ with $k = 1, 2, \ldots$, and

$$\det \begin{pmatrix} \dfrac{\partial^2 e_0^1}{\partial x_1 \partial x_2} & \dfrac{\partial^2 e_0^1}{\partial x_2^2} \\[2ex] \dfrac{\partial^2 e_0^2}{\partial x_1 \partial x_2} & \dfrac{\partial^2 e_0^2}{\partial x_2^2} \end{pmatrix} = -\frac{2\pi r}{L} \sin^2 \frac{2\pi x_1}{L} \, h_{11}''(x_2) \neq 0,$$

for any $(x_1, x_2) = ((2k+1)L/4, 0)$ or $(x_1, x_2) = ((2k+1)L/4, 1)$ with $k = 1, 2, \ldots$. Therefore, the vector field (5.31) is D-regular, and consequently, the vector fields e in (5.28) are D-regular for any $R_c < R < R_c + \varepsilon$ for some $\varepsilon > 0$.

Next we show that the following subspace of H

$$E = \{ (u, T) \in H_1 \mid \int_0^L \int_0^1 u_1 \, dx = 0 \}$$

is invariant for (2.4)-(2.8). In fact, we can verify that

$$\int_0^L \int_0^1 P[(u \cdot \nabla)u] \cdot i \, dx = 0 \quad \forall u \in H_1, \tag{5.32}$$

where $i = (0, 1)^t$ is the unit vector in the x_1 direction, and P the Leray projection. Indeed, by the Helmholtz decomposition, we have

$$[(u \cdot \nabla)u] \cdot i = P[(u \cdot \nabla)u] \cdot i + \frac{\partial \phi}{\partial x_1},$$

for some $\phi \in H^1(\Omega)$. Hence,

$$\int_0^L \int_0^1 P[(u \cdot \nabla)u] \cdot i \, dx = \int_0^L \int_0^1 (u \cdot \nabla)u_1 \, dx = 0.$$

The invariance of E for (2.4)-(2.8) implies that for the vector field e given in (5.28), we have

$$\int_0^1 v_1 \, dx_2 = 0. \tag{5.33}$$

Hence in the Fourier expansion of e in (5.5) and (5.6), the coefficients of $\widetilde{\psi}_{0j}$ are zero. By the connection lemma in [10], it follows from (5.33) that the vector field $e = (e_1, e_2)$ of (5.28) is topologically equivalent to the vector field e_0 given by (5.31), which has the topological structure as shown in Fig. 4.3.

For any initial value $\psi_0 = (u_0, T_0) \in H \setminus E$,

$$\psi_0 = \sum_k \alpha_k \widetilde{\psi}_{0k} + \Phi_0,$$

$$\Phi_0 \in E,$$

$$\widetilde{\psi}_{0k} = (\sin \pi x_2, 0, 0), \qquad k = 2m+1, m = 0, 1, \ldots.$$

By (5.32), we see that for the operator G in (2.12),

$$(G(\psi), \widetilde{\psi}_{0k}) = 0,$$

which implies that the solution of (2.4)-(2.8) with (2.11) has the form

$$\psi(t, \psi_0) = \sum_k \alpha_k e^{t\beta_{0k}} \widetilde{\psi}_{0k} + \widetilde{\Phi}(t, \psi_0), \qquad (5.34)$$

where $\widetilde{\Phi} \in E$, and $\beta_{0k} < 0$ near $R = R_c$.

Let

$$K = \min\{ \ k \ | \ \alpha_k \neq 0 \text{ and } k \text{ is odd}\},$$

and let $\psi(t, \psi_0) = (u(t, \psi_0), T(t, \psi_0))$ be the solution of (2.4)-(2.8) given by (5.34). Then by (5.27), the vector field $u(t, \psi_0)$ is topologically equivalent to the following vector field for any $t > t_0$ with $t_0 > 0$ sufficiently large,

$$\widetilde{u} = e + (\alpha_k e^{-t\beta_{0k}} \sin k\pi x_2, 0), \qquad (5.35)$$

where e is as in (5.28).

The vector field e is D-regular and topologically equivalent to the vector field as shown in Fig. 4.3. Therefore, using the method for breaking saddle connections in [10], it is easy to show that the vector field $\widetilde{u}$ given by (5.35) is topologically equivalent to the structure as shown in Fig. 4.2(a) if $\alpha_k < 0$, and to the structure as shown in Fig. 4.2(b) if $\alpha_k > 0$, for any $t > t_0$ sufficiently large.

Thus the proof of Thm. 4.1 is complete.

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