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SELF-SIMILAR SOLUTIONS OF THE NON-STRICTLY HYPERBOLIC WHITHAM EQUATIONS*

VIRGIL U. PIERCE† AND FEI-RAN TIAN‡

Abstract. We study the Whitham equations for the fifth order KdV equation. The equations are neither strictly hyperbolic nor genuinely nonlinear. We are interested in the solution of the Whitham equations when the initial values are given by a step function. We classify the step-like initial data into eight different types. We construct self-similar solutions for each type.

Key words. Zero dispersion limit, Whitham equations, non-stricitly hyperbolic equations

AMS subject classifications. 35L65, 35L67, 35Q05, 35Q15, 35Q53, 35Q58

1. Introduction

It is known that the solution of the KdV equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0 \tag{1.1}$$

has a weak limit as $\epsilon \to 0$ while the initial values

$$u(x,0;\epsilon) = u_0(x)$$

are fixed.

The weak limit is described by hyperbolic equations. It satisfies the Burgers equation

$$u_t + (3u^2)_x = 0 \tag{1.2}$$

until its solution develops shocks. Immediately after, the weak limit is governed by the Whitham equations [4, 5, 12, 13]

$$u_{it} + \lambda_i(u_1,u_2,u_3)u_{ix} = 0, \quad i = 1,2,3, \tag{1.3}$$

where the λ_i's are given by formulae (2.12). Equations (1.3) form a 3×3 system of hyperbolic equations. After the breaking of the solution of (1.3), the weak limit is described by a 5×5 system of hyperbolic equations similar to (1.3). Similarly, after the solution of the 5×5 system breaks down, the weak limit is characterized by a 7×7 system of hyperbolic equations. In other words, for general initial data $u_0(x)$, one must construct the weak limit by patching together solutions of (1.2), (1.3), 5×5, 7×7, etc systems in the x-t plane.

The KdV equation (1.1) is just the first member of an infinite sequence of equations, the second of which is the so-called fifth order KdV equation

$$u_t + 30u^2 u_x + 20\epsilon^2 u_x u_{xx} + 10\epsilon^2 u u_{xxx} + \epsilon^4 u_{xxxxx} = 0. \tag{1.4}$$

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†Department of Mathematics, Ohio State University, 231 W. 18th Avenue, Columbus, OH 43210, USA (vpierce@math.ohio-state.edu).

‡Department of Mathematics, Ohio State University, 231 W. 18th Avenue, Columbus, OH 43210, USA (tian@math.ohio-state.edu).

The solution of the fifth order KdV equation (1.4) also has a weak limit as $\epsilon \to 0$. As in the KdV case, this weak limit satisfies the Burgers type equation

$$u_t + (10u^3)_x = 0 \tag{1.5}$$

until the solution of (1.5) forms a shock. After the breaking of the solution of (1.5), the limit is governed by equations similar to (1.3), namely,

$$u_{it} + \mu_i(u_1, u_2, u_3) u_{ix} = 0, \quad i = 1, 2, 3, \tag{1.6}$$

where μ_i's are given in (2.18). They will also be called the Whitham equations. As in the KdV case, after the solution of (1.6) breaks down, the weak limit is described by a 5×5 system of hyperbolic equations.

In this paper, we are interested in the solution of the Whitham equation (1.6) for the fifth order KdV (1.4) with a step-like initial function

$$u_0(x) = \begin{cases} a, & x < 0, \\ b, & x > 0. \end{cases} \quad a \neq b. \tag{1.7}$$

For such an initial function with $a > 0$, $b < a$ or $a < 0$, $b > a$, the solution of the Burgers type equation (1.5) has already developed a shock at the initial time, $t = 0$. Hence, immediately after $t = 0$, the Whitham equations (1.6) kick in. Solutions of (1.6) occupy some domains of the space-time while solutions of (1.5) occupy other domains. These solutions are matched on the boundaries of the domains.

Equations (1.2) and equations (1.5) are prototypes in the theory of hyperbolic conservation laws [6]. Their solutions will generally develop shocks in finite times. The solutions can be extended beyond the singularities as the entropy solutions. The entropy solution of the Burgers equation (1.2) with initial function (1.7) is simple: it is either a rarefaction wave or a single shock wave. The Burgers type equation (1.5) is more complicated, as its flux function changes convexity at $u = 0$. Its entropy solution with step-like initial data (1.7) can be a rarefaction wave, a single shock wave or a combination of both [6].

Solutions of equations (1.2) or equations (1.5), in the theory of the zero dispersion limit, are not extended as weak or entropy solutions after the formation of singularities. Instead, they are extended to match the Whitham solutions of (1.3) or (1.6). For initial data (1.7), the resulting solutions of the Whitham equations (1.6) will be seen to be more complex than those of (1.3) in the KdV case. Indeed, there are eight types of different solutions in the former case while there is only one type of solution in the latter case.

The KdV case with the step-like initial data (1.7) was first studied by Gurevich and Pitaevskii [2]. The Burgers solution of (1.2) develops a shock only for $a > b$. Moreover, because of the Galilean invariance of the KdV equation, the corresponding initial function is equivalent to the case $a = 1$, $b = 0$. In this case, Gurevich and Pitaevskii found that it was enough to use the Burgers solution of (1.2) and the Whitham solution of (1.3) to cover the whole x-t plane, without going to the 5×5 or 7×7 system. Namely, the space-time is divided into three parts

$$(1)\ \frac{x}{t} < -6, \quad (2)\ -6 < \frac{x}{t} < 4, \quad (3)\ \frac{x}{t} > 4.$$

The solution of (1.2) occupies the first and third parts,

$$u(x,t) \equiv 1 \quad \text{when } \tfrac{x}{t} < -6, \quad u(x,t) \equiv 0 \quad \text{when } \tfrac{x}{t} > 4. \tag{1.8}$$

The Whitham solution of (1.3) lives in the second part,

$$u_1(x,t) \equiv 1, \quad \frac{x}{t} = \lambda_2(1,u_2,0), \quad u_3(x,t) \equiv 0, \tag{1.9}$$

when $-6 < x/t < 4$.

Whether the second equation of (1.9) can be inverted to give u_2 as a function of the self-similarity variable x/t hinges on whether

$$\frac{\partial \lambda_2}{\partial u_2}(1,u_2,0) \neq 0.$$

Indeed, Levermore [7] has proved the genuine nonlinearity of the Whitham equations (1.3), i.e.,

$$\frac{\partial \lambda_i}{\partial u_i}(u_1,u_2,u_3) > 0, \quad i = 1,2,3, \tag{1.10}$$

for $u_1 > u_2 > u_3$.

For the fifth order KdV (1.4), equations (1.6), in general, are not genuinely nonlinear, i.e., a property like (1.10) is not available. Hence, solutions like (1.8) and (1.9) need to be modified.

Our construction of solutions of the Whitham equation (1.6) makes use of the non-strict hyperbolicity of the equations. For KdV, it is known that the Whitham equations (1.3) are strictly hyperbolic, namely:

$$\lambda_1(u_1,u_2,u_3) > \lambda_2(u_1,u_2,u_3) > \lambda_3(u_1,u_2,u_3)$$

for $u_1 > u_2 > u_3$. For the fifth order KdV (1.4), different eigenspeeds of (1.6), $\mu_i(u_1,u_2,u_3)$'s, may coalesce in the region $u_1 > u_2 > u_3$.

For the fifth order KdV with step-like initial function (1.7) where $a = 1$ and $b = 0$, the space time is divided into four regions (see Figure 1.1)

$$(1)\ \frac{x}{t} < -15, \quad (2)\ -15 < \frac{x}{t} < \alpha, \quad (3)\ \alpha < \frac{x}{t} < 16, \quad (4)\ \frac{x}{t} > 16,$$

where α is determined by (3.15). In the first and fourth regions, the solution of (1.5) governs the evolution:

$$u(x,t) \equiv 1 \quad \text{where } x/t < -15 \text{ and } u(x,t) \equiv 0 \quad \text{where } x/t > 16\ .$$

The Whitham solution of (1.6) lives in the second and third regions; namely:

$$u_1(x,t) \equiv 1, \quad \frac{x}{t} = \mu_2(1,u_2,u_3), \quad \frac{x}{t} = \mu_3(1,u_2,u_3), \tag{1.11}$$

when $-15 < x/t < \alpha$, and

$$u_1(x,t) \equiv 1, \quad \frac{x}{t} = \mu_2(1,u_2,0), \quad u_3(x,t) \equiv 0,$$

when $\alpha < x/t < 16$.

Equations (1.11) yield

$$\mu_2(1,u_2,u_3) = \mu_3(1,u_2,u_3)$$

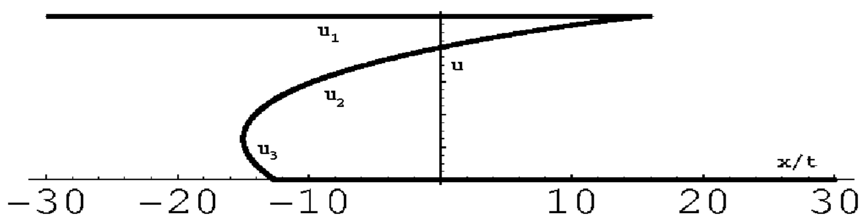

FIG. 1.1. *Self-Similar solution of the Whitham equations for $a=1$ and $b=0$ of type II.*

on a curve in the region $0<u_3<u_2<1$. This implies the non-strict hyperbolicity of the Whitham equations (1.6) for the fifth order KdV.

The organization of the paper is as follows. In Section 2, we will study the eigenspeeds, $\mu_i(u_1,u_2,u_3)$'s, of the Whitham equations (1.6). In Section 3, we will construct the self-similar solution of the Whitham equations for the initial function (1.7) with $a=1$, $b=0$. In Section 4, we will use the self-similar solution of Section 3 to construct the minimizer of a variational problem for the zero dispersion limit of the fifth order KdV. In Section 5, we will consider all the other possible step-like initial data (1.7). We find that there are eight different types of initial data. We construct self-similar solutions for each type.

2. The Whitham equations

In this section we define the eigenspeeds of the Whitham equations for both the KdV (1.1) and fifth order KdV (1.4). We first introduce the polynomials of ξ for $n=0,1,2,\dots$ [1, 3, 10]:

$$P_n(\xi,u_1,u_2,u_3)=\xi^{n+1}+a_{n,1}\xi^n+\cdots+a_{n,n+1}, \tag{2.1}$$

where the coefficients, $a_{n,1},a_{n,2},\dots,a_{n,n+1}$ are uniquely determined by the two conditions

$$\frac{P_n(\xi,u_1,u_2,u_3)}{\sqrt{(\xi-u_1)(\xi-u_2)(\xi-u_3)}}=\xi^{n-1/2}+\mathcal{O}(\xi^{-3/2}) \quad \text{for large } |\xi| \tag{2.2}$$

and

$$\int_{u_3}^{u_2}\frac{P_n(\xi,u_1,u_2,u_3)}{\sqrt{(\xi-u_1)(\xi-u_2)(\xi-u_3)}}d\xi=0. \tag{2.3}$$

Here the sign of the square root is given by $\sqrt{(\xi-u_1)(\xi-u_2)(\xi-u_3)}>0$ for $\xi>u_1$ and the branch cuts are along $(-\infty,u_3)$ and (u_2,u_1).

In particular,

$$P_0(\xi,u_1,u_2,u_3)=\xi+a_{0,1}, \quad P_1(\xi,u_1,u_2,u_3)=\xi^2-\frac{1}{2}(u_1+u_2+u_3)\xi+a_{1,2}, \tag{2.4}$$

where

$$a_{0,1} = (u_1 - u_3)\frac{E(s)}{K(s)} - u_1,$$
$$a_{1,2} = \frac{1}{3}(u_1u_2 + u_1u_3 + u_2u_3) + \frac{1}{6}(u_1 + u_2 + u_3)a_{0,1}.$$

Here

$$s = \frac{u_2 - u_3}{u_1 - u_3}$$

and $K(s)$ and $E(s)$ are complete elliptic integrals of the first and second kind.

$K(s)$ and $E(s)$ have some well-known properties [8, 9]. They have the expansions

$$K(s) = \frac{\pi}{2}[1 + \frac{s}{4} + \frac{9}{64}s^2 + \cdots + (\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n})^2 s^n + \cdots], \tag{2.5}$$
$$E(s) = \frac{\pi}{2}[1 - \frac{s}{4} - \frac{3}{64}s^2 - \cdots - \frac{1}{2n-1}(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n})^2 s^n - \cdots], \tag{2.6}$$

for $|s| < 1$. They also have the asymptotics

$$K(s) \approx \frac{1}{2}\log\frac{16}{1-s}, \tag{2.7}$$
$$E(s) \approx 1 + \frac{1}{4}(1-s)[\log\frac{16}{1-s} - 1], \tag{2.8}$$

as s is close to 1. Furthermore,

$$\frac{dK(s)}{ds} = \frac{E(s) - (1-s)K(s)}{2s(1-s)}, \tag{2.9}$$
$$\frac{dE(s)}{ds} = \frac{E(s) - K(s)}{2s}. \tag{2.10}$$

It immediately follows from (2.5) and (2.6) that

$$\frac{1}{1-\frac{s}{2}} < \frac{K(s)}{E(s)} < \frac{1-\frac{s}{2}}{1-s} \qquad for\ 0 < s < 1. \tag{2.11}$$

The eigenspeeds of the Whitham equations (1.3) are defined in terms of P_0 and P_1 of (2.4),

$$\lambda_i(u_1,u_2,u_3) = 12\frac{P_1(u_i,u_1,u_2,u_3)}{P_0(u_i,u_1,u_2,u_3)}, \quad i = 1,2,3,$$

which give

$$\lambda_1(u_1,u_2,u_3) = 2(u_1 + u_2 + u_3) + 4(u_1 - u_2)\frac{K(s)}{E(s)},$$
$$\lambda_2(u_1,u_2,u_3) = 2(u_1 + u_2 + u_3) + 4(u_2 - u_1)\frac{sK(s)}{E(s) - (1-s)K(s)},$$
$$\lambda_3(u_1,u_2,u_3) = 2(u_1 + u_2 + u_3) + 4(u_2 - u_3)\frac{K(s)}{E(s) - K(s)}. \tag{2.12}$$

Using (2.11), we obtain

$$\lambda_1 - 2(u_1+u_2+u_3) > 0, \tag{2.13}$$
$$\lambda_2 - 2(u_1+u_2+u_3) < 0, \tag{2.14}$$
$$\lambda_3 - 2(u_1+u_2+u_3) < 0, \tag{2.15}$$

for $u_1 > u_2 > u_3$. In view of (2.5-2.8), we find that λ_1, λ_2 and λ_3 have behavior:

(1) At $u_2 = u_3$:

$$\begin{aligned} &\lambda_1(u_1,u_2,u_3) = 6u_1, \\ &\lambda_2(u_1,u_2,u_3) = \lambda_3(u_1,u_2,u_3) = 12u_3 - 6u_1. \end{aligned} \tag{2.16}$$

(2) At $u_1 = u_2$:

$$\begin{aligned} &\lambda_1(u_1,u_2,u_3) = \lambda_2(u_1,u_2,u_3) = 4u_1 + 2u_3, \\ &\lambda_3(u_1,u_2,u_3) = 6u_3. \end{aligned} \tag{2.17}$$

The eigenspeeds of the Whitham equations (1.6) are

$$\mu_i(u_1,u_2,u_3) = 80\frac{P_2(u_i,u_1,u_2,u_3)}{P_0(u_i,u_1,u_2,u_3)}, \quad i = 1,2,3. \tag{2.18}$$

They can be expressed in terms of λ_1, λ_2 and λ_3 of the KdV.

LEMMA 2.1. *[10] The eigenspeeds, μ_i's, satisfy:*

1.

$$\mu_i(u_1,u_2,u_3) = \frac{1}{2}[\lambda_i - 2(u_1+u_2+u_3)]\frac{\partial q}{\partial u_i}(u_1,u_2,u_3) + q(u_1,u_2,u_3), \tag{2.19}$$

where $q(u_1,u_2,u_3)$ is the solution of the boundary value problem of the Euler-Poisson-Darboux equations:

$$\begin{aligned} 2(u_i - u_j)\frac{\partial^2 q}{\partial u_i \partial u_j} &= \frac{\partial q}{\partial u_i} - \frac{\partial q}{\partial u_j}, \quad i,j = 1,2,3\ ; i \neq j, \\ q(u,u,u) &= 30u^2. \end{aligned} \tag{2.20}$$

2.

$$\frac{\partial \mu_i}{\partial u_j} = \frac{\frac{\partial \lambda_i}{\partial u_j}}{\lambda_i - \lambda_j}[\mu_i - \mu_j], \qquad i \neq j. \tag{2.21}$$

The solution of (2.20) is a symmetric quadratic function of u_1, u_2 and u_3

$$q(u_1,u_2,u_3) = 6(u_1^2 + u_2^2 + u_3^2) + 4(u_1u_2 + u_1u_3 + u_2u_3). \tag{2.22}$$

For KdV, λ_i's satisfy [8]

$$\frac{\partial \lambda_3}{\partial u_3} < \frac{3}{2}\frac{\lambda_2 - \lambda_3}{u_2 - u_3} < \frac{\partial \lambda_2}{\partial u_2}$$

for $u_3 < u_2 < u_1$. Similar results also hold for the fifth order KdV.

LEMMA 2.2.

$$\frac{\partial \mu_2}{\partial u_2} > \frac{3}{2}\,\frac{\mu_2-\mu_3}{u_2-u_3} \quad \text{if} \quad \frac{\partial q}{\partial u_2} > 0, \tag{2.23}$$

$$\frac{\partial \mu_3}{\partial u_3} < \frac{3}{2}\,\frac{\mu_2-\mu_3}{u_2-u_3} \quad \text{if} \quad \frac{\partial q}{\partial u_3} > 0, \tag{2.24}$$

for $u_3 < u_2 < u_1$.

Proof. We use (2.19) to calculate

$$\begin{aligned}\frac{\partial \mu_3}{\partial u_3} &= \frac{1}{2}\frac{\partial \lambda_3}{\partial u_3}\frac{\partial q}{\partial u_3} + \frac{1}{2}[\lambda_3 - 2(u_1+u_2+u_3)]\frac{\partial^2 q}{\partial u_3^2} \\ &< \frac{3}{4}\frac{\lambda_2-\lambda_3}{u_2-u_3}\frac{\partial q}{\partial u_3} + \frac{1}{2}[\lambda_3 - 2(u_1+u_2+u_3)]\frac{\partial^2 q}{\partial u_3^2},\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\mu_2 - \mu_3 &= \frac{1}{2}(\lambda_2-\lambda_3)\frac{\partial q}{\partial u_3} + \frac{1}{2}[\lambda_3 - 2(u_1+u_2+u_3)]\left(\frac{\partial q}{\partial u_2} - \frac{\partial q}{\partial u_3}\right) \\ &= \frac{1}{2}(\lambda_2-\lambda_3)\frac{\partial q}{\partial u_3} + \frac{1}{2}[\lambda_3 - 2(u_1+u_2+u_3)]2(u_2-u_3)\frac{\partial^2 q}{\partial u_2 \partial u_3} \\ &= \frac{2}{3}(u_2-u_3)\left(\frac{3}{4}\frac{\lambda_2-\lambda_3}{u_2-u_3}\frac{\partial q}{\partial u_3} + \frac{3}{2}[\lambda_3 - 2(u_1+u_2+u_3)]\frac{\partial^2 q}{\partial u_2 \partial u_3}\right),\end{aligned} \tag{2.26}$$

where we have used equation (2.20)

$$\frac{\partial q}{\partial u_2} - \frac{\partial q}{\partial u_3} = 2(u_2-u_3)\frac{\partial^2 q}{\partial u_2 \partial u_3}.$$

It follows from formula (2.22) for q that

$$3\frac{\partial^2 q}{\partial u_2 \partial u_3} = \frac{\partial^2 q}{\partial u_3^2},$$

which, along with with (2.25) and (2.26), proves (2.23).

Inequality (2.24) can be proved in the same way. ∎

The following calculations are useful in the subsequent sections.

Using formula (2.19) for μ_2 and μ_3 and formulae (2.12) for λ_2 and λ_3, we obtain

$$\mu_2(u_1,u_2,u_3) - \mu_3(u_1,u_2,u_3) = \frac{2(u_2-u_3)K}{(K-E)[E-(1-s)K]}M(u_1,u_2,u_3), \tag{2.27}$$

where

$$M(u_1,u_2,u_3) = [\frac{\partial q}{\partial u_3} + (1-s)\frac{\partial q}{\partial u_2}]E - (1-s)(\frac{\partial q}{\partial u_2} + \frac{\partial q}{\partial u_3})K.$$

We then use (2.9), (2.10) and (2.22) to calculate

$$\frac{\partial M(u_1,u_2,u_3)}{\partial u_2} = \frac{10(u_1-3u_2-u_3)}{u_1-u_3}\,(E-K). \tag{2.28}$$

We next consider

$$F(u_1,u_2,u_3):=\frac{\mu_2(u_1,u_2,u_3)-\mu_3(u_1,u_2,u_3)}{u_2-u_3}. \tag{2.29}$$

Using formula (2.19) for μ_2 and μ_3 and formulae (2.12) for λ_2 and λ_3, we obtain

$$\begin{aligned} F &= -2\frac{(1-s)K}{E-(1-s)K}\frac{\partial q}{\partial u_2}+2\frac{K}{K-E}\frac{\partial q}{\partial u_3} \\ &= -4\frac{s(1-s)K}{E-(1-s)K}(u_1-u_3)\frac{\partial^2 q}{\partial u_2\partial u_3}+2[\frac{K}{K-E}-\frac{(1-s)K}{E-(1-s)K}]\frac{\partial q}{\partial u_3}, \end{aligned}$$

where we have used equations (2.20) in the last equality. Finally, we use the expansions (2.5-2.6) for K and E to obtain

$$\begin{aligned} F(u_1,u_2,u_3) &= -4[(2-\frac{7}{4}s+\cdots)(u_1-u_3)\frac{\partial^2 q}{\partial u_2\partial u_3}+(-\frac{3}{4}+O(s^2))\frac{\partial q}{\partial u_3}] \\ &= -16[(2-\frac{7}{4}s+\cdots)(u_1-u_3)+(-\frac{3}{4}+O(s^2))(u_1+u_2+3u_3)], \end{aligned} \tag{2.30}$$

where we have used formula (2.22) for q in the last equality.

3. A Self-similar solution

In this section, we construct the self-similar solution of the Whitham equations (1.6) for the initial function (1.7) with $a=1$ and $b=0$. We will study all the other step-like initial data in Section 5.

THEOREM 3.1. *(see Figure 1.1) For the step-like initial data $u_0(x)$ of (1.7) with $a=1, b=0$, the solution of the Whitham equations (1.6) is given by*

$$u_1=1, \quad x=\mu_2(1,u_2,u_3)t, \quad x=\mu_3(1,u_2,u_3)t \tag{3.1}$$

for $-15t<x\le \alpha t$ and by

$$u_1=1, \quad x=\mu_2(1,u_2,0)t, \quad u_3=0 \tag{3.2}$$

for $\alpha t\le x<16t$, where $\alpha=\mu_2(1,u^*,0)$ and u^* is the unique solution u_2 of $\mu_2(1,u_2,0)=\mu_3(1,u_2,0)$ in the interval $0<u_2<1$. Outside the region $-15t<x<16t$, the solution of the Burgers type equation (1.5) is given by

$$u\equiv 1 \quad x\le -15t \tag{3.3}$$

and

$$u\equiv 0 \quad x\ge 16t. \tag{3.4}$$

The boundaries $x=-15t$ and $x=16t$ are called the trailing and leading edges, respectively. They separate the solutions of the Whitham equations and Burgers type equations. The Whitham solution matches the Burgers type solution in the following fashion (see Figure 1.1):

$$u_1=\text{the Burgers type solution defined outside the region}, \tag{3.5}$$

$$u_2=u_3, \tag{3.6}$$

at the trailing edge;

$$u_1 = u_2, \tag{3.7}$$

$$u_3 = \text{the Burgers type solution defined outside the region,} \tag{3.8}$$

at the leading edge.

The proof of Theorem 3.1 is based on a series of lemmas.

We first show that the solutions defined by formulae (3.1) and (3.2) indeed satisfy the Whitham equations (1.6) [1, 11].

Lemma 3.2.

1. *The functions u_1, u_2 and u_3 determined by equations (3.1) give a solution of the Whitham equations (1.6) as long as u_2 and u_3 can be solved from (3.1) as functions of x and t.*
2. *The functions u_1, u_2 and u_3 determined by equations (3.2) give a solution of the Whitham equations (1.6) as long as u_2 can be solved from (3.2) as a function of x and t.*

Proof. (1) u_1 obviously satisfies the first equation of (1.6). To verify the second and third equations, we observe that

$$\frac{\partial \mu_2}{\partial u_3} = \frac{\partial \mu_3}{\partial u_2} = 0 \tag{3.9}$$

on the solution of (3.1). To see this, we use (2.21) to calculate

$$\frac{\partial \mu_2}{\partial u_3} = \frac{\frac{\partial \lambda_2}{\partial u_3}}{\lambda_2 - \lambda_3} (\mu_2 - \mu_3) = 0.$$

The second part of (3.9) can be shown in the same way.

We then calculate the partial derivatives of the second equation of (3.1) with respect to x and t.

$$1 = \frac{\partial \mu_2}{\partial u_2} t u_{2x}, \quad 0 = \frac{\partial \mu_2}{\partial u_2} t u_{2t} + \mu_2,$$

which give the second equation of (1.6).

The third equation of (1.6) can be verified in the same way.

(2) The second part of Lemma 3.2 can easily be proved. □

We now determine the trailing edge. Eliminating x and t from the last two equations of (3.1) yields

$$\mu_2(1, u_2, u_3) - \mu_3(1, u_2, u_3) = 0. \tag{3.10}$$

Since it degenerates at $u_2 = u_3$, we replace (3.10) by

$$F(1, u_2, u_3) := \frac{\mu_2(1, u_2, u_3) - \mu_3(1, u_2, u_3)}{u_2 - u_3} = 0. \tag{3.11}$$

Here, the function F is also defined in (2.29).

Therefore, at the trailing edge where $u_2 = u_3$, i.e., $s = 0$, equation (3.11), in view of the expansion (2.30), becomes

$$2(1 - u_3) - \frac{3}{4}(1 + 4u_3) = 0,$$

which gives $u_2=u_3=1/4$.

LEMMA 3.3. *Equation (3.11) has a unique solution satisfying $u_2=u_3$. The solution is $u_2=u_3=1/4$. The rest of equations (3.1) at the trailing edge are $u_1=1$ and $x/t=\mu_2(1,1/4,1/4)=-15$.*

Having located the trailing edge, we now solve equations (3.1) in the neighborhood of the trailing edge. We first consider equation (3.11). We use (2.30) to differentiate F at the trailing edge $u_1=1$, $u_2=u_3=1/4$

$$\frac{\partial F(1,\frac{1}{4},\frac{1}{4})}{\partial u_2}=\frac{\partial F(1,\frac{1}{4},\frac{1}{4})}{\partial u_3}=40,$$

which show that equation (3.11) or equivalently (3.10) can be inverted to give u_3 as a decreasing function of u_2

$$u_3=A(u_2) \tag{3.12}$$

in a neighborhood of $u_2=u_3=1/4$.

We now extend the solution (3.12) of equation (3.10) in the region $1>u_2>1/4>u_3>0$ as far as possible. We deduce from Lemma 2.2 that

$$\frac{\partial \mu_2}{\partial u_2}>0, \quad \frac{\partial \mu_3}{\partial u_3}<0 \tag{3.13}$$

on the solution of (3.10). Because of (3.9) and (3.13), solution (3.12) of equation (3.10) can be extended as long as $1>u_2>1/4>u_3>0$.

There are two possibilities: (1) u_2 touches 1 before or simultaneously as u_3 reaches 0 and (2) u_3 touches 0 before u_2 reaches 1.

It follows from (2.17) and (2.19) that

$$\mu_2(1,1,u_3)>\mu_3(1,1,u_3) \quad \text{for } 0\le u_3<1.$$

This shows that (1) is impossible. Hence, u_3 will touch 0 before u_2 reaches 1. When this happens, equation (3.10) becomes

$$\mu_2(1,u_2,0)-\mu_3(1,u_2,0)=0. \tag{3.14}$$

LEMMA 3.4. *Equation (3.14) has a simple zero in the region $0<u_2<1$, counting multiplicities. Denoting the zero by u^*, then $\mu_2(1,u_2,0)-\mu_3(1,u_2,0)$ is positive for $u_2>u^*$ and negative for $u_2<u^*$.*

Proof. We now use (2.27) and (2.28) to prove the lemma. In equation (2.27), $K-E$ and $E-(1-s)K$ are all positive for $0<s<1$ in view of (2.11). By (2.28),

$$\frac{\partial M(1,u_2,0)}{\partial u_2}=10(3u_2-1)[K-E] \quad \text{for } 0<u_2<1.$$

Since $M(1,u_2,0)$ vanishes at $u_2=0$ and is positive at $u_2=1$ in view of (2.5-2.8), we conclude from the above derivative that $M(1,u_2,0)$ has a simple zero in $0<u_2<1$. This zero is exactly u^* and the rest of the theorem can be proved easily. ∎

Having solved equation (3.10) for u_3 as a decreasing function of u_2 for $1/4 \le u_2 \le u^*$, we turn to equations (3.1). Because of (3.13), the second equation of (3.1) gives u_2 as an increasing function of x/t, for $-15 \le x/t \le \alpha$, where

$$\alpha = \mu_2(1, u^*, 0). \tag{3.15}$$

Consequently, u_3 is a decreasing function of x/t in the same interval.

LEMMA 3.5. *The last two equations of (3.1) can be inverted to give u_2 and u_3 as increasing and decreasing functions, respectively, of the self-similarity variable x/t in the interval $-15 \le x/t \le \alpha$, where $\alpha = \mu_2(1, u^*, 0)$ and u^* is given in Lemma 3.4.*

We now turn to equations (3.2). We want to solve the second equation when $x/t > \alpha$ or equivalently when $u_2 > u^*$. According to Lemma 3.4, $\mu_2(1, u_2, 0) - \mu_3(1, u_2, 0) > 0$ for $u^* < u_2 < 1$, which, together with (2.23), shows that

$$\frac{\partial \mu_2(1, u_2, 0)}{\partial u_2} > 0.$$

Hence, the second equation of (3.2) can be solved for u_2 as an increasing function of x/t as long as $u^* < u_2 < 1$. When u_2 reaches 1, we have

$$x/t = \mu_2(1, 1, 0) = 16,$$

where we have used (2.17) and (2.19) in the last equality. We have therefore proved the following result.

LEMMA 3.6. *The second equation of (3.2) can be inverted to give u_2 as an increasing function of x/t in the interval $\alpha \le x/t \le 16$.*

We are ready to conclude the proof of Theorem 3.1.

The Burgers type solutions (3.3) and (3.4) are trivial.

According to Lemma 3.5, the last two equations of (3.1) determine u_2 and u_3 as functions of x/t in the region $-15 \le x/t \le \alpha$. By the first part of Lemma 3.2, the resulting u_1, u_2 and u_3 satisfy the Whitham equations (1.6). Furthermore, the boundary conditions (3.5) and (3.6) are satisfied at the trailing edge $x = -15t$.

Similarly, by Lemma 3.6, the second equation of (3.2) determines u_2 as a function of x/t in the region $\alpha \le x/t \le 16$. It then follows from the second part of Lemma 3.2 that u_1, u_2 and u_3 of (3.2) satisfy the Whitham equations (1.6). They also satisfy the boundary conditions (3.7) and (3.8) at the leading edge $x = 16t$.

We have therefore completed the proof of Theorem 3.1.

A graph of the Whitham solution is given in Figure 1.1. It is obtained by plotting the exact solutions of (3.1) and (3.2).

4. The Minimization problem

The zero dispersion limit of the solution of the fifth order KdV equation (1.4) with step-like initial function (1.7), $a = 1$, $b = 0$, is also determined by a minimization problem with constraints [4, 5, 12]

$$\underset{\{\psi \ge 0,\ \psi \in L^1\}}{\text{Minimize}} \{-\frac{1}{2\pi}\int_0^1 \int_0^1 \log\left|\frac{\eta-\mu}{\eta+\mu}\right| \psi(\eta)\psi(\mu) d\eta d\mu + \int_0^1 [\eta x - 16\eta^5 t]\psi(\eta) d\eta\}. \tag{4.1}$$

In this section, we will use the self-similar solution of Section 3 to construct the minimizer. We first define a linear operator

$$L\psi(\eta) = \frac{1}{2\pi}\int_0^1 log\left(\frac{\eta-\mu}{\eta+\mu}\right)^2 \psi(\mu) d\mu.$$

The variational conditions are

$$L\psi = x\eta - 16t\eta^5 \quad \text{where } \psi > 0, \tag{4.2}$$

$$L\psi \le x\eta - 16t\eta^5 \quad \text{where } \psi = 0. \tag{4.3}$$

The constraint for the minimization problem is

$$\psi \ge 0. \tag{4.4}$$

The minimizer of (4.1) is given explicitly:

THEOREM 4.1. *The minimizer of the variational problem (4.1) is as follows:*

1. *For* $x \le -15t$,

$$\psi(\eta) = \frac{-x\eta + 80t\eta(\eta^4 - \frac{1}{2}\eta^2 - \frac{1}{8})}{\sqrt{1-\eta^2}}.$$

2. *For* $-15t < x < \alpha t$,

$$\psi(\eta) = \begin{cases} -\frac{-x\eta P_0(\eta^2,1,u_2,u_3) + 80t\eta P_2(\eta^2,1,u_2,u_3)}{\sqrt{(1-\eta^2)(u_2-\eta^2)(u_3-\eta^2)}}, & \eta < \sqrt{u_3}, \\ 0, & \sqrt{u_3} < \eta < \sqrt{u_2}\,, \\ \frac{-x\eta P_0(\eta^2,1,u_2,0) + 80t\eta P_2(\eta^2,1,u_2,0)}{\sqrt{(1-\eta^2)(\eta^2-u_2)(\eta^2-u_3)}}, & \sqrt{u_2} < \eta < 1, \end{cases}$$

where P_0 *and* P_2 *are defined in (2.1) and* u_2 *and* u_3 *are determined by equations (3.1).*

3. *For* $\alpha t < x < 16t$,

$$\psi(\eta) = \begin{cases} 0, & \eta < \sqrt{u_2}, \\ \frac{-xP_0(\eta^2,1,u_2,0) + 80tP_2(\eta^2,1,u_2,0)}{\sqrt{(1-\eta^2)(\eta^2-u_2)}}, & \sqrt{u_2} < \eta < 1, \end{cases}$$

where u_2 *is determined by (3.2).*

4. *For* $x \ge 16t$,

$$\psi(\eta) \equiv 0.$$

Proof. We extend the function ψ defined on $[0,1]$ to the entire real line by setting $\psi(\eta) = 0$ for $\eta > 1$ and taking ψ to be odd. In this way, the operator L is connected to the Hilbert transform H on the real line [4]:

$$L\psi(\eta) = \int_0^\eta H\psi(\mu)d\mu \quad \text{where } H\psi(\eta) = \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{\psi(\mu)}{\eta - \mu} d\mu.$$

We verify case (4) first. Clearly, $\psi(\eta) = 0$ satisfies the constraints (4.4). We now check the variational conditions (4.2-4.3). Since $\psi = 0$,

$$L\psi = 0 \le x\eta - 16t\eta^5,$$

where the inequality follows from $x \ge 16t$ and $0 \le \eta \le 1$. Hence, variational conditions (4.2-4.3) are satisfied.

Next we consider case (1). We write $\psi(\eta)$ as the real part of $g_1(\eta)$ for real η, where

$$g_1=\sqrt{-1}(x-80t\eta^4)+\frac{\sqrt{-1}[-x\eta+80t\eta(\eta^4-\frac{1}{2}\eta^2-\frac{1}{8})]}{\sqrt{\eta^2-1}}.$$

The function g_1 is analytic in the upper half complex plane $Im(\eta)>0$ and $g_1(\eta)\approx O(1/\eta^2)$ for large $|\eta|$. Hence, $H\psi(\eta)=Im[g_1(\eta)]=x-80t\eta^4$ on $0\le\eta\le 1$, where H is the Hilbert transform [4]. We then have for $0\le\eta\le 1$

$$L\psi(\eta)=\int_0^\eta H\eta(\mu)d\mu=x\eta-16t\eta^5,$$

which shows that the variational conditions are satisfied. Since $15+80(\eta^4-\frac{1}{2}\eta^2-\frac{1}{8})=80(\eta^2-\frac{1}{4})^2\ge 0$, it follows from $x\le -15t$ that $\psi\ge 0$. Hence, the constraint (4.4) is verified.

We now turn to case (2). By Lemma 3.5, the last two equations of (3.1) determine u_2 and u_3 as functions of the self-similarity variable x/t in the interval $-15\le x/t\le\alpha$.

We write $\psi=Re(g_2)$ for real η, where

$$g_2=\sqrt{-1}(x-80t\eta^4)+\frac{\sqrt{-1}[-x\eta P_0(\eta^2,1,u_2,u_3)+80t\eta P_2(\eta^2,1,u_2,u_3)]}{\sqrt{(\eta^2-1)(\eta^2-u_2)(\eta^2-u_3)}}.$$

The function g_2 is analytic in $Im(\eta)>0$ and $g_2(\eta)\approx O(1/\eta^2)$ for large $|\eta|$ in view of the asymptotics (2.2) for P_0 and P_2. Hence, taking the imaginary part of g_2 yields

$$H\psi(\eta)=\begin{cases} x-80t\eta^4, & 0<\eta<\sqrt{u_3},\\ x-80t\eta^4-\frac{[-xP_0(\eta^2,1,u_2,0)+80tP_2(\eta^2,1,u_2,0)]\eta}{\sqrt{(1-\eta^2)(u_2-\eta^2)(\eta^2-u_3)}}, & \sqrt{u_3}<\eta<\sqrt{u_2},\\ x-80t\eta^4, & \sqrt{u_2}<\eta<1.\end{cases}$$

We then have

$$L\psi(\eta)=\begin{cases} x\eta-16t\eta^5, & 0<\eta<\sqrt{u_3},\\ x\eta-16t\eta^5-\int_{\sqrt{u_3}}^{\eta}\frac{[-xP_0+80tP_2]\mu}{\sqrt{(1-\mu^2)(u_2-\mu^2)(\mu^2-u_3)}}d\mu, & \sqrt{u_3}<\eta<\sqrt{u_2},\\ x\eta-16t\eta^5, & \sqrt{u_2}<\eta<1.\end{cases}$$

where we have used

$$\int_{\sqrt{u_3}}^{\sqrt{u_2}}\frac{[-xP_0+80tP_2]\mu}{\sqrt{(1-\mu^2)(u_2-\mu^2)(\mu^2-u_3)}}d\mu=0, \tag{4.5}$$

which is a consequence of (2.3) for P_0 and P_2.

We study the zeros of $-xP_0+80tP_2$. It has two zeros at $\eta=\sqrt{u_2}$ and $\eta=\sqrt{u_3}$. This follows from (2.18) and (3.1). It also has a zero between $\sqrt{u_2}$ and $\sqrt{u_3}$ because of (4.5). Since it is a cubic polynomial of η^2, $-xP_0+80tP_2$ has no more than three zeros on the positive η axis and furthermore these three positive zeros are simple.

Since the leading term in $-xP_0+80tP_2$ is $80t\eta^6$, the polynomial is positive for $\eta>\sqrt{u_2}$ and negative for $0\le\eta<\sqrt{u_3}$. This proves $\psi\ge 0$; so (4.4) is verified. Since $-xP_0+80tP_2$ changes sign at each simple zero, it follows from (4.5) that

$$\int_{\sqrt{u_3}}^{\eta}\frac{[-xP_0+80tP_2]\mu}{\sqrt{(1-\mu^2)(u_2-\mu^2)(\mu^2-u_3)}}d\mu>0$$

for $\sqrt{u_3} < \eta < \sqrt{u_2}$. This verifies the variational conditions (4.2) and (4.3).

We finally consider case (3). By Lemma 3.6, the second equation of (3.2) determines u_2 as an increasing function of x/t in the interval $\alpha \le x/t \le 16$.

We write $\psi = Re(g_3)$ for real η, where

$$g_3 = \sqrt{-1}(x - 80t\eta^4) + \frac{\sqrt{-1}[-xP_0(\eta^2,1,u_2,0) + 80tP_2(\eta^2,1,u_2,0)]}{\sqrt{(\eta^2-1)(\eta^2-u_2)}}.$$

The function g_3 is analytic in $Im(\eta) > 0$ and $g_3(\eta) \approx O(1/\eta^2)$ for large $|\eta|$ in view of the asymptotics (2.2) for P_0 and P_2. Hence, taking the imaginary part of g_3 yields

$$H\psi(\eta) = \begin{cases} x - 80t\eta^4 - \frac{-xP_0(\eta^2,1,u_2,0)+80tP_2(\eta^2,1,u_2,0)}{\sqrt{(1-\eta^2)(u_2-\eta^2)}}, & 0 < \eta < \sqrt{u_2}, \\ x - 80t\eta^4, & \sqrt{u_2} < \eta < 1. \end{cases}$$

We then have

$$L\psi(\eta) = \begin{cases} x\eta - 16t\eta^5 - \int_0^\eta \frac{-xP_0+80tP_2}{\sqrt{(1-\mu^2)(u_2-\mu^2)}} d\mu, & 0 < \eta < \sqrt{u_2}, \\ x\eta - 16t\eta^5, & \sqrt{u_2} < \eta < 1. \end{cases}$$

where we have used

$$\int_0^{\sqrt{u_2}} \frac{-xP_0(\mu^2,1,u_2,0) + 80tP_2(\mu^2,1,u_2,0)}{\sqrt{(1-\mu^2)(u_2-\mu^2)}} d\mu = 0, \tag{4.6}$$

which is a consequence of (2.3) for P_0 and P_2.

The function $-xP_0(\eta^2,1,u_2,0) + 80tP_2(\eta^2,1,u_2,0)$ has two zeros on the positive η-axis. One is at $\eta = \sqrt{u_2}$, in view of (2.18) and (3.2). The other is between 0 and $\sqrt{u_2}$, in view of (4.6). At $\eta = 0$, the function has a positive value. To see this,

$$-xP_0(0,1,u_2,0) + 80tP_2(0,1,u_2,0) = P_0(0,1,u_2,0)[-x + t\mu_3(1,u_2,0)]. \tag{4.7}$$

According to Lemma 3.4, $\mu_2(1,u_2,0) > \mu_3(1,u_2,0)$ when $u_2 > u^*$ or equivalently when $\alpha < x/t < 16$. It follows from formula (2.4) and inequality (2.11) that $P_0(0,1,u_2,0) < 0$. Hence, the right hand side of (4.7) is bigger than

$$P_0(0,1,u_2,0)[-x + t\mu_2(1,u_2,0)] = 0,$$

where the equality comes from (3.2). Since it is a cubic polynomial in η^2 and since it is positive for large $\eta > 0$, the function $-xP_0(\eta^2,1,u_2,0) + 80tP_2(\eta^2,1,u_2,0)$ can have at most two zeros on the positive η-axis. Hence, the above two zeros are all simple zeros.

It now becomes straight forward to check the variational conditions (4.2-4.3) and the constraint (4.4), just as we do in case (2). □

5. Other step like initial data

In this section, we will classify all types of step-like initial data (1.7) for equation (1.4). When $a = 0$, since $b \ne 0$, the solution of (1.5) will never develop a shock. We therefore study the cases $a > 0$ and $a < 0$. In the former case, it is easy to check that, when $b > a$, the solution of equation (1.5) will never develop a shock; accordingly, we will restrict to $b < a$. Similarly, in the latter case, we will confine ourselves to $b > a$.

We will only present our proofs briefly, since they are, more or less, similar to those in Section 3.

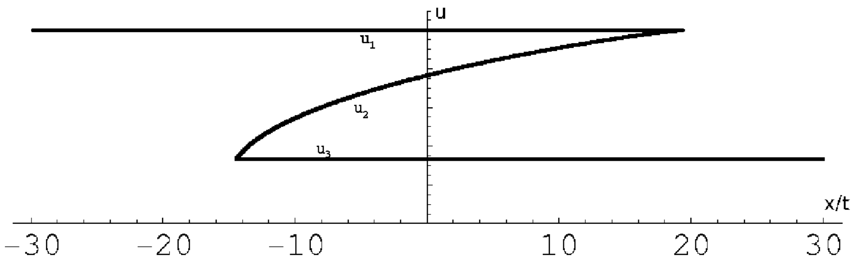

FIG. 5.1. *Self-similar solution of the Whitham equations for $a=1$ and $b=1/3$ of type I.*

5.1. Type I: $a>0$, $a/4\leq b<a$.

THEOREM 5.1. *(see Figure 5.1) For the step-like initial data (1.7) with $a>0$, $a/4\leq b<a$, the solution of the Whitham equations (1.6) is given by*

$$u_1=a, \quad x=\mu_2(a,u_2,b)t, \quad u_3=b$$

for $\mu_2(a,b,b)<x/t<\mu_2(a,a,b)$, where $\mu_2(a,b,b)=-10a^2-40ab+80b^2$ and $\mu_2(a,a,b)=16a^2+8ab+6b^2$. Outside this interval, the solution of (1.5) is given by

$$u\equiv a \quad x/t\leq\mu_2(a,b,b)$$

and

$$u\equiv b \quad x/t\geq\mu_2(a,a,b).$$

Proof. It suffices to show that $\mu_2(a,u_2,b)$ is an increasing function of u_2 for $b<u_2<a$. By (2.28), we have

$$\frac{dM(a,u_2,b)}{du_2}=\frac{10(3u_2+b-a)}{a-b}[K-E]>0$$

for $b<u_2<a$, where we have used $a/4\leq b<a$ in the inequality. Since $M(a,u_2,b)=0$ at $u_2=b$, this implies that $M(a,u_2,b)>0$ for $b<u_2<a$. It then follows from (2.27) that $\mu_2(a,u_2,b)-\mu_3(a,u_2,b)>0$. By Lemma 2.2, we conclude that

$$\frac{d\mu_2(a,u_2,b)}{du_2}>0$$

for $b<u_2<a$. ∎

5.2. Type II: $a>0$, $-2a/3<b<a/4$. Theorem 3.1 is a special case of the following theorem.

THEOREM 5.2. *(see Figure 1.1) For the step-like initial data (1.7) with $a>0$, $-2a/3<b<a/4$, the solution of the Whitham equations (1.6) is given by*

$$u_1=a, \quad x=\mu_2(a,u_2,u_3)t, \quad x=\mu_3(a,u_2,u_3)t$$

for $-15a^2 < x/t \le \mu_2(a,u^{**},b)$ *and by*

$$u_1 = a, \quad x = \mu_2(a,u_2,b)t, \quad u_3 = b$$

for $\mu_2(a,u^{**},b) \le x/t < 16a^2+8ab+6b^2$, *where* u^{**} *is the unique solution* u_2 *of* $\mu_2(a,u_2,b) = \mu_3(a,u_2,b)$ *in the interval* $b < u_2 < a$. *Outside the region* $-15a^2 < x/t < 16a^2+8ab+6b^2$, *the solution of the Burgers type equation (1.5) is given by*

$$u \equiv a \qquad x/t \le -15a^2$$

and

$$u \equiv b \qquad x/t \ge 16a^2+8ab+6b^2.$$

Proof. The trailing edge is determined by

$$F(a,u_2,u_3) = 0 \tag{5.1}$$

when $u_2 = u_3$. Here F is given by (2.29). In view of the expansion (2.30), the above equation when $u_2 = u_3$, i.e., $s = 0$, reduces to

$$2(a-u_3) - \frac{3}{4}(a+4u_3) = 0,$$

which gives $u_2 = u_3 = a/4$ at the trailing edge.

Having located the trailing edge, we solve equation (5.1) in the neighborhood of $u_2 = u_3 = a/4$. We use the expansion (2.30) to calculate

$$\frac{\partial F(a,\frac{a}{4},\frac{a}{4})}{\partial u_2} = \frac{\partial F(a,\frac{a}{4},\frac{a}{4})}{\partial u_3} = 40,$$

which implies that equation (5.1) can be solved for u_3 as a decreasing function of u_2 near $u_2 = u_3 = a/4$.

The solution of

$$\mu_2(a,u_2,u_3) - \mu_3(a,u_2,u_3) = 0 \tag{5.2}$$

can be extended as long as $a > u_2 > a/4 > u_3 > b$. To see this, we need to show that

$$\frac{\partial \mu_2(a,u_2,u_3)}{\partial u_3} = 0, \quad \frac{\partial \mu_3(a,u_2,u_3)}{\partial u_2} = 0, \quad \frac{\partial \mu_2(a,u_2,u_3)}{\partial u_2} > 0, \quad \frac{\partial \mu_3(a,u_2,u_3)}{\partial u_3} < 0$$

on the solution of (5.2). The proof of the equalities is the same as that of (3.9) in Section 3. To prove the inequalities, in view of Lemma 2.2, it is enough to show that

$$\frac{\partial q(a,u_2,u_3)}{\partial u_2} > 0, \quad \frac{\partial q(a,u_2,u_3)}{\partial u_3} > 0.$$

We use formulae (2.19) to rewrite equation (5.2) as

$$\frac{1}{2}[\lambda_2 - 2(a+u_2+u_3)]\frac{\partial q(a,u_2,u_3)}{\partial u_2} = \frac{1}{2}[\lambda_3 - 2(a+u_2+u_3)]\frac{\partial q(a,u_2,u_3)}{\partial u_3},$$

which, together with inequalities (2.14) and (2.15), proves that $\frac{\partial q}{\partial u_2}$ and $\frac{\partial q}{\partial u_3}$ have the same sign on the solution of (5.2). On the other hand, we calculate from (2.22)

$$\frac{\partial q(a,u_2,u_3)}{\partial u_2} = 4(a+3u_2+u_3) > 0$$

for $a > u_2 > a/4 > u_3 > b > -2a/3$.

We now extend the solution of (5.2) as far as possible in the region $a > u_2 > a/4 > u_3 > b$. There are two possibilities: (1) u_2 touches a before or simultaneously as u_3 reaches b and (2) u_3 touches b before u_2 reaches a.

Possibility (1) is impossible. To see this, we use (2.17) and (2.19) to calculate

$$\mu_2(a,a,u_3) - \mu_3(a,a,u_3) = 2(a-u_3)\frac{\partial q(a,a,u_3)}{\partial u_3} = 8(a-u_3)(2a+3u_3), \qquad (5.3)$$

which, in view of $b > -2a/3$, is positive for $b \le u_3 < a$.

Therefore, u_3 will touch b before u_2 reaches a. When this happens, we have $\mu_2(a,u_2,b) - \mu_3(a,u_2,b) = 0$. In the same way as we prove Lemma 3.4, we can show that this equation has a unique solution u_2 in the interval $b < u_2 < a$.

The rest of the proof is similar to that of Theorem 3.1. □

5.3. Type III: $a > 0$, $b = -2a/3$.

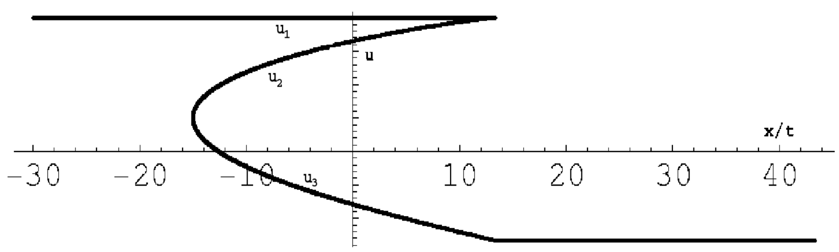

FIG. 5.2. *Self-similar solution of the Whitham equations for $a=1$ and $b=-2/3$ of type III.*

THEOREM 5.3. *(see Figure 5.2) For the step-like initial data (1.7) with $a > 0$, $b = -2a/3$, the solution of the Whitham equations (1.6) is given by*

$$u_1 = a, \quad x = \mu_2(a,u_2,u_3)t, \quad x = \mu_3(a,u_2,u_3)t$$

for $-15a^2 < x/t < 40a^2/3$. Outside the region, the solution of the Burgers type equation (1.5) is given by

$$u \equiv a \quad x/t \le -15a^2$$

and

$$u \equiv b \quad x/t \ge 40a^2/3.$$

Proof. It suffices to show that u_2 and u_3 of $\mu_2(a,u_2,u_3)-\mu_3(a,u_2,u_3)=0$ reaches a and $b=-2a/3$, respectively, simultaneously. To see this, we deduce from equation (5.3) that

$$\mu_2(a,a,-2a/3)-\mu_3(a,a,-2a/3)=8(a-2a/3)[2a+3(-2a/3)]=0. \tag{5.4}$$

□

5.4. Type IV: $a>0$**,** $b<-2a/3$**.**

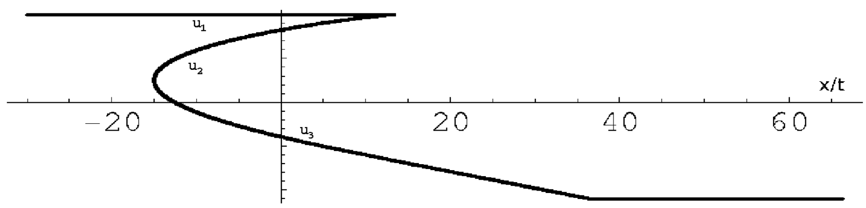

FIG. 5.3. *Self-similar solution of the Whitham equations for $a=1$ and $b=-1.1$ of type IV.*

THEOREM 5.4. *(see Figure 5.3) For the step-like initial data (1.7) with $a>0$, $b<-2a/3$, the solution of the Whitham equations (1.6) is given by*

$$u_1=a, \quad x=\mu_2(a,u_2,u_3)t, \quad x=\mu_3(a,u_2,u_3)t$$

for $-15a^2<x/t<40a^2/3$. Outside the region, the solution of the Burgers type equation (1.5) is given by

$$u\equiv a \quad x/t\le -15a^2$$

and

$$u=\begin{cases} -\sqrt{\frac{x}{30t}} & 40a^2/3\le x/t\le 30b^2 \\ b & x/t\ge 30b^2 \end{cases}.$$

Proof. By the calculation (5.4), when u_2 of $\mu_2(a,u_2,u_3)-\mu_3(a,u_2,u_3)=0$ touches a, the corresponding u_3 reaches $-2a/3$, which is above b. Hence, equations

$$x=\mu_2(a,u_2,u_3)t, \quad x=\mu_3(a,u_2,u_3)t$$

can be inverted to give u_2 and u_3 as functions of x/t in the region $\mu_2(a,a/4,a/4)<x/t\le\mu_2(a,a,-2a/3)$. To the right of this region, the Burgers type equation (1.5) has a rarefaction wave solution. □

5.5. Type V: $a<0$**,** $b\le -a/4$**.**

THEOREM 5.5. *(see Figure 5.4) For the step-like initial data (1.7) with $a<0$, $a<b\le -a/4$, the solution of the Whitham equations (1.6) is given by*

$$u_1=b, \quad x=\mu_2(b,u_2,a)t, \quad u_3=a$$

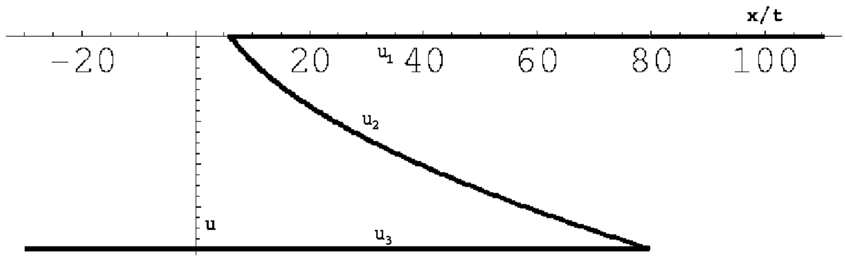

FIG. 5.4. *Self-Similar solution of the Whitham equations for $a=-1$ and $b=0$ of type V.*

for $\mu_2(b,b,a)<x/t<\mu_2(b,a,a)$, *where* $\mu_2(b,b,a)=6a^2+8ab+16b^2$ *and* $\mu_2(b,a,a)=80a^2-40ab-10b^2$. *Outside this interval, the solution of (1.5) is given by*

$$u\equiv a \quad x/t\le\mu_2(b,b,a)$$

and

$$u\equiv b \quad x/t\ge\mu_2(b,a,a).$$

Proof. It suffices to show that $\mu_2(a,u_2,b)$ is a decreasing function of u_2 for $a<u_2<b$. By (2.19), we have

$$\frac{\partial\mu_2(b,u_2,a)}{\partial u_2}=\frac{1}{2}\frac{\partial\lambda_2}{\partial u_2}\frac{\partial q}{\partial u_2}+\frac{1}{2}[\lambda_2-2(b+u_2+a)]\frac{\partial^2 q}{\partial u_2^2}.$$

The second term is negative because of (2.14) and $\frac{\partial^2 q}{\partial u_2^2}=12>0$. The first term is also negative. Its first factor is positive in view of (1.10). The second factor

$$\frac{\partial q}{\partial u_2}=4(b+3u_2+a)<0$$

for $a<u_2<b$ because of $b\le -a/4$. □

5.6. Type VI: $a<0$, $-a/4<b<-2a$.

THEOREM 5.6. *(see Figure 5.5) For the step-like initial data (1.7) with* $a<0$, $-a/4<b<-2a$, *the solution of the Whitham equations (1.6) is given by*

$$x=\mu_1(u_1,u_2,a)t,\quad x=\mu_2(u_1,u_2,a)t,\quad u_3=a \tag{5.5}$$

for $5a^2<x/t\le\mu_2(b,u^{***},a)$ *and by*

$$u_1=b,\quad x=\mu_2(b,u_2,a)t,\quad u_3=a \tag{5.6}$$

for $\mu_2(b,u^{***},a)\le x/t<80a^2-40ab-10b^2$, *where* u^{***} *is the unique solution* u_2 *of* $\mu_1(b,u_2,a)=\mu_2(b,u_2,a)$ *in the interval* $a<u_2<b$. *Outside the region* $5a^2<x/t<80a^2-40ab-10b^2$, *the solution of the Burgers type equation (1.5) is given by*

$$u\equiv a \quad x/t\le 5a^2$$

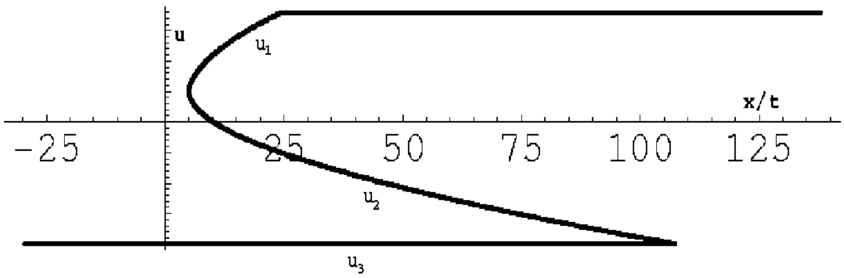

FIG. 5.5. *Self-Similar solution of the Whitham equations for $a=-1$ and $b=1.2$ of type VI.*

and

$$u \equiv b \qquad x/t \geq 80a^2 - 40ab - 10b^2.$$

Proof. We first locate the "leading" edge, i.e., the solution of equation (5.5) at $u_1 = u_2$. Eliminating x/t from the first two equations of (5.5) yields

$$\mu_1(u_1,u_2,a) - \mu_2(u_1,u_2,a) = 0. \tag{5.7}$$

Since it degenerates at $u_1 = u_2$, we replace (5.7) by

$$G(u_1,u_2,a) := \frac{\mu_1(u_1,u_2,a) - \mu_2(u_1,u_2,a)}{(u_1-u_2)K(s)} = 0. \tag{5.8}$$

Using formulae (2.19) for μ_1 and μ_2 and formulae (2.12) for λ_1 and λ_2, we write

$$G(u_1,u_2,a) = \frac{2}{E[E-(1-s)K]}\{(\frac{\partial q}{\partial u_1} + s\frac{\partial q}{\partial u_2})E - (1-s)\frac{\partial q}{\partial u_1}K\}.$$

In view of (2.7) and (2.8), equation (5.8) reduces to

$$\frac{\partial q(u_1,u_2,a)}{\partial u_1} + \frac{\partial q(u_1,u_2,a)}{\partial u_2} = 0$$

at the "leading" edge $u_1 = u_2$. This gives

$$u_1 = u_2 = -\frac{a}{4}.$$

Having located the "leading" edge, we solve equation (5.8) near $u_1 = u_2 = -a/4$. We calculate

$$\frac{\partial G(-a/4,-a/4,a)}{\partial u_1} = \frac{\partial G(-a/4,-a/4,a)}{\partial u_2} = 32.$$

These show that equation (5.8) gives u_1 as a decreasing function of u_2

$$u_1 = B(u_2) \tag{5.9}$$

in a neighborhood of $u_1 = u_2 = -a/4$.

We now extend the solution (5.9) of equation (5.7) as far as possible in the region $a < u_2 < -a/4 < u_1 < b$. We use formula (2.19) to calculate

$$\frac{\partial \mu_1}{\partial u_1} = \frac{1}{2}\frac{\partial \lambda_1}{\partial u_1}\frac{\partial q}{\partial u_1} + \frac{1}{2}[\lambda_1 - 2(u_1+u_2+a)]\frac{\partial^2 q}{\partial u_1^2},$$
$$\frac{\partial \mu_2}{\partial u_2} = \frac{1}{2}\frac{\partial \lambda_2}{\partial u_2}\frac{\partial q}{\partial u_2} + \frac{1}{2}[\lambda_2 - 2(u_1+u_2+a)]\frac{\partial^2 q}{\partial u_2^2}.$$

In view of (1.10), (2.13) and (2.14), we have

$$\frac{\partial \mu_1}{\partial u_1} > 0 \quad \text{if} \quad \frac{\partial q}{\partial u_1} > 0,$$
$$\frac{\partial \mu_2}{\partial u_2} < 0 \quad \text{if} \quad \frac{\partial q}{\partial u_2} < 0.$$

We claim that

$$\frac{\partial q}{\partial u_1} > 0, \quad \frac{\partial q}{\partial u_2} < 0 \tag{5.10}$$

on the solution of (5.7) in the region $a < u_2 < -a/4 < u_1 < b$. To see this, we use formula (2.19) to rewrite equation (5.7) as

$$\frac{1}{2}[\lambda_1 - 2(u_1+u_2+a)]\frac{\partial q}{\partial u_1} = \frac{1}{2}[\lambda_2 - 2(u_1+u_2+a)]\frac{\partial q}{\partial u_2}.$$

This, together with

$$\frac{\partial q}{\partial u_1} - \frac{\partial q}{\partial u_2} = 2(u_1 - u_2)\frac{\partial^2 q}{\partial u_1 \partial u_2} = 8(u_1 - u_2) > 0$$

for $u_1 > u_2$, and inequalities (2.13) and (2.14), proves (5.10).

Hence, the solution (5.9) can be extended as long as $a < u_2 < -a/4 < u_1 < b$.

There are two possibilities: (1) u_1 touches b before u_2 reaches a and (2) u_2 touches a before or simultaneously as u_1 reaches a.

Possibility (2) is impossible. To see this, we use (2.16), (2.19) and (2.22) to calculate

$$\mu_1(u_1,a,a) - \mu_2(u_1,a,a) = 40(u_1 - a)(u_1 + 2a), \tag{5.11}$$

which is negative for $-a/4 < u_1 \le b < -2a$.

Therefore, u_1 will touch b before u_2 reaches a. When this happens, we have

$$\mu_1(b,u_2,a) - \mu_2(b,u_2,a) = 0. \tag{5.12}$$

Lemma 5.7. *Equation (5.12) has a simple zero, counting multiplicities, in the interval* $a < u_2 < b$. *Denoting this zero by* u^{***}, *then* $\mu_1(b,u_2,a) - \mu_2(b,u_2,a)$ *is positive for* $u_2 > u^{***}$ *and negative for* $u_2 < u^{***}$.

Proof. We write

$$\mu_1(b,u_2,a) - \mu_2(b,u_2,a) = \frac{2(b-u_2)K}{E[E-(1-s)K]}\{(\frac{\partial q}{\partial u_1} + s\frac{\partial q}{\partial u_2})E - (1-s)\frac{\partial q}{\partial u_1}K\}. \tag{5.13}$$

Denote the parenthesis of (5.13) by $N(b,u_2,a)$. Since $E-(1-s)K>0$ for $a<u_2<b$, the left hand side has a zero iff $N(b,u_2,a)$ on the right has one.

We now calculate

$$\frac{\partial N(b,u_2,a)}{\partial u_2}=\frac{30E(s)}{b-a}[u_2-\frac{a-b}{3}].$$

Since $N(b,u_2,a)$ is zero at $u_2=a$ and positive at $u_2=b$, we conclude from the above derivative that $N(b,u_2,a)$ has a simple zero in $a<u_2<b$. □

We now continue to prove Theorem 5.6. Having solved equation (5.7) for u_1 as a decreasing function of u_2 for $u^{***}<u_2<-a/4$, we can then use the last two equations of (5.5) to determine u_1 and u_2 as functions of x/t in the interval $\mu_2(-a/4,-a/4,a)< x/t<\mu_2(b,u^{***},a)$.

We finally turn to equations (5.6). We want to solve the second equation of (5.6), $x/t=\mu_2(b,u_2,a)$, for $u_2<u^{***}$. It is enough to show that $\mu_2(b,u_2,a)$ is a decreasing function of u_2 for $u_2<u^{***}$.

According to Lemma 5.7, $\mu_1(b,u_2,a)-\mu_2(b,u_2,a)<0$ for $u_2<u^{***}$. Using formula (2.19) for μ_1 and μ_2, we have

$$\frac{1}{2}[\lambda_1-2(b+u_2+a)]\frac{\partial q}{\partial u_1}<\frac{1}{2}[\lambda_2-2(b+u_2+a)]\frac{\partial q}{\partial u_2}.$$

This, together with

$$\frac{\partial q}{\partial u_1}-\frac{\partial q}{\partial u_2}=2(b-u_2)\frac{\partial^2 q}{\partial u_1\partial u_2}=8(b-u_2)>0$$

for $u_1>u_2$, and inequalities (2.13) and (2.14), proves

$$\frac{\partial q(b,u_2,a)}{\partial u_2}<0$$

for $u_2<u^{***}$. Hence,

$$\frac{\partial \mu_2}{\partial u_2}=\frac{1}{2}\frac{\partial\lambda_2}{\partial u_2}\frac{\partial q}{\partial u_2}+\frac{1}{2}[\lambda_2-2(b+u_2+a)]\frac{\partial^2 q}{\partial u_2^2}<0.$$

□

5.7. Type VII: $a<0$, $b=-2a$.

THEOREM 5.8. *(see Figure 5.6) For the step-like initial data (1.7) with $a<0$, $b=-2a$, the solution of the Whitham equations (1.6) is given by*

$$x=\mu_1(u_1,u_2,a)t,\quad x=\mu_2(u_1,u_2,a)t,\quad u_3=a$$

for $5a^2<x/t<120a^2$. Outside the region, the solution of the Burgers type equation (1.5) is given by

$$u\equiv a\quad x/t\le 5a^2$$

and

$$u\equiv b\quad x/t\ge 120a^2.$$

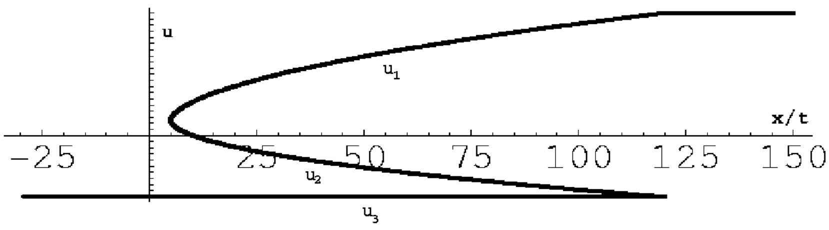

FIG. 5.6. *Self-Similar solution of the Whitham equations for $a=-1$ and $b=2$ of type VII.*

Proof. It suffices to show that u_1 and u_2 of $\mu_1(u_1,u_2,a)-\mu_2(u_1,u_2,a)=0$ reaches $b=-2a$ and a, respectively, simultaneously. To see this, we deduce from equation (5.11) that

$$\mu_1(u_1,a,a)-\mu_3(u_1,a,a)=8(u_1-a)(u_1+2a) \tag{5.14}$$

is negative for $u_1<b$ and vanish when $u_1=b=-2a$. □

5.8. Type VIII: $a<0$, $b>-2a$.

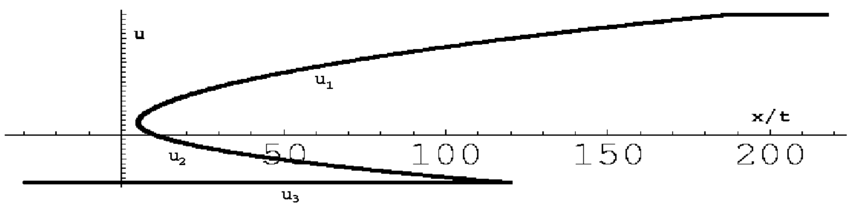

FIG. 5.7. *Self-Similar solution of the Whitham equations for $a=-1$ and $b=2.5$ of type VIII.*

THEOREM 5.9. *(see Figure 5.7) For the step-like initial data (1.7) with $a<0$, $b>-2a$, the solution of the Whitham equations (1.6) is given by*

$$x=\mu_1(u_1,u_2,a)t, \quad x=\mu_2(u_1,u_2,a)t, \quad u_3=a$$

for $5a^2<x/t<120a^2$. Outside the region, the solution of the Burgers type equation (1.5) is given by

$$u\equiv a, \quad x/t\le 5a^2,$$

and

$$u=\begin{cases}\sqrt{\frac{x}{30t}}, & 120a^2\le x/t\le 30b^2,\\ b, & x/t\ge 30b^2.\end{cases}$$

Proof. By the calculation (5.14), when u_2 of $\mu_2(u_1,u_2,a)-\mu_3(u_1,u_2,a)=0$ touches a, the corresponding u_3 reaches $-2a$, which is below b. Hence, equations

$$x=\mu_2(a,u_2,u_3)t, \quad x=\mu_3(a,u_2,u_3)t$$

can be inverted to give u_2 and u_3 as functions of x/t in the region $\mu_2(-a/4,-a/4,a)< x/t<\mu_2(-2a,-2a,a)$. To the right of this region, the Burgers type equation (1.5) has a rarefaction wave solution. □

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