KINETIC MODELS OF OPINION FORMATION[*]

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Abstract. We introduce and discuss certain kinetic models of (continuous) opinion formation involving both exchange of opinion between individual agents and diffusion of information. We show conditions which ensure that the kinetic model reaches non trivial stationary states in case of lack of diffusion in correspondence of some opinion point. Analytical results are then obtained by considering a suitable asymptotic limit of the model yielding a Fokker-Planck equation for the distribution of opinion among individuals.

Key words. Sociophysics, Boltzmann equation, opinion formation

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1. Introduction

Microscopic models of both social and political phenomena describing collective behaviors and self–organization in a society have been recently introduced and analyzed by several authors [11, 15, 18, 24, 25, 27, 29]. The leading idea is that collective behaviors of a society composed by a sufficiently large number of individuals (agents) can be hopefully described using the laws of statistical mechanics as it happens in a physical system composed of many interacting particles. The details of the social interactions between agents then characterize the emerging statistical phenomena.

Among others, the modeling of opinion formation attracted the interest of a increasing number of researchers (cfr. [8, 15, 18, 27] and the references therein). The starting point of a large part of these models, however, is represented by a cellular automata, where the lattice points are the agents, and where any of the agents of a community is initially associated with a random distribution of numbers, one of which is the opinion. Hence society is modeled as a graph, where each agent interacts with his neighborhoods in iterative way.

Very recently, other attempts have been successfully applied [1, 22], with the aim to describe formation of opinion by means of mean fields model equations. These models are in general described by systems of ordinary differential equations or partial differential equations of diffusive type, that can in some case be treated analytically to give explicit steady states. In [1], attention has been focused on two aspects of opinion formation, which in principle could be responsible for the formation of coherent structures. The first one is the remarkably simple compromise process, in which pairs of agents reach a fair compromise after exchanging opinions [2, 3, 8, 10, 12, 26, 30].

The second one is the diffusion process, which allows individual agents to change their opinions in a random diffusive fashion. While the compromise process has its basis on the human tendency to settle conflicts, diffusion accounts for the possibility that people may change opinion through a global access to information. In the present time, this aspect is gaining in importance due to the emerging of new possibilities (among them electronic mail and web navigation [20]).

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Following this line of thought, we consider here a class of kinetic models of opinion formation, based on two-body interactions involving both compromise and diffusion properties in exchanges between individuals. Compromise and diffusion will be quantified by two parameters, which are mainly responsible of the behavior of the model, and allow for a rigorous asymptotic analysis. In consequence of our assumptions on the microscopic interaction, in various relevant cases the model will satisfy mass and momentum conservation, which are the starting point for studying the asymptotic behavior.

In this direction, we shall show that the kinetic model gives in a suitable asymptotic limit (hereafter called quasi-invariant opinion limit) a partial differential equation of Fokker-Planck type for the distribution of opinion among individuals. Similar diffusion equations were obtained recently in [22] as the mean field limit of the Ochrombel simplification of the Sznajd model [27].

The equilibrium state of the Fokker-Planck equation can be computed explicitly and reveals formation of picks in correspondence to the points where diffusion is missing.

The mathematical methods we use are close to those used in the context of kinetic theory of granular gases, where the limit procedure is known as quasi-elastic asymptotics [17, 28]. We mention here that a similar asymptotic analysis was performed on a kinetic model of a simple market economy with a constant growth mechanism [7, 21], showing formation of steady states with Pareto tails [19]. In this context, the mean field approximation leads to the same Fokker-Planck equation [5, 23], showing consistency between kinetic and stochastic approaches.

The paper is organized as follows. In the next Section we introduce the binary interaction between agents, which is at the basis of the kinetic model. The main properties of the model are discussed in Section 3. These properties justify the quasi-invariant opinion limit procedure, performed in Section 4. The limit procedure is then illustrated by several choices of the diffusion in Section 5.

2. Description of the kinetic model

The goal of the forthcoming kinetic model of opinion formation, is to describe the evolution of the distribution of opinions in a society by means of *microscopic* interactions among agents or individuals which exchange information. To fix ideas, we associate opinion with a variable w which varies continuously from -1 to 1, where -1 and 1 clearly denote the two (extreme) opposite opinions. We will moreover assume that interactions do not destroy the bounds, which corresponds to impose that the extreme opinions can not be crossed. This crucial rule emphasizes the difference between the present *social* interactions, where not all outcomes are permitted, and the classical interactions between molecules, familiar to people working in kinetic theory of rarefied gases [6].

Let $\mathcal{I} = [-1, +1]$ denote the interval of possible opinions. From a microscopic view point, we describe the binary interaction by the rules

$$w' = w - \gamma P(|w|)(w - w_*) + \eta D(|w|)$$
$$w'_* = w_* - \gamma P(|w_*|)(w_* - w) + \eta_* D(|w_*|) \qquad (2.1)$$

where the pair (w, w_*), with $w, w_* \in \mathcal{I}$ denotes the opinions of two arbitrary individuals before the interaction and (w', w'_*) their opinions after exchanging information between them and with the exterior. In (2.1) we will not allow opinions to cross boundaries, and thus the interaction takes place only if both $w', w'_* \in \mathcal{I}$. In (2.1) the

coefficient $\gamma \in (0,1/2)$ is a given constant, while η and η_* are random variables with the same distribution with variance σ^2 and zero mean, taking values on a set $\mathcal{B} \subseteq IR$. The constant γ and the variance σ^2 measure respectively the compromise propensity and the modification of opinion due to diffusion. Finally, the functions $P(\cdot)$ and $D(\cdot)$ describe the local relevance of the compromise and diffusion for a given opinion.

Let us describe the details of the interaction in the right hand side. The first part is related to the compromise propensity of the agents, and the last contains the diffusion effects of external events. Note that the pre-interaction opinion w increases (getting closer to w_*) when $w_* > w$ and decreases in the opposite situation. The presence of both the functions $P(\cdot)$ and $D(\cdot)$ is linked to the hypothesis that the availability to the change of opinion is linked to the opinion itself, and decreases as soon as one gets closer to extremal opinions. This corresponds to the realistic idea that extremal opinions are more difficult to change. We will present later on various realizations of these functions. In all cases, however, we assume that both $P(|w|)$ and $D(|w|)$ are non increasing with respect to $|w|$, and in addition $0 \leq P(|w|) \leq 1$, $0 \leq D(|w|) \leq 1$.

In absence of the diffusion contribution ($\eta, \eta_* \equiv 0$), (2.1) implies

$$
\begin{aligned}
w' + w'_* &= w + w_* + \gamma(w - w_*)(P(|w|) - P(|w_*|)) \\
w' - w'_* &= (1 - \gamma(P(|w|) + P(|w_*|)))(w - w_*).
\end{aligned}
\tag{2.2}
$$

Thus, unless the function $P(\cdot)$ is assumed constant, $P = 1$, the total *momentum* is not conserved and it can increase or decrease depending on the opinions before the interaction. If $P(\cdot)$ is assumed constant, the conservation law is reminiscent of analogous conservations which take place in kinetic theory. In such a situation, thanks to the bounds on the coefficient γ, equations (2.1) correspond to a granular gas like interaction (or to a traffic flow model [13]) where the stationary state is a Dirac delta centered in the average opinion (usually referred to as synchronized traffic state in traffic flow modelling). This behavior is a consequence of the fact that, in a single interaction, the compromise propensity implies that the difference of opinion is diminishing, with $|w' - w'_*| = (1 - 2\gamma)|w - w_*|$. Thus all agents will end up in the society with exactly the same opinion. Note that in this elementary case a constant part of the relative opinion is restituted after the interaction. This property does not remain true if the function P depends on the opinion variable. In this case

$$
|w' - w'_*| = (1 - \gamma(P(|w|) + P(|w_*|)))|w - w_*|.
$$

In fact, since $\gamma \in (0,1/2)$ and $0 \leq P(|w|) \leq 1$, $0 \leq D(|w|) \leq 1$,

$$
0 \leq \varepsilon(w, w_*) = 1 - \gamma(P(|w|) + P(|w_*|)) \leq 1.
$$

Hence the general case corresponds to a granular gas interaction with a variable coefficient of restitution [28].

We remark moreover that, in absence of diffusion, the lateral bounds are not violated, since

$$
\begin{aligned}
w' &= (1 - \gamma P(|w|))w + \gamma P(|w|)w_* \\
w'_* &= (1 - \gamma P(|w_*|))w_* + \gamma P(|w_*|)w
\end{aligned}
\tag{2.3}
$$

implies

$$
\max\{|w'|, |w'_*|\} \leq \max\{|w|, |w_*|\}.
$$

Let $f(w,t)$ denote the distribution of opinion $w \in \mathcal{I}$ at time $t \geq 0$. A direct application of standard methods of kinetic theory of binary interactions [6] allows us to recover the time evolution of f as a balance between bilinear gain and loss of opinion terms, described by the integro-differential equation of Boltzmann type

$$\frac{\partial f}{\partial t} = \int_{\mathcal{B}^2} \int_{\mathcal{I}} \left('\beta \frac{1}{J} f('w) f('w_*) - \beta f(w) f(w_*) \right) dw_* \, d\eta \, d\eta_*, \qquad (2.4)$$

where $('w,'w_*)$ are the pre-interaction opinions that generate the couple (w,w_*) of opinions after the interaction. In (2.4) J is the Jacobian of the transformation of (w,w_*) into (w',w'_*), while the kernels $'\beta$ and β are related to the details of the binary interaction.

As usual in classical kinetic theory of rarefied gases, the interaction integral on the right-hand side of (2.4) represents the instantaneous variation of the distribution of opinion, due to the binary exchanges of information. The presence of the Jacobian J, guarantees that equation (2.4) preserves the mass (total opinion), for any choice of the rate function β. The transition rate is taken of the form

$$\beta_{(w,w_*) \to (w',w'_*)} = \Theta(\eta)\Theta(\eta_*)\chi(|w'| \leq 1)\chi(|w'_*| \leq 1), \qquad (2.5)$$

where $\chi(A)$ is the indicator function of the set A, and $\Theta(\cdot)$ is a symmetric probability density with zero mean and variance σ^2. The rate function $\beta_{(w,w_*) \to (w',w'_*)}$ characterizes the effects of external events on opinion through the distribution of the random variables Θ and Θ_* and takes into account the hypothesis that bounds can not be violated. We remark that in principle the support $\mathcal{B}$ of the symmetric random variable is a subset of $\mathcal{I}$, to prevent diffusion to generate a complete change of opinion. This property can be weakened by assuming for example diffusion as a random variable normally distributed but well concentrated on zero.

For a general probability density $\Theta(\cdot)$, the rate function β depends on the opinion variables (w,w_*) through the indicator functions χ. This fact reminds us of a similar property of the classical Boltzmann equation [6, 9], where the rate function depends on the relative velocity. As we shall see, a simplified situation occurs when a suitable choice of the function $D(\cdot)$ in (2.1) coupled with a small support $\mathcal{B}$ of random variables implies that both $|w'| \leq 1$ and $|w'_*| \leq 1$, and the kernel β does not depend on the opinion variables (w,w_*). In this case the kinetic equation (2.4) corresponds to the classical Boltzmann equation for Maxwell molecules [4], which presents several mathematical simplifications. In all cases however, methods borrowed from kinetic theory of rarefied gas can be used to study the evolution of the function f.

3. Simplifications and main properties of the model

The main problem in opinion dynamics is the formation of stationary profiles for the opinion. In the kinetic picture this corresponds to an investigation of the large time behavior of the density of opinion $f(w,t)$. To investigate in detail the large-time behavior, a preliminary analysis of equation (2.4) is needed. We will start this analysis by introducing some notations and by discussing the main properties of the kinetic equation.

Let $Q(f,f)$ denote the interaction integral,

$$Q(f,f)(w) = \int_{\mathcal{B}^2} \int_{\mathcal{I}} \left('\beta \frac{1}{J} f('w) f('w_*) - \beta f(w) f(w_*) \right) dw_* \, d\eta \, d\eta_*. \qquad (3.1)$$

Let $\mathcal{M}_0(A)$ be the space of all probability measures taking values in $A \subseteq IR$ and by

$$\mathcal{M}_p(A) = \left\{ \Theta \in \mathcal{M}_0 : \int_A |w|^p d\Theta(w) < +\infty, p \geq 0 \right\}, \tag{3.2}$$

the space of all Borel probability measures of finite momentum of order p, equipped with the topology of the weak convergence of the measures.

Let $\mathcal{F}_s(\mathcal{I})$, be the class of all real functions h on $\mathcal{I}$ such that $h(\pm 1) = h'(\pm 1) = 0$, and $h^{(m)}(v)$ is Hölder continuous of order δ,

$$\|h^{(m)}\|_\delta = \sup_{v \neq w} \frac{|h^{(m)}(v) - h^{(m)}(w)|}{|v - w|^\delta} < \infty, \tag{3.3}$$

the integer m and the number $0 < \delta \leq 1$ are such that $m + \delta = s$, and $h^{(m)}$ denotes the m-th derivative of h.

In the rest of the paper we will assume that the symmetric probability density $\Theta(\eta)$ which characterizes the diffusion of information belongs to $\mathcal{M}_{2+\alpha}$, for some $\alpha > 0$. Moreover, to simplify computations, we assume that this density is obtained from a given random variable Y with zero mean and unit variance, that belongs to $\mathcal{M}_{2+\alpha}$. Thus, Θ of variance σ^2 is the density of σY. By this assumption, we can easily obtain the dependence on σ of the moments of Θ. In fact, for any $p > 0$ such that the p-th moment of Y exists,

$$\int_{IR} |\eta|^p \Theta(\eta) d\eta = E(|\sigma Y|^p) = \sigma^p E(|Y|^p).$$

By a weak solution of the initial value problem for equation (2.4), corresponding to the initial probability density $f_0(w) \in \mathcal{M}_0(\mathcal{I})$, we shall mean any probability density $f \in C^1(IR_+, \mathcal{M}_0(\mathcal{I}))$ satisfying the weak form of the equation

$$\frac{d}{dt} \int_{\mathcal{I}} \phi(w) f(w, t) \, dw = (Q(f, f), \phi) =$$
$$\int_{\mathcal{I}^2} \int_{\mathcal{B}^2} \beta_{(w, w_*) \to (w', w'_*)} f(w) f(w_*) (\phi(w') - \phi(w)) dw_* dw d\eta d\eta_*, \tag{3.4}$$

for $t > 0$ and all $\phi \in \mathcal{F}_p(\mathcal{I})$, and such that for all $\phi \in \mathcal{F}_p(\mathcal{I})$

$$\lim_{t \to 0} \int_{\mathcal{I}} \phi(w) f(w, t) \, dw = \int_{\mathcal{I}} \phi(w) f_0(w) \, dw. \tag{3.5}$$

The form (3.4) is easier to handle, and it is the starting point to explore the evolution of macroscopic quantities (moments). By symmetry reasons, we can alternatively use the symmetric form

$$\frac{d}{dt} \int_{\mathcal{I}} f(w) \phi(w) \, dw = \frac{1}{2} \int_{\mathcal{I}^2} \int_{\mathcal{B}^2} \beta_{(w, w_*) \to (w', w'_*)} f(w) f(w_*)$$
$$(\phi(w') + \phi(w'_*) - \phi(w) - \phi(w_*)) dw_* \, dw \, d\eta \, d\eta_*. \tag{3.6}$$

Existence of a weak solution to the initial value problem for equation (2.4) can be easily obtained by using methods first applied to the Boltzmann equation [6]. On the other hand, for a general kernel (2.5), it appears extremely difficult to describe in detail the large-time behavior of the solution.

For this reason, at first we restrict our analysis to the cases in which the kernel β does not depend on the opinion variables (Maxwellian case). In this direction, let us briefly discuss the importance of the support $\mathcal{B}$ of the probability density function $\Theta(\eta)$ in connection with the possible simplification of the kernel. To clarify the point, let us set

$$D(|w|) = 1 - |w|.$$

This function satisfies all the requirements we fixed in Section 2. Then, since

$$(1 - \gamma P(|w|))w + \gamma P(|w|)w_* + (1 - |w|)\eta \leq (1 - \gamma P(|w|))w + \gamma P(|w|) + (1 - |w|)\eta,$$

in order that $|w'| \leq 1$ it suffices that

$$(1 - \gamma P(|w|))w + \gamma P(|w|) + (1 - |w|)\eta \leq 1, \tag{3.7}$$

or, what is the same

$$(1 - |w|)\eta \leq (1 - \gamma P(|w|))(1 - w). \tag{3.8}$$

Since $P(|w|) \leq 1$, bound (3.8) is verified for all $w \geq 0$, as soon as $\eta \leq 1 - \gamma$. An analogous result holds if $w \leq 0$. Hence, if $D(|w|) = 1 - |w|$ and $\mathcal{B} = (-(1 - \gamma), 1 - \gamma)$, both w' and w'_* belong to the right interval.

REMARK 3.1. Any choice of $D(|w|)$ or $\mathcal{B}$ which is suitable to preserve the lateral bounds of extreme opinions allows us to study in detail the dynamics of the model with a significant simplification. In these cases, the kernel β defined in (2.5) simplifies to

$$\beta_{(w,w_*) \to (w',w'_*)} = \beta(\eta, \eta_*) = \Theta(\eta)\Theta(\eta_*).$$

In the rest of the paper we will limit ourselves to such types of kernels. As briefly explained in Section 2, this assumption is the analogue to Maxwell molecules interaction in the Boltzmann equation [6].

From (3.4) (or equivalently from (3.6)) conservation of the total opinion is obtained for $\phi(w) = 1$, which represents in general the only conservation property satisfied by the system. The choice $\phi(w) = w$ is of particular interest since it gives the time evolution of the average opinion. We have

$$\frac{d}{dt} \int_{\mathcal{I}} w f(w,t)\, dw = \int_{\mathcal{I}^2} \int_{\mathcal{B}^2} \beta(\eta, \eta_*) f(w) f(w_*) \gamma (P(|w|)w_* - P(|w|)w)\, dw_* dw\, d\eta\, d\eta_*$$

$$+ \int_{\mathcal{I}^2} \int_{\mathcal{B}^2} \beta(\eta, \eta_*) f(w) f(w_*) \eta D(|w|)\, dw_* dw\, d\eta\, d\eta_*.$$

The first integral on the right–hand side represents the contribution of the exchange of information to the variation of momentum. In case $P(|w|) = 1$, this contribution disappears, since, by symmetry,

$$\int_{\mathcal{I}^2} \int_{\mathcal{B}^2} \beta(\eta, \eta_*) f(w) f(w_*) \gamma (w_* - w)\, dw_* dw\, d\eta\, d\eta_* = 0.$$

In this case

$$\frac{d}{dt}\int_{\mathcal{I}} w f(w,t)\,dw$$

$$=\int_{\mathcal{I}^2}\int_{\mathcal{B}^2}\eta\Theta(\eta)\Theta(\eta_*)\chi(|w'|\leq 1)\chi(|w'_*|\leq 1)D(|w|)f(w)f(w_*)dw_*dw\,d\eta\,d\eta_*=0, \quad (3.9)$$

since the mean value of the density Θ is zero. This shows that P constant implies that the average opinion is conserved. The situation changes when P is not constant. In this case, the time evolution of the average opinion is given by

$$\frac{d}{dt}\int_{\mathcal{I}} w f(w,t)\,dw=\gamma\int_{\mathcal{I}} P(|w|)f(w)\,dw\int_{\mathcal{I}} w f(w)\,dw-\gamma\int_{\mathcal{I}} wP(|w|)f(w)\,dw. \quad (3.10)$$

Note that equation (3.10) is not closed.

Let us fix now $\phi(w)=w^2$. We have

$$\frac{d}{dt}\int_{\mathcal{I}} w^2 f(w,t)\,dw=\frac{1}{2}\int_{\mathcal{I}^2}\int_{\mathcal{B}^2}\Theta(\eta)\Theta(\eta_*)f(w)f(w_*)(w'^2+w'^2_*-w^2-w^2_*)dw_*dw\,d\eta\,d\eta_*.$$
$$(3.11)$$

Keeping in mind that Θ has zero mean and variance σ^2, by easy computations one shows that

$$\frac{1}{2}\int_{\mathcal{I}^2}\int_{\mathcal{B}^2}\Theta(\eta)\Theta(\eta_*)f(w)f(w_*)(w'^2+w'^2_*-w^2-w^2_*)dw_*dw\,d\eta\,d\eta_*=$$

$$\gamma^2\int_{\mathcal{I}^2} P(|w|)^2(w-w_*)^2 f(w)f(w_*)\,dw\,dw_*-$$

$$2\gamma\int_{\mathcal{I}^2} P(|w|)w(w-w_*)f(w)f(w_*)\,dw\,dw_*+\sigma^2\int_{\mathcal{I}} D(|w|)^2 f(w)\,dw. \quad (3.12)$$

The choice $P(|w|)=1$ leads to the simpler evolution equation

$$\frac{d}{dt}\int_{\mathcal{I}} w^2 f(w,t)\,dw=-2\gamma(1-\gamma)\left[\int_{\mathcal{I}} w^2 f(w)\,dw-m^2\right]+\sigma^2\int_{\mathcal{I}} D(|w|)^2 f(w)\,dw, \quad (3.13)$$

where m is the constant value of the average opinion

$$m=\int_{\mathcal{I}} w f(w,t)\,dw.$$

Since $|w|\leq 1$, the boundedness of the mass implies that all moments are bounded. This implies that, in all cases, we can draw conclusions on the large–time convergence of the class of probability densities $\{f(w,t)\}_{t\geq 0}$. By virtue of Prokhorov theorem (cfr. [14]) the existence of a uniform bound on moments implies in fact that this class is tight, so that any sequence $\{f(w,t_n)\}_{n\geq 0}$ contains an infinite subsequence which converges weakly to some probability measure f_∞.

4. The quasi-invariant opinion limit

The analysis of the previous section shows that in general it is quite difficult both to study in detail the evolution of the opinion density, and to describe its asymptotic behavior. For a general kernel one has in addition to take into account that the mean opinion is varying in time. As is usual in kinetic theory, however, particular asymptotics of the equation result in simplified models (generally of Fokker-Planck type), for which it is relatively easy to find steady states, and to prove their stability. These asymptotics are particularly relevant in case they are able to describe with a good approximation the stationary profiles of the kinetic equation. In order to give a physical basis to these asymptotics, let us discuss the interaction rule (2.1) from a slightly different point of view. For the moment, we will assume $P(|w|) = 1$, so that conservation both of mass and momentum holds. The case of a general $P(|w|)$ will be treated subsequently. Let us denote by $E(X)$ the mathematical expectation of the random variable X. Then the following properties follow from (2.1)

$$E[w' + w'_*] = w + w_*, \qquad E[w' - w'_*] = (1 - 2\gamma)(w - w_*). \qquad (4.1)$$

The first equality in (4.1) describes the property of mean conservation of opinion. The second refers to the compromise propensity, which plays in favor of the decrease (in mean) of the distance of opinions after the interaction. This tendency is a universal consequence of the rule (2.1), in that it holds whatever distribution one assigns to Θ, namely to the random variable which accounts for the effects of the external word in opinion formation.

The second property in (4.1) is analogous to the similar one that holds in a collision between molecules in a granular gas. There the quantity $e = 2\gamma$ is called "coefficient of restitution", and describes the peculiar fact that energy is dissipated [28].

We consider now the situation in which most of the interactions produce a very small exchange of opinion ($\gamma \to 0$), while at the same time both properties (4.1) remain at a macroscopic level. This corresponds to pretend that, while $\gamma \to 0$,

$$\int_{\mathcal{I}^2} (w + w_*) f(w) f(w_*) dw dw_* = 2 \int_{\mathcal{I}} w f(w) dw = 2m(t) \qquad (4.2)$$

remains constant, and

$$\frac{1}{2} \int_{\mathcal{I}^2} (w - w_*)^2 f(w) f(w_*) dw dw_* = \int_{\mathcal{I}} w^2 f(w) dw - m_0^2 = C_f(t) \qquad (4.3)$$

varies with time, and decays to zero when the diffusion is not present (i.e. $\sigma = 0$).

Since in our case the kernel β does not depend on the opinion variables, (3.9) implies that $m(t) = m_0$ independently of the value of γ. Moreover, using the computations of Section 3, one obtains that $C_f(t)$ varies with law

$$\frac{dC_f(t)}{dt} = -2\gamma(1 - \gamma)C_f(t) + \sigma^2 \int_{\mathcal{I}} D(|w|)^2 f(w) dw. \qquad (4.4)$$

Hence, if we set

$$\tau = \gamma t, \quad g(w, \tau) = f(w, t), \qquad (4.5)$$

which implies $f_0(w) = g_0(w)$, it follows

$$\frac{dC_g(\tau)}{d\tau} = -2(1 - \gamma)C_g(\tau) + \frac{\sigma^2}{\gamma} \int_{\mathcal{I}} D(|w|)^2 f(w) dw. \qquad (4.6)$$

Letting now both $\gamma \to 0$ and $\sigma \to 0$ in such a way that $\sigma^2/\gamma = \lambda$, (4.6) becomes in the limit

$$\frac{dC_g(\tau)}{d\tau} = -2C_g(\tau) + \lambda \int_{\mathcal{I}} D(|w|)^2 f(w)\,dw. \qquad (4.7)$$

This argument shows that the value of the ratio σ^2/γ is of paramount importance to get asymptotics which maintain memory of the microscopic interactions. It is remarkable that, thanks to (4.5), $t = \tau/\gamma$, so that the limit $\gamma \to 0$ describes the large-time behavior of $f(v,t)$. On the other hand, since $f(w,t) = g(w,\tau)$ the large-time behavior of $f(w,t)$ is close to the large-time behavior of $g(w,\tau)$.

REMARK 4.1. The balance of $\gamma \to 0$ and $\sigma \to 0$ in such a way that $\sigma^2/\gamma = \lambda$, allows us to recover in the limit the contributions due both to compromise propensity and the diffusion. Other limits can be considered, which are diffusion dominated ($\sigma^2/\gamma = \infty$) or compromise dominated ($\sigma^2/\gamma = 0$). As we shall present in Section 5, however, the formation of an asymptotic profile for the opinion is linked to the first balance [1].

In the remainder of this Section, we shall present the rigorous derivation of a Fokker-Planck model, starting from the Boltzmann equation for the opinion density $g(w,\tau)$, when both $\gamma \to 0$ and $\sigma \to 0$ in such a way that $\sigma^2/\gamma \to \lambda$. For the sake of simplicity, we will assume that $P(|w|) = 1$. This type of analysis is close to the one described in [7] for a kinetic model of wealth distribution in an open economy.

The scaled density $g(v,\tau) = f(v,t)$ satisfies the equation (in weak form)

$$\frac{d}{d\tau}\int_{\mathcal{I}} g(w)\phi(w)\,dw = \frac{1}{\gamma}\int_{\mathcal{I}^2}\int_{\mathcal{B}^2} \Theta(\eta)\Theta(\eta_*)g(w)g(w_*)(\phi(w') - \phi(w))\,dw_*\,dw\,d\eta\,d\eta_*.$$
$$(4.8)$$

Given $0 < \delta \le \alpha$, let us set $\phi \in \mathcal{F}_{2+\delta}(\mathcal{I})$.

By (2.1),

$$w' - w = \gamma(w_* - w) + \eta D(|w|).$$

Then, if we use a second order Taylor expansion of ϕ around w

$$\phi(w') - \phi(w) = (\gamma(w_* - w) + \eta D(|w|))\phi'(w) + \frac{1}{2}(\gamma(w_* - w) + \eta D(|w|))^2 \phi''(\tilde{w}),$$

where, for some $0 \le \theta \le 1$

$$\tilde{w} = \theta w' + (1 - \theta)w.$$

Inserting this expansion in the collision operator, we get

$$\frac{d}{d\tau}\int_{\mathcal{I}} g(w)\phi(w)\,dw = \frac{1}{\gamma}\int_{\mathcal{I}^2}\int_{\mathcal{B}^2}\Theta(\eta)\Theta(\eta_*)[(\gamma(w_* - w) + \eta D(|w|))\phi'(w) +$$

$$+\frac{1}{2}(\gamma(w_* - w) + \eta D(|w|))^2\phi''(w)]g(w)g(w_*)\,dw_*\,dw\,d\eta\,d\eta_* + R(\gamma,\sigma), \qquad (4.9)$$

where

$$R(\gamma,\sigma) = \frac{1}{2\gamma}\int_{\mathcal{I}^2}\int_{\mathcal{B}^2}\Theta(\eta)\Theta(\eta_*)(\gamma(w_* - w) + \eta D(|w|))^2 \cdot$$
$$\cdot(\phi''(\tilde{w}) - \phi''(w))g(w)g(w_*)\,dw_*\,dw\,d\eta\,d\eta_*.$$

Since $\phi \in \mathcal{F}_{2+\delta}(\mathcal{I})$, and $|\tilde{w} - w| = \theta|w' - w|$

$$|\phi''(\tilde{w}) - \phi''(w)| \leq \|\phi''\|_\delta |\tilde{w} - w|^\delta \leq \|\phi''\|_\delta |w' - w|^\delta. \tag{4.10}$$

Hence

$$|R(\gamma,\sigma)| \leq \frac{\|\phi''\|_\delta}{2\gamma} \int_{\mathcal{I}^2} \int_{\mathcal{B}^2} \Theta(\eta)\Theta(\eta_*) \cdot$$
$$\cdot |\gamma(w_* - w) + \eta D(|w|)|^{2+\delta} g(w)g(w_*) dw_* \, dw \, d\eta \, d\eta_*.$$

By virtue of the inequality

$$|\gamma(w_* - w) + \eta D(|w|)|^{2+\delta} \leq 2^{1+\delta} \left(|\gamma(w_* - w)|^{2+\delta} + |\eta D(|w|)|^{2+\delta} \right) \leq$$

$$\leq 2^{2+\delta}\gamma^{1+\delta} + 2^{1+\delta}|\eta|^{2+\delta},$$

we finally obtain the bound

$$|R(\gamma,\sigma)| \leq 2^{1+\delta}\|\phi''\|_\delta \left(\gamma^{1+\delta} + \frac{1}{2\gamma} \int_{\mathcal{B}} |\eta|^{2+\delta}\Theta(\eta)d\eta \right). \tag{4.11}$$

Since Θ is a probability density with zero mean and $\lambda\gamma$ variance, and Θ belongs to $\mathcal{M}_{2+\alpha}$, for $\alpha > \delta$,

$$\int_{\mathcal{I}} |\eta|^{2+\delta}\Theta(\eta)d\eta = E\left(\left| \sqrt{\lambda\gamma} Y \right|^{2+\delta} \right) = (\lambda\gamma)^{1+\delta/2} E\left(|Y|^{2+\delta} \right),$$

and $E\left(|Y|^{2+\delta} \right)$ is bounded. Using this equality in (4.11) one shows that $R(\gamma,\sigma)$ converges to zero as both γ and σ converge to zero, in such a way that $\sigma^2 = \lambda\gamma$. Within the same scaling,

$$\lim_{\gamma \to 0} \frac{1}{\gamma} \int_{\mathcal{I}^2} \int_{\mathcal{B}^2} \Theta(\eta)\Theta(\eta_*)[(\gamma(w_* - w) + \eta D(|w|))\phi'(w) +$$

$$+ \frac{1}{2}(\gamma(w_* - w) + \eta D(|w|))^2 \phi''(w)]g(w)g(w_*) dw_* \, dw \, d\eta \, d\eta_* =$$

$$\int_{\mathcal{I}} \left[(m - w)\phi'(w) + \frac{\lambda}{2} D(|w|)^2 \phi''(w) \right] g(w)dw. \tag{4.12}$$

Considering that $\phi \in \mathcal{F}_s(\mathcal{I})$, we can integrate back by parts. This shows that the right-hand side of (4.12) coincides with the weak form of the Fokker-Planck equation

$$\frac{\partial g}{\partial \tau} = \frac{\lambda}{2} \frac{\partial^2}{\partial w^2} \left(D(|w|)^2 g \right) + \frac{\partial}{\partial w} ((w - m)g). \tag{4.13}$$

Last, since the solution to the kinetic model conserves mass and momentum, while the second moment is uniformly bounded in time, conservation of both mass and momentum pass to the limit. We remark that these conservations are difficult to prove directly on the Fokker-Planck equation, due to the fact that $\phi(v) = v$ does not belong to $\mathcal{F}_s(\mathcal{I})$. Hence we proved:

THEOREM 4.2. *Let the probability density $f_0 \in \mathcal{M}_0(\mathcal{I})$, and let the symmetric random variable Y which characterizes the kernel have a density in $\mathcal{M}_{2+\alpha}$, with $\alpha > \delta$. Then, as $\gamma \to 0$, $\sigma \to 0$ in such a way that $\sigma^2 = \lambda\gamma$ the weak solution to the Boltzmann equation for the scaled density $g_\gamma(v,\tau) = f(v,t)$, with $\tau = \gamma t$ converges, up to extraction of a subsequence, to a probability density $g(w,\tau)$. This density is a weak solution of the Fokker-Planck equation (4.13), and it is such that the average opinion is conserved.*

5. Other Fokker-Planck models of opinion formation

Theorem 4.2 can be generalized in many ways. Always remaining with the simplification of Remark 3.1, we can choose a general function $P(|w|)$ in the interaction rule (2.1). The main difference with respect to the proof of Theorem 4.2 is the evaluation of the first order term into (4.9), which, using the fact that the mean value of Θ is zero, reads

$$\int_{\mathcal{I}^2} P(|w|)(w_* - w)\phi'(w)g(w)g(w_*)dw_* \, dw = \int_{\mathcal{I}} P(|w|)(m(\tau) - w)\phi'(w)g(w) \, dw, \quad (5.1)$$

where $m(\tau)$ is the value of the average opinion at time $\tau \geq 0$,

$$m(\tau) = \int_{\mathcal{I}} w g(w,\tau) dw.$$

Note that, by (4.5)

$$m(\tau) = \int_{\mathcal{I}} w g(w,\tau) dw = \int_{\mathcal{I}} w f(w,t) dw. \quad (5.2)$$

Hence by (3.10) the evolution of $m(\tau)$ obeys the law

$$\frac{dm(\tau)}{d\tau} = m(\tau) \int_{\mathcal{I}} P(|w|)g(w,\tau) \, dw - \int_{\mathcal{I}} w P(|w|)g(w,\tau) \, dw. \quad (5.3)$$

Finally, in the limit $\gamma \to 0$ we obtain that $g(w,\tau)$ satisfies the Fokker-Planck equation

$$\frac{\partial g}{\partial \tau} = \frac{\lambda}{2} \frac{\partial^2}{\partial w^2} \left(D(|w|)^2 g \right) + \frac{\partial}{\partial w} \left(P(|w|)(w - m(t))g \right). \quad (5.4)$$

REMARK 5.1. The presence of a general propensity function $P(|w|)$ introduces a difficult to treat nonlinearity into the Fokker-Planck equation. The nonlinearity is due to the fact that the average opinion is no more constant, and the evaluation of the drift term requires the evaluation of (5.3).

While to our knowledge the Fokker-Planck equation (5.4) has never been considered before, linked pure diffusion and drift equations have been recently introduced in [22]. These equations, in our picture, refer to diffusion dominated ($\sigma^2/\gamma = \infty$) or compromise dominated ($\sigma^2/\gamma = 0$) limits. Looking at the proof of Theorem (4.2), it is almost immediate to conclude that the diffusion dominated limit takes into account only the second-order term into the Taylor expansion. To verify this, suppose that

$$\frac{\sigma^2}{\gamma^\alpha} \to \lambda, \quad \alpha < 1.$$

Then we can set

$$\tau = \gamma^\alpha t, \quad g(w,\tau) = f(w,t), \quad (5.5)$$

where now $g(w,\tau)$ satisfies

$$\frac{d}{d\tau} \int_{\mathcal{I}} g(w)\phi(w) \, dw = \frac{1}{\gamma^\alpha} \int_{\mathcal{I}^2} \int_{\mathcal{B}^2} \Theta(\eta)\Theta(\eta_*)[(\gamma(w_* - w) + \eta D(|w|))\phi'(w) +$$

$$+\frac{1}{2}(\gamma(w_* - w) + \eta D(|w|))^2 \phi''(w)]g(w)g(w_*)dw_* dw d\eta d\eta_* + R(\gamma, \sigma), \qquad (5.6)$$

with obvious meaning of the remainder. Since $\alpha < 1$, the first order term in the Taylor expansion vanishes in the limit, and g satisfies the diffusion equation

$$\frac{\partial g}{\partial \tau} = \frac{\lambda}{2} \frac{\partial^2}{\partial w^2}\left(D(|w|)^2 g\right). \qquad (5.7)$$

The choice

$$D(|w|) = \sqrt{1 - w^2}, \quad \lambda = 2, \qquad (5.8)$$

brings to the diffusion equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2}{\partial w^2}\left[(1 - w^2)g\right]. \qquad (5.9)$$

This diffusion equation has been derived in a mean field approximation [22] to describe the evolution of the Sznajd model in Ochrombel simplification [18] on a complete graph of N nodes in case of two opinions. The same equation has been shown to describe the former model in case of a large number q of opinions. The variable now represents the mean distribution of occupation numbers and the limit is taken as both N and q tend to infinity at the same rate.

Likewise, the compromise dominated ($\sigma^2/\gamma = 0$) limit can be considered. In this case,

$$\frac{\sigma^2}{\gamma^\alpha} \to \lambda, \quad \alpha > 1,$$

and we can set

$$\tau = \gamma t, \quad g(w, \tau) = f(w, t). \qquad (5.10)$$

The diffusion part disappears in the limit and we obtain the pure drift equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial w}\left(P(|w|)(w - m(t))g\right). \qquad (5.11)$$

Note that, due to (5.10) the evolution of the mean opinion $m(t)$ obeys the law (5.2). The choice

$$P(|w|) = 1 - w^2$$

has been considered in [22]. In this case

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial w}\left((1 - w^2)(w - m(t))g\right), \qquad (5.12)$$

where

$$\frac{dm(\tau)}{d\tau} = -m(\tau)\int_{\mathcal{I}} w^2 g(w, \tau)\,dw + \int_{\mathcal{I}} w^3 g(w, \tau)\,dw. \qquad (5.13)$$

Thus, our equation differs from the pure drift in magnetization obtained in [22] as the mean field limit of the Sznajd model [27] in case of two opinions. There the first-order partial differential equation reads

$$\frac{\partial g}{\partial \tau} = -\frac{\partial}{\partial w}\left((1 - w^2)wg\right). \qquad (5.14)$$

Note that the sign in front of the drift (5.14) is now opposite to the sign in (5.12), and the equation is now linear in g, even if the evolution of the mean opinion is not closed.

6. Stationary solutions of the Fokker-Planck opinion model

In this Section we analyze in some detail various cases of the interaction dynamics in the Boltzmann equation from which it is possible to derive a Fokker-Planck equation with an explicitly computable steady state. The structure of the steady state then represents the formation of opinion consequent to the choice of the interaction dynamics. In most cases, we are forced to suppose $P(|w|) = 1$, which implies conservation of the average opinion, and Fokker-Planck (4.13) as underlying quasi-invariant opinion limit. For any of these choices, we briefly discuss the link between $D(|w|)$ and the maximal support $\mathcal{B}$ of the diffusion variable.

The first model of the diffusion dependence on opinion we propose is

$$D(|w|) = 1 - w^2.$$

Since $P(|w|) = 1$, the interaction rules are

$$w' = w - \gamma(w - w_*) + \eta(1 - w^2)$$

$$w'_* = w_* - \gamma(w_* - w) + \eta_*(1 - w_*^2).$$

In this case $|\eta|(1 + |w|) \leq 1 - \gamma$ implies $|w'| \leq 1$. Hence, for a given opinion $\mathcal{B} = (-(1-\gamma)/(1+|w|), (1-\gamma)/(1+|w|))$, which shows that when the opinion w is close to zero and γ is small, the effects of the global access to information can move the opinion towards extremals. This possibility reduces as soon as $|w|$ increases. The steady state distribution of opinion is a solution to

$$\frac{\lambda}{2} \frac{\partial}{\partial w} \left((1 - w^2)^2 g \right) + (w - m)g = 0 \tag{6.1}$$

where m is a given constant (the average initial opinion) $-1 < m < 1$. The solution to (6.1) is easily found as

$$g_\infty(w) = c_{m,\lambda} (1 + w)^{-2 + m/(2\lambda)} (1 + w)^{-2 - m/(2\lambda)} \exp\left\{ -\frac{1 - mw}{\lambda(1 - w^2)} \right\}, \tag{6.2}$$

where the constant $c_{m,\lambda}$ is such that the mass of g_∞ is equal to one. Note that the presence of the exponential assures that $g_\infty(\pm 1) = 0$. The solution is regular, but not symmetric unless $m = 0$. Hence, the initial opinion distribution reflects on the steady state through the mean opinion. In any case, the stationary distribution has two picks (on the right and on the left of zero) with intensity depending on λ.

A similar result is expected from the choice

$$D(|w|) = 1 - |w|.$$

As discussed in Section 2 $|\eta| \leq 1 - \gamma$ implies $|w'| \leq 1$, and $\mathcal{B} = (-(1-\gamma), 1-\gamma)$. Once more, the support of the random variable, for γ small, covers the whole domain of opinions. The steady state distribution of opinion is a solution to

$$\frac{\lambda}{2} \frac{\partial}{\partial w} \left((1 - |w|)^2 g \right) + (w - m)g = 0 \tag{6.3}$$

where m is a given constant (the average initial opinion) $-1 < m < 1$. The solution to (6.1) is easily found as

$$g_\infty(w) = c_{m,\lambda} (1 - |w|)^{-2 - 2/\lambda} \exp\left\{ -\frac{1 - mw/|w|}{2\lambda(1 - |w|)} \right\}, \tag{6.4}$$

where, as usual, the constant $c_{m,\lambda}$ is such that the mass of g_∞ is equal to one. We remark that the low regularity of $D(|w|)$ reflects on the steady solution, which has a jump in $w=0$. The jump disappears only when the mean $m=0$, and only in this case do we have a symmetric distribution. As in the first case, the presence of the exponential assures us that $g_\infty(\pm 1)=0$, and the initial opinion distribution reflects on the steady state through the mean opinion.

Last, we consider

$$D(|w|) = \sqrt{1-w^2}$$

in the Fokker-Planck model (4.13). The steady state distribution of opinion solves

$$\frac{\lambda}{2}\frac{\partial}{\partial w}\left((1-w^2)g\right) + (w-m)g = 0, \tag{6.5}$$

and equals

$$g_\infty(w) = c_{m,\lambda}\left(\frac{1}{1+w}\right)^{1-(1+m)/\lambda}\left(\frac{1}{1-w}\right)^{1-(1-m)/\lambda}. \tag{6.6}$$

As usual, the constant $c_{m,\lambda}$ is such that the mass of g_∞ is equal to one. Since $-1<m<1$, g_∞ is integrable on $\mathcal{I}$. Differently from the previous cases, however $g_\infty(w)$ tends to infinity as $w\to\pm 1$, and it has no peaks inside the interval $\mathcal{I}$. The explanation comes out from a deep insight into the connection between $D(|w|)$ and the support $\mathcal{B}$ in this case.

It is immediate to verify that, in order to satisfy the constraint in (2.1), one can not choose directly $D(|w|)=\sqrt{1-w^2}$. Within this choice, in fact, choosing $w^*=1$, the first equality in (2.1) gives for η the upper bound

$$\eta \le (1-\gamma)\frac{(1-w)}{\sqrt{1-w^2}}.$$

Since the right-hand side converges to zero as $w\to 1$, it follows that there is no way to satisfy the constraint. A different choice which gives in the limit the Fokker–Planck equation with $D(|w|)=\sqrt{1-w^2}$, is the following. For any given γ, we set

$$D(|w|) = \sqrt{(1-(1+\gamma^p)w^2)_+}, \quad p>0,$$

where f_+ denotes as usual the positive part of f, that is $f_+ = f$ if $f>0$, while $f_+=0$ if $f\le 0$. In this case, one can show that it is sufficient for η to satisfy the condition

$$|\eta| \le a_\gamma = \frac{1-\gamma}{\sqrt{1+\gamma^p}}\gamma^{p/2}, \tag{6.7}$$

to respect the constraint on post-interaction opinions. To give an example, let us assume that Θ is uniformly distributed on the interval $-a_\gamma, a_\gamma$. Then σ^2 behaves like $\gamma^{(3p)/2}$, and it is enough to set $p=2/3$ to obtain $\lambda=1$. The previous discussion shows that the choice $D(|w|)=\sqrt{1-w^2}$ in the Fokker-Planck equation (4.13) corresponds to a kinetic interaction in which diffusion is of the order of γ^p, where p is taken so that σ^2/γ tends to a finite limit λ as $\gamma\to 0$. In this case, the *smallness* of the interval of diffusion produces the peaks on $w=\pm 1$.

A interesting feature of the Fokker-Planck equation

$$\frac{\partial g}{\partial \tau} = \frac{\lambda}{2}\frac{\partial^2}{\partial w^2}\left((1-w^2)g\right) + \frac{\partial}{\partial w}\left((w-m)g\right), \tag{6.8}$$

is that it leads to close evolution of moments. We remark that equation (6.8) is to be studied with the conservation of both mass and momentum.

7. Conclusions

We introduced and discussed here some kinetic models of opinion formation based on binary interactions involving both compromise and diffusion properties in exchanges between individuals. A suitable scaling of compromise and diffusion allows us to derive Fokker-Planck equations for which it is easy to recover the stationary distribution of opinion. Among these Fokker-Planck equations, one is emerging (6.8) and takes the role of the analogous one obtained in [5, 7] for the evolution of wealth. The main feature of this equation is that moments can be evaluated in closed form. Further numerical studies are in progress to understand the evolution of opinion density for various choices of the underlying functions $P(|w|)$ and $D(|w|)$. In particular, when $P(|w|)$ is not linear, the evolution of moments is far from being completely understood.

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