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ON THE L^2-WELL POSEDNESS OF AN INITIAL BOUNDARY VALUE PROBLEM FOR THE 3D LINEAR ELASTICITY*

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Abstract. In a recent paper, we analyzed the L^2-well posedness of an initial boundary value problem (ibvp) for the two-dimensional system of the linear elasticity under the uniform Kreiss-Lopatinskii condition. The present work is devoted to studying the analog of this problem in the three-dimensional case, when the Majda-Osher's analysis cannot be applied. The well-posedness is achieved by constructing an everywhere smooth non-degenerate dissipative Kreiss symmetrizer of the ibvp: this is done by adapting to the present situation the techniques already implemented for the two-dimensional linear elasticity. Compared with the latter case, some further technical difficulties have to be accounted for.

Key words. Linear elasticity, initial boundary value problems, dissipative symmetrizers.

AMS subject classifications. 35L50, 74B05

1. Introduction

We are concerned with the system of linear elasticity in three space dimension (3D). This system reads as follows

$$\begin{aligned} &\partial_t F + \nabla z = 0, \\ &\partial_t z + \mathrm{div} T = 0, \end{aligned} \tag{1.1}$$

where the unknowns $z = z(x,t) \in \mathbb{R}^3$ and $F = F(x,t) \in \mathbf{M}_{3\times 3}(\mathbb{R})$ (for $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ and $t > 0$) represent respectively the opposite of the material velocity and the infinitesimal deformation tensor, while the stress tensor T is defined by

$$T := \lambda(F + F^T) + \mu(\mathrm{Tr}F)I_3 \tag{1.2}$$

and λ, μ are given positive constants (the so-called *Lamé coefficients*). A detailed analysis of the elasticity model can be found, for instance, in the books of P. G. Ciarlet [2] and C. Dafermos [3].

Since in (1.1) the skew-symmetric part of F decouples from the rest, we may reduce to the system describing the evolution of z and the symmetric part of F. The latter system is Friedrichs symmetrizable, since it admits the quadratic energy

$$Q := \frac{1}{2}|z|^2 + \frac{\lambda}{4}|F + F^T|^2 + \frac{\mu}{2}(\mathrm{Tr}F)^2. \tag{1.3}$$

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Setting $c_P := \sqrt{2\lambda+\mu}$ (the velocity of *pressure waves*), the choice of new variables

$$u := \begin{pmatrix} \frac{c_P\sqrt{\lambda(2\lambda+3\mu)}}{\sqrt{\lambda+\mu}} F_{1,1} \\ c_P\sqrt{\lambda}(F_{1,2}+F_{2,1}) \\ \frac{\sqrt{\lambda}\mu}{\sqrt{\lambda+\mu}} F_{1,1} + 2\sqrt{\lambda(\lambda+\mu)} F_{2,2} \\ c_P\sqrt{\lambda}(F_{1,3}+F_{3,1}) \\ c_P\sqrt{\lambda}(F_{2,3}+F_{3,2}) \\ \mu F_{1,1} + \mu F_{2,2} + c_P^2 F_{3,3} \\ c_P z \end{pmatrix} \tag{1.4}$$

puts this system in the symmetric form

$$Lu := \partial_t u + A^1\partial_1 u + A^2\partial_2 u + A^3\partial_3 u = 0. \tag{1.5}$$

For $\xi = (\eta,\xi_3)^T = (\xi_1,\xi_2,\xi_3)^T$, agreeing with usual notations for matrices, we compute

$$A(\xi) = A^1\xi_1 + A^2\xi_2 + A^3\xi_3 = \begin{pmatrix} \mathbf{0}_3 & a_{1,2}(\eta) \\ a_{1,2}(\eta)^T & a_2(\eta)+\xi_3 a^2 \end{pmatrix}; \tag{1.6}$$

hereafter, we will write $\mathbf{0}_{m\times n}$ for the zero matrix of size $m\times n$, in particular we set $\mathbf{0}_n := \mathbf{0}_{n\times n}$ and $0_n := \mathbf{0}_{n\times 1}$; furthermore

$$a_{1,2}(\eta) := (\mathbf{0}_3\ \Phi(\eta)\ 0_3) \quad \Phi(\eta) := \begin{pmatrix} \Lambda\xi_1 & 0 \\ \sqrt{\lambda}\xi_2 & \sqrt{\lambda}\xi_1 \\ \Xi\xi_1 & \Theta\xi_2 \end{pmatrix}, \tag{1.7}$$

with $\Lambda := \frac{\sqrt{\lambda(2\lambda+3\mu)}}{\sqrt{\lambda+\mu}}$, $\Xi := \frac{\mu\sqrt{\lambda}}{c_P\sqrt{\lambda+\mu}}$, $\Theta := \frac{2\sqrt{\lambda(\lambda+\mu)}}{c_P}$,

$$a_2(\eta) := \begin{pmatrix} \mathbf{0}_3 & b(\eta)^T \\ b(\eta) & \mathbf{0}_3 \end{pmatrix}, \quad b(\eta) = \begin{pmatrix} \mathbf{0}_2 & \frac{\mu}{c_P}\eta \\ \sqrt{\lambda}\eta^T & 0 \end{pmatrix} \tag{1.8}$$

and

$$a^2 := \begin{pmatrix} \mathbf{0}_3 & b^2 \\ b^2 & \mathbf{0}_3 \end{pmatrix}, \quad b^2 := \begin{pmatrix} \sqrt{\lambda} & 0 & 0 \\ 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & c_P \end{pmatrix}. \tag{1.9}$$

We are interested in the well posedness of an initial boundary value problem (ibvp) for (1.5) on the half-space $\mathbb{R}^3_+ := \mathbb{R}^2\times\mathbb{R}^+$; setting for brevity $y=(x_1,x_2)$, the ibvp reads as follows:

$$\begin{aligned} &Lu(y,x_3,t) = f(y,x_3,t), \quad y\in\mathbb{R}^2, x_3,t>0, \\ &Bu(y,0,t) = g(y,t), \quad y\in\mathbb{R}^2, t>0, \\ &u(y,x_3,0) = a(y,x_3), \quad y\in\mathbb{R}^2, x_3>0. \end{aligned} \tag{1.10}$$

Here f,g,a are given smooth functions, while B is a given 3×9 real matrix with $\mathrm{rank}B=3$. Since $\mathrm{rank}A^3=6$, the ibvp (1.10) is uniformly characteristic in the sense of [5]. In [5], Majda and Osher studied the well posedness of an ibvp for a wide class of symmetric hyperbolic linear systems obeying several structural hypotheses. However, Majda-Osher's analysis does not encompass the 3D elasticity system (1.5)-(1.9); indeed, differently from the two-dimensional (2D) case (cf. [7]), the matrix

$A(\eta,0)$ (which can be written down by setting $\xi_3=0$ in (1.6)) does not fulfill the Assumption 1.1 of Majda-Osher's [5] where the upper-left block of $A(\eta,0)$ is required to have simple eigenvalues.We assume that $\mathrm{Ker}A^3=\mathbb{R}^3\times\{0_6\}\subset\mathrm{Ker}B$, which yields $B=(\mathbf{0}_3\,B_2)$ with $B_2\in\mathbf{M}_{3\times 6}(\mathbb{R})$ of full rank. This last assumption, which is called *reflexivity* by T. Ohkubo [8], is natural for characteristic ibvps, since for L^2 solutions u the best control of boundary terms that we expect is that of A^3u; as a matter of fact, this restriction is also justified by [5].We study the strong L^2-well posedness of (1.10), assuming that the matrix B satisfies the well-known *uniform Kreiss-Lopatinskii condition* (written (UKL), for the sake of brevity). Recall that the characteristic ibvp (1.10) is said to fulfill the (UKL) condition provided that there exists a positive constant C for which the estimate below

$$|A^3V|\le C|BV|,\quad V\in\mathcal{E}_-(\tau,\eta),$$

holds true for all pairs $(\tau,\eta)\in\mathbb{C}\times\mathbb{R}^2$, with $\Re\tau>0$. For any (τ,η) as before, we mean by $\mathcal{E}_-(\tau,\eta)$ the *stable subspace* of the system

$$(\tau I_9+iA(\eta,0))V+A^3\frac{dV}{dx_3}=0,$$

obtained by taking the Fourier-Laplace transform of (1.5) with respect to (y,t). We refer to [4] (see also [9] Chapter 14) for the precise statement of the (UKL) condition. The main result of the paper can be stated as follows

THEOREM 1.1. *Let us consider the ibvp* (1.10)*; let the matrix* $B\in\mathbf{M}_{3\times 9}(\mathbb{R})$ *satisfy the reflexivity and the (UKL) condition. Then for every data* $f\in L^2(\mathbb{R}^3_+\times(0,T))$, $g\in L^2(\mathbb{R}^2\times(0,T))$ *and* $a\in L^2(\mathbb{R}^3_+)$*, with arbitrary* $T>0$*, there exists one, and only one, solution* $u\in L^2(\mathbb{R}^3_+\times(0,T))$ *of* (1.10) *such that:*

a. $u\in C([0,T];L^2(\mathbb{R}^3_+))$*;*

b. A^3u *admits a trace* γ_0A^3u *on the boundary* $\partial\mathbb{R}^3_+\equiv\mathbb{R}^2$ *of* $\mathbb{R}^3_+$ *of class* $L^2(\mathbb{R}^2\times(0,T))$.

Finally, for every positive number γ*, the following a priori estimate holds true*

$$e^{-2\gamma T}\|u(T)\|^2_{L^2}+\|u\|^2_{\gamma,T}\le C\left(\|a\|^2_{L^2}+\int_0^T e^{-2\gamma t}\left(\frac{1}{\gamma}\|f(t)\|^2_{L^2}+\|g(t)\|^2_{L^2}\right)dt\right),\tag{1.11}$$

where the constant $C>0$ *does not depend on* f,g,a *and* γ,T*. In* (1.11) $\|.\|_{L^2}$ *denotes either the norm in* $L^2(\mathbb{R}^3_+)$ *or that in* $L^2(\mathbb{R}^2)$*; moreover we have set*

$$\|u\|^2_{\gamma,T}:=\int_0^T\int_{\mathbb{R}^2}e^{-2\gamma t}|(\gamma_0A^2u)(y,t)|^2dydt+\gamma\int_0^T\int_0^{+\infty}\int_{\mathbb{R}^2}e^{-2\gamma t}|u(y,x_3,t)|^2dydx_3dt.\tag{1.12}$$

The preceding theorem is precisely the 3D counterpart of Theorem 1.1. in [7]. To prove this result, we look for the existence of a *dissipative Kreiss symmetrizer* of (1.10) (cf. [1, 4, 6]). Recall that a dissipative symmetrizer consists of a smooth matrix-valued function $(\tau,\eta)\mapsto K(\tau,\eta)\in\mathbf{M}_{9\times 9}(\mathbb{C})$, defined on the set $\mathcal{X}:=\{(\tau,\eta)\in\mathbb{C}\times\mathbb{R}^2:\Re\tau\ge 0,\,|\tau|+|\eta|\neq 0\}$, fullfilling the following assumptions:

i. $\Sigma(\tau,\eta):=K(\tau,\eta)A^3$ is Hermitian for every $(\tau,\eta)\in\mathcal{X}$;

ii. $\Sigma(\tau,\eta)$ must be non positive on $\mathrm{Ker}B$ and its restriction to $\mathrm{Ker}B$ vanishes only on $\mathrm{Ker}A^3$, uniformly in $\mathcal{X}$;

iii. For $P(\tau,\eta):=K(\tau,\eta)(\tau I_9+iA(\eta,0))$, there exists a positive number c_0 such that:

$$\Re P \geq c_0(\Re\tau)I_9, \qquad \forall(\tau,\eta)\in\mathcal{X}. \tag{1.13}$$

As in the case of a non-characteristic strictly hyperbolic ibvp, considered by Kreiss [4], a dissipative symmetrizer $K(\tau,\eta)$ turns out to be a fundamental tool in order to investigate the well posedness of ibvps in more general situations. Besides the already mentioned work of Majda and Osher [5], let us quote a recent result by D. Serre (cf. [1], §6.2) where the existence of a dissipative symmetrizer is proved for a general characteristic Friedrichs symmetric ibvp, fulfilling the (UKL) condition, under an auxiliary assumption due to T. Ohkubo. Namely Ohkubo considers in [8] a Friedrichs symmetric system like (1.5)-(1.6) for which $a_2(\eta)$ vanishes identically. In fact the assumption made by Ohkubo is slightly more general and it is satisfied by many relevant physical examples such as the curl operator, Maxwell system and the shallow water equations. However the system of linear elasticity does not meet the Ohkubo assumption; indeed for the linear elasticity we get the nontrivial $a_2(\eta)$ displayed in (1.8). Analogously to the case of a non-characteristic ibvp, actually we are led to find a symmetrizer $K(\tau,\eta)$ which is homogeneous of degree zero in (τ,η). Therefore, it will be enough to build $K(\tau,\eta)$ in the unit hemi-sphere of $\mathcal{X}$, namely the set of pairs $(\tau,\eta)\in\mathcal{X}$ such that $\Re\tau\geq 0$ and $|\tau|^2+|\eta|^2=1$. By a compactness argument, we still reduce to define $K(\tau,\eta)$, with properties i.-iii., locally in a neighborhood of each point of the unit hemi-sphere. As in the 2D case, studied in [7], we see that the arguments used in [1] to build the dissipative symmetrizer of a characteristic ibvp, under the Ohkubo hypothesis, work as well in order to make a dissipative symmetrizer of (1.10) near the "interior points" $(\tau,\eta)\in\mathcal{X}$, with $\Re\tau>0$, and the "boundary points" $(\tau,\eta)\in\mathcal{X}$, with $\Re\tau=0$ and $\tau\neq 0$. It is just in the vicinity of the "central points" $(0,\eta)$, with $\eta\neq 0$, that the aforesaid Ohkubo assumption plays a fundamental role in the analysis performed in [1]. The next section is devoted to presenting an alternative strategy of construction of the symmetrizer near the latter critical points.

2. Construction of a dissipative symmetrizer near the central points

Throughout this section, we will assume that the matrix $B\in\mathbf{M}_{3\times 9}(\mathbb{R})$ satisfies the reflexivity and the (UKL) condition. Remember that we are looking for a smooth function $(\tau,\eta)\mapsto K(\tau,\eta)\in\mathbf{M}_{9\times 9}(\mathbb{C})$, defined in an open neighborhood of each point $(0,\eta_0)$, with $\eta_0\neq 0$, displaying properties i.-iii., listed at the end of the preceding section. Due to the homogeneity, actually we restrict our construction to the points $(0,\eta_0)$ such that $|\eta_0|=1$. Agreeing with the notations of §1 and setting also $\eta=(\xi_1,\xi_2)^T$, we write blockwise the matrices A^3 and $\tau I_9+iA(\eta,0)$ of the linear elasticity as

$$A^3=\begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_{3\times 2} & 0_3 & \mathbf{0}_{3\times 2} & 0_3 \\ \mathbf{0}_{3\times 2}^T & \mathbf{0}_2 & 0_2 & \sqrt{\lambda}I_2 & 0_2 \\ 0_3^T & 0_2^T & 0 & 0_2^T & c_P \\ \mathbf{0}_{3\times 2}^T & \sqrt{\lambda}I_2 & 0_2 & \mathbf{0}_2 & 0_2 \\ 0_3^T & 0_2^T & c_P & 0_2^T & 0 \end{pmatrix}, \tag{2.1}$$

$$\tau I_9+iA(\eta,0)=\begin{pmatrix} \tau I_3 & \mathbf{0}_{3\times 2} & 0_3 & i\Phi(\eta) & 0_3 \\ \mathbf{0}_{3\times 2}^T & \tau I_2 & 0_2 & \mathbf{0}_2 & i\sqrt{\lambda}\eta \\ 0_3^T & 0_2^T & \tau & i\frac{\mu}{c_P}\eta^T & 0 \\ i\Phi(\eta)^T & \mathbf{0}_2 & i\frac{\mu}{c_P}\eta & \tau I_2 & 0_2 \\ 0_3^T & i\sqrt{\lambda}\eta^T & 0 & 0_2^T & \tau \end{pmatrix}. \tag{2.2}$$

In this way, the above matrices keep exactly the same expressions as in the 2D case (cf. [7]). It is worthwhile remarking that for an arbitrary $\eta \neq 0_2$ we have $\mathrm{Ker}\,\Phi(\eta) = \{0_2\}$; this implies the existence of a 2×3 real matrix $\Psi(\eta)$ such that $\Psi(\eta)\Phi(\eta) = I_2$. Requiring that the matrix $\Sigma = KA^3$ is Hermitian leads to the next expression of K

$$K = \begin{pmatrix} k_{\mathrm{I,I}} & \mathbf{0}_{3\times 6} \\ K_1 & K_2 \end{pmatrix} \tag{2.3}$$

with suitable $k_{\mathrm{I,I}} \in \mathbf{M}_{3\times 3}(\mathbb{C})$, $K_1 \in \mathbf{M}_{6\times 3}(\mathbb{C})$ and $K_2 \in \mathbf{M}_{6\times 6}(\mathbb{C})$; consequently, Σ itself reduces to

$$\Sigma = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_{3\times 6} \\ \mathbf{0}^T_{3\times 6} & \Sigma_2 \end{pmatrix}, \tag{2.4}$$

where $\Sigma_2 := K_2 a^2$ must be Hermitian too. Later on, we will adopt for K the same blockwise structure that we used to express the matrices (2.1), (2.2); namely, by referring to (2.3), we will write K_1, K_2 as

$$K_1 = \begin{pmatrix} k_{\mathrm{II,I}} \\ k_{\mathrm{III,I}} \\ k_{\mathrm{IV,I}} \\ k_{\mathrm{V,I}} \end{pmatrix}, \quad K_2 = \begin{pmatrix} k_{\mathrm{II,II}} & k_{\mathrm{II,III}} & k_{\mathrm{II,IV}} & k_{\mathrm{II,V}} \\ k_{\mathrm{III,II}} & k_{\mathrm{III,III}} & k_{\mathrm{III,IV}} & k_{\mathrm{III,V}} \\ k_{\mathrm{IV,II}} & k_{\mathrm{IV,III}} & k_{\mathrm{IV,IV}} & k_{\mathrm{IV,V}} \\ k_{\mathrm{V,II}} & k_{\mathrm{V,III}} & k_{\mathrm{V,IV}} & k_{\mathrm{V,V}} \end{pmatrix}; \tag{2.5}$$

the blocks $k_{i,j}$, $i = \mathrm{II},\dots,\mathrm{V}, j = \mathrm{I},\dots,\mathrm{V}$, together with $k_{\mathrm{I,I}}$, are complex matrices (to be determined appropriately), whose sizes are equal to that of the corresponding blocks in (2.1), (2.2). Agreeing with (2.3), (2.5), we split the variable $z = (z_1,\dots,z_9)^T$ of $\mathbb{C}^9$ as $z = (z_I, z_{\mathrm{II}}, z_{\mathrm{III}}, z_{\mathrm{IV}}, z_{\mathrm{V}})^T$, with $z_{\mathrm{I}} \in \mathbb{C}^3$, $z_{\mathrm{II}}, z_{\mathrm{IV}} \in \mathbb{C}^2$ and $z_{\mathrm{III}}, z_{\mathrm{V}} \in \mathbb{C}$. Requiring that Σ_2 is Hermitian produces the next conditions on the blocks $k_{i,j}$; they are formally analogous to the conditions that we already got in the 2D case (cf. [7]).

$$\begin{aligned}
&k_{\mathrm{II,IV}} = k^*_{\mathrm{II,IV}}, \qquad k_{\mathrm{IV,II}} = k^*_{\mathrm{IV,II}} \quad \text{(i.e. } k_{\mathrm{II,IV}} \text{ and } k_{\mathrm{IV,II}} \text{ are Hermitian)},\\
&k_{\mathrm{III,V}}, k_{\mathrm{V,III}} \in \mathbb{R}\\
&\sqrt{\lambda}k_{\mathrm{III,IV}} = c_P k^*_{\mathrm{II,V}}, \quad k_{\mathrm{IV,IV}} = k^*_{\mathrm{II,II}},\\
&\sqrt{\lambda}k_{\mathrm{V,IV}} = c_P k^*_{\mathrm{II,III}}, \quad c_P k_{\mathrm{IV,V}} = \sqrt{\lambda}k^*_{\mathrm{III,II}},\\
&k_{\mathrm{V,V}} = \overline{k}_{\mathrm{III,III}}, \qquad \sqrt{\lambda}k_{\mathrm{V,II}} = c_P k^*_{\mathrm{IV,III}}.
\end{aligned} \tag{2.6}$$

In view of the inclusion $\mathrm{Ker}A^3 \subset \mathrm{Ker}B$, the assumption ii. reduces to the existence of a positive constant ϵ_0 such that the inequality

$$\Sigma_{2|\mathrm{Ker}B_2} \leq -\epsilon_0 I_6, \tag{2.7}$$

holds true for all $(\tau,\eta) \in \mathcal{X}$ belonging to a neighborhood of each point $(0,\eta_0)$ with $|\eta_0| = 1$ on the unit hemi-sphere. Setting $z' = (z_{\mathrm{II}}, z_{\mathrm{III}}, z_{\mathrm{IV}}, z_{\mathrm{V}})^T$, we find also

$$\begin{aligned}
(z')^*\Sigma_2 z' &= z^*_{\mathrm{II}}\sqrt{\lambda}k_{\mathrm{II,IV}}z_{\mathrm{II}} + c_P k_{\mathrm{III,V}}|z_{\mathrm{III}}|^2 + z^*_{\mathrm{IV}}\sqrt{\lambda}k_{\mathrm{IV,II}}z_{\mathrm{IV}} + c_P k_{\mathrm{V,III}}|z_{\mathrm{V}}|^2\\
&+ 2\Re(z^*_{\mathrm{II}}c_P k_{\mathrm{II,V}}z_{\mathrm{III}}) + 2\Re(z^*_{\mathrm{II}}\sqrt{\lambda}k_{\mathrm{II,II}}z_{\mathrm{IV}}) + 2\Re(z^*_{II}c_P k_{\mathrm{II,III}}z_{\mathrm{V}})\\
&+ 2\Re(\overline{z}_{\mathrm{III}}\sqrt{\lambda}k_{\mathrm{III,II}}z_{\mathrm{IV}}) + 2\Re(\overline{z}_{\mathrm{III}}c_P k_{\mathrm{III,III}}z_{\mathrm{V}}) + 2\Re(z^*_{\mathrm{IV}}c_P k_{\mathrm{IV,III}}z_{\mathrm{V}}),
\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
z^*Pz &= \tau z_{\mathrm{I}}^* k_{\mathrm{I,I}} z_{\mathrm{I}} + z_{\mathrm{II}}^*(\tau k_{\mathrm{II,I}} + ik_{\mathrm{II,IV}}\Phi(\eta)^T)z_{\mathrm{I}} + \overline{z}_{\mathrm{III}}(\tau k_{\mathrm{III,I}} + i\frac{c_P}{\sqrt{\lambda}}k_{\mathrm{II,V}}^*\Phi(\eta)^T)z_{\mathrm{I}} \\
&+ z_{\mathrm{IV}}^*(\tau k_{\mathrm{IV,I}} + ik_{\mathrm{II,II}}^*\Phi(\eta)^T)z_{\mathrm{I}} + \overline{z}_{\mathrm{V}}(\tau k_{\mathrm{V,I}} + i\frac{c_P}{\sqrt{\lambda}}k_{\mathrm{II,III}}^*\Phi(\eta)^T)z_{\mathrm{I}} + z_{\mathrm{II}}^*(\tau k_{\mathrm{II,II}} \\
&+ i\sqrt{\lambda}k_{\mathrm{II,V}}\eta^T)z_{\mathrm{II}} + \overline{z}_{\mathrm{III}}(\tau k_{\mathrm{III,II}} + i\sqrt{\lambda}k_{\mathrm{III,V}}\eta^T)z_{\mathrm{II}} + z_{\mathrm{IV}}^*(\tau k_{\mathrm{IV,II}} + i\frac{\lambda}{c_P}k_{\mathrm{III,II}}^*\eta^T)z_{\mathrm{II}} \\
&+ \overline{z}_{\mathrm{V}}(\tau\frac{c_P}{\sqrt{\lambda}}k_{\mathrm{IV,III}}^* + i\sqrt{\lambda}\overline{k}_{\mathrm{III,III}}\eta^T)z_{\mathrm{II}} + z_{\mathrm{II}}^*(\tau k_{\mathrm{II,III}} + i\frac{\mu}{c_P}k_{\mathrm{II,IV}}\eta)z_{\mathrm{III}} + (\tau k_{\mathrm{III,III}} \\
&+ i\frac{\mu}{\sqrt{\lambda}}k_{\mathrm{II,V}}^*\eta)|z_{\mathrm{III}}|^2 + z_{\mathrm{IV}}^*(\tau k_{\mathrm{IV,III}} + i\frac{\mu}{c_P}k_{\mathrm{II,II}}^*\eta)z_{\mathrm{III}} + \overline{z}_{\mathrm{V}}(\tau k_{\mathrm{V,III}} + i\frac{\mu}{\sqrt{\lambda}}k_{\mathrm{II,III}}^*\eta)z_{\mathrm{III}} \\
&+ z_{\mathrm{I}}^* ik_{\mathrm{I,I}}\Phi(\eta)z_{\mathrm{IV}} + z_{\mathrm{II}}^*(ik_{\mathrm{II,I}}\Phi(\eta) + i\frac{\mu}{c_P}k_{\mathrm{II,III}}\eta^T + \tau k_{\mathrm{II,IV}})z_{\mathrm{IV}} + \overline{z}_{\mathrm{III}}(ik_{\mathrm{III,I}}\Phi(\eta) \\
&+ i\frac{\mu}{c_P}k_{\mathrm{III,III}}\eta^T + \tau\frac{c_P}{\sqrt{\lambda}}k_{\mathrm{II,V}}^*)z_{\mathrm{IV}} + z_{\mathrm{IV}}^*(ik_{\mathrm{IV,I}}\Phi(\eta) + i\frac{\mu}{c_P}k_{\mathrm{IV,III}}\eta^T + \tau k_{\mathrm{II,II}}^*)z_{\mathrm{IV}} \\
&+ \overline{z}_{\mathrm{V}}(ik_{\mathrm{V,I}}\Phi(\eta) + i\frac{\mu}{c_P}k_{\mathrm{V,III}}\eta^T + \tau\frac{c_P}{\sqrt{\lambda}}k_{\mathrm{II,III}}^*)z_{\mathrm{IV}} + z_{\mathrm{II}}^*(i\sqrt{\lambda}k_{\mathrm{II,II}}\eta + \tau k_{\mathrm{II,V}})z_{\mathrm{V}} \\
&+ \overline{z}_{\mathrm{III}}(i\sqrt{\lambda}k_{\mathrm{III,II}}\eta + \tau k_{\mathrm{III,V}})z_{\mathrm{V}} + z_{\mathrm{IV}}^*(i\sqrt{\lambda}k_{\mathrm{IV,II}}\eta + \tau\frac{\sqrt{\lambda}}{c_P}k_{\mathrm{III,II}}^*)z_{\mathrm{V}} + (ic_Pk_{\mathrm{IV,III}}^*\eta \\
&+ \tau\overline{k}_{\mathrm{III,III}})|z_{\mathrm{V}}|^2. \qquad (2.9)
\end{aligned}
$$

Analogously to the 2D case, we go to specialize the values of the blocks $k_{i,j}$ of $K(\tau,\eta)$; namely for any $\tau = \gamma + i\rho$, with $\gamma \ge 0$, and $\eta \in \mathbb{R}^2$ such that $|\tau|^2 + |\eta|^2 = 1$, $|\tau|$ is sufficiently small and η ranges over a small neighborhood $\mathcal{V}_0$ of a given η_0 with $|\eta_0| = 1$, we set

$$
\begin{aligned}
&k_{\mathrm{I,I}} = (h + \chi\overline{\tau})I_3, \quad k_{\mathrm{II,II}} = (h + \chi\overline{\tau})I_2, \quad k_{\mathrm{III,III}} = h + \chi\overline{\tau}, \\
&k_{\mathrm{II,I}} = \gamma\Phi(\eta)^T, \quad k_{\mathrm{III,I}} = k_{\mathrm{V,I}} = \gamma\eta^T\Phi(\eta)^T, \quad k_{\mathrm{IV,I}} = G_{M,N}(\eta)\Psi(\eta), \\
&k_{\mathrm{II,IV}} = -\rho\gamma I_2, \quad k_{\mathrm{II,III}} = k_{\mathrm{II,V}} = -\frac{\sqrt{\lambda}}{c_P}\rho\gamma\eta, \\
&k_{\mathrm{III,II}} = 0_2^T, \quad k_{\mathrm{III,V}} = -\frac{\mu}{c_P\sqrt{\lambda}}\rho\gamma, \\
&k_{\mathrm{IV,II}} = -AI_2, \quad k_{\mathrm{IV,III}} = iN\eta, \quad k_{\mathrm{V,III}} = -A\frac{c_P\sqrt{\lambda}}{\mu}. \qquad (2.10)
\end{aligned}
$$

Here h, χ, A, M, N are positive constants to be chosen in a suitable way, while for given positive M, N and $\eta \ne 0_2$, $G_{M,N}(\eta)$ is the squared matrix

$$
G_{M,N}(\eta) = -i\begin{pmatrix} M\xi_1^2 + \left(M - \frac{\mu}{c_P}N\right)\xi_2^2 & \frac{\mu}{c_P}N\xi_1\xi_2 \\ \frac{\mu}{c_P}N\xi_1\xi_2 & \left(M - \frac{\mu}{c_P}N\right)\xi_1^2 + M\xi_2^2 \end{pmatrix} \qquad (2.11)
$$

and $\Psi(\eta)$ is a left inverse of $\Phi(\eta)$. The blocks $k_{i,j}$ of K which are not explicitly listed in (2.10) are determined by positions (2.10) themselves through the relations in (2.6). Putting (2.10) into (2.8) we get for $(z')^*\Sigma_2 z'$ the following expression

$$
(z')^*\Sigma_2 z' = S(z') - AQ(z') + R_\chi(z'), \qquad (2.12)
$$

where

$$
\begin{aligned}
S(z') &:= -\sqrt{\lambda}\rho\gamma|z_{\mathrm{II}}|^2 - \frac{\mu}{c_P\sqrt{\lambda}}\rho\gamma|z_{\mathrm{III}}|^2 + 2\Re(-\sqrt{\lambda}\rho\gamma z_{\mathrm{II}}^*\eta z_{\mathrm{III}}) \\
&+ 2\Re\,(z_{\mathrm{II}}^*\sqrt{\lambda}hz_{\mathrm{IV}}) + 2\Re(-\sqrt{\lambda}\rho\gamma z_{\mathrm{II}}^*\eta z_{\mathrm{V}}) + 2\Re(\overline{z}_{\mathrm{III}}c_Phz_{\mathrm{V}}) + 2\Re(ic_PNz_{\mathrm{IV}}^*\eta z_{\mathrm{V}}); \qquad (2.13)
\end{aligned}
$$

$$Q(z'):=\sqrt{\lambda}|z_{\mathrm{IV}}|^2+\frac{c_P^2\sqrt{\lambda}}{\mu}|z_{\mathrm{V}}|^2; \tag{2.14}$$

$$R_\chi(z'):=2\Re(\sqrt{\lambda}\chi\bar{\tau}z_{\mathrm{II}}^* z_{\mathrm{IV}})+2\Re(c_P\chi\bar{\tau}\bar{z}_{\mathrm{III}}z_{\mathrm{V}}). \tag{2.15}$$

For any $\eta\neq 0_2$, let us define the vector space $H(\eta)$ by

$$H(\eta):=\sum_{\substack{\xi\in\mathbb{R}\eta+\mathbb{R}\mathbf{e}_3\\ \xi\neq 0_3}}\mathrm{Ker}A(\xi), \tag{2.16}$$

where, hereafter, we set $\mathbf{e}_1=(1,0,0)^T$, $\mathbf{e}_2:=(0,1,0)^T$, $\mathbf{e}_3:=(0,0,1)^T$ and η denotes also the three-dimensional vector $\xi_1\mathbf{e}_1+\xi_2\mathbf{e}_2$. For Friedrichs symmetric systems like (1.5), $H(\eta)$ is shown to be an isotropic subspace of all the matrices $A(\xi)$, for any ξ belonging to the vector plane $\mathbb{R}\eta+\mathbb{R}\mathbf{e}_3$; in particular this implies that the dimension of $H(\eta)$ is not larger than 6 (cf. [1], §6.1.4-5). Since $H(\eta)$ contains $\mathrm{Ker}A^3=\mathbb{R}^3\times\{0_6\}$, for any $\eta\neq 0_2$ it can be splitted as $\mathbb{R}^3\times H_1(\eta)$, where $H_1(\eta)$ is an isotropic subspace, of dimension not larger that 3, for both the matrices $a_2(\eta)$ and a^2 defined in (1.8), (1.9). In the case of the 3D linear elasticity system, a direct computation shows that for any $\eta\neq 0_2$ and $\xi=\eta+\xi_3\mathbf{e}_3$ we have that $\mathrm{Ker}A(\xi)$, hence $H(\eta)$, is included in $\mathbb{R}^6\times\{0_3\}$ (cf. (1.6), (1.7) and recall that, for every $\eta\neq 0_2$, $\Phi(\eta)$ is injective). Actually, it can be shown that, for any non zero η, the dimension of $H(\eta)$ is maximal, so that $H(\eta)=\mathbb{R}^6\times\{0_3\}$; indeed the system of linearly independent vectors

$$\begin{pmatrix}\mathbf{e}_1\\0_2\\0\\0_3\end{pmatrix},\quad \begin{pmatrix}\mathbf{e}_2\\0_2\\0\\0_3\end{pmatrix},\quad \begin{pmatrix}\mathbf{e}_3\\0_2\\0\\0_3\end{pmatrix},\quad \begin{pmatrix}\Psi(\eta)^T\eta\\0_2\\-\frac{c_P}{\mu}\\0_3\end{pmatrix},\quad \begin{pmatrix}0_3\\\eta^\perp\\0\\0_3\end{pmatrix},\quad \begin{pmatrix}z_{\mathrm{I}}^0\\\eta\\-\frac{\sqrt{\lambda}}{c_P}|\eta|^2\\0_3\end{pmatrix}, \tag{2.17}$$

where $\eta^\perp:=(-\xi_2,\xi_1)^T$ and $z_{\mathrm{I}}^0:=\sqrt{\lambda}(\frac{\mu}{c_P^2}|\eta|^2-1)\Psi(\eta)^T\eta$, provides a basis of $H(\eta)$ (remember that $\Psi(\eta)$ is a left inverse of $\Phi(\eta)$). Analogously to the 2D case, we can also check that there are not non trivial vectors $U\in H(\eta)^\perp$ fulfilling

$$A(\eta+\xi_3\mathbf{e}_3)U\in H(\eta)^\perp, \tag{2.18}$$

for $\eta\neq 0_2$ and non real $\xi_3=-i\sigma$, $\sigma\neq 0$. It has been shown in [1] (see Proposition 6.6 there) that, in view of the preceding properties, the matrix $B=(\mathbf{0}_3\,B_2)$ satisfies the (UKL) condition near a central point $(0,\eta_0)$, with $|\eta_0|=1$, if, and only if,

$$\mathbb{C}^6=\mathrm{Ker}B_2\oplus H_1(\eta), \tag{2.19}$$

where, hereafter, $H_1(\eta)$ denotes both the real space and its complexification. For the 3D linear elasticity system, we easily compute

$$H_1(\eta)=\left\{\begin{pmatrix}z_{\mathrm{II}}\\z_{\mathrm{III}}\\0_2\\0\end{pmatrix}\right\},\quad \mathrm{Ker}a_{1,2}(\eta)=\left\{\begin{pmatrix}z_{\mathrm{II}}\\z_{\mathrm{III}}\\0_2\\z_{\mathrm{V}}\end{pmatrix}\right\},\quad \text{for } \eta\neq 0_2, \tag{2.20}$$

as z_{II} and $z_{\mathrm{III}},z_{\mathrm{V}}$ span $\mathbb{C}^2$ and $\mathbb{C}$ respectively, so that $H_1(\eta)\subset\mathrm{Ker}\,a_{1,2}(\eta)$; actually the latter inclusion holds true in the framework of a general Friedrichs symmetric system with characteristic boundary (cf. [1], §6.1.4 again). Let $H_1(\eta)^\perp$ be the subspace of $\mathrm{Ker}\, a_{1,2}(\eta)$ orthogonal to $H_1(\eta)$; thus we get the following decomposition

$$\mathbb{C}^6 = H_1(\eta) \oplus^\perp H_1(\eta)^\perp \oplus^\perp R(a_{1,2}(\eta)^T), \tag{2.21}$$

where $\oplus^\perp$ stands for the orthogonal direct sum operator. As we already specified, let us assume that η runs through a small neighborhood $\mathcal{V}_0$ of a given point η_0 with $|\eta_0| = 1$; following the analysis performed in [1] for the Ohkubo's case (cf. also [7]), from (2.19) we show the existence of a linear operator $D = D(\eta)$, acting from $H_1(\eta)^\perp \oplus^\perp R(a_{1,2}(\eta)^T)$ to $H_1(\eta)$, depending smoothly thus boundedly on $\frac{\eta}{|\eta|}$, such that

$$\mathrm{Ker} B_2 = \{r + Dr; r \in H_1(\eta)^\perp \oplus^\perp R(a_{1,2}(\eta)^T)\}. \tag{2.22}$$

For the 3D linear elasticity, we compute explicitly $H_1(\eta)^\perp \oplus^\perp R(a_{1,2}(\eta)^T) = \{0_3\} \times \mathbb{C}^3$. Therefore we may conclude, like in 2D, that the quadratic form $Q(z')$ in (2.14) is positive definite on $\mathrm{Ker} B_2$; more precisely there exists a positive constant $\epsilon > 0$ such that for all $z' \in \mathrm{Ker} B_2$, $\eta \in \mathcal{V}_0$ and $|\rho|, \gamma \geq 0$ sufficiently small

$$Q(z') \geq \epsilon |z'|^2. \tag{2.23}$$

Straightforward computations give also the following estimates for the quadratic forms $S(z')$ and $R_\chi(z')$

$$\begin{aligned} S(z') &\leq c^* |z'|^2, \\ R_\chi(z') &\leq C_1 \chi |\tau| |z'|^2, \end{aligned} \tag{2.24}$$

as $z', \eta, \gamma, |\rho|$ range as before, $C_1 := \max\{c_P, \sqrt{\lambda}\}$ and c^* depends only (increasingly) on N, h. Analogously to the 2D case, estimates (2.23), (2.24) and formula (2.12) lead to prove the following result about Σ_2.

LEMMA 2.1. *For given $h, N, \delta_0 > 0$ there exists a constant $A > 0$ such that for every $\chi > 0$ we find $\sigma_0 = \sigma_0(\chi) > 0$ for which*

$$(z')^* \Sigma_2 z' \leq -\delta_0 |z'|^2, \qquad z' \in \mathrm{Ker} B_2, \quad \eta \in \mathcal{V}_0, \quad |\tau| < \sigma_0, \quad \Re\tau \geq 0. \tag{2.25}$$

The remaining part of the construction will be devoted to seeking for suitable values of the constants h, M, N involved in (2.10), in such a way that the corresponding $K = K(\tau, \eta)$ satisfies the estimate (1.13). Later on, for a given matrix $W = (w_{i,j})$, we will set $\|W\| := \max_{i,j} |w_{i,j}|$. It is worthwhile emphasizing some simple facts about the matrices $G_{M,N}(\eta)$, $\Phi(\eta)$ and $\Psi(\eta)$ previously introduced (cf. (1.7), (2.11)). Firstly, let us assume that $M > \frac{\mu}{c_P} N$; then the matrix $G_{M,N}(\eta)$ fulfills

$$\|G_{M,N}(\eta)\| \leq M |\eta|^2.$$

On the other hand, when η belongs to a small neighborhood $\mathcal{V}_0$ of some point η_0, with $|\eta_0| = 1$, the matrix $\Psi(\eta)$ may be constructed in such a way that it depends smoothly, thus boundedly, on η. Therefore, its norm, as well as the norm of $\Phi(\eta)$, is bounded from above as $\eta \in \mathcal{V}_0$. The computations we need here are very similar to the ones performed in the 2D case, due to the formal analogy between (2.9) and its 2D counterpart (cf. [7]). However, for convenience of the reader, let us sketch here below the main arguments leading to the desired result. Firstly, we plug the expressions (2.10) in (2.9) and take the real part of the resulting z^*Pz. From the values of $k_{\mathrm{I,I}}, k_{\mathrm{II,II}}, k_{\mathrm{III,III}}$ we find immediately

$$\begin{aligned}
&\Re(z^*_{\mathrm{IV}} i k^*_{\mathrm{II,II}} \Phi(\eta)^T z_{\mathrm{I}}) + \Re(z^*_{\mathrm{I}} i k_{\mathrm{I,I}} \Phi(\eta) z_{\mathrm{IV}}) = 0,\\
&\Re(\overline{z}_{\mathrm{V}} i\sqrt{\lambda}\overline{k}_{\mathrm{III,III}} \eta^T z_{\mathrm{II}}) + \Re(z^*_{\mathrm{II}} i\sqrt{\lambda} k_{\mathrm{II,II}} \eta z_{\mathrm{V}}) = 0,\\
&\Re(z^*_{\mathrm{IV}} i\tfrac{\mu}{c_P} k^*_{\mathrm{II,II}} \eta z_{\mathrm{III}}) + \Re(\overline{z}_{\mathrm{III}} i\tfrac{\mu}{c_P} k_{\mathrm{III,III}} \eta^T z_{\mathrm{IV}}) = 0.
\end{aligned}$$

On the other hand, from (2.10) we derive directly $\sqrt{\lambda} k_{\mathrm{III,V}} I_2 = \frac{\mu}{c_P} k_{\mathrm{II,IV}}$, $\frac{\mu}{c_P} k_{\mathrm{V,III}} I_2 = \sqrt{\lambda} k_{\mathrm{IV,II}}$. Hence we get

$$\begin{aligned}
&\Re(\overline{z}_{\mathrm{III}} i\sqrt{\lambda} k_{\mathrm{III,V}} \eta^T z_{\mathrm{II}}) + \Re(z^*_{\mathrm{II}} i\tfrac{\mu}{c_P} k_{\mathrm{II,IV}} \eta z_{\mathrm{III}}) = 0,\\
&\Re(\overline{z}_{\mathrm{V}} i\tfrac{\mu}{c_P} k_{\mathrm{V,III}} \eta^T z_{\mathrm{IV}}) + \Re(z^*_{\mathrm{IV}} i\sqrt{\lambda} k_{\mathrm{IV,II}} \eta z_{\mathrm{V}}) = 0.
\end{aligned}$$

Eventually, by the help of the Cauchy-Schwarz and Young inequalities, the remaining terms of $\Re(z^*Pz)$ are estimated as follows:

$$\begin{aligned}
&\Re(\tau z^*_{\mathrm{I}} k_{\mathrm{I,I}} z_{\mathrm{I}}) = (h\gamma + \chi|\tau|^2)|z_{\mathrm{I}}|^2;\\
&\Re(z^*_{\mathrm{II}}(\tau k_{\mathrm{II,I}} + i k_{\mathrm{II,IV}} \Phi(\eta)^T) z_{\mathrm{I}}) \geq -\tfrac{\gamma^2\|\Phi(\eta)\|}{2}(|z_{\mathrm{I}}|^2 + |z_{\mathrm{II}}|^2);\\
&\Re(\overline{z}_{\mathrm{III}}(\tau k_{\mathrm{III,I}} + i\tfrac{c_P}{\sqrt{\lambda}} k^*_{\mathrm{II,V}} \Phi(\eta)^T) z_{\mathrm{I}}) \geq -\tfrac{\gamma^2|\eta|\|\Phi(\eta)\|}{2}(|z_{\mathrm{I}}|^2 + |z_{\mathrm{III}}|^2);\\
&\Re(z^*_{\mathrm{IV}} \tau k_{\mathrm{IV,I}} z_{\mathrm{I}}) \geq -\tfrac{M|\eta|^2\|\Psi(\eta)\|}{2}(\chi_1|\tau|^2|z_{\mathrm{I}}|^2 + \tfrac{1}{\chi_1}|z_{\mathrm{IV}}|^2);\\
&\Re(\overline{z}_{\mathrm{V}}(\tau k_{\mathrm{V,I}} + i\tfrac{c_P}{\sqrt{\lambda}} k^*_{\mathrm{II,III}} \Phi(\eta)^T) z_{\mathrm{I}}) \geq -\tfrac{\gamma^2|\eta|\|\Phi(\eta)\|}{2}(|z_{\mathrm{I}}|^2 + |z_{\mathrm{V}}|^2);\\
&\Re(z^*_{\mathrm{II}}(\tau k_{\mathrm{II,II}} + i\sqrt{\lambda} k_{\mathrm{II,V}} \eta^T) z_{\mathrm{II}}) = (h\gamma + \chi|\tau|^2)|z_{\mathrm{II}}|^2;\\
&\Re(z^*_{\mathrm{IV}} \tau k_{\mathrm{IV,II}} z_{\mathrm{II}}) \geq -\tfrac{A}{2}(\chi_2|\tau|^2|z_{\mathrm{II}}|^2 + \tfrac{1}{\chi_2}|z_{\mathrm{IV}}|^2);\\
&\Re(\overline{z}_{\mathrm{V}} \tau \tfrac{c_P}{\sqrt{\lambda}} k^*_{\mathrm{IV,III}} z_{\mathrm{II}}) \geq -\tfrac{c_P N|\eta|}{2\sqrt{\lambda}}(\chi_3|\tau|^2|z_{\mathrm{II}}|^2 + \tfrac{1}{\chi_3}|z_{\mathrm{V}}|^2);\\
&\Re(z^*_{\mathrm{II}} \tau k_{\mathrm{II,III}} z_{\mathrm{III}}) \geq -\tfrac{\sqrt{\lambda}|\rho||\tau||\eta|\gamma}{2c_P}(|z_{\mathrm{II}}|^2 + |z_{\mathrm{III}}|^2);\\
&\Re((\tau k_{\mathrm{III,III}} + i\tfrac{\mu}{\sqrt{\lambda}} k^*_{\mathrm{II,V}} \eta)|z_{\mathrm{III}}|^2) = (\gamma h + \chi|\tau|^2)|z_{\mathrm{III}}|^2;
\end{aligned}$$

$$\begin{aligned}
&\Re(z^*_{\mathrm{IV}} \tau k_{\mathrm{IV,III}} z_{\mathrm{III}}) \geq -\tfrac{N|\eta|}{2}(\chi_4|\tau|^2|z_{\mathrm{III}}|^2 + \tfrac{1}{\chi_4}|z_{\mathrm{IV}}|^2);\\
&\Re(\overline{z}_{\mathrm{V}}(\tau k_{\mathrm{V,III}} + i\tfrac{\mu}{\sqrt{\lambda}} k^*_{\mathrm{II,III}} \eta) z_{\mathrm{III}}) \geq -\tfrac{\sqrt{\lambda} c_P A}{2\mu}(\chi_5|\tau|^2|z_{\mathrm{III}}|^2 + \tfrac{1}{\chi_5}|z_{\mathrm{V}}|^2)\\
&\qquad -\tfrac{\mu|\rho|\gamma|\eta|^2}{2c_P}(|z_{\mathrm{III}}|^2 + |z_{\mathrm{V}}|^2);\\
&\Re(z^*_{\mathrm{II}}(i k_{\mathrm{II,I}} \Phi(\eta) + i\tfrac{\mu}{c_P} k_{\mathrm{II,III}} \eta^T + \tau k_{\mathrm{II,IV}}) z_{\mathrm{IV}}) \geq -\tfrac{\|\Phi(\eta)\|^2}{2}(\gamma^2|z_{\mathrm{II}}|^2 + |z_{\mathrm{IV}}|^2)\\
&\qquad -\tfrac{\mu\sqrt{\lambda}|\rho|\gamma|\eta|^2}{2c_P^2}(|z_{\mathrm{II}}|^2 + |z_{\mathrm{IV}}|^2) - \tfrac{|\tau||\rho|\gamma}{2}(|z_{\mathrm{II}}|^2 + |z_{\mathrm{IV}}|^2);\\
&\Re(\overline{z}_{\mathrm{III}}(i k_{\mathrm{III,I}} \Phi(\eta) + \tau\tfrac{c_P}{\sqrt{\lambda}} k^*_{\mathrm{II,V}}) z_{\mathrm{IV}}) \geq -\tfrac{|\eta|\|\Phi(\eta)\|^2}{2}(\gamma^2|z_{\mathrm{III}}|^2 + |z_{\mathrm{IV}}|^2)\\
&\qquad -\tfrac{\gamma|\rho||\tau|}{2}(|z_{\mathrm{III}}|^2 + |z_{\mathrm{IV}}|^2);\\
&\Re(z^*_{\mathrm{IV}}(i k_{\mathrm{IV,I}} \Phi(\eta) + i\tfrac{\mu}{c_P} k_{\mathrm{IV,III}} \eta^T + \tau k^*_{\mathrm{II,II}}) z_{\mathrm{IV}})\\
&\qquad = \Big(\Big(M - \tfrac{\mu}{c_P} N\Big)|\eta|^2 + h\gamma + \chi(\gamma^2 - \rho^2)\Big)|z_{\mathrm{IV}}|^2;\\
&\Re(\overline{z}_{\mathrm{V}}(i k_{\mathrm{V,I}} \Phi(\eta) + \tau\tfrac{c_P}{\sqrt{\lambda}} k^*_{\mathrm{II,III}}) z_{\mathrm{IV}}) \geq -\tfrac{|\eta|\|\Phi(\eta)\|^2\gamma}{2}(|z_{\mathrm{IV}}|^2 + |z_V|^2)\\
&\qquad -\tfrac{\gamma|\rho||\tau|}{2}(|z_{\mathrm{IV}}|^2 + |z_{\mathrm{V}}|^2);\\
&\Re(z^*_{\mathrm{II}} \tau k_{\mathrm{II,V}} z_{\mathrm{V}}) \geq -\tfrac{\sqrt{\lambda}|\rho|\gamma|\tau|}{2c_P}(|z_{\mathrm{II}}|^2 + |z_{\mathrm{V}}|^2);\\
&\Re(\overline{z}_{\mathrm{III}} \tau k_{\mathrm{III,V}} z_{\mathrm{V}}) \geq -\tfrac{\mu|\tau||\rho|\gamma}{2c_P\sqrt{\lambda}}(|z_{\mathrm{III}}|^2 + |z_{\mathrm{V}}|^2);\\
&\Re\big((i c_P k^*_{\mathrm{IV,III}} \eta + \tau\overline{k}_{\mathrm{III,III}})|z_{\mathrm{V}}|^2\big) = (c_P N|\eta|^2 + \gamma h + \chi(\gamma^2 - \rho^2))|z_{\mathrm{V}}|^2.
\end{aligned}$$

In the above inequalities χ_j, $j = 1,\dots,5$, are positive constants to be precised later together with the values of h, M, N. Analogously to the 2D case, gathering the latter estimates leads us to obtain

$$\Re(z^*Pz) \geq c_{\mathrm{I}}|z_{\mathrm{I}}|^2 + c_{\mathrm{II}}|z_{\mathrm{II}}|^2 + c_{\mathrm{III}}|z_{\mathrm{III}}|^2 + c_{\mathrm{IV}}|z_{\mathrm{IV}}|^2 + c_{\mathrm{V}}|z_{\mathrm{V}}|^2, \tag{2.26}$$

where η belongs to $\mathcal{V}_0$, $|\rho|, \gamma \geq 0$ are sufficiently small and $c_j = c_j(\tau,\eta)$ are given continuous functions of (τ,η). Let us set for brevity $\phi_0 := \|\Phi(\eta_0)\|$, $\psi_0 := \|\Psi(\eta_0)\|$; in view of the inequalities listed just above, the functions $c_{\mathrm{IV}}(\tau,\eta), c_{\mathrm{V}}(\tau,\eta)$ will satisfy

$$\begin{aligned}
&c_{\mathrm{IV}}(\tau,\eta_0) \geq M - \frac{\mu}{c_P}N - \phi_0^2 - \frac{M\psi_0}{2\chi_1} - \frac{A}{2\chi_2} - \frac{N}{2\chi_4} + \chi(\gamma^2 - \rho^2) - \frac{\sqrt{\lambda}\gamma|\rho|}{2c_P^2}\\
&- \frac{3\gamma|\rho|}{2}|\tau| - \frac{\phi_0^2}{2}\gamma,\\
&c_{\mathrm{V}}(\tau,\eta_0) \geq c_P N - \frac{c_P N}{2\sqrt{\lambda}\chi_3} - \frac{\sqrt{\lambda}c_P A}{2\mu\chi_5} + \chi(\gamma^2 - \rho^2) - \frac{\phi_0^2}{2}\gamma^2 - \frac{\mu\gamma|\rho|}{c_P} - \frac{\phi_0}{2}\gamma - \frac{\gamma|\rho|}{2}|\tau|\\
&- \frac{\sqrt{\lambda}\gamma|\rho|}{2c_P}|\tau| - \frac{\mu\gamma|\rho|}{2c_P\sqrt{\lambda}}|\tau|.
\end{aligned} \tag{2.27}$$

The latter inequalities suggest to choose the constants M,N so that

$$\widetilde{C} := M - \frac{\mu}{c_P}N - \phi_0^2 > 0. \tag{2.28}$$

Once M,N have been fixed, for given $h,\delta_0 > 0$, Lemma 2.1 allows to find also a positive constant A for which (2.25) is satisfied with arbitrary $\chi > 0$ provided that $|\tau|$ is sufficiently small; now, let us assume that χ_j, $j = 1,\dots,5$, are large enough so that

$$\begin{aligned}
\nu_{\mathrm{IV}} &:= 2\widetilde{C} - \frac{M\psi_0}{\chi_1} - \frac{A}{\chi_2} - \frac{N}{\chi_4} > 0,\\
\nu_{\mathrm{V}} &:= 2c_P N - \frac{c_P N}{\sqrt{\lambda}\chi_3} - \frac{\sqrt{\lambda}c_P A}{\mu\chi_5} > 0,
\end{aligned} \tag{2.29}$$

and observe that the terms in the right-hand sides of (2.27), which are not involved in (2.29), are $O(|\tau|)$; thus by shrinking $|\tau|$, if it is necessary, we get

$$c_j(\tau,\eta_0) > \frac{\nu_j}{4}, \quad j = \mathrm{IV},\mathrm{V}. \tag{2.30}$$

Besides (2.27), the functions $c_j(\tau,\eta)$, for $j = \mathrm{I},\mathrm{II},\mathrm{III}$, will satisfy

$$\begin{aligned}
&c_{\mathrm{I}}(\tau,\eta_0) \geq \left(h - \frac{3\phi_0}{2}\gamma\right)\gamma + \left(\chi - \frac{M\psi_0}{2}\chi_1\right)|\tau|^2,\\
&c_{\mathrm{II}}(\tau,\eta_0) \geq \left(h - \frac{\phi_0}{2}\gamma - \frac{\sqrt{\lambda}|\rho|}{2c_P}|\tau| - \frac{\phi_0^2}{2}\gamma - \frac{\mu\sqrt{\lambda}|\rho|}{2c_P^2} - \frac{|\rho|}{2}|\tau| - \frac{\sqrt{\lambda}|\rho|}{2c_P}|\tau|\right)\gamma\\
&+ \left(\chi - \frac{A}{2}\chi_2 - \frac{c_P N}{2\sqrt{\lambda}}\chi_3\right)|\tau|^2,\\
&c_{\mathrm{III}}(\tau,\eta_0) \geq \left(h - \frac{\phi_0}{2}\gamma - \frac{\sqrt{\lambda}|\rho|}{2c_P}|\tau| - \frac{\mu|\rho|}{2c_P} - \frac{\phi_0^2}{2}\gamma - \frac{|\rho|}{2}|\tau| - \frac{\mu|\rho|}{2c_P\sqrt{\lambda}}|\tau|\right)\gamma\\
&+ \left(\chi - \frac{N}{2}\chi_4 - \frac{\sqrt{\lambda}c_P A}{2\mu}\chi_5\right)|\tau|^2.
\end{aligned} \tag{2.31}$$

If we take a positive χ so that

$$\chi > \frac{1}{2}\left\{ M\psi_0\chi_1, A\chi_2 + \frac{c_P N}{\sqrt{\lambda}}\chi_3, N\chi_4 + \frac{\sqrt{\lambda}c_P A}{\mu}\chi_5 \right\}, \tag{2.32}$$

from (2.31) we derive

$$c_j(\tau,\eta_0) > \frac{h}{2}\gamma, \quad j = \mathrm{I},\mathrm{II},\mathrm{III}, \tag{2.33}$$

provided that $\gamma \geq 0$ and $|\rho|$ are sufficiently small. Because of the continuity of $c_j(\tau,\eta)$ with respect to η, inequalities (2.30), (2.33) hold true, replacing η_0 with any point η into some small neighborhood of η_0; therefore we derive from (2.26)

$$\Re(z^* P z) > \frac{h}{2}\gamma(|z_{\mathrm{I}}|^2 + |z_{\mathrm{II}}|^2 + |z_{\mathrm{III}}|^2) + \frac{\nu_{\mathrm{IV}}}{4}|z_{\mathrm{IV}}|^2 + \frac{\nu_{\mathrm{V}}}{4}|z_{\mathrm{V}}|^2 > C^*\gamma|z|^2, \tag{2.34}$$

for every $z \in \mathbb{C}^9$ and (τ,η) belonging to a small neighborhood of $(0,\eta_0)$, with $|\eta_0| = 1$, on the unit hemi-sphere of $\mathcal{X}$; $C^* := \min\left\{\frac{h}{2}, \frac{\nu_{\mathrm{IV}}}{4}, \frac{\nu_{\mathrm{V}}}{4}\right\}$ is independent of γ,ρ,η. The above estimate just provides (1.13). Let us now summarize the basic steps in the choice of the constants h,χ,A,M,N involved in (2.10).

1. Firstly, we choose the positive constants M, N satisfying (2.28).
2. For fixed $\delta_0, h > 0$, by means of Lemma 2.1 we find also $A > 0$ such that (2.25) holds true; we emphasize that the given value of A does not depend on χ, provided $|\tau|$ is small enough.
3. After estimating $\Re(z^* P z)$ by means of (2.26), we choose the positive constants χ_j, $j = 1,\dots,5$, from (2.27), (2.31) in such a way that inequalities (2.29) hold true; these inequalities, as well as (2.25), are achieved independently of the value of χ, provided that $|\rho|,\gamma$ (then $|\tau|$) are sufficiently small.
4. Finally, in view of estimates (2.31), we take χ fulfilling (2.32). Thus we get estimates (2.33) that, jointly with (2.30), yield (2.34).

Agreeing with the blockwise structure introduced at the beginning of this section (cf. (2.3), (2.5)), the matrix-valued function $K = K(\tau,\eta)$ that we have just built near the central points $(0,\eta_0)$ of the unit hemi-sphere $|\tau|^2 + |\eta|^2 = 1$, $\Re\tau \geq 0$, takes the following form

$$K(\tau,\eta) = \begin{pmatrix} (h+\chi\overline{\tau})I_3 & \mathbf{0}_{3\times 2} & 0_3 & \mathbf{0}_{3\times 2} & 0_3 \\ \gamma\Phi(\eta)^T & (h+\chi\overline{\tau})I_2 & -\frac{\sqrt{\lambda}}{c_P}\rho\gamma\eta & -\rho\gamma I_2 & -\frac{\sqrt{\lambda}}{c_P}\rho\gamma\eta \\ \gamma\eta^T\Phi(\eta)^T & 0_2^T & h+\chi\overline{\tau} & -\rho\gamma\eta^T & -\frac{\mu}{c_P\sqrt{\lambda}}\rho\gamma \\ G_{M,N}(\eta)\Psi(\eta) & -AI_2 & iN\eta & (h+\chi\tau)I_2 & 0_2 \\ \gamma\eta^T\Phi(\eta)^T & -i\frac{c_P}{\sqrt{\lambda}}N\eta^T & -A\frac{c_P\sqrt{\lambda}}{\mu} & -\rho\gamma\eta^T & h+\chi\tau \end{pmatrix}, \tag{2.35}$$

where h,χ,A,M,N are positive constants to be chosen as it was previously explained, $\Phi(\eta)$, $G_{M,N}(\eta)$ are defined by (1.7), (2.11) and $\Psi(\eta)$ is a 2×3 real matrix such that $\Psi(\eta)\Phi(\eta) = I_2$. Once a function $K = K(\tau,\eta)$, displaying properties i.-iii., has been made locally in a neighborhood of each point of the unit hemi-sphere of $\mathcal{X}$, by use of a smooth partition of unity and exploiting a homogeneity argument, the global existence of a dissipative symmetrizer for the ibvp (1.10) plainly follows. When a Kreiss symmetrizer $K(\tau,\eta)$ of (1.10) has been constructed, the result of Theorem 1.1 is achieved by standard arguments. We refer to [1], Chapter 4, for a detailed proof of the non-characteristic counterpart of Theorem 1.1; we summarize here only the basic steps of this proof in our framework. Firstly, under the hypotheses of Theorem 1.1, we find that any function $u \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ compactly supported in $\overline{\mathbb{R}^3_+} \times \mathbb{R}$ obeys the following estimate

$$e^{-2\gamma T}\|u(T)\|_{L^2} + \gamma \iint\limits_{\mathbb{R}^3_+ \times \mathbb{R}} e^{-2\gamma t}|u(y,x_3,t)|^2 dy\, dx_3\, dt + \iint\limits_{\mathbb{R}^2 \times \mathbb{R}} e^{-2\gamma t}|A^3 u(y,0,t)|^2 dy\, dt$$

$$\leq C \left(\frac{1}{\gamma} \iint\limits_{\mathbb{R}^3_+ \times \mathbb{R}} e^{-2\gamma t}|Lu(y,x_3,t)|^2 dy\, dx_3\, dt + \iint\limits_{\mathbb{R}^2 \times \mathbb{R}} e^{-2\gamma t}|Bu(y,0,t)|^2 dy\, dt \right) \tag{2.36}$$

for all real T and $\gamma > 0$, where the positive constant C does not depend on γ, T and u. By a duality argument relying on the above estimate for an "adjoint" ibvp, one shows the existence of a solution to the ibvp (1.10) in the weighted space $L^2_\gamma(\mathbb{R}^3_+ \times \mathbb{R})$ for every $\gamma > 0$. Let us recall that, for a given positive γ, $L^2_\gamma(\mathbb{R}^3_+ \times \mathbb{R})$ is the space of all measurable functions $u(y,x_3,t)$ for which the norm $\|u\|^2_\gamma := \iint\limits_{\mathbb{R}^3_+ \times \mathbb{R}} e^{-2\gamma t}|u(y,x_3,t)|^2 dy\, dx_3\, dt$ is finite. The uniqueness of the solution into the space $L^2(\mathbb{R}^3_+ \times (0,T))$, for a finite $T > 0$, then follows by arguing directly on estimates (2.36). Eventually, the a priori estimates (1.11) are derived from (2.36) themselves, by a density argument.

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