COMMUN. MATH. SCI.
Vol. 5, No. 2, pp. 243–252

NEW EXACT SOLUTIONS FOR THE CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION*

YAN-ZE PENG† AND E. V. KRISHNAN‡

Abstract. The algebraic method is developed to obtain new exact solutions, including stationary wave solutions and traveling wave solutions, for the cubic-quintic nonlinear Schrödinger (NLS) equation. Specifically, we present two general solution formulae, which degenerate to the corresponding solution of the cubic NLS equation, when the quintic nonlinear term is absent. It is expected that they are useful in correlative physics fields.

Key words. The cubic-quintic nonlinear Schrödinger equation, the stationary wave solution, traveling wave solution

AMS subject classifications. 35Q35, 35B20, 37K45

1. Introduction

There is a huge variety of methods available for constructing exact solutions of nonlinear partial differential equations (PDEs). Some of the most important methods are the inverse scattering transformation [1], the bilinear method [2], symmetry reductions [3, 4], Bäcklund and Darboux transformations [5], the singular manifold method [6] and so on. Recently, searching for periodic wave solutions to nonlinear PDEs in terms of the Jacobi elliptic functions has aroused great interest [7-11] because of the elegant properties of the elliptic functions [12-14].

The cubic-quintic nonlinear Schrödinger equation,

$$iu_t + u_{xx} + \delta|u|^2 u - \varepsilon|u|^4 u = 0, \tag{1.1}$$

appears in many physics fields: the optical pulse propagations in dielectric media of non-Kerr type [15], the nuclear hydrodynamics with Skyrme forces [16], etc. Also, it is used to describe the boson gas with two and three body interactions [17]. So it is important to search for the exact solutions of Eq. (1.1). Some traveling wave solutions with linear phase have been reported [18-19]. The authors in [20] discussed exact solutions of Eq. (1.1) and the relation to blowup. In this paper, the algebraic method is developed to obtain the stationary periodic wave solutions and the traveling periodic wave solutions with linear phase and with nonlinear phase. Under the long wave limit, periodic waves degenerate to the corresponding solitary waves.

2. Stationary wave solutions

First, we look for a stationary solution for Eq. (1.1) of the form

$$u(x,t) = v(x)\mathrm{e}^{-i\Omega t}, \tag{2.1}$$

where Ω is the constant to be determined, and v a real function. The substitution of Eq. (2.1) into Eq. (1.1) yields

$$v'' + \Omega v + \delta v^3 - \varepsilon v^5 = 0, \tag{2.2}$$

*Received: September 30, 2006; accepted (in revised version): January 26, 2007. Communicated by Jack Xin.

†Corresponding author, School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, P. R. China (yanzepeng@163.com).

‡Department of Mathematics and Statistics, Sultan Qaboos University, P. O. Box 36, Al-Khod 123, Muscat, Sultanate of Oman (krish@squ.edu.om).

where the prime denotes the derivative with respect to x. This is a special case of the Lienard equation. Kong [21] and Feng [22] studied its exact solutions. Following their methods, we obtain the following exact solutions of Eq. (1.1).

(1)

$$u(x,t)=\pm\Big[\frac{-4C_0\Omega e^{-2\sqrt{-\Omega}x}}{(C_0 e^{-2\sqrt{-\Omega}x}+\delta/2)^2+4/3\Omega\varepsilon}\Big]^{1/2} e^{-i\Omega t}, \tag{2.3}$$

where C_0 is an arbitrary positive constant and which is valid for $\Omega<0$ and $\varepsilon\leq 0$.

(2)

$$u(x,t)=\pm\Big[\frac{4\sqrt{3\Omega^2/(3\delta^2+16\Omega\varepsilon)}\operatorname{sech}^2\sqrt{-\Omega}x}{2+(-1+\sqrt{3}\delta/\sqrt{3\delta^2+16\Omega\varepsilon})\operatorname{sech}^2\sqrt{-\Omega}x}\Big]^{1/2} e^{-i\Omega t}, \tag{2.4}$$

which is valid for $\Omega<0$, $\delta>0$ and $\varepsilon\leq 0$ or $\Omega<0$, $\delta\leq 0$ and $\varepsilon<0$.

(3)

$$u(x,t)=\pm\sqrt{-\frac{2\Omega}{\delta}(1\pm\tanh\sqrt{-\Omega}x)e^{-i\Omega t}}, \tag{2.5}$$

which is valid for $\Omega<0$, $\delta>0$ and $3\delta^2+16\Omega\varepsilon=0$.

In what follows, we will obtain some periodic wave solutions of Eq. (2.2), thus of Eq. (1.1), in terms of the Jacobi elliptic functions, by means of the mapping method proposed recently by the author [23-24]. Integrating Eq.(2.2) once, we have

$$v'^{\,2}+\Omega v^2+\frac{1}{2}\delta v^4-\frac{1}{3}\varepsilon v^6=C, \tag{2.6}$$

with C being the integration constant. It is convenient to introduce $w=v^2$. In terms of w, Eq. (2.6) has the form

$$w'^{\,2}+4\Omega w^2+2\delta w^3-\frac{4}{3}\varepsilon w^4=4Cw. \tag{2.7}$$

Now, we assume Eq. (2.7) has solutions of the form

$$w=A_0+A_1 f, \tag{2.8}$$

where f satisfies the auxiliary ordinary differential equation

$$f'^{\,2}=pf^2+\frac{1}{2}qf^4+r. \tag{2.9}$$

Eq. (2.8) establishes a mapping relation between the solution of Eq. (2.7) and that of Eq. (2.9). Due to the entry of parameters p, q and r, Eq. (2.9) is more flexible than Eq. (2.7). Substituting Eq. (2.8) with Eq. (2.9) into Eq. (2.7) and equating the coefficients of like powers of f, one obtains

$$\begin{aligned}
&A_0=\frac{3\delta}{8\varepsilon},\quad A_1=\pm\frac{1}{2}\sqrt{\frac{3q}{2\varepsilon}},\\
&\Omega=-\frac{1}{4}p-\frac{49\delta^2}{144\varepsilon},\\
&C=2\Omega A_0+\frac{3}{2}\delta A_0^2-\frac{4}{3}\varepsilon A_0^3,
\end{aligned} \tag{2.10}$$

with the constraint of the parameters

$$101\delta^4-144p\delta^2\varepsilon-384qr\varepsilon^2=0. \tag{2.11}$$

Thus, we obtain the exact stationary wave solution of Eq. (1.1)

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{2}\sqrt{\frac{3q}{2\varepsilon}}f(x)\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.12}$$

where f satisfies Eq. (2.9) and Ω is given by the Eq. (2.10), with the constraint (2.11) between the model parameters. In order to give the specific expression of f, the following discussion is meaningful and interesting.

Case 1. $p=-(1+m^2),\ q=2m^2,\ r=1$.

In this case, Eq. (2.9) has the periodic wave solution $f=\mathrm{sn}(x|m)$. Throughout the paper, sn, cn and dn denote the Jacobi elliptic functions, and $0<m<1$ is the modulus of the elliptic function, this notation is standard. So, Eq. (1.1) has the stationary periodic wave solution

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{2}\sqrt{\frac{3}{\varepsilon}}m\mathrm{sn}(x|m)\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.13}$$

where $\Omega=\frac{1}{4}(1+m^2)-\frac{49\delta^2}{144\varepsilon}$, with the constraints of the model parameters, $\varepsilon>0$ and $101\delta^4+144(1+m^2)\delta^2\varepsilon-768m^2\varepsilon^2=0$. As $m\to 1$, Eq. (2.13) degenerates to the stationary kink or anti-kink solution

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{2}\sqrt{\frac{3}{\varepsilon}}\tanh(x)\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.14}$$

where $\Omega=\frac{1}{2}-\frac{49\delta^2}{144\varepsilon}$, which is valid for $\varepsilon>0$ and $101\delta^4+288\delta^2\varepsilon-768\varepsilon^2=0$.

Case 2. $p=-\frac{1}{2}(2-m^2),\ q=\frac{1}{2}m^2,\ r=\frac{1}{4}$.

The solution of Eq. (2.9) reads $f=\mathrm{sn}(x|m)/(1+\mathrm{dn}(x|m))$. Thus, Eq. (1.1) has the stationary periodic wave solution

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{4}\sqrt{\frac{3}{\varepsilon}}m\frac{\mathrm{sn}(x|m)}{1+\mathrm{dn}(x|m)}\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.15}$$

where $\Omega=\frac{1}{8}(2-m^2)-\frac{49\delta^2}{144\varepsilon}$, which is valid for $\varepsilon>0$ and $101\delta^4+72(2-m^2)\delta^2\varepsilon-48m^2\varepsilon^2=0$. As $m\to 1$, the corresponding stationary shock wave solution of Eq. (1.1) reads

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{4}\sqrt{\frac{3}{\varepsilon}}\frac{\tanh(x)}{1+\mathrm{sech}(x)}\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.16}$$

where $\Omega=\frac{1}{8}-\frac{49\delta^2}{144\varepsilon}$, which is valid for $\varepsilon>0$ and $101\delta^4+72\delta^2\varepsilon-48\varepsilon^2=0$.

Case 3. $p=-\frac{1}{2}(2m^2-1),\ q=\frac{1}{2},\ r=\frac{1}{4}$.

We have $f=\mathrm{cn}(x|m)/(\sqrt{1-m^2}\mathrm{sn}(x|m)+\mathrm{dn}(x|m))$. The corresponding solution of Eq. (1.1) reads

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{4}\sqrt{\frac{3}{\varepsilon}}\frac{\mathrm{cn}(x|m)}{\sqrt{1-m^2}\mathrm{sn}(x|m)+\mathrm{dn}(x|m)}\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.17}$$

where $\Omega=\frac{1}{8}(2m^2-1)-\frac{49\delta^2}{144\varepsilon}$, with the constraints of the model parameters, $\varepsilon>0$ and $101\delta^4+72(2m^2-1)\delta^2\varepsilon-48\varepsilon^2=0$.

Case 4. $p=-\frac{1}{2}(2-m^2)$, $q=\frac{1}{2}m^4$, $r=\frac{1}{4}$.

Eq. (2.9) has the solution $f=\mathrm{cn}(x|m)/(\sqrt{1-m^2}+\mathrm{dn}(x|m))$. Hence, Eq. (1.1) has the stationary wave solution

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{4}\sqrt{\frac{3}{\varepsilon}}m^2\frac{\mathrm{cn}(x|m)}{\sqrt{1-m^2}+\mathrm{dn}(x|m)}\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.18}$$

where $\Omega=\frac{1}{8}(2-m^2)-\frac{49\delta^2}{144\varepsilon}$, which is valid for $\varepsilon>0$ and $101\delta^4+72(2-m^2)\delta^2\varepsilon-48m^4\varepsilon^2=0$.

As $m\to 1$, Eqs. (2.17) and (2.18) degenerate to continuous wave solutions.

Case 5. $p=2m^2-1$, $q=-2m^2$, $r=1-m^2$.

The solution of Eq. (2.9) is $f=\mathrm{cn}(x|m)$. Therefore, the stationary periodic wave solution reads

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{2}\sqrt{\frac{-3}{\varepsilon}}m\mathrm{cn}(x|m)\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.19}$$

with $\Omega=-\frac{1}{4}(2m^2-1)-\frac{49\delta^2}{144\varepsilon}$, which is valid for $\varepsilon<0$ and $101\delta^4-144(2m^2-1)\delta^2\varepsilon+768m^2(1-m^2)\varepsilon^2=0$.

Case 6. $p=2-m^2$, $q=-2$, $r=-(1-m^2)$.

Eq. (2.9) has the solution $f=\mathrm{dn}(x|m)$. Hence, we obtain the stationary periodic wave solution for Eq. (1.1)

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{2}\sqrt{\frac{-3}{\varepsilon}}\mathrm{dn}(x|m)\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.20}$$

with $\Omega=-\frac{1}{4}(2-m^2)-\frac{49\delta^2}{144\varepsilon}$, which is valid for $\varepsilon<0$ and $101\delta^4-144(2-m^2)\delta^2\varepsilon-768(1-m^2)\varepsilon^2=0$.

Case 7. $p=\frac{1}{2}(1+m^2)$, $q=-\frac{1}{2}$, $r=-\frac{1}{4}(1-m^2)^2$.

The solution of Eq. (2.9) reads $f=m\mathrm{cn}(x|m)+\mathrm{dn}(x|m)$. The corresponding solution for Eq. (1.1) is

$$u(x,t)=\pm\Big[\frac{3\delta}{8\varepsilon}\pm\frac{1}{4}\sqrt{\frac{-3}{\varepsilon}}(m\mathrm{cn}(x|m)+\mathrm{dn}(x|m))\Big]^{1/2}\mathrm{e}^{-i\Omega t}, \tag{2.21}$$

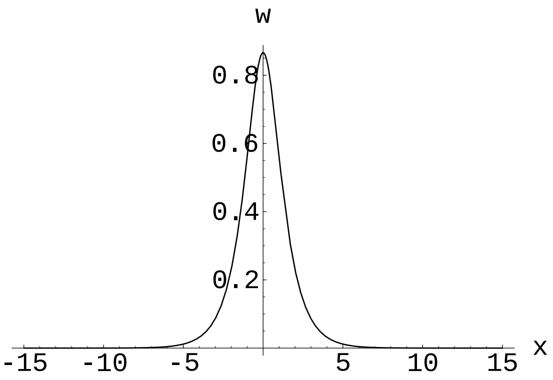

FIG. 3.1. *The graph of Eq. (2.21) for $w \equiv |u|^2$ with $\delta = 0$, $m = 1$.*

with $\Omega = -\frac{1}{8}(1+m^2) - \frac{49\delta^2}{144\varepsilon}$, which is valid for $\varepsilon < 0$ and $101\delta^4 - 72(1+m^2)\delta^2\varepsilon - 48(1-m^2)^2\varepsilon^2 = 0$.

Eqs. (2.19), (2.20) and (2.21) are invalid for $m \to 1$ since the model parameters δ and ε are real.

3. Properties of stationary solutions

In this section, we discuss the property of stationary solutions when $\delta \geq 0$ and $\varepsilon < 0$, taking Case 7 as an example. When $\delta = 0$ and $\varepsilon < 0$, the NLS is critical in the sense that if the initial data has negative energy and the initial mass exceeds that of the ground state (i.e. stationary solution when $v(x)$ is localized and positive), solutions blow up in finite time. The stationary solution (ground state) in this case provides the critical mass (L^2 norm) for blow up. Without loss of generality, we take $\varepsilon = -1$. We draw the figures of Eq. (2.21) for $w \equiv |u|^2$ with $\delta = 0$, $m = 1$, $\delta = 0.486804$, $m = 0.5$ and $\delta = 0.608578$, $m = 0.25$, respectively. After computation, it is found that the maximum of w is 0.866025, 0.466968 and 0.313049 in figures 3.1, 3.2, and 3.3, respectively. The maximum of w is decreasing with the increase of δ. Although the authors in [20] discussed the blowup phenomenon of exact solutions of Eq. (1.1), the solutions obtained above do not develop singularity at a finite point, i.e. for any fixed $t = t_0$, there exists x_0 at which the solutions blow up.

4. Traveling wave solutions

4.1. Traveling wave solutions with linear phase. In order to obtain the traveling wave solution with linear phase for Eq. (1.1), we make the ansatz

$$u(x,t) = v(\xi)\mathrm{e}^{i(Kx-\Omega t)}, \quad \xi = k(x-ct). \tag{4.1}$$

It is assumed that $k > 0$ without loss of generality. Substituting Eq. (4.1) into Eq. (1.1) and taking $c = 2K$, one obtains the differential equation for v

$$k^2 v'' + (\Omega - K^2)v + \delta v^3 - \varepsilon v^5 = 0. \tag{4.2}$$

Eq. (4.2) is the same as Eq. (2.2). So making the transformation

$$\begin{aligned} &\Omega \to \frac{\Omega - K^2}{k^2}, \quad \delta \to \frac{\delta}{k^2}, \quad \varepsilon \to \frac{\varepsilon}{k^2}, \\ &x \to k(x-2Kt), \quad \mathrm{e}^{-i\Omega t} \to \mathrm{e}^{i(Kx-\Omega t)} \end{aligned} \tag{4.3}$$

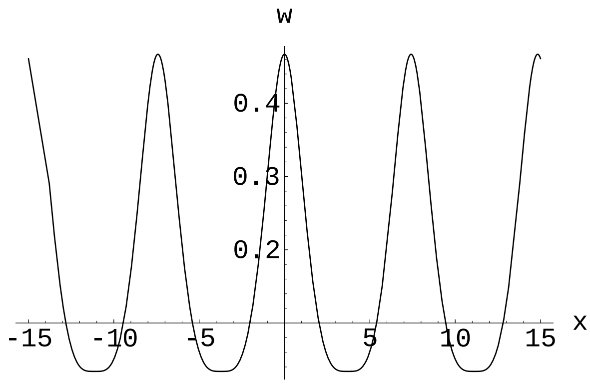

FIG. 3.2. *The graph of Eq. (2.21) for* $w \equiv |u|^2$ *with* $\delta = 0.486804$, $m = 0.5$.

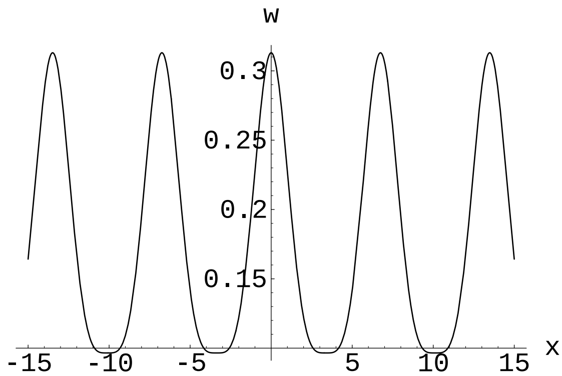

FIG. 3.3. *The graph of Eq. (2.21) for* $w \equiv |u|^2$ *with* $\delta = 0.608578$, $m = 0.25$.

in the stationary wave solutions (2.3)-(2.5) and (2.13)-(2.21), we obtain the corresponding traveling wave solution with linear phase for Eq. (1.1).

4.2. Traveling wave solutions with nonlinear phase. Now we introduce the amplitude $\phi(x,t)$ and the phase $\psi(x,t)$. Assume

$$u(x,t) = \phi(x,t)\mathrm{e}^{i\psi(x,t)}, \tag{4.4}$$

where both $\phi(x,t)$ and $\psi(x,t)$ are real functions. Substituting Eq. (4.4) into Eq. (1.1) and separating the real and imaginary parts, one has

$$\begin{gathered} \phi_t + 2\psi_x\phi_x + \psi_{xx}\phi = 0, \\ -\psi_t\phi + \phi_{xx} - \psi_x^2\phi + \delta\phi^3 - \varepsilon\phi^5 = 0, \end{gathered} \tag{4.5}$$

which is equivalent to the original Eq. (1.1). We restrict ourselves to traveling wave solutions in this paper. That is, we set

$$\begin{aligned} \phi(x,t) &= \phi(\xi), \quad \xi = x - vt, \\ \psi(x,t) &= Kx - \Omega t + \theta(\xi). \end{aligned} \tag{4.6}$$

In the above, v, K and Ω are real constants. With Eq. (4.6), Eq. (4.5) reduces to a set of coupled ordinary differential equations,

$$(2K-v+2\theta')\phi'+\theta''\phi=0,$$
$$\phi''+[\Omega+v\theta'-(K+\theta')^2]\phi+\delta\phi^3-\varepsilon\phi^5=0, \tag{4.7}$$

where the prime means the differential with respect to ξ. Integrating the first of Eq. (4.7), we have

$$\theta'=C_1\phi^{-2}-\frac{1}{2}(2K-v), \tag{4.8}$$

where C_1 is the constant of integration. Using Eq. (4.8) in the second of Eq. (4.7), we get a closed differential equation for the amplitude $\phi(\xi)$,

$$\phi''-C_1^2\phi^{-3}+(\Omega-vK+\frac{v^2}{4})\phi+\delta\phi^3-\varepsilon\phi^5=0. \tag{4.9}$$

The integration of Eq. (4.9) yields

$$\phi'^{\,2}+C_1^2\phi^{-2}+(\Omega-vK+\frac{v^2}{4})\phi^2+\frac{1}{2}\delta\phi^4-\frac{1}{3}\varepsilon\phi^6=C_2, \tag{4.10}$$

with C_2 being the integral constant. It is convenient to introduce $\Phi(\xi)=\phi^2(\xi)$, which is nothing but the number density $\rho(x,t)=u^*(x,t)u(x,t)$ for the traveling waves. In terms of $\Phi(\xi)$, Eq. (4.10) has a simpler form,

$$\Phi'^{\,2}=-4C_1^2+4C_2\Phi-4(\Omega-vK+\frac{v^2}{4})\Phi^2-2\delta\Phi^3+\frac{4}{3}\varepsilon\Phi^4. \tag{4.11}$$

If $C_1=0$, Eq. (4.11) reduces to Eq. (2.7). Then, it follows from Eq. (4.8) that one can only obtain the traveling wave solutions with linear phase for Eq. (1.1). So, we must study the exact solutions of Eq. (4.11) with $C_1\neq 0$. Hereafter, we assume that the roots of the polynomial at the right hand of Eq. (4.11) are all real and distinct, that is

$$\Phi'^{\,2}=\frac{4}{3}\varepsilon(\Phi-a)(\Phi-b)(\Phi-c)(\Phi-d),$$
$$a<b<c<d. \tag{4.12}$$

The constants a, b, c and d satisfy the relations

$$a+b+c+d=\frac{3\delta}{2\varepsilon},$$
$$ab+bc+cd+da+ac+bd=-\frac{3(\Omega-vK+v^2/4)}{\varepsilon},$$
$$abc+bcd+cda+dab=-\frac{3C_2}{\varepsilon},$$
$$abcd=-\frac{3C_1^2}{\varepsilon}. \tag{4.13}$$

The bounded traveling wave solutions are classified into two types depending on the sign of ε. Note that the last one of Eq. (4.13) and $\Phi=\phi^2>0$.

4.2.1. $\varepsilon > 0$. The case $a < 0 < b < c < d$ is considered. Other choices do not lead to a bounded solution. Possible solutions lie in the region $b < \Phi < c$. In this case, the solution of Eq. (4.12) reads

$$\begin{aligned}\Phi(\xi) &= \frac{c(d-b) - d(c-b)\mathrm{sn}^2[\alpha(\xi-\xi_0)|m]}{(d-b)-(c-b)\mathrm{sn}^2[\alpha(\xi-\xi_0)|m]}\\ &= \frac{b(d-c)+d(c-b)\mathrm{cn}^2[\alpha(\xi-\xi_0)|m]}{(d-c)+(c-b)\mathrm{cn}^2[\alpha(\xi-\xi_0)|m]},\end{aligned} \tag{4.14}$$

where

$$\alpha = \sqrt{\frac{\varepsilon}{3}(c-a)(d-b)}, \quad m = \sqrt{\frac{(d-a)(c-b)}{(d-b)(c-a)}}, \tag{4.15}$$

and ξ_0 is the integral constant. A well-known formula, $\mathrm{sn}^2(x|m)+\mathrm{cn}^2(x|m) = 1$, has been used. The solution (4.14) corresponds to a bright solution train. The function $\theta(\xi)$ is then found to be

$$\begin{aligned}\theta(\xi) = &\frac{C_1}{d}\int_{\xi_0}^{\xi} \frac{(d-b)(d-c)}{c(d-b)-d(c-b)\mathrm{sn}^2[\alpha(\xi-\xi_0)|m]} d\xi\\ &+[\frac{C_1}{d} - \frac{1}{2}(2K-v)]\xi + \theta_0,\end{aligned} \tag{4.16}$$

which is an elliptic integral of the third kind, and θ_0 is a constant of integration. The formulas (4.4), (4.6), (4.14) and (4.16) provide the general bounded traveling wave solution of Eq. (1.1) with $\varepsilon > 0$. To see this explicitly, we consider a special case, $m = 1$ (the infinite periodic case). From Eq. (4.15) and $a < 0 < b$, one sees that $a = b = 0$ (The case $c = d$ can only result in the trivial result). Eq. (4.13) reduces to

$$\begin{aligned}&C_1 = 0, \quad C_2 = 0,\\ &c = \frac{3\delta - \sqrt{9\delta^2 + 48\varepsilon(\Omega - vK + v^2/4)}}{4\varepsilon},\\ &d = \frac{3\delta + \sqrt{9\delta^2 + 48\varepsilon(\Omega - vK + v^2/4)}}{4\varepsilon}.\end{aligned} \tag{4.17}$$

Thus, it follows from Eqs. (4.14), (4.16) and (4.6) that

$$\begin{aligned}\phi(x,t) &= \left\{\frac{cd\ \mathrm{sech}^2[\sqrt{-(\Omega - vK + v^2/4)}(x-vt-x_0)]}{d-c+c\ \mathrm{sech}^2[\sqrt{-(\Omega - vK + v^2/4)}(x-vt-x_0)]}\right\}^{1/2},\\ \psi(x,t) &= \frac{1}{2}vx - (\Omega - vK + \frac{1}{2}v^2)t + \theta_0,\end{aligned} \tag{4.18}$$

where c and d are given by Eq. (4.17), and which is valid for $\delta > 0$, $\varepsilon > 0$ and $-9\delta^2/(48\varepsilon) < \Omega - vK + v^2/4 < 0$. Hence, Eqs. (4.4) and (4.18) constitute a bright solitary wave solution of Eq. (1.1) with $\delta > 0$ and $\varepsilon > 0$. As $\varepsilon \to 0$, Eq. (4.18) degenerates to

$$\phi = \sqrt{\frac{2}{\delta}} k\ \mathrm{sech}[k(x-vt-x_0)], \tag{4.19}$$

with $k = \sqrt{-(\Omega - vK + v^2/4)}$, while ψ remains to be unchanged, and which is a bright solitary wave solution for the cubic nonlinear Schrödinger equation with $\delta > 0$, a well-known result.

4.2.2. $\varepsilon < 0$. We consider the case $a < b < 0 < c < d$, and the other cases may be considered in a similar way. The possible solution lies in the region $c < \Phi < d$, and reads

$$\Phi(\xi) = \frac{d(c-a) - a(c-d)\mathrm{sn}^2[\beta(\xi-\xi_0)|m]}{(c-a) - (c-d)\mathrm{sn}^2[\beta(\xi-\xi_0)|m]}, \tag{4.20}$$

where

$$\beta = \sqrt{-\frac{\varepsilon}{3}(d-b)(c-a)}, \quad m = \sqrt{\frac{(a-b)(c-d)}{(d-b)(c-a)}}, \tag{4.21}$$

and ξ_0 is the integral constant. The function $\theta(\xi)$ is

$$\begin{aligned}\theta(\xi) = &\frac{C_1}{a}\int_{\xi_0}^{\xi} \frac{(c-a)(a-d)}{d(c-a) - a(c-d)\mathrm{sn}^2[\beta(\xi-\xi_0)|m]} d\xi \\ &+ \Big[\frac{C_1}{a} - \frac{1}{2}(2K - v)\Big]\xi + \theta_0,\end{aligned} \tag{4.22}$$

where θ_0 is the integral constant. The formulas (4.4), (4.6), (4.20) and (4.22) provide a general bounded traveling wave solution of Eq. (1.1) with $\varepsilon < 0$. As $m \to 1$, it follows from Eq. (4.21) and $b < 0 < c$ that $b = c = 0$ (The case $a = d$ can only result in the trivial result). Then Eq. (4.13) reduces to

$$\begin{aligned} &C_1 = 0, \quad C_2 = 0, \\ &a = \frac{3\delta + \sqrt{9\delta^2 + 48\varepsilon(\Omega - vK + v^2/4)}}{4\varepsilon}, \\ &d = \frac{3\delta - \sqrt{9\delta^2 + 48\varepsilon(\Omega - vK + v^2/4)}}{4\varepsilon}, \end{aligned} \tag{4.23}$$

with $\delta > 0$. Notice that $\delta < 0$ will bring on a contradiction. Then, from Eqs. (4.20), (4.22) and (4.6) we obtain

$$\begin{aligned} &\phi(x,t) = \Big\{ \frac{ad\ \mathrm{sech}^2[\sqrt{-(\Omega - vK + v^2/4)}(x - vt - x_0)]}{a - d\ \tanh^2[\sqrt{-(\Omega - vK + v^2/4)}(x - vt - x_0)]} \Big\}^{1/2}, \\ &\psi(x,t) = \frac{1}{2}vx - (\Omega - vK + \frac{1}{2}v^2)t + \theta_0, \end{aligned} \tag{4.24}$$

where a and d are given by Eq. (4.23), and which is valid for $\delta > 0$, $\varepsilon < 0$ and $\Omega - vK + v^2/4 < 0$. As $\varepsilon \to 0$, Eq. (4.24) degenerates to Eq. (4.19), an expected result.

5. Conclusion

In this paper, an algebraic method is devised to search for exact solutions, including stationary wave solutions and traveling wave solutions, for the cubic-quintic NLS Eq. (1.1). All solutions obtained in this paper are classified into two types depending on the sign of quintic nonlinear term, i.e., the sign of ε. The solutions (2.5), (2.13)-(2.18) and (4.14) are valid for $\varepsilon > 0$ while (2.3), (2.4), (2.19)-(2.21) and (4.20) for $\varepsilon < 0$. Note that cases (2)-(4) and (7) in Sec.2 are new results for the solutions of Eq. (2.9), which are not given in [23-24]. The solutions (4.18) and (4.24) can only degenerate to the solution of the cubic NLS equation with anomalous dispersion ($\delta > 0$). It is an open problem how to obtain from Eq. (4.11) the solution of the cubic-quintic

NLS equation (1.1) which degenerates to the solution of the cubic NLS equation with normal dispersion ($\delta < 0$).

There is a rich literature on extended cubic NLS (including cubic-quintic NLS) that arise in optical physics, and exact solitary type solutions are studied, see e.g. [25], and among others the recent book by F. Abdullaev, et al, on cubic-quintic media [26]. Thus, concrete applications of these exact solutions are omitted.

REFERENCES

[1] C. S. Gardner, J. M. Green, M. D. Kruskal and R. M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett., 19, 1095-1097, 1967.

[2] R. Hirota, *Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons*, Phys. Rev. Lett., 27, 1192-1194, 1971.

[3] P. J. Olver, *Application of Lie Group to Differential Equation*, New York, Springer, 1986.

[4] P. A. Clarkson and M. D. Kruskal, *New similarity reductions of the Boussinesq equation*, J. Math. Phys., 30, 2201-2213, 1989.

[5] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Berlin, Springer, 1991.

[6] J. Weiss, M. Tabor and G. Carnevale, *The Painlevé property for partial differential equations*, J. Math. Phys., 24, 522-526, 1983.

[7] N. Akhmediev and A. Ankiewicz, *First order exact solutions of the nonlinear Schrödinger equation in the normal dispersion regime*, Phys. Rev. A, 47, 3213-3221, 1993.

[8] K. W. Chow, K. Nakkeeran and B. A. Malomed, *Periodic waves in bimodal optical fibers*, Opt. Commun., 219, 251-259, 2003.

[9] Y. Z. Peng, *Periodic waves and periodic solitons and their interactions for a (2+1)-dimensional KdV equation*, Progress of Theoretical Physics, 113, 927-933, 2005.

[10] E. V. Krishnan and Y. Z. Peng, *Squared Jacobian elliptic function solutions for the (2+1)-D Korteweg de-Vries equation*, Advances in Theoretical and Applied Mathematics, 1, 9-25, 2006.

[11] Y. Z. Peng and E. V. Krishnan, *Exact traveling wave solutions for a class of nonlinear partial differential equations*, Int. J. Pure Appl. Math. Sci., 3, 11-20, 2006.

[12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematial Functions*, Dover, New York, 1972.

[13] V. Prasolov and Y. Solovyev, *Elliptic Functions and Elliptic Integrals*, American Mathematical Society, Providence, 1997.

[14] K. Chamdrasekharan, *Elliptic Functions*, Springer-Verlag, Berlin, 1985.

[15] A. Kumar, S. N. Sarkar and A. K. Ghatak, *Effect of fifth-order non-linearity in refractive index on Gaussian pulse propagation in lossy optical fibers*, Opt. Lett., 11, 321-323, 1986.

[16] V. G. Kartavenko, *Soliton-like solutions in nuclear hydrodynamics*, Sov. J. Nucl. Phys., 40, 240-246, 1984.

[17] I. V. Barashenkov and V. G. Makhankov, *Soliton-like bubbles in the system of interacting bosons*, Phys. Lett. A, 128, 52-56, 1988.

[18] Kh. I. Pushkarov, D. I. Pushkarov and I. V. Tomov, *Self-action of light beams in nonlinear media: soliton solution*, Opt. Quant. El., 11, 471-498, 1979.

[19] D. E. Pelinvosky, Y. S. Kivshar and V. V. Afanasjev, Instability-induced dynamics of dark solitons, Phys. Rev. E, 54, 2015-2032, 1996.

[20] D. Pathria and J. L. Morris, *Exact solutions for a generalized nonlinear Schrodinger equation*, Phys. Scr., 39, 673-679, 1989.

[21] D. X. Kong, *Explicit exact solutions for the Lienard equation and its applications*, Phys. Lett. A, 196, 301-306, 1995.

[22] Z. S. Feng, *On explicit exact solutions for the Lienard equation and its applications*, Phys. Lett. A, 293, 50-56, 2002.

[23] Y. Z. Peng, *A mapping method for obtaining exact traveling wave solutions to nonlinear evolution equations*, Chin. J. Phys., 41, 103-110, 2003.

[24] Y. Z. Peng, *Exact solutions for some nonlinear partial differential equations*, Phys. Lett. A, 314, 401-408, 2003.

[25] R. Radhakrishnan, A. Kundu and M. Lakshmanan, *Coupled nonlinear Schrödinger equations with cubic-quintic nonlinearity: integrability and soliton interaction in non-Kerr media*, Phys. Rev. E, 60, 3314-3323, 1999.

[26] F. Abdullaev and V. V. Konotop, *Optical pulse evolution toward light bullet and vortex solution*, NATO Science Series II: Mathematics, Physics and Chemistry, Springer, 153, 2005.