A STOCHASTIC EVOLUTION EQUATION ARISING FROM THE FLUCTUATIONS OF A CLASS OF INTERACTING PARTICLE SYSTEMS *

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Abstract. In an earlier paper, we studied the approximation of solutions $V(t)$ to a class of SPDEs by the empirical measure $V^n(t)$ of a system of n interacting diffusions. In the present paper, we consider a central limit type problem, showing that $\sqrt{n}(V^n - V)$ converges weakly, in the dual of a nuclear space, to the unique solution of a stochastic evolution equation. Analogous results in which the diffusions that determine V^n are replaced by their Euler approximations are also discussed.

1. Introduction

In [26], we considered a class of nonlinear stochastic partial differential equations (SPDE) of the form

$$dv(t,x) = \left(\frac{1}{2} \sum_{i,j=1}^{d} \partial_{x_i} \partial_{x_j} \left[a_{ij}(x,v(t,\cdot))v(t,x) \right] - \sum_{i=1}^{d} \partial_{x_i} \left[b_i(x,v(t,\cdot))v(t,x) \right] \right.$$

$$+ d(x,v(t,\cdot))v(t,x) \Bigg) dt$$

$$- \int_U \left(\beta\left(x,v(t,\cdot),u\right)v(t,x) + \sum_{i=1}^{d} \partial_{x_i} \left[\alpha_i(x,v(t,\cdot),u) \right] \right) W(dudt). \quad (1.1)$$

The natural interpretation of v is as the density of a mass distribution V evolving in time, and in fact, since v will not have the regularity presumed in (1.1), to rigorously formulate the equation, we must use a weak form

$$\langle \phi, V(t) \rangle - \langle \phi, V(0) \rangle$$

$$= \int_0^t \langle \phi d(\cdot, V(s)) + L(V(s))\phi, V(s) \rangle \, ds$$

$$+ \int_{U \times [0,t]} \left\langle \phi \beta(\cdot, V(s), u) + \nabla \phi^T \alpha(\cdot, V(s), u), V(s) \right\rangle W(duds). \quad (1.2)$$

where

$$L(v)\phi(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x,v) \partial_{x_i} \partial_{x_j} \phi(x) + \sum_i b_i(x,v) \partial_{x_i} \phi(x).$$

Equations in this class arise in a variety of settings, including nonlinear filtering with both the Zakai and the Kushner-FKK equations being of this form. Other examples include McKean-Vlasov equations [30] and classes of SPDEs considered by Kotelenez [23] and Dawson and Vaillancourt [9].

In [26], we established a representation of the solution of (1.1) in terms of weighted empirical measures of the form

$$V(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} A_i(t) \delta_{X_i(t)}, \tag{1.3}$$

where δ_x is the Dirac measure at x and the limit exists in the weak* topology on $\mathcal{M}(\mathbb{R}^d)$. To be precise, let U be a Polish space and μ be a σ-finite measure on U. W will be a space-time Gaussian white noise on $U \times [0, \infty)$ with covariance measure $\mu(du)dt$, namely

$$E(W(A \times [0,t])W(B \times [0,s])) = \mu(A \cap B)(t \wedge s).$$

For $1 \le i$, $j \le d$, a_{ij}, b_i, d and α_i, β will be real functions on $\mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d)$ and on $\mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times U$ respectively. Here $\mathcal{M}(\mathbb{R}^d)$ is the collection of all finite measures on $\mathbb{R}^d$. Dot notation will represent a function in that variable alone. For example, $d(\cdot, v)$ is the real-valued function on $\mathbb{R}^d$ with $v \in \mathcal{M}(\mathbb{R}^d)$ fixed. b and α will denote the vectors with components b_i and α_i and a will denote the matrix $((a_{ij}))$.

The particle system $\{X_i, A_i, V\}$ giving the solution is governed by the following equations:

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V(s))dB_i(s) + \int_0^t c(X_i(s), V(s))ds$$
$$+ \int_{U \times [0,t]} \alpha(X_i(s), V(s), u)W(duds) \tag{1.4}$$

and

$$A_i(t) = A_i(0) + \int_0^t A_i(s)\gamma^T(X_i(s), V(s))dB_i(s) + \int_0^t A_i(s)d(X_i(s), V(s))ds$$
$$+ \int_{U \times [0,t]} A_i(s)\beta(X_i(s), V(s), u)W(duds), \tag{1.5}$$

where the B_i are independent, standard $\mathbb{R}^d$-valued Brownian motions, independent of W, and $\{(X_i(0), A_i(0))\}$ is an exchangeable sequence of random variables in $\mathbb{R}^d \times \mathbb{R}$ that is independent of $\{B_i\}$ and W. Here σ, c and γ are related to a, b, α, and β by

$$a(x,v) = \sigma(x,v)\sigma^T(x,v) + \int_U \alpha(x,v,u)\alpha^T(x,v,u)\mu(du)$$

and

$$b(x,v) = c(x,v) + \sigma(x,v)\gamma(x,v) + \int_U \beta(x,v,u)\alpha(x,v,u)\mu(du).$$

The representation of the solution given by (1.3) suggests that the solution can be approximated by the weighted empirical measure

$$V^n(t) = \frac{1}{n} \sum_{i=1}^{n} A_i^n(t) \delta_{X_i^n(t)}, \tag{1.6}$$

of a finite particle system satisfying

$$X_i^n(t) = X_i(0) + \int_0^t \sigma(X_i^n(s), V^n(s)) dB_i(s) + \int_0^t c(X_i^n(s), V^n(s)) ds$$

$$+ \int_{U \times [0,t]} \alpha(X_i^n(s), V^n(s), u) W(duds) \qquad (1.7)$$

$$A_i^n(t) = A_i(0) + \int_0^t A_i^n(s) \gamma^T(X_i^n(s), V^n(s)) dB_i(s) + \int_0^t A_i^n(s) d(X_i^n(s), V^n(s)) ds$$

$$+ \int_{U \times [0,t]} A_i^n(s) \beta(X_i^n(s), V^n(s), u) W(duds), \qquad (1.8)$$

for $i = 1, 2, \cdots, n$.

In [27], it was shown that for an appropriate metric $\widetilde{\rho}$ on $\mathcal{M}(\mathbb{R}^d)$, $\{\sqrt{n}\widetilde{\rho}(V^n(t), V(t))\}_{n \geq 1}$ is stochastically bounded, that is, for each $\epsilon > 0$, there is a constant K_ϵ such that

$$\sup_n \mathbb{P}\left(\sqrt{n}\widetilde{\rho}(V^n(t), V(t)) > K_\epsilon\right) < \epsilon. \qquad (1.9)$$

From (1.9), we see that the convergence rate has an upper bound of the order of $\frac{1}{\sqrt{n}}$. A natural question to ask is whether $\frac{1}{\sqrt{n}}$ is the right order. To this end, we study the convergence of the process $S_n(t) = \sqrt{n}(V^n(t) - V(t))$ and show that for an appropriate space $\Phi_{-\kappa}$ of distributions, S_n converges in distribution in $C_{\Phi_{-\kappa}}[0, \infty)$. We characterize the limit S as the unique solution of a stochastic evolution equation of the form

$$\langle \phi, S(t) \rangle = \langle \phi, S(0) \rangle + \langle \phi, M(t) \rangle + \int_0^t \langle F_1(V(s)) \phi, S(s) \rangle ds$$

$$+ \int_{U \times [0,t]} \langle F_2(V(s), u) \phi, S(s) \rangle W(duds), \qquad (1.10)$$

where F_1 and F_2 are linear in ϕ and M is a distribution-valued martingale. F_1 and F_2 are defined in Assumption (S6) in Section 4 in terms of appropriate differentials of the coefficients of (1.2), reflecting the fact that we are rescaling the deviation of V^n from V.

This type of problem has been studied by various authors in the McKean-Vlasov setting, that is, $\alpha = 0$ and $A_i^n(t) \equiv 1$ (cf. Hitsuda and Mitoma [19] and the references therein). Comparing the present results with those of [19], here the process V is not deterministic and the process S is not Gaussian. In [19], the limit S is characterized by its covariance structure which, because S is Gaussian, uniquely determines its distribution. A stochastic evolution equation is also derived in that paper. The uniqueness of the solution to that stochastic evolution equation is proved by Mitoma [33].

Another new feature in this paper is that the driving martingale M in the evolution equation (1.10) is not Gaussian and has to be defined by the particle system $\{X_i, A_i, V\}$ itself. The main difficulty in establishing the uniqueness of the solution of (1.10) comes from the addition of the last term in (1.10) which does not appear in [19] and [33].

Limits of empirical measure processes for systems of interacting diffusions have been studied by various authors (see, for example, Chiang, Kallianpur and Sundar [4], Graham [18], Kallianpur and Xiong [22], Méléard [31], and Morien [34]) since the pioneering work by McKean [30]. Typically, the driving processes in the models are assumed to be independent, and the limit is then a deterministic, measure-valued function.

Florchinger and Le Gland [14] consider particle approximations for stochastic partial differential equations in a setting that, in the notation above, corresponds to taking $\gamma = \sigma = 0$ and the other coefficients independent of V. Florchinger and Le Gland were motivated by approximations to the Zakai equation of nonlinear filtering. Del Moral [10] specifically studies this example. Kotelenez [23] introduces a model of n-particles with the same driving process for each particle and studies the empirical process as the solution of an SPDE. His model corresponds to taking $\gamma = \sigma = d = \beta = 0$, but the other coefficients are allowed to depend on V. In particular, the weights A_i are constants. Dawson and Vaillancourt [9] consider a model given as a solution of a martingale problem that corresponds to taking $A_i(t) \equiv 1$ in the current model. Bernard, Talay, and Tubaro [1] consider a system with time-varying weights and a deterministic limit.

The paper is organized as follows: In the next section, we derive key estimates on the magnitude of the A_i^n and on the error in the approximation of (X_i, A_i) by (X_i^n, A_i^n). In Section 3, we prove that $\{S_n\}$ is a tight sequence of Φ'-valued processes (Φ' being a conuclear space defined later). Then, in Section 4, we show that the limit S of $\{S_n\}$ is the unique solution of (1.10).

If one wants to use the finite system to simulate the solution of the SPDE, then the finite system must also be approximated. The simplest approach is to use an Euler approximation. In the last section of this paper, we analyze this approximation in the simplest setting, assuming that W is a one-dimensional Brownian motion (that is, U consists of a single point). Letting $V^{n,1/n}$ denote the weighted empirical measure for the Euler scheme approximating the finite system (cf. (5.1-5.3)), we consider the process $\widetilde{S}_n(t) = \sqrt{n}(V^{n,1/n}(t) - V^n(t))$. We prove tightness for $\{\widetilde{S}_n\}$ and characterize its limit as the unique solution of another stochastic evolution equation. Finally, we combine the two parts and derive a stochastic evolution equation for the limit of $\sqrt{n}(V^{n,1/n} - V)$.

2. Preliminaries

In this section, we state the main results of [26] and [27] needed in the present paper for the convenience of the reader. The following assumptions were made in [26] for the existence and uniqueness of solutions of the SPDE (1.1).

(S1) There exists a constant K such that for each $x \in \mathbb{R}^d$, $\nu \in \mathcal{M}(\mathbb{R}^d)$

$$|\sigma(x,\nu)|^2 + |c(x,\nu)|^2 + \int_U |\alpha(x,\nu,u)|^2 \mu(du)$$

$$+ |\gamma(x,\nu)|^2 + |d(x,\nu)|^2 + \int_U \beta(x,\nu,u)^2 \mu(du) \le K^2.$$

(S2) For each $x_1, x_2 \in \mathbb{R}^d$, $\nu_1, \nu_2 \in \mathcal{M}(\mathbb{R}^d)$

$$|\sigma(x_1,\nu_1) - \sigma(x_2,\nu_2)|^2 + |c(x_1,\nu_1) - c(x_2,\nu_2)|^2$$

$$+|\gamma(x_1,\nu_1)-\gamma(x_1,\nu_1)|^2+\int_U |\alpha(x_1,\nu_1,u)-\alpha(x_2,\nu_2,u)|^2\mu(du)$$

$$+|d(x_1,\nu_1)-d(x_2,\nu_2)|^2+\int_U |\beta(x_1,\nu_1,u)-\beta(x_2,\nu_2,u)|^2\mu(du)$$

$$\leq K^2(|x_1-x_2|^2+\rho(\nu_1,\nu_2)^2),$$

where

$$\rho(\nu_1,\nu_2)=\sup\{|\langle\phi,\nu_1\rangle-\langle\phi,\nu_2\rangle|:\phi\in\mathbb{B}_1\}$$

and

$$\mathbb{B}_1=\left\{\phi:|\phi(x)-\phi(y)|\leq|x-y|,|\phi(x)|\leq 1,\forall x,y\in\mathbb{R}^d\right\}.$$

By the same proof as in Proposition 2.1 of [26], we have the following result.

PROPOSITION 2.1. *Suppose that Assumption (S1) holds and p is a positive number.*
i) If

$$\mathbb{E}e^{p|X_1(0)|}<\infty, \tag{2.1}$$

then

$$\sup_{1\leq n\leq\infty}\mathbb{E}\sup_{0\leq s\leq T}e^{p|X_i^n(s)|}<\infty. \tag{2.2}$$

ii) If

$$\mathbb{E}|A_1(0)|^p<\infty, \tag{2.3}$$

then

$$\sup_{1\leq n\leq\infty}\mathbb{E}\sup_{0\leq s\leq T}|A_i^n(s)|^p<\infty. \tag{2.4}$$

iii) If

$$\mathbb{E}|A_1(0)|e^{r|X_1(0)|}<\infty, \tag{2.5}$$

then

$$\sup_{1\leq n\leq\infty}\mathbb{E}\sup_{0\leq s\leq T}|A_i^n(s)|e^{r|X_i^n(s)|}<\infty. \tag{2.6}$$

REMARK 2.2. *If (2.1) and (2.3) hold, then (2.5) holds with $r=p-1$.*
    A weaker form of the following assumption was used in [27].
    (S3) There exist constants $\lambda>1$ and $K>0$ such that for any *iid* sequence
        (ξ_i,η_i), $i=1,2,\cdots$ and $x\in\mathbb{R}^d$,

$$\mathbb{E}\left|\sigma\left(x,\frac{1}{n}\sum_{i=1}^n \xi_i\delta_{\eta_i}\right)-\sigma(x,\mu)\right|^{2\lambda}\leq\frac{K\mathbb{E}\xi_1^{2\lambda}}{n^\lambda},$$

    where $\mu(\cdot)=\mathbb{E}[\xi_1 1_{\eta_1\in\cdot}]$, and a similar inequality holds for the other coefficients.

REMARK 2.3. *If $\sigma(x,\mu) = \int \sigma_1(x,y)\mu(dy)$ or $\sigma(x,\mu) = \sigma_1(x)$, then (S3) usually holds. For example, if $|\sigma_1(x,y)| \le K$, then (S3) holds.*

The following estimate is the key for the proof of the tightness of $\{S_n\}$.

THEOREM 2.4. *Under the assumptions (S1)-(S3), there exists a constant $c_1(T,m)$ such that*

$$\mathbb{E}\sup_{t \le T}\left(|X_i^n(t \wedge \eta_m^n) - X_i(t \wedge \eta_m^n)|^{2\lambda} + \left(\frac{1}{n}\sum_{j=1}^n |A_j^n(t \wedge \eta_m^n) - A_j(t \wedge \eta_m^n)|^\lambda \right)^2 \right)$$

$$\le \frac{c_1(T,m)}{n^\lambda},$$

where

$$\eta_m^n = \inf\left\{ t : \frac{1}{n}\sum_{i=1}^n A_i^n(t)^2 > m^2 \ \ or \ \ \lim_{k \to \infty} \frac{1}{k}\sum_{i=1}^k A_i(t)^2 > m^2 \right\}.$$

Proof: By Doob's inequality and Holder's inequality, we have

$$\mathbb{E}\sup_{r \le t}|X_i^n(r \wedge \eta_m^n) - X_i(r \wedge \eta_m^n)|^{2\lambda}$$

$$\le 3^{2\lambda}\left(\frac{2\lambda}{2\lambda-1}\right)^{2\lambda} \mathbb{E}\int_0^t |\sigma(X_i^n(s),V^n(s)) - \sigma(X_i(s),V(s))|^{2\lambda}1_{s \le \eta_m^n}\,ds\,t^{\lambda-1}$$

$$+3^{2\lambda}t^{2\lambda-1}\mathbb{E}\int_0^t |c(X_i^n(s),V^n(s)) - c(X_i(s),V(s))|^{2\lambda}1_{s \le \eta_m^n}\,ds$$

$$+3^{2\lambda}\left(\frac{2\lambda}{2\lambda-1}\right)^{2\lambda}t^{\lambda-1}$$

$$\times \mathbb{E}\int_0^t \left(\int_U |\alpha(X_i^n(s),V^n(s),u) - \alpha(X_i(s),V(s),u)|^2\mu(du)\right)^\lambda 1_{s \le \eta_m^n}\,ds.$$

$$(2.7)$$

Let

$$\widetilde{V}^n(t) = \frac{1}{n}\sum_{i=1}^n A_i(t)\delta_{X_i(t)} \qquad \text{and} \qquad \widetilde{V}_i^n(t) = \frac{1}{n-1}\sum_{j=1,\ j \ne i}^n A_j(t)\delta_{X_j(t)}.$$

Then

$$\mathbb{E}|\sigma(X_i^n(s),V^n(s)) - \sigma(X_i(s),V(s))|^{2\lambda}1_{s \le \eta_m^n}$$

$$\le 3^{2\lambda}\mathbb{E}|\sigma(X_i^n(s),V^n(s)) - \sigma(X_i(s),\widetilde{V}^n(s))|^{2\lambda}1_{s \le \eta_m^n}$$

$$+3^{2\lambda}\mathbb{E}|\sigma(X_i(s),\widetilde{V}^n(s)) - \sigma(X_i(s),\widetilde{V}_i^n(s))|^{2\lambda}$$

$$+3^{2\lambda}\mathbb{E}|\sigma(X_i(s),\widetilde{V}_i^n(s)) - \sigma(X_i(s),V(s))|^{2\lambda}1_{s \le \eta_m^n}$$

$$\le 3^{2\lambda}K^{2\lambda}\mathbb{E}\left(|X_i^n(s) - X_i(s)|^2 + \rho(V^n(s),\widetilde{V}^n(s))^2\right)^\lambda 1_{s \le \eta_m^n}$$

$$+3^{2\lambda}K^{2\lambda}\mathbb{E}\rho(\widetilde{V}^n(s),\widetilde{V}_i^n(s))^{2\lambda}1_{s \le \eta_m^n}$$

$$+3^{2\lambda}\mathbb{E}\left[\mathbb{E}\left[|\sigma(X_i(s),\frac{1}{n-1}\sum_{j=1,\ j \ne i}^n A_j(s)\delta_{X_j(s)}) - \sigma(X_i(s),V(s))^{2\lambda}|\,|W,X_i]\right].$$

$$(2.8)$$

Note that, similar to the arguments in the proof of Theorem 2.1 in [26], we have

$$\rho(V^n(s), \widetilde{V}^n(s))$$

$$\leq \frac{1}{n}\sum_{j=1}^{n} A_j^n(s)|X_j^n(s) - X_j(s)| + \frac{1}{n}\sum_{j=1}^{n}|A_j^n(s) - A_j(s)|$$

$$\leq (\frac{1}{n}\sum_{j=1}^{n} A_j^n(s)^2)^{1/2}(\frac{1}{n}\sum_{j=1}^{n}|X_j^n(s) - X_j(s)|^2)^{1/2} + \frac{1}{n}\sum_{j=1}^{n}|A_j^n(s) - A_j(s)|. \quad (2.9)$$

Similarly,

$$\rho(\widetilde{V}^n(s), \widetilde{V}_i^n(s)) \leq \frac{1}{n}A_i(s) + \frac{1}{n(n-1)}\sum_{j=1}^{n} A_j(s).$$

Let

$$f_m^n(t) = \mathbb{E}\sup_{r\leq t}|X_i^n(r\wedge\eta_m^n) - X_i(r\wedge\eta_m^n)|^{2\lambda},$$

and

$$g_m^n(t) = \mathbb{E}\sup_{r\leq t}\left(\frac{1}{n}\sum_{j=1}^{n}|A_j^n(r\wedge\eta_m^n) - A_j(r\wedge\eta_m^n)|^{\lambda}\right)^2.$$

Then, for the right hand side of (2.8),

$$\text{1st term} \leq 3^{2\lambda}K^{2\lambda}2^{\lambda}\left(\mathbb{E}|X_i^n(s\wedge\eta_m^n) - X_i(s\wedge\eta_m^n)|^{2\lambda}\right.$$

$$\left. + 2^{2\lambda}m^{2\lambda}\mathbb{E}\frac{1}{n}\sum_{j=1}^{n}|X_j^n(s\wedge\eta_m^n) - X_j(s\wedge\eta_m^n)|^{2\lambda} + 2^{2\lambda}g_m^n(s)\right)$$

$$\leq 18^{\lambda}K^{2\lambda}\left(f_m^n(s) + 4^{\lambda}m^{2\lambda}f_m^n(s) + 4^{\lambda}g_m^n(s)\right)$$

and

$$\text{2nd term} \leq 3^{2\lambda}K^{2\lambda}2^{2\lambda}\left(\frac{1}{n^{2\lambda}}\mathbb{E}\sup_{r\leq t}|A_i(r)|^{2\lambda} + \frac{1}{(n-1)^{2\lambda}}m^{2\lambda}\right).$$

Since, conditioning on (W, X_i), (A_j, X_j), $j\neq i$, are *iid*, we have

$$\text{3rd term} \leq 3^{2\lambda}\mathbb{E}\left(\frac{K^{2\lambda}}{(n-1)^{\lambda}}\mathbb{E}\left(A_1(s)^{2\lambda}|W, X_i\right)\right) = \frac{3^{2\lambda}K^{2\lambda}}{(n-1)^{\lambda}}\mathbb{E}A_1(s)^{2\lambda}.$$

Hence, the first term on the right hand side of (2.7) is dominated by

$$\left(\frac{6\lambda}{2\lambda-1}\right)^{2\lambda}T^{\lambda-1}\int_0^t 18^{\lambda}K^{2\lambda}\left((1+2^{\lambda}m^{2\lambda})f_m^n(s) + 2^{\lambda}g_m^n(s)\right)ds$$

$$+ \left(\frac{6\lambda}{2\lambda-1}\right)^{2\lambda}\left[3^{2\lambda}K^{2\lambda}2^{2\lambda}\left(\frac{1}{n^{2\lambda}}\mathbb{E}\sup_{r\leq t}|A_i(r)|^{2\lambda} + \frac{1}{(n-1)^{2\lambda}}m^{2\lambda}\right)\right.$$

$$\left. + \frac{3^{2\lambda}K^{2\lambda}}{(n-1)^{\lambda}}\mathbb{E}A_1(s)^{2\lambda}\right]T^{\lambda}.$$

Similar estimates hold for other terms on the right hand side of (2.7). Therefore, there exist constants $c_2(T,m)$ and $c_3(T,m)$ such that

$$f_m^n(t) \leq c_2(T,m) \int_0^t (f_m^n(s) + g_m^n(s)) ds + \frac{c_3(T,m)}{n^\lambda}. \qquad (2.10)$$

By similar arguments as in (2.7) of [26] and (2.10) above, we have

$$g_m^n(t) \leq c_4(T,m) \int_0^t (f_m^n(s) + g_m^n(s)) ds + \frac{c_5(T,m)}{n^\lambda}.$$

Therefore

$$f_m^n(t) + g_m^n(t) \leq (c_2 + c_4) \int_0^t (f_m^n(s) + g_m^n(s)) ds + \frac{c_3 + c_5}{n^\lambda}.$$

By Gronwall's inequality, we have

$$f_m^n(t) + g_m^n(t) \leq \frac{c_1}{n^\lambda}$$

by taking $c_1 = \frac{e^{(c_2+c_4)T}(c_3+c_5)}{c_2+c_4}$. ∎

3. Tightness

In this section, we prove tightness for $\{S_n\}$ in an appropriate space. For simplicity of notation, we restrict our calculations to space dimension $d=1$ in the rest of this paper.

As in Hitsuda and Mitoma [19], we use the modified Schwartz space Φ. Let $\rho(x) = C \exp\left(-1/(1-|x|^2)\right) 1_{|x|<1}$ where C is a constant such that $\int \rho(x) dx = 1$. Let

$$\psi(x) = \int e^{-|y|} \rho(x-y) dy.$$

Then for any integer k, we have $|e^{(k)}(x)| \leq c_6(k)(1 + e^{|x|})$. Let

$$\Phi = \{\phi : \phi\psi \in \mathcal{S}\},$$

where $\mathcal{S}$ is the Schwartz space. For $\kappa = 0, 1, 2, \ldots$, define

$$\|\phi\|_\kappa^2 = \sum_{0 \leq k \leq \kappa} \int_{\mathbb{R}} (1 + |x|^2)^{2\kappa} \left| \frac{d^k}{dx^k}(\phi(x)\psi(x)) \right|^2 dx.$$

Let Φ_κ be the completion of Φ with respect to $\|\cdot\|_\kappa$. Then Φ_κ is a Hilbert space with inner product

$$\langle \phi_1, \phi_2 \rangle_\kappa = \sum_{0 \leq k \leq \kappa} \int_{\mathbb{R}} (1 + |x|^2)^{2\kappa} \left(\frac{d^k}{dx^k}(\phi_1(x)\psi(x)) \right) \left(\frac{d^k}{dx^k}(\phi_2(x)\psi(x)) \right) dx.$$

Note that $\Phi_\kappa \supset \Phi_{\kappa+1}$ and that Φ_0 is $L^2(\mu_\psi)$, where $\mu_\psi(dx) = \psi^2(x) dx$. For $\widehat{\phi} \in \Phi_0$ and $\phi \in \Phi_\kappa$,

$$\langle \widehat{\phi}, \phi \rangle \equiv \langle \widehat{\phi}, \phi \rangle_0 = \int_{\mathbb{R}} \widehat{\phi}(x) \phi(x) \psi^2(x) dx$$

defines a continuous linear functional on Φ_κ with norm

$$\|\widehat{\phi}\|_{-\kappa} = \sup_{\phi \in \Phi_\kappa} \frac{|\langle \widehat{\phi}, \phi \rangle|}{\|\phi\|_\kappa},$$

and we let $\Phi_{-\kappa}$ denote the completion of Φ_0 with respect to this norm. Then $\Phi_{-\kappa}$ is a representation of the dual of Φ_κ. If $\{\phi_j^\kappa\}$ is a complete, orthonormal system for Φ_κ, then the inner product for $\Phi_{-\kappa}$ can be written as

$$\langle \widehat{\phi}_1, \widehat{\phi}_2 \rangle_{-\kappa} = \sum_{j=1}^\infty \langle \widehat{\phi}_1, \phi_j^\kappa \rangle \langle \widehat{\phi}_2, \phi_j^\kappa \rangle. \tag{3.1}$$

By a slight modification of Theorem 7, page 82, of [17], these norms determine a nuclear space, so in particular, for each κ there exists a $\kappa' > \kappa$ such that the embedding $T_\kappa^{\kappa'} : \Phi_{\kappa'} \to \Phi_\kappa$ is a Hilbert-Schmidt operator. The adjoint $T_\kappa^{\kappa'*} : \Phi_{-\kappa} \to \Phi_{-\kappa'}$ is also Hilbert-Schmidt. $\Phi' = \cup_{k=0}^\infty \Phi_{-k}$ gives a representation of the dual of Φ. (See [17], page 59.) We prove tightness for $\{S_n\}$ in $C_{\Phi_{-\kappa}}[0, \infty)$ for an appropriate κ.

THEOREM 3.1. *Suppose that (S1)-(S3) hold and that (2.1) and (2.3) hold for*

$$p = max(4\lambda, \lambda/(\lambda - 1)).$$

Then there exists κ such that $\{S_n\}$ is tight in $C_{\Phi_{-\kappa}}[0, \infty)$.
Proof: Let

$$\eta_m^{n,p} = \inf \left\{ t : \frac{1}{n} \sum_{i=1}^n A_i^n(t)^p \geq m^p \text{ or } \frac{1}{n} \sum_{i=1}^n \left(1 + e^{p|X_i^n(t)|} \vee e^{p|X_i(t)|} \right) \geq m^p \right\}.$$

Then for $T > 0$, we have

$$\sup_n \mathbb{P}(\eta_m^{n,p} \leq T) \leq \frac{2}{m^p} \sup_{1 \leq n \leq \infty} \left(1 + \mathbb{E} \sup_{0 \leq s \leq T} A_i^n(s)^p + \mathbb{E} \sup_{0 \leq s \leq T} e^{p|X_i^n(s)|} \right),$$

and since by Proposition 2.1, for each $T > 0$, the right side goes to zero as $m \to \infty$, it is enough to prove tightness for $\{S_n(\cdot \wedge \eta_m^{n,p})\}$.

By Itô's formula, we have

$$\langle \phi, V^n(t) \rangle - \langle \phi, V^n(0) \rangle$$

$$= \frac{1}{n} \sum_{i=1}^n \int_0^t A_i^n(s) \Big(\phi(X_i^n(s)) \gamma(X_i^n(s), V^n(s))$$

$$+ \phi'(X_i^n(s)) \sigma(X_i^n(s), V^n(s)) \Big) dB_i(s)$$

$$+ \int_0^t \langle \phi d(\cdot, V^n(s)) + L(V^n(s))\phi, V^n(s) \rangle ds$$

$$+ \int_{U \times [0,t]} \langle \phi \beta(\cdot, V^n(s), u) + \phi' \alpha(\cdot, V^n(s), u), V^n(s) \rangle W(duds), \tag{3.2}$$

where

$$L(v)\phi(x) = \frac{1}{2} a(x, v)\phi''(x) + b(x, v)\phi'(x).$$

Hence, by (1.2) and (3.2), we have

$$
\begin{aligned}
\langle \phi, & S_n(t) - S_n(0) \rangle \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_0^t A_i^n(s) \Big(\phi(X_i^n(s)) \gamma(X_i^n(s), V^n(s)) \\
&\qquad\qquad\qquad + \phi'(X_i^n(s)) \sigma(X_i^n(s), V^n(s)) \Big) dB_i(s) \\
&\quad + \int_0^t \sqrt{n} \Big(\langle \phi d(\cdot, V^n(s)) + L(V^n(s)) \phi, V^n(s) \rangle \\
&\qquad\qquad\qquad - \langle \phi d(\cdot, V(s)) + L(V(s)) \phi, V(s) \rangle \Big) ds \\
&\quad + \int_{U \times [0,t]} \sqrt{n} \Big(\langle \phi \beta(\cdot, V^n(s), u) + \phi' \alpha(\cdot, V^n(s), u), V^n(s) \rangle \\
&\qquad\qquad - \langle \phi \beta(\cdot, V(s), u) + \phi' \alpha(\cdot, V(s), u), V(s) \rangle \Big) W(du\,ds). \quad (3.3)
\end{aligned}
$$

Note that $\langle \phi, S_n(t) \rangle = M_\phi^{1,n}(t) + A_\phi^n(t) + M_\phi^{2,n}(t)$ is a semimartingale with respect to the filtration $\{\mathcal{F}_t\}$ generated by W and the B_i. Setting

$$
\begin{aligned}
G_\phi^n(s,u) = \sqrt{n} \Big(& \langle \phi \beta(\cdot, V^n(s), u) + \phi' \alpha(\cdot, V^n(s), u), V^n(s) \rangle \\
& - \langle \phi \beta(\cdot, V(s), u) + \phi' \alpha(\cdot, V(s), u), V(s) \rangle \Big),
\end{aligned}
$$

we have

$$
\begin{aligned}
\Big[M_\phi^{1,n} \Big]_t \\
= \frac{1}{n} \sum_{i=1}^{n} \int_0^t A_i^n(s)^2 \Big(\phi(X_i^n(s)) \gamma(X_i^n(s), V^n(s)) + \phi'(X_i^n(s)) \sigma(X_i^n(s), V^n(s)) \Big)^2 ds
\end{aligned}
$$

and

$$
\Big[M^{2,n} \Big]_t = \int_0^t \int_U G_\phi^n(s,u)^2 \mu(du) dt.
$$

Let

$$
\begin{aligned}
H_\phi^{1,n}(s) &= \frac{1}{n} \sum_{i=1}^{n} A_i^n(s)^2 \Big(\phi(X_i^n(s)) \gamma(X_i^n(s), V^n(s)) + \phi'(X_i^n(s)) \sigma(X_i^n(s), V^n(s)) \Big)^2 \\
H_\phi^{2,n}(s) &= \int_U G_\phi^n(s,u)^2 \mu(du) \\
H_\phi^{3,n}(s) &= n \left(\langle \phi d(\cdot, V^n(s)) + L(V^n(s)) \phi, V^n(s) \rangle - \langle \phi d(\cdot, V(s)) + L(V(s)) \phi, V(s) \rangle \right)^2.
\end{aligned}
$$

It follows, for example, that for $t, h > 0$, we have

$$
\mathbb{E}[\langle \phi, S_n(t+h) - S_n(t) \rangle^2 | \mathcal{F}_t] \le \mathbb{E}[\int_t^{t+h} 3(H_\phi^{1,n}(s) + H_\phi^{2,n}(s) + h H_\phi^{3,n}(s)) ds | \mathcal{F}_t], \quad (3.4)
$$

and, applying Doob's inequality,

$$
\mathbb{E}[\sup_{s \le t} \langle \phi, S_n(s) - S_n(0) \rangle^2] \le \mathbb{E}[\int_0^t 12(H_\phi^{1,n}(s) + H_\phi^{2,n}(s) + t H_\phi^{3,n}(s)) ds]. \quad (3.5)
$$

We need to estimate each of the $H_\phi^{k,n}$.

Let $\widetilde{\phi} = \phi\psi$ and $|\widetilde{\phi}|_\kappa = \sup_{x,0 \le k \le \kappa} |(d^k/dx^k)\widetilde{\phi}(x)|$. Then $|\widetilde{\phi}|_\kappa \le \text{const} \|\phi\|_{\kappa+1}$. It is easy to see that there exists a constant c_7 such that

$$|\phi(x)| \le c_7 |\widetilde{\phi}|_0 e^{|x|}$$

and

$$|\phi(x)| + |\phi'(x)| \le c_7 |\widetilde{\phi}|_1 e^{|x|}.$$

Hence

$$\mathbb{E}\left| \frac{1}{n} \sum_{i=1}^n A_i^n(s)^2 \left(\phi(X_i^n(s))\gamma(X_i^n(s), V^n(s)) + \phi'(X_i^n(s))\sigma(X_i^n(s), V^n(s)) \right)^2 \right|^\lambda 1_{\{\eta_m^{n,p} \ge s\}}$$

$$\le \mathbb{E}\left| \frac{1}{n} \sum_{i=1}^n A_i^n(s)^2 K^2 c_7 e^{2|X_i^n(s)|} |\widetilde{\phi}|_1^2 \right|^\lambda 1_{\{\eta_m^{n,p} \ge s\}}$$

$$\le K^{2\lambda} c_7^\lambda |\widetilde{\phi}|_1^{2\lambda} \mathbb{E}\left| \frac{1}{n} \sum_{i=1}^n A_i^n(s)^4 \frac{1}{n} \sum_{i=1}^n e^{4|X_i^n(s)|} \right|^{\frac{\lambda}{2}} 1_{\{\eta_m^{n,p} \ge s\}}$$

$$\le K^{2\lambda} c_7^\lambda |\widetilde{\phi}|_1^{2\lambda} \left| m^4 2^3 m^4 \right|^{\frac{\lambda}{2}}$$

$$\equiv c_8(m, \lambda, s) |\widetilde{\phi}|_1^{2\lambda},$$

and we have

$$\mathbb{E}[H_\phi^{1,n}(s)^\lambda 1_{\{\eta_m^{n,p} \ge s\}}] \le c_8(m, \lambda, s) |\widetilde{\phi}|_1^{2\lambda} \le c_8(m, \lambda, s) \|\phi\|_2^{2\lambda}. \qquad (3.6)$$

Observing that

$$\sqrt{n}\left(\langle \phi d(\cdot, V^n(s))), V^n(s) \rangle - \langle \phi d(\cdot, V(s))), V(s) \rangle \right)$$

$$= \sqrt{n}\left\langle \phi d(\cdot, V^n(s))), V^n(s) - \widetilde{V}^n(s) \right\rangle$$

$$+ \sqrt{n}\left\langle \phi(d(\cdot, V^n(s)) - d(\cdot, \widetilde{V}^n(s)), \widetilde{V}^n(s) \right\rangle$$

$$+ \sqrt{n}\left\langle \phi(d(\cdot, \widetilde{V}^n(s)) - d(\cdot, V(s))), \widetilde{V}^n(s) \right\rangle$$

$$+ \sqrt{n}\left\langle \phi d(\cdot, V(s)), \widetilde{V}^n(s) - V(s) \right\rangle,$$

we have

$$\left| \sqrt{n}\left\langle \phi d(\cdot, V^n(s))), V^n(s) - \widetilde{V}^n(s) \right\rangle \right|$$

$$= \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(A_i^n(s)\phi(X_i^n(s))d(X_i^n(s), V^n(s)) - A_i(s)\phi(X_i(s))d(X_i(s), V^n(s)) \right) \right|$$

$$\le \frac{1}{\sqrt{n}} \sum_{i=1}^n |\phi(X_i^n(s)) - \phi(X_i(s))||d(X_i^n(s), V^n(s))|A_i^n(s)$$

$$+ \frac{1}{\sqrt{n}} \sum_{i=1}^n |\phi(X_i(s))||d(X_i^n(s), V^n(s)) - d(X_i(s), V^n(s))|A_i^n(s)$$

$$+ \frac{1}{\sqrt{n}} \sum_{i=1}^n |A_i^n(s) - A_i(s)||\phi(X_i(s))d(X_i(s), V^n(s))|$$

$$\leq \frac{K}{\sqrt{n}} \sum_{i=1}^{n} |\phi'(\theta X_i^n(s) + (1-\theta)X_i(s))||X_i^n(s) - X_i(s)|A_i^n(s)$$

$$+ \frac{K}{\sqrt{n}} \sum_{i=1}^{n} |\phi(X_i(s))||X_i^n(s) - X_i(s)|A_i^n(s)$$

$$+ \frac{K}{\sqrt{n}} \sum_{i=1}^{n} |\phi(X_i(s))||A_i^n(s) - A_i(s)|$$

$$\leq \frac{2K}{\sqrt{n}} \sum_{i=1}^{n} c_7 \left(e^{|X_i^n(s)|} \vee e^{|X_i(s)|} \right) |\widetilde{\phi}|_1 |X_i^n(s) - X_i(s)|A_i^n(s)$$

$$+ \frac{K}{\sqrt{n}} \sum_{i=1}^{n} c_7 e^{|X_i(s)|} |\widetilde{\phi}|_0 |A_i^n(s) - A_i(s)|,$$

and hence

$$\mathbb{E} \left| \sqrt{n} \left\langle \phi d(\cdot, V^n(s))), V^n(s) - \widetilde{V}^n(s) \right\rangle \right|^{2\lambda} 1_{\{\eta_m^{n,p} \geq s\}}$$

$$\leq 2^{2\lambda-1} |\widetilde{\phi}|_1^{2\lambda} (2\sqrt{n} c_7 K)^{2\lambda} \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^{n} |X_i^n(s) - X_i(s)|^{2\lambda} \right.$$

$$\times \left(\frac{1}{n} \sum_{i=1}^{n} \left| \left(e^{|X_i^n(s)|} \vee e^{|X_i(s)|} \right) A_i^n(s) \right|^{\frac{2\lambda}{2\lambda-1}} \right)^{2\lambda-1} 1_{\{\eta_m^{n,p} \geq s\}} \right)$$

$$+ 2^{2\lambda-1} |\widetilde{\phi}|_0^{2\lambda} n^\lambda K^{2\lambda} c_7^{2\lambda} \mathbb{E} \left(\left(\frac{1}{n} \sum_{i=1}^{n} |A_i^n(s) - A_i(s)|^\lambda \right)^2 \right.$$

$$\times \left(\frac{1}{n} \sum_{i=1}^{n} e^{\frac{\lambda}{\lambda-1}|X_i(s)|} \right)^{\lambda-1} 1_{\{\eta_m^{n,p} \geq s\}} \right)$$

$$\leq c_9(m, \lambda, s) |\widetilde{\phi}|_1^{2\lambda}.$$

As

$$\left| \sqrt{n} \left\langle \phi(d(\cdot, V^n(s)) - d(\cdot, \widetilde{V}^n(s)), \widetilde{V}^n(s) \right\rangle \right|$$

$$= \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i(s) \phi(X_i(s))(d(X_i(s), V^n(s)) - d(X_i(s), \widetilde{V}^n(s))) \right|$$

$$\leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i(s) |\phi(X_i(s))| K \rho(V^n(s), \widetilde{V}^n(s)),$$

we have

$$\mathbb{E} \left| \sqrt{n} \left\langle \phi(d(\cdot, V^n(s)) - d(\cdot, \widetilde{V}^n(s)), \widetilde{V}^n(s) \right\rangle \right|^{2\lambda} 1_{\{\eta_m^{n,p} \geq s\}}$$

$$\leq K^{2\lambda} n^\lambda \mathbb{E} \left| \frac{1}{n} \sum_{i=1}^{n} A_i(s) c_7 e^{|X_i(s)|} |\widetilde{\phi}|_0 \rho(V^n(s), \widetilde{V}^n(s)) \right|^{2\lambda} 1_{\{\eta_m^{n,p} \geq s\}}$$

$$\leq K^{2\lambda} n^\lambda c_7^{2\lambda} |\widetilde{\phi}|_0^{2\lambda} \mathbb{E} \left[\left(\frac{1}{n} \sum_{i=1}^{n} A_i(s)^2 \frac{1}{n} \sum_{i=1}^{n} e^{2|X_i(s)|} \right)^\lambda \right.$$

$$\times 2^{2\lambda-1}\left(m^{2\lambda}(\frac{1}{n}\sum_{i=1}^{n}|X_i^n(s)-X_i(s)|^2)^{\lambda}+(\frac{1}{n}\sum_{i=1}^{n}|A_i^n(s)-A_i(s)|)^{2\lambda}\right)1_{\{\eta_m^{n,p}\geq s\}}\bigg]$$

$$\leq K^{2\lambda}n^{\lambda}c_7^{2\lambda}|\widetilde{\phi}|_0^{2\lambda}\left(m^2c(m,2)\right)^{\lambda}2^{2\lambda-1}$$

$$\times\mathbb{E}\left[m^{2\lambda}\frac{1}{n}\sum_{i=1}^{n}|X_i^n(s)-X_i(s)|^{2\lambda}+\left(\frac{1}{n}\sum_{i=1}^{n}|A_i^n(s)-A_i(s)|^{\lambda}\right)^2 1_{\{\eta_m^{n,p}\geq s\}}\right]$$

$$\leq c_{10}(m,\lambda,s)|\widetilde{\phi}|_0^{2\lambda}.$$

As

$$\left|\sqrt{n}\left\langle\phi(d(\cdot,\widetilde{V}^n(s))-d(\cdot,V(s))),\widetilde{V}^n(s)\right\rangle\right|$$
$$=\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}A_i(s)\phi(X_i(s))(d(X_i(s),\widetilde{V}^n(s)))-d(X_i(s),V(s)))\right|,$$

we have

$$\mathbb{E}\left|\sqrt{n}\left\langle\phi(d(\cdot,\widetilde{V}^n(s))-d(\cdot,V(s))),\widetilde{V}^n(s)\right\rangle\right|^{2\lambda}1_{\{\eta_m^{n,p}\geq s\}}$$

$$\leq c_7^{2\lambda}K^{2\lambda}n^{\lambda}|\widetilde{\phi}|_0^{2\lambda}\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}A_i(s)^{\frac{2\lambda}{2\lambda-1}}e^{\frac{2\lambda}{2\lambda-1}|X_i(s)|}\right)^{2\lambda-1}\right.$$

$$\left.\times\frac{1}{n}\sum_{i=1}^{n}\left|d(X_i(s),\widetilde{V}^n(s))-d(X_i(s),V(s))\right|^{2\lambda}1_{\{\eta_m^{n,p}\geq s\}}\right]$$

$$\leq c_{11}(m,\lambda,s)|\widetilde{\phi}|_0^{2\lambda},$$

where the last inequality follows by arguments similar to the estimate for the third term on the right side of (2.8).

Let

$$N_n=\sum_{i=1}^{n}(A_i(s)\phi(X_i(s))d(X_i(s),V(s))-\langle\phi d(\cdot,V(s)),V(s)\rangle),\qquad n\geq 1.$$

Then $\{N_n:n=1,2,\cdots\}$ is a discrete-time $\mathbb{P}(\cdot|W)$-martingale with, using the notation of Burkholder [3],

$$S_n(N)^2=\sum_{i=1}^{n}(A_i(s)\phi(X_i(s))d(X_i(s),V(s))-\langle\phi d(\cdot,V(s)),V(s)\rangle)^2.$$

(Do not confuse $S_n(N)$ here with our process S_n.) By Theorem 3.2 in Burkholder [3], there exists a constant C_λ such that

$$\mathbb{E}(N_n^{2\lambda}|W)\leq C_\lambda\mathbb{E}(S_n(N)^{2\lambda}|W).$$

Therefore

$$
\begin{aligned}
\mathbb{E}\Big|\sqrt{n}\Big\langle \phi d(\cdot,V(s)),\widetilde{V}^n(s)-V(s)\Big\rangle\Big|^{2\lambda} \\
&= \frac{1}{n^\lambda}\mathbb{E}(\mathbb{E}(N_n^{2\lambda}|W)) \\
&\leq \frac{1}{n^\lambda}C_\lambda \mathbb{E}\Big[\mathbb{E}\Big[\Big(\sum_{i=1}^n (A_i(s)\phi(X_i(s))d(X_i(s),V(s)) - \langle \phi d(\cdot,V(s)),V(s)\rangle)^2\Big)^\lambda\Big|W\Big]\Big] \\
&= C_\lambda \mathbb{E}\Big(|A_1(s)\phi(X_1(s))d(X_1(s),V(s)) - \langle \phi d(\cdot,V(s)),V(s)\rangle|^{2\lambda}\Big) \\
&\leq c_{12}(m,\lambda,s)|\widetilde{\phi}|_0^{2\lambda},
\end{aligned}
$$

and we have

$$
\mathbb{E}\big[\big|\sqrt{n}(\langle \phi d(\cdot,V^n(s))),V^n(s)\rangle - \langle \phi d(\cdot,V(s))),V(s)\rangle)\big|^{2\lambda}\mathbf{1}_{\{\eta_m^{n,p}\geq s\}}\big] \leq c_{13}(m,\lambda,s)|\widetilde{\phi}|_1^{2\lambda}.
$$

Similar arguments give

$$
\mathbb{E}\big[\big(\sqrt{n}|\langle L(V^n(s))\phi,V^n(s)\rangle - \langle L(V(s))\phi,V(s)\rangle|\big)^{2\lambda}\mathbf{1}_{\{\eta_m^{n,p}\geq s\}}\big] \leq c_{14}(m,\lambda,s)|\widetilde{\phi}|_3^{2\lambda},
$$

where the estimate in terms of the higher derivatives is required because of the differential operator, and we have

$$
\mathbb{E}[H_\phi^{3,n}(s)^\lambda \mathbf{1}_{\{\eta_m^{n,p}\geq s\}}] \leq c_{15}(m,\lambda,s)|\widetilde{\phi}|_3^{2\lambda} \leq c_{15}(m,\lambda,s)\|\phi\|_4^{2\lambda}. \tag{3.7}
$$

Finally, again applying similar arguments, we can show

$$
\mathbb{E}[H_\phi^{2,n}(s)^\lambda \mathbf{1}_{\{\eta_m^{n,p}\geq s\}}] \leq c_{16}(m,\lambda,s)|\widetilde{\phi}|_2^{2\lambda} \leq c_{16}(m,\lambda,s)\|\phi\|_3^{2\lambda}. \tag{3.8}
$$

Without loss of generality, we can assume that all of the $c_l(m,\lambda,s)$ are nondecreasing in s. Applying (3.6), (3.7), and (3.8), (3.5) gives

$$
\mathbb{E}[\sup_{s\leq t}\langle \phi, S_n(s\wedge\eta_m^{n,p}) - S_n(0)\rangle^2] \leq c_{17}(m,p,t)\|\phi\|_4^2.
$$

For κ sufficiently large, the embedding $T_{\kappa-1}^4:\Phi_{\kappa-1}\to\Phi_4$ is Hilbert-Schmidt and hence, if $\{\phi_k\}$ is an orthonormal basis for $\Phi_{\kappa-1}$, $\sum_{k=1}^\infty\|\phi_k\|_4^2<\infty$. (See [17], Lemma 1 and Theorem 2, pages 33-34.) Consequently,

$$
\begin{aligned}
\mathbb{E}[\sup_{s\leq t}\|S_n(s\wedge\eta_m^{n,p}) - S_n(0)\|_{-(\kappa-1)}^2] &\leq \mathbb{E}[\sum_{k=1}^\infty \sup_{s\leq t}\langle \phi_k, S_n(s\wedge\eta_m^{n,p}) - S_n(0)\rangle^2] \\
&\leq c_{17}(m,p,t)\sum_{k=1}^\infty\|\phi_k\|_4^2 < \infty.
\end{aligned}
$$

It follows that for each $t\geq 0$ and $\epsilon>0$, there exists $k_{t,\epsilon}>0$ such that

$$
\sup_n \mathbb{P}\{\sup_{s\leq t}\|S_n(s\wedge\eta_m^{n,p})\|_{-(\kappa-1)} > k_{t,\epsilon}\} \leq \epsilon.
$$

But $\{\psi\in\Phi_{-\kappa}:\|\psi\|_{-(\kappa-1)}\leq k_{t,\epsilon}\}$ is a compact subset of $\Phi_{-\kappa}$, so $\{S_n(\cdot\wedge\eta_m^{n,p})\}$ satisfies the compact containment condition in $\Phi_{-\kappa}$.

Similarly, by (3.4), for $t < t+h \le T$, $h < 1$,

$$\mathbb{E}[\|S_n((t+h)\wedge\eta_m^{n,p}) - S_n(t\wedge\eta_m^{n,p})\|^2_{-(\kappa-1)}|\mathcal{F}_t]$$

$$\le \sum_{k=1}^{\infty} \mathbb{E}[\int_t^{t+h} (H_{\phi_k}^{1,n}(s) + H_{\phi_k}^{2,n}(s) + hH_{\phi_k}^{3,n}(s))\mathbf{1}_{\{\eta_m^{n,p}\ge s\}}ds|\mathcal{F}_t]$$

$$\le h^{(\lambda-1)/\lambda}\mathbb{E}[\sum_{k=1}^{\infty}(\int_0^T (H_{\phi_k}^{1,n}(s) + H_{\phi_k}^{2,n}(s) + hH_{\phi_k}^{3,n}(s))^\lambda \mathbf{1}_{\{\eta_m^{n,p}\ge s\}}ds)^{1/\lambda}|\mathcal{F}_t].$$

Then

$$\mathbb{E}[\sum_{k=1}^{\infty}(\int_0^T (H_{\phi_k}^{1,n}(s) + H_{\phi_k}^{2,n}(s) + hH_{\phi_k}^{3,n}(s))^\lambda \mathbf{1}_{\{\eta_m^{n,p}\ge s\}}ds)^{1/\lambda}]$$

$$= \mathbb{E}[\sum_{k=1}^{\infty}\|\phi_k\|_4^2(\int_0^T \left((H_{\phi_k}^{1,n}(s) + H_{\phi_k}^{2,n}(s) + hH_{\phi_k}^{3,n}(s))/\|\phi_k\|_4^2\right)^\lambda \mathbf{1}_{\{\eta_m^{n,p}\ge s\}}ds)^{1/\lambda}]$$

$$\le \mathbb{E}[\sum_{k=1}^{\infty}\|\phi_k\|_4^2(\int_0^T \left((H_{\phi_k}^{1,n}(s) + H_{\phi_k}^{2,n}(s) + hH_{\phi_k}^{3,n}(s))/\|\phi_k\|_4^2\right)^\lambda \mathbf{1}_{\{\eta_m^{n,p}\ge s\}}ds)^{1/\lambda}]$$

$$\le (\sum_{k=1}^{\infty}\|\phi_k\|_4^2)^{(\lambda-1)/\lambda}$$

$$\times \left(\sum_{k=1}^{\infty}\|\phi_k\|_4^2 \int_0^T \mathbb{E}[\left((H_{\phi_k}^{1,n}(s) + H_{\phi_k}^{2,n}(s) + hH_{\phi_k}^{3,n}(s))/\|\phi_k\|_4^2\right)^\lambda \mathbf{1}_{\{\eta_m^{n,p}\ge s\}}]ds\right)^{1/\lambda}$$

$$< \infty.$$

Since

$$\|S_n((t+h)\wedge\eta_m^{n,p}) - S_n(t\wedge\eta_m^{n,p})\|^2_{-\kappa} \le \|S_n((t+h)\wedge\eta_m^{n,p}) - S_n(t\wedge\eta_m^{n,p})\|^2_{-(\kappa-1)},$$

we have verified the conditions of Theorem 4.20 of [24] (Theorem 3.8.6 of [13]) with

$$\gamma_n(\delta) = \delta^{(\lambda-1)/\lambda}\sum_{k=1}^{\infty}(\int_0^T (H_{\phi_k}^{1,n}(s) + H_{\phi_k}^{2,n}(s) + \delta H_{\phi_k}^{3,n}(s))^\lambda \mathbf{1}_{\{\eta_m^{n,p}\ge s\}}ds)^{1/\lambda}.$$

Note that since the S_n are continuous, tightness of $\{S_n\}$ in $D_{\Phi_{-k}}[0,\infty)$ implies tightness in $C_{\Phi_{-k}}[0,\infty)$. ∎

The same argument gives tightness for $\{M^{1,n}\}$, and we have the following additional result.

LEMMA 3.1. *Under the conditions of Theorem 3.1, $\{M^{1,n}\}$ is tight in $C_{\Phi_{-\kappa}}[0,\infty)$.*

4. Characterization of the limit
We need the following additional assumptions.
(S4) There exists $\delta > 0$ such that

$$|\sigma(x,\nu)^T z|^2 - \delta \int_U |z\cdot\alpha(x,\nu,u)|^2\mu(du) \ge 0$$

$\forall x,z \in \mathbb{R}^d$, $\nu \in \mathcal{M}(\mathbb{R}^d)$.

(S5) The coefficients σ, c, d, a, b, γ, α, and β are differentiable with respect to the measure in the sense that, for example, there exists a bounded, continuous function ∂d on $\mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$ such that for $\nu_1, \nu_2 \in \mathcal{M}(\mathbb{R}^d)$,

$$d(x,\nu_2) - d(x,\nu_1) = \int_0^1 \int_{\mathbb{R}^d} \partial d(x,(1-r)\nu_1 + r\nu_2, y)(\nu_2(dy) - \nu_1(dy))dr.$$

(S6) For κ given by Theorem 3.1, $\phi \in \Phi_{\kappa+l+2}$, $\nu_1, \nu_2 \in \mathcal{M}(\mathbb{R}^d)$, and $u \in U$,

$$
\begin{aligned}
F_1&(\nu_1,\nu_2)\phi \\
&\equiv d(\cdot,\nu_2)\phi + L(\nu_2)\phi \\
&\quad + \int_{\mathbb{R}^d} \int_0^1 \Big(\phi(x)\partial d(x,r\nu_2 + (1-r)\nu_1,\cdot) \\
&\qquad\qquad\qquad + \partial L(r\nu_2 + (1-r)\nu_1,\cdot)\phi(x)\Big)dr\nu_1(dx) \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
F_2&(\nu_1,\nu_2,u)\phi \\
&\equiv \phi\beta(\cdot,\nu_2,u) + \nabla\phi^T \alpha(\cdot,\nu_2,u) \\
&\quad - \int_{\mathbb{R}^d} \int_0^1 \Big(\phi(x)\partial\beta(x,r\nu_2 + (1-r)\nu_1,u,\cdot) \\
&\qquad\qquad\qquad + \nabla\phi^T(x)\partial\alpha(x,r\nu_2 + (1-r)\nu_1,u,\cdot)\Big)dr\nu_1(dx) \quad (4.2)
\end{aligned}
$$

are in $\Phi_{\kappa+l}$ for $0 \leq l \leq 2$.
For $\phi \in \Phi_{\kappa+2}$, the mappings

$$(\nu_1,\nu_2,v) \in \mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d) \times \Phi_{-\kappa} \to \langle F_1(\nu_1,\nu_2)\phi, v\rangle \in \mathbb{R} \qquad (4.3)$$

and

$$(\nu_1,\nu_2,v) \in \mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d) \times \Phi_{-\kappa} \to \langle F_2(\nu_1,\nu_2,\cdot)\phi, v\rangle \in L^2(U,\mu) \qquad (4.4)$$

are continuous.

(S7) For each $\nu \in \mathcal{M}(\mathbb{R}^d)$, the mappings from $x \in \mathbb{R}^d$ to $a_{ij}(x,\nu)$, $b_i(x,\nu)$, $d(x,\nu) \in \mathbb{R}$ and $\alpha_i(x,\nu,\cdot)$, $\beta(x,\nu,\cdot) \in L^2(U,\mu)$ have bounded derivatives with respect to x up to order $q \equiv \kappa+2$. For each $x \in \mathbb{R}^d$, $u \in U$ and $\nu \in \mathcal{M}(\mathbb{R}^d)$, $\partial a_{ij}(x,\nu,\cdot)$, $\partial b_i(x,\nu,\cdot)$, $\partial d(x,\nu,\cdot)$, $\partial\alpha_i(x,\nu,u,\cdot)$, $\partial\beta(x,\nu,u,\cdot)$ are in Φ_q, and there exists a constant K such that

$$
\sum_{i,j} \|\partial a_{ij}(x,\nu,\cdot)\|_q^2 + \sum_i \|\partial b_i(x,\nu,\cdot)\|_q^2 + \|\partial d(x,\nu,\cdot)\|_q^2
$$
$$
+ \int_U \left(\sum_i \|\partial\alpha_i(x,\nu,u,\cdot)\|_q^2 + \|\partial\beta(x,\nu,u,\cdot)\|_q^2 \right) \mu(du) \leq K.
$$

REMARK 4.1. *If $\nu_1 = \nu_2 = \nu$, we write $F_i(\nu)$ rather than $F_i(\nu,\nu)$. Condition (S6) implies smoothness and growth conditions on the coefficients of the differential operators. Continuity of the mapping*

$$(\nu_1,\nu_2) \in \mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d) \to F_1(\nu_1,\nu_2)\phi \in \Phi_k$$

would imply (4.3).

Continuing to restrict the calculations to dimension $d = 1$, by (3.3),

$$\langle \phi, S_n(t) \rangle = \langle \phi, S_n(0) \rangle + \langle \phi, M^{1,n}(t) \rangle + \int_0^t \langle F_1(V(s), V^n(s)) \phi, S_n(s) \rangle \, ds$$

$$+ \int_{U \times [0,t]} \langle F_2(V(s), V^n(s), u) \phi, S_n(s) \rangle W(du \, ds), \tag{4.5}$$

where

$$M^{1,n}_\phi(t) = \langle \phi, M^{1,n}(t) \rangle$$

$$= n^{-1/2} \sum_{i=1}^n \int_0^t A_i^n(s) \Big(\phi(X_i^n(s)) \gamma(X_i^n(s), V^n(s))$$

$$+ \phi'(X_i^n(s)) \sigma(X_i^n(s), V^n(s)) \Big) dB_i(s),$$

$$\sqrt{n} \left(\langle \phi d(\cdot, V^n(s)) + L(V^n(s)) \phi, V^n(s) \rangle - \langle \phi d(\cdot, V(s)) + L(V(s)) \phi, V(s) \rangle \right)$$

$$= \langle \phi d(\cdot, V^n(s)) + L(V^n(s)) \phi, S_n(s) \rangle$$

$$+ \sqrt{n} \langle \phi d(\cdot, V^n(s)) + L(V^n(s)) \phi - \phi d(\cdot, V(s)) + L(V(s)) \phi, V(s) \rangle$$

$$= \langle \phi d(\cdot, V^n(s)) + L(V^n(s)) \phi, S_n(s) \rangle$$

$$+ \int_{\mathbb{R}} \int_0^1 \Big\langle \phi(x) \partial d(x, rV^n(s) + (1-r)V(s), \cdot)$$

$$+ \partial L(rV^n(s) + (1-r)V(s), \cdot) \phi(x), S_n(s) \Big\rangle dr V(s, dx)$$

$$= \langle F_1(V(s), V^n(s)) \phi, S_n(s) \rangle,$$

and

$$\sqrt{n} \Big(\langle \phi \beta(\cdot, V^n(s), u) + \phi' \alpha(\cdot, V^n(s), u), V^n(s) \rangle - \langle \phi \beta(\cdot, V(s), u) + \phi' \alpha(\cdot, V(s), u), V(s) \rangle \Big)$$

$$= \langle \phi \beta(\cdot, V^n(s), u) + \phi' \alpha(\cdot, V^n(s), u), S_n(s) \rangle$$

$$- \int_{\mathbb{R}} \int_0^1 \Big\langle \phi(x) \partial \beta(x, rV^n(s) + (1-r)V(s), u, \cdot)$$

$$+ \phi'(x) \partial \alpha(x, V(s), u, \cdot), S_n(s) \Big\rangle dr V(s, dx)$$

$$= \langle F_2(V(s), V^n(s), u) \phi, S_n(s) \rangle.$$

Let $\mathbb{H} = L^2(U, \mu)$. In the terminology of Kurtz and Protter [25], we define a $\mathbb{R} \times \mathbb{H}^\#$-semimartingale Y by setting

$$Y(a, h, t) = at + B^h(t) = at + \int_{U \times [0,t]} h(u) W(du \, ds),$$

for $a \in \mathbb{R}$ and $h \in \mathbb{H}$. Let

$$U_n(t) = S_n(0) + M^{1,n}(t),$$

and for $\phi \in \Phi_{\kappa+2}$, let $\langle F(V(s), V^n(s)) \phi, S_n(s) \rangle$ denote the $\mathbb{R} \times \mathbb{H}$-valued process given by

$$\langle F(V(s), V^n(s)) \phi, S_n(s) \rangle = (\langle F_1(V(s), V^n(s)) \phi, S_n(s) \rangle, \langle F_2(V(s), V^n(s), u) \phi, S_n(s) \rangle).$$

Then (4.5) can be rewritten in the notation of Kurtz and Protter [25] as

$$\langle \phi, S_n(t) \rangle = \langle \phi, U_n(t) \rangle + \langle F(V(\cdot), V^n(\cdot)) \phi, S_n \rangle \cdot Y(t).$$

Note that for each $\phi \in C^1(\mathbb{R})$, $h \in \mathbb{H}$, and $1 \le i \le n$, $\left[M_\phi^{1,n}, B^h \right]_t = 0$,

$$\left[M_\phi^{1,n}, B_i \right]_t$$
$$= \frac{1}{n} \int_0^t (A_i^n(s))^2 \left(\phi(X_i^n(s)) \gamma(X_i^n(s), V^n(s)) + \phi'(X_i^n(s)) \sigma(X_i^n(s), V^n(s)) \right)^2 ds,$$

and

$$\left[M_\phi^{1,n} \right]_t = \int_0^t \left\langle \left(\phi \gamma(\cdot, V^n(s)) + \phi' \sigma(\cdot, V^n(s)) \right)^2, V_2^n(s) \right\rangle ds,$$

where $V_2^n(s) = \frac{1}{n} \sum_{j=1}^n A_j^n(s)^2 \delta_{X_j^n(s)}$.

We should emphasize that we are proving convergence *in distribution* for $\{S_n\}$. The limit will not "live" on the original probability space. To be precise, for a countable dense subset $\{h_j\} \subset \mathbb{H}$, the sequence

$$\{ (V^n, M^{1,n}, S_n, \{Y(h_j)\}, \{B_i\}, \{X_i\}, \{A_i\}) \}$$

is relatively compact in $C_{\mathcal{M}(\mathbb{R}) \times (\Phi_{-\kappa})^2 \times (\mathbb{R}^\infty)^4}[0, \infty)$. Denoting a limit point by

$$(V, M, S, \{Y(h_i)\}, \{B_i\}, \{X_i\}, \{A_i\})$$

(even though these are *not* the V, Y, $\{B_i\}$, $\{X_i\}$, $\{A_i\}$ on the original probability space, they will have the same distribution), M (and hence S) will not be adapted to the filtration $\{\mathcal{F}_t^{Y, \{B_i\}}\}$ generated by Y and $\{B_i\}$. Note that $\{Y(h_i)\}$ determines $Y(h)$ (and hence B^h) for all $h \in \mathbb{H}$ and the $Y(h)$ determine W.

For any limit point, M will be a $\Phi_{-\kappa}$-valued local martingale with $\left[M_\phi, B^h \right]_t = 0$ for every $\phi \in \Phi$ and $h \in \mathbb{H}$ and

$$[M_\phi]_t = \int_0^t \left\langle \left(\phi \gamma(\cdot, V(s)) + \phi' \sigma(\cdot, V(s)) \right)^2, V_2(s) \right\rangle ds,$$

where

$$V_2(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n A_i(t)^2 \delta_{X_i(t)}.$$

LEMMA 4.1. *For $\phi \in \Phi_\kappa$,*

$$\mathbb{E}[e^{i \langle \phi, S(0) + M(t) \rangle} | W]$$
$$= \exp \left\{ -\frac{1}{2} \left(\langle \phi^2, V_2(0) \rangle - \langle \phi, V(0) \rangle^2 + \int_0^t \langle (\phi \gamma(\cdot, V(s)) + \phi' \sigma(\cdot, V(s)))^2, V_2(s) \rangle ds \right) \right\}$$

and

$$\mathbb{E}[e^{i \langle \phi, M(t+r) - M(t) \rangle} | \sigma(W) \vee \mathcal{F}_t^M]$$
$$= \exp \left\{ -\frac{1}{2} \int_t^{t+r} \langle (\phi \gamma(\cdot, V(s)) + \phi' \sigma(\cdot, V(s)))^2, V_2(s) \rangle ds \right\},$$

which determine the joint distribution of W and M.

Proof: Define

$$\widetilde{M}_\phi^n(t) = \left\langle \phi, \widetilde{M}^n(t) \right\rangle$$

$$= n^{-1/2} \sum_{i=1}^n \int_0^t A_i(s) \Big(\phi(X_i(s))\gamma(X_i(s), V(s))$$

$$+ \phi'(X_i(s))\sigma(X_i(s), V(s)) \Big) dB_i(s),$$

and observe that

$$\left[M_\phi^{1,n} - \widetilde{M}_\phi^n \right]_t$$

$$= \frac{1}{n} \sum_{i=1}^n \int_0^t \Big(A_i^n(s)\phi(X_i^n(s))\gamma(X_i^n(s), V^n(s)) + \phi'(X_i^n(s))\sigma(X_i^n(s), V^n(s))$$

$$- A_i(s)\phi(X_i(s))\gamma(X_i(s), V(s)) - \phi'(X_i(s))\sigma(X_i(s), V(s)) \Big)^2 ds.$$

converges to zero. It follows that $M^{1,n}$ and $\widetilde{M}^n$ must have the same limit. Again, to be precise, one should say that any limit point of

$$\{(V^n, M^{1,n}, \widetilde{M}^n, S_n, \{Y(h_j)\}, \{B_i\}, \{X_i\}, \{A_i\})\}$$

will be of the form

$$(V, M, M, S, \{Y(h_i)\}, \{B_i\}, \{X_i\}, \{A_i\}).$$

For a $\sigma(W)$ measurable random variable Z, exchangeability implies

$$\mathbb{E}[e^{i\langle \phi, S(0)+M(t)\rangle} Z]$$

$$= \lim_{n\to\infty} \mathbb{E}[e^{i\langle \phi, S_n(0)+\widetilde{M}^n(t)\rangle} Z]$$

$$= \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[\exp\{i\frac{1}{\sqrt{n}}(A_1(0)\phi(X_1(0)) - \langle \phi, V(0)\rangle + \widetilde{M}_{\phi,1}\}|W]^n Z]$$

$$= \mathbb{E}[\exp\{-\frac{1}{2}\big(\langle \phi^2, V_2(0)\rangle - \langle \phi, V(0)\rangle^2 + \int_0^t \langle(\phi\gamma(\cdot, V(s)) + \phi'\sigma(\cdot, V(s)))^2, V_2(s)\rangle ds\}Z]$$

where

$$\widetilde{M}_{\phi,1}(t) = \int_0^t A_1(s)\Big(\phi(X_1(s))\gamma(X_1(s), V(s)) + \phi'(X_1(s))\sigma(X_1(s), V(s)) \Big) dB_1(s).$$

The proof of the second identity is similar. ∎

Let $U(t) = S(0) + M(t)$ and

$$\langle F(V(s))\phi, S(s)\rangle = (\langle F_1(V(s))\phi, S(s)\rangle, \langle F_2(V(s), u)\phi, S(s)\rangle).$$

Then (S_n, U_n) is relatively compact in $C_{\Phi_{-\kappa} \times \Phi_{-\kappa}}[0, \infty)$, and by the continuity assumptions on F_1 and F_2 and Theorem 5.5 in [25], for any limit point (S, U), we have

$$\langle \phi, S(t)\rangle = \langle \phi, U(t)\rangle + \langle F(V(\cdot))\phi, S\rangle \cdot Y(t).$$

Specifically, any limit point of $\{S_n\}$ satisfies (1.10).

To prove uniqueness for the solution to (1.10), suppose that S_1 and S_2 are solutions and set $\xi = S_1 - S_2$. Then ξ satisfies

$$\langle \phi, \xi(t) \rangle = \int_0^t \langle F_1(V(s))\phi, \xi(s) \rangle \, ds + \int_{U \times [0,t]} \langle F_2(V(s),u)\phi, \xi(s) \rangle \, W(duds). \qquad (4.6)$$

We adapt arguments of Rozovskii [36] to establish that $\xi \equiv 0$ is the unique solution to (4.6) and hence establish uniqueness for (1.10).

LEMMA 4.2.   *Suppose that the assumptions (S1)-(S7) hold. Then $\xi = 0$ a.s.*
Proof: Let $q = \kappa + 2$, and for $\nu \in \mathcal{M}(\mathbb{R})$, define $\rho_\nu \equiv \int_{\mathbb{R}} \psi(x)^{-1} \nu(dx) < \infty$.   By Lemma A.6,

$$2 \langle v, F_1^*(\nu)v \rangle_{-q} + \int_U \| F_2^*(\nu,u)v \|_{-q}^2 \mu(du) \leq c_{18} \rho_\nu^2 \| v \|_{-q}^2, \qquad (4.7)$$

for all $v \in \Phi_{-\kappa}$.

Note that $\xi(t)$ takes values in $\Phi_{-\kappa} \subset \Phi_{-q}$. Let $\{\phi_j^q\}$ be an orthonormal basis for Φ_q. Applying Itô's formula, we have

$$\begin{aligned}
\langle \phi_j^q, \xi(t) \rangle^2 = \int_0^t & 2 \langle \phi_j^q, \xi(t) \rangle \langle F_1(V(s))\phi_j^q, \xi(s) \rangle \, ds \\
& + \int_{U \times [0,t]} 2 \langle \phi_j^q, \xi(t) \rangle \langle F_2(V(s),u)\phi_j^q, \xi(s) \rangle \, W(duds) \\
& + \int_0^t \int_U \langle F_2(V(s),u)\phi_j^q, \xi(s) \rangle^2 \mu(du) ds.
\end{aligned}$$

By Proposition 2.1, if (2.1) and (2.3) hold, then (2.6) holds giving $\mathbb{E}[\sup_{t \leq T} \rho_{V(t)}] < \infty$. Let $\tau_k = \inf\{t : \rho_{V(t)} \geq k\}$.   Stopping the processes at τ_k, taking expectations, and summing over j, (4.7) gives

$$\begin{aligned}
\mathbb{E}\|\xi(t \wedge \tau_k)\|_{-q}^2 = \mathbb{E} \int_0^{t \wedge \tau_k} & \left(2 \langle \xi(s), F_1^*(V(s))\xi(s) \rangle_{-q} + \int_U \| F_2^*(V(s),u)\xi(s) \|_{-q}^2 \mu(du) \right) ds \\
& \leq \int_0^t c_{18} k^2 \mathbb{E}\|\xi(s \wedge \tau_k)\|_{-q}^2 ds.
\end{aligned}$$

Then uniqueness follows from Gronwall's inequality and the fact that $\tau_k \to \infty$ as $k \to \infty$.   ∎

Finally, we have our main result.

THEOREM 4.2.   *Under assumptions (S1)-(S7), we have $S_n \Rightarrow S$ and S is the unique solution to the stochastic evolution equation (1.10).*

5. CLT for Euler scheme

Now we consider the CLT for the Euler scheme used in [27]. Let $\eta_\delta(s) = \left[\frac{s}{\delta} \right] \delta$, and for some partition $\{U_k\}$ of U and $u_k \in U_k$, define $\xi_\delta(u) = u_k$, $u \in U_k$, $k = 1, 2, \ldots$.

Let $\{(X_i^{n,\delta}, A_i^{n,\delta}), i = 1, \ldots, n\}$ satisfy

$$
\begin{aligned}
X_i^{n,\delta}(t) = X_i(0) &+ \int_0^t \sigma(X_i^{n,\delta}(\eta_\delta(s)), V^{n,\delta}(\eta_\delta(s))) dB_i(s) \\
&+ \int_0^t c(X_i^{n,\delta}(\eta_\delta(s)), V^{n,\delta}(\eta_\delta(s))) ds \\
&+ \int_{U \times [0,t]} \alpha(X_i^{n,\delta}(\eta_\delta(s)), V^{n,\delta}(\eta_\delta(s)), \xi_\delta(u)) W(duds),
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
A_i^{n,\delta}(t) = A_i(0) &+ \int_0^t A_i^{n,\delta}(s) \gamma(X_i^{n,\delta}(\eta_\delta(s)), V^{n,\delta}(\eta_\delta(s))) dB_i(s) \\
&+ \int_0^t A_i^{n,\delta}(s) d(X_i^{n,\delta}(\eta_\delta(s)), V^{n,\delta}(\eta_\delta(s))) ds \\
&+ \int_{U \times [0,t]} A_i^{n,\delta}(s) \beta(X_i^{n,\delta}(\eta_\delta(s)), V^{n,\delta}(\eta_\delta(s)), \xi_\delta(u)) W(duds),
\end{aligned} \tag{5.2}
$$

where

$$
V^{n,\delta}(t) = \frac{1}{n} \sum_{i=1}^n A_i^{n,\delta}(t) \delta_{X_i^{n,\delta}(t)}. \tag{5.3}
$$

In this paper, we only analyze the simplest case in which W is a one-dimensional Brownian motion, that is, U consists of a single point.

Modifying Theorem 3.3 in [27] in a way similar to the proof of Theorem 2.4 of the current paper, we have the following result.

THEOREM 5.1. *Under the assumptions (S1)-(S5), we have*

$$
\mathbb{E} \sup_{0 \le t \le T} \left(\left| X_i^{n,\delta}(t) - X_i^n(t) \right|^{2\lambda} + \left(\frac{1}{n} \sum_{j=1}^n |A_j^{n,\delta}(t) - A_j^n(t)|^\lambda \right)^2 \right) 1_{t < \eta_m^{n,\delta}} \le c(T,m)\delta^\lambda,
$$

where

$$
\eta_m^{n,\delta} = \inf \left\{ t : \frac{1}{n} \sum_{i=1}^n A_i^n(t)^2 > m^2 \text{ or } \frac{1}{n} \sum_{i=1}^n A_i^{n,\delta}(t)^2 > m^2 \right\}.
$$

Applying the same arguments as those in Section 3, we can prove that the sequence $\widetilde{S}_n \equiv \sqrt{n}(V^{n,1/n} - V^n)$ is tight. Now we characterize its limit points.

Note that $\widetilde{S}_n(0) = 0$. As in (3.3), we have

$$
\begin{aligned}
&\left\langle \phi, \widetilde{S}_n(t) \right\rangle \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \Bigg\{ A_i^{n,1/n}(s) \Big[\phi(X_i^{n,1/n}(s)) \gamma(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s))) \\
&\qquad + \phi'(X_i^{n,1/n}(s)) \sigma(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s))) \Big] \\
&\qquad - A_i^n(s) [\phi(X_i^n(s)) \gamma(X_i^n(s), V^n(s)) + \phi'(X_i^n(s)) \sigma(X_i^n(s), V^n(s))] \Bigg\} dB_i(s) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \Bigg\{ A_i^{n,1/n}(s) \Big[\phi(X_i^{n,1/n}(s)) d(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s))) \\
&\qquad + L(V^{n,1/n}(\eta_{\frac{1}{n}}(s))) \phi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s))) \Big] \\
&\qquad - A_i^n(s) [\phi(X_i^n(s)) d(X_i^n(s), V^n(s)) + L(V^n(s)) \phi(X_i^n(s))] \Bigg\} ds \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \Bigg\{ A_i^{n,1/n}(s) \Big[\phi(X_i^{n,1/n}(s)) \beta(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s))) \\
&\qquad + \phi'(X_i^{n,1/n}(s)) \alpha(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s))) \Big] \\
&\qquad - A_i^n(s) [\phi(X_i^n(s)) \beta(X_i^n(s), V^n(s)) + \phi'(X_i^n(s)) \alpha(X_i^n(s), V^n(s))] \Bigg\} dW(s) \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}
\tag{5.4}
$$

LEMMA 5.1. *Let $\xi_i^n(t)$ be processes satisfying*

$$
\frac{1}{n} \sum_{i=1}^n \xi_i^n(t)^2 \le K, \qquad \forall t, \forall \omega.
$$

Then

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \xi_i^n(s)(B_i(s) - B_i(\eta_{\frac{1}{n}}(s)))^2 ds \to 0,
\tag{5.5}
$$

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \xi_i^n(s)(s - \eta_{\frac{1}{n}}(s))^2 ds \to 0,
\tag{5.6}
$$

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \xi_i^n(s)(W(s) - W(\eta_{\frac{1}{n}}(s)))^2 ds \to 0,
\tag{5.7}
$$

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \xi_i^n(\eta_{\frac{1}{n}}(s))(B_i(s) - B_i(\eta_{\frac{1}{n}}(s))) ds \to 0,
\tag{5.8}
$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{0}^{t}\xi_{i}^{n}(\eta_{\frac{1}{n}}(s))(s-\eta_{\frac{1}{n}}(s))ds\to 0, \tag{5.9}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{0}^{t}\xi_{i}^{n}(\eta_{\frac{1}{n}}(s))(W(s)-W(\eta_{\frac{1}{n}}(s)))ds\to 0, \tag{5.10}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{0}^{t}\xi_{i}^{n}(\eta_{\frac{1}{n}}(s))(B_{i}(s)-B_{i}(\eta_{\frac{1}{n}}(s)))dW(s)\to 0, \tag{5.11}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{0}^{t}\xi_{i}^{n}(\eta_{\frac{1}{n}}(s))(s-\eta_{\frac{1}{n}}(s))dW(s)\to 0. \tag{5.12}$$

Proof: Note that

$$\mathbb{E}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{0}^{t}\xi_{i}^{n}(s)(B_{i}(s)-B_{i}(\eta_{\frac{1}{n}}(s)))^{2}ds\right|^{2}$$

$$\leq n\mathbb{E}\int_{0}^{t}\frac{1}{n}\sum_{i=1}^{n}\xi_{i}^{n}(s)^{2}\frac{1}{n}\sum_{i=1}^{n}(B_{i}(s)-B_{i}(\eta_{\frac{1}{n}}(s)))^{4}ds$$

$$\leq K\sum_{i=1}^{n}\int_{0}^{t}3(s-\eta_{\frac{1}{n}}(s))^{2}ds$$

$$\leq 3Kn\sum_{j=0}^{[nt]}\int_{\frac{j}{n}}^{\frac{j+1}{n}}(s-\frac{j}{n})^{2}ds$$

$$= Kn[nt]\frac{1}{n^{3}}\to 0.$$

This proves (5.5). (5.6), (5.7) and (5.9) can be proved similarly.

For $k=1,\ 2,\ \cdots$, let

$$M_{k}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{0}^{\frac{k}{n}}\xi_{i}^{n}(\eta_{\frac{1}{n}}(s))(B_{i}(s)-B_{i}(\eta_{\frac{1}{n}}(s)))ds.$$

Then M_{k} is a discrete-time, square integrable martingale with quadratic variation process

$$[M]_{k}=\sum_{j=1}^{k}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_{i}^{n}(\eta_{\frac{1}{n}}(s))\int_{\frac{j-1}{n}}^{\frac{j}{n}}(B_{i}(s)-B_{i}(\eta_{\frac{1}{n}}(s)))ds\right)^{2}.$$

Hence

$$\mathbb{E}M_{[nt]}^2 = \sum_{j=1}^{[nt]} \mathbb{E} \left(\int_{\frac{j-1}{n}}^{\frac{j}{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^n(\frac{j-1}{n})(B_i(s) - B_i(\frac{j-1}{n}))ds \right)^2$$

$$= \sum_{j=1}^{[nt]} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left(\xi_i^n(\frac{j-1}{n})^2 \right) (s_1 \wedge s_2 - \frac{j-1}{n})ds_1 ds_2$$

$$\leq K \sum_{j=1}^{[nt]} \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} s_1 \wedge s_2 ds_1 ds_2$$

$$= 2K[nt] \int_0^{\frac{1}{n}} s_1(\frac{1}{n} - s_1)ds_1$$

$$= \frac{K[nt]}{3n^3} \to 0.$$

This proves (5.8). (5.10) can be proved similarly.

Finally,

$$\mathbb{E} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_0^t \xi_i^n(\eta_{\frac{1}{n}}(s))(B_i(s) - B_i(\eta_{\frac{1}{n}}(s)))dW(s) \right|^2$$

$$= \mathbb{E} \int_0^t \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^n(\eta_{\frac{1}{n}}(s))(B_i(s) - B_i(\eta_{\frac{1}{n}}(s))) \right)^2 ds$$

$$= \int_0^t \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\xi_i^n(\eta_{\frac{1}{n}}(s))^2(s - \eta_{\frac{1}{n}}(s))ds$$

$$\leq K \sum_{j=0}^{[nt]} \int_{\frac{j}{n}}^{\frac{j+1}{n}} (s - \frac{j}{n})ds$$

$$= K([nt]+1)\frac{1}{2n^2} \to 0,$$

which proves (5.11). (5.12) can be proved similarly. ∎

LEMMA 5.2. *Let*

$$\widetilde{W}^n(t) = \int_0^t \sqrt{2n}(W(s) - W(\eta_{\frac{1}{n}}(s)))dW(s).$$

Then $\widetilde{W}^n \Rightarrow \widetilde{W}$ and $\widetilde{W}$ is a one-dimensional Brownian motion independent of W.
Proof: It is clear that $\widetilde{W}^n$ is a sequence of martingales and

$$\left[\widetilde{W}^n \right]_t = \int_0^t 2n(W(s) - W(\eta_{\frac{1}{n}}(s)))^2 ds$$

$$= \frac{1}{nt} \sum_{j=1}^{nt} \left[2n^2t \int_{\frac{j-1}{n}}^{\frac{j}{n}} (W(s) - W(\frac{j-1}{n}))^2 ds \right]$$

$$\to 2n^2t \int_0^{\frac{1}{n}} sds = t.$$

By (5.10), we have

$$\left[\widetilde{W}^n, W\right]_t = \int_0^t \sqrt{n}(W(s) - W(\eta_{\frac{1}{n}}(s)))ds \to 0,$$

and the lemma follows by the martingale central limit theorem ∎

Note that

$$X_i^{n,1/n}(s) - X_i^{n,1/n}(\eta_{\frac{1}{n}}(s))$$
$$= \sigma(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s)))(B_i(s) - B_i(\eta_{\frac{1}{n}}(s)))$$
$$+ c(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s)))(s - \eta_{\frac{1}{n}}(s))$$
$$+ \alpha(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s)))(W(s) - W(\eta_{\frac{1}{n}}(s))) \qquad (5.13)$$

and

$$A_i^{n,1/n}(s) = A_i^{n,1/n}(\eta_{\frac{1}{n}}(s)) \exp\left(\gamma(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s)))(B_i(s) - B_i(\eta_{\frac{1}{n}}(s)))\right.$$
$$+ D(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s)))(s - \eta_{\frac{1}{n}}(s))$$
$$\left. + \beta(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s)))(W(s) - W(\eta_{\frac{1}{n}}(s)))\right),$$

where $D = d - \frac{1}{2}(\gamma^2 + \beta^2)$. (5.5)-(5.7) then justify the replacement of $A_i^{n,1/n}(s)$ and $X_i^{n,1/n}(s)$ by $A_i^{n,1/n}(\eta_{\frac{1}{n}}(s))$ and $X_i^{n,1/n}(\eta_{\frac{1}{n}}(s))$ in the calculations below, where the notation $\approx$ means that the difference converges to zero in probability.

LEMMA 5.3. *Let* $\psi : \mathbb{R}^2 \to \mathbb{R}$ *be bounded and continuous and have bounded, continuous first derivative* $\partial_2\psi$ *with respect to the second variable. Then*

$$\int_0^t \frac{1}{\sqrt{n}} \sum_{i=1}^n A_i^{n,1/n}(\eta_{\frac{1}{n}}(s)) \left\langle \psi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), \cdot), V^{n,1/n}(s) - V^{n,1/n}(\eta_{\frac{1}{n}}(s)) \right\rangle dW(s)$$
$$\approx \int_0^t \left\langle \alpha(\cdot, V(s))\partial_2\psi(*, \cdot) + \beta(\cdot, V(s))\psi(*, \cdot), V(s) \otimes V(s) \right\rangle d\widetilde{W}^n(s).$$

Proof: Note that

$$\int_0^t \frac{1}{\sqrt{n}} \sum_{i=1}^n A_i^{n,1/n}(\eta_{\frac{1}{n}}(s)) \left\langle \psi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), \cdot), V^{n,1/n}(s) - V^{n,1/n}(\eta_{\frac{1}{n}}(s)) \right\rangle dW(s)$$
$$= \int_0^t \frac{1}{\sqrt{n}} \sum_{i=1}^n A_i^{n,1/n}(\eta_{\frac{1}{n}}(s)) \frac{1}{n} \sum_{j=1}^n A_j^{n,1/n}(s)$$
$$(\psi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), X_j^{n,1/n}(s)) - \psi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), X_j^{n,1/n}(\eta_{\frac{1}{n}}(s))))dW(s)$$
$$+ \int_0^t \frac{1}{\sqrt{n}} \sum_{i=1}^n A_i^{n,1/n}(\eta_{\frac{1}{n}}(s)) \frac{1}{n} \sum_{j=1}^n (A_j^{n,1/n}(s) - A_j^{n,1/n}(\eta_{\frac{1}{n}}(s)))$$

$$\psi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)),X_j^{n,1/n}(\eta_{\frac{1}{n}}(s)))dW(s)$$

$$\approx \int_0^t \frac{1}{n}\sum_{i=1}^n A_i^{n,1/n}(\eta_{\frac{1}{n}}(s))\frac{1}{n}\sum_{j=1}^n A_j^{n,1/n}(s)\partial_2\psi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)),X_j^{n,1/n}(\eta_{\frac{1}{n}}(s)))$$

$$\alpha(X_j^{n,1/n}(\eta_{\frac{1}{n}}(s)),V^{n,1/n}(\eta_{\frac{1}{n}}(s)))d\widetilde{W}^n(s)$$

$$+\int_0^t \frac{1}{n}\sum_{i=1}^n A_i^{n,1/n}(\eta_{\frac{1}{n}}(s))\frac{1}{n}\sum_{j=1}^n A_i^{n,1/n}(\eta_{\frac{1}{n}}(s))\beta(X_j^{n,1/n}(\eta_{\frac{1}{n}}(s)),V^{n,1/n}(\eta_{\frac{1}{n}}(s)))$$

$$\psi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)),X_j^{n,1/n}(\eta_{\frac{1}{n}}(s))d\widetilde{W}^n(s).$$

The conclusion of the lemma then follows. ∎

THEOREM 5.2. *For $\phi \in \Phi_\kappa$, define*

$$\widetilde{M}_\phi(t) \equiv \left\langle \phi, \widetilde{M}(t)\right\rangle$$

$$=\int_0^t \langle(\alpha\partial_1\beta)(\cdot,V(s))\phi,V(s)\rangle\,d\widetilde{W}(s)$$

$$+\int_0^t \langle\alpha(\cdot,V(s))\partial_2\partial_3\beta(*,V(s),\cdot)+\beta(\cdot,V(s))\partial_2\beta(*,V(s),\cdot))\phi(*),V(s)\otimes V(s)\rangle\,d\widetilde{W}(s)$$

$$+\int_0^t \langle(\alpha\partial_1\alpha)(\cdot,V(s))\phi,V(s)\rangle\,d\widetilde{W}(s)$$

$$+\int_0^t \langle\alpha(\cdot,V(s))\partial_2\partial_3\alpha(*,V(s),\cdot)+\beta(\cdot,V(s))\partial_2\alpha(*,V(s),\cdot))\phi'(*),V(s)\otimes V(s)\rangle\,d\widetilde{W}(s),$$

where ∂_1 and ∂_3 are derivatives with respect to the corresponding variables and ∂_2 refers to the operator defined in (S5). Then $\widetilde{M}_\phi$ is a martingale satisfying $[W,\widetilde{M}_\phi]_t=0$. Let $\widetilde{S}$ be a limit point of $\{\widetilde{S}_n\}$. Then $\widetilde{S}$ is the unique solution of

$$\left\langle\phi,\widetilde{S}(t)\right\rangle = \left\langle\phi,\widetilde{M}(t)\right\rangle + \int_0^t \left\langle F_1(V(s))\phi,\widetilde{S}(s)\right\rangle ds + \int_0^t \left\langle F_2(V(s))\phi,\widetilde{S}(s)\right\rangle dW(s).$$

$$(5.14)$$

Proof: Recall that I_1, I_2, I_3 are defined by (5.4). It is easy to see that $I_1 \to 0$. Note that

$$I_2 = \frac{1}{\sqrt{n}}\sum_{i=1}^n \int_0^t A_i^{n,1/n}(s)\phi(X_i^{n,1/n}(s))$$

$$\left[d(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)),V^{n,1/n}(\eta_{\frac{1}{n}}(s)))-d(X_i^{n,1/n}(s),V^{n,1/n}(s))\right]ds$$

$$+\frac{1}{\sqrt{n}}\sum_{i=1}^n \int_0^t A_i^{n,1/n}(s)\left[L(V^{n,1/n}(\eta_{\frac{1}{n}}(s)))\phi(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)))\right.$$

$$\left.-L(V^{n,1/n}(s))\phi(X_i^{n,1/n}(s))\right]ds$$

$$+\sqrt{n}\int_0^t \left(\left\langle d(\cdot, V^{n,1/n}(s))\phi + L(V^{n,1/n}(s)\phi, V^{n,1/n}(s)) \right\rangle \right.$$

$$\left. - \left\langle d(\cdot, V^n(s))\phi + L(V^n(s)\phi, V^n(s)) \right\rangle \right) ds$$

$$\equiv I_{21} + I_{22} + I_{23},$$

By Lemma 5.1, it is easy to show that $I_{21} \to 0$ and $I_{22} \to 0$, and by the definition of F_1 in (4.1),

$$I_{23} = \int_0^t \left\langle F_1(V^n(s), V^{n,1/n}(s))\phi, \widetilde{S}_n(s) \right\rangle ds$$

which converges to the second term on the right of (5.14).

Similarly

$$I_3 = \int_0^t \frac{1}{\sqrt{n}} \sum_{i=1}^n A_i^{n,1/n}(s)\phi(X_i^{n,1/n}(s))$$

$$\left[\beta(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s))) - \beta(X_i^n(s), V^n(s)) \right] dW(s)$$

$$+ \int_0^t \frac{1}{\sqrt{n}} \sum_{i=1}^n A_i^{n,1/n}(s)\phi'(X_i^{n,1/n}(s))$$

$$\left[\alpha(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)), V^{n,1/n}(\eta_{\frac{1}{n}}(s))) - \alpha(X_i^n(s), V^n(s)) \right] dW(s)$$

$$+ \int_0^t \left\langle F_2(V^n(s), V^{n,1/n}(s)\phi, \widetilde{S}_n(s) \right\rangle dW(s)$$

$$\equiv I_{31} + I_{32} + I_{33}.$$

By (5.13) and Lemma 5.3, we have

$$I_{31} \approx \int_0^t \frac{1}{\sqrt{n}} \sum_{i=1}^n A_i^{n,1/n}(s)\phi(X_i^{n,1/n}(s) \left(\partial_1 \beta(X_i^{n,1/n}(s), V^{n,1/n}(s))(X_i^{n,1/n}(\eta_{\frac{1}{n}}(s)) - X_i^{n,1/n}(s)) \right.$$

$$\left. + \left\langle \partial_2 \beta(X_i^{n,1/n}(s), V^{n,1/n}(s), \cdot), V^{n,1/n}(\eta_{\frac{1}{n}}(s))) - V^{n,1/n}(s) \right\rangle \right) dW(s)$$

$$\approx \int_0^t \langle (\alpha \partial_1 \beta)(\cdot, V(s))\phi, V(s) \rangle d\widetilde{W}^n(s)$$

$$+ \int_0^t \langle \alpha(\cdot, V(s))\partial_2 \partial_3 \beta(*, V(s), \cdot) + \beta(\cdot, V(s))\partial_2 \beta(*, V(s), \cdot))\phi(*), V(s) \otimes V(s) \rangle d\widetilde{W}^n(s)$$

and

$$I_{32} \approx \int_0^t \langle (\alpha \partial_1 \alpha)(\cdot, V(s))\phi, V(s) \rangle d\widetilde{W}^n(s)$$

$$+ \int_0^t \langle \alpha(\cdot, V(s))\partial_2 \partial_3 \alpha(*, V(s), \cdot)$$

$$+ \beta(\cdot, V(s))\partial_2 \alpha(*, V(s), \cdot))\phi'(*), V(s) \otimes V(s) \rangle d\widetilde{W}^n(s).$$

Then $I_{31} + I_{32}$ converges to $\left\langle \phi, \widetilde{M} \right\rangle$ and I_{33} converges to the third term on the right of (5.14).

Uniqueness of the solution follows from Lemma 4.2 giving the desired result. ∎

Finally, we combine the results of Sections 4 and 5.

THEOREM 5.3.   *Let* $S_n^*(t) = \sqrt{n}(V^{n,1/n}(t) - V(t))$.   *Then* $S_n^* \Rightarrow S^*$ *where* S^* *is the unique solution of the stochastic evolution equation*

$$\langle \phi, S^*(t) \rangle = \langle \phi, S(0) \rangle + \Big\langle \phi, M(t) + \widetilde{M}(t) \Big\rangle + \int_0^t \langle F_1(V(s))\phi, S^*(s) \rangle \, ds$$
$$+ \int_0^t \langle F_2(V(s)\phi, S^*(s) \rangle \, dW(s), \quad (5.15)$$

$\phi \in \Phi_\kappa$.

REMARK 5.4.   *Comparing (5.15) to (1.10), we see that the difference lies in the additional martingale driving term* $\widetilde{M}$. $\widetilde{M}$ *arises directly from the discrete-time approximation of the driving Brownian motion* W *through the limit in Lemma 5.2.*

Appendix A. Proof of monotonicity.

We can represent Φ_{-q} as the space of equivalence classes of $(q+1)$-tuples $v = \{(v_0, v_1, \cdots, v_q)\}$, where $v_j \in L_q \equiv L^2(\mathbb{R}, (1+x^2)^{2q} dx)$, such that

$$\langle v, f \rangle = \sum_{k=0}^q \int_{\mathbb{R}} (1+x^2)^{2q} v_k(x) \frac{d^k}{dx^k}(f(x)\psi(x)) dx. \qquad (A.1)$$

The $(q+1)$-tuples u and v are equivalent if the right side of (A.1) does not change when v is replaced by u. Then

$$\|v\|_{-q}^2 = \inf \left\{ \sum_{k=0}^q \|u_k\|_{L_q}^2 : u = (u_0, \ldots, u_q) \sim v \right\}. \qquad (A.2)$$

By the Riesz representation theorem, for each $v \in \Phi_{-q}$, there exists a unique $\phi \equiv \theta_q v \in \Phi_q$ such that

$$\langle v, f \rangle = \langle \phi, f \rangle_q = \sum_{k=0}^q \langle (\phi\psi)^{(k)}, (f\psi)^{(k)} \rangle_{L_q}.$$

It follows that

$$v \sim \{(\phi\psi, (\phi\psi)', \cdots, (\phi\psi)^{(q)})\}$$

and

$$\|v\|_{-q}^2 = \sum_{k=0}^q \|(\phi\psi)^{(k)}\|_{L_q}^2.$$

In particular, the infimum in (A.2) is achieved. For each $u = \{(u_0, u_1, \cdots, u_q)\} \in \Phi_{-q}$,

$$\langle u, v \rangle_{-q} = \sum_{k=0}^q \Big\langle u_k, (\phi\psi)^{(k)}) \Big\rangle_{L_q}$$

does not depend on the choice of the $(q+1)$-tuple in the class of u. Note also, that since Φ_{q+2} is dense in Φ_q, $\{v \in \Phi_{-q} : \theta_q v \in \Phi_{q+2}\}$ is dense in Φ_{-q}.

LEMMA A.1. *Suppose that f and its derivatives up to order q are bounded. Then for $\gamma, \phi \in \Phi_{q+1}$,*

$$\sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (f\gamma'\psi)^{(k)} (\phi\psi)^{(k)} dx$$

$$= \int_{\mathbb{R}} (1+x^2)^{2q} f(\gamma\psi)^{(q+1)} (\phi\psi)^{(q)} dx + O(\|\gamma\|_q \|\phi\|_q)$$

$$= -\int_{\mathbb{R}} (1+x^2)^{2q} f(\gamma\psi)^{(q)} (\phi\psi)^{(q+1)} dx + O(\|\gamma\|_q \|\phi\|_q),$$

and hence for $\gamma = \phi$,

$$\sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (f\phi'\psi)^{(k)} (\phi\psi)^{(k)} dx = O(\|\phi\|_q^2).$$

In addition

$$\sup_{\gamma \in \Phi_q} \frac{\sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (f\gamma'\psi)^{(k)} (\phi\psi)^{(k)} dx}{\|\gamma\|_q}$$

$$\leq \sqrt{\int_{\mathbb{R}} (1+x^2)^{2q} |f(\phi\psi)^{(q+1)}|^2 dx} + c_{19} \sum_{1 \leq i,j \leq q} \sqrt{\int_{\mathbb{R}} (1+x^2)^{2q} |f^{(i)} (\phi\psi)^{(j)}|^2 dx}$$

$$(A.3)$$

Proof: Let $f_1 = \frac{f\psi'}{\psi}$ and note that if f and its derivatives are bounded, then f_1 and its derivatives are bounded. Then

$$\sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (f\gamma'\psi)^{(k)} (\phi\psi)^{(k)} dx$$

$$= \sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (f(\gamma\psi)' - f_1\gamma\psi)^{(k)} (\phi\psi)^{(k)} dx$$

$$= \sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (f(\gamma\psi)')^{(k)} (\phi\psi)^{(k)} dx - \sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (f_1\gamma\psi)^{(k)} (\phi\psi)^{(k)} dx$$

$$= \int_{\mathbb{R}} (1+x^2)^{2q} f(\gamma\psi)^{(q+1)} (\phi\psi)^{(q)} dx + \sum_{l=0}^{q-1} \binom{q}{l} \int_{\mathbb{R}} (1+x^2)^{2q} f^{(q-l)} (\gamma\psi)^{(l+1)} (\phi\psi)^{(q)} dx$$

$$\quad + \sum_{k=0}^{q-1} \int_{\mathbb{R}} (1+x^2)^{2q} (f(\gamma\psi)')^{(k)} (\phi\psi)^{(k)} dx$$

$$\quad - \sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (f_1\gamma\psi)^{(k)} (\phi\psi)^{(k)} dx$$

$$= \int_{\mathbb{R}} (1+x^2)^{2q} f(\gamma\psi)^{(q+1)} (\phi\psi)^{(q)} dx + O(\|\gamma\|_q \|\phi\|_q).$$

Integrating the first term in the fourth expression above by parts gives

$$-\int_{\mathbb{R}}(1+x^2)^{2q}f(\gamma\psi)^{(q)}\left((\phi\psi)^{(q+1)}+\frac{4qx}{1+x^2}(\phi\psi)^{(q)}\right)dx$$

$$-\int_{\mathbb{R}}(1+x^2)^{2q}f'(\gamma\psi)^{(q)}(\phi\psi)^{(q)}dx$$

$$+\sum_{l=0}^{q-1}\binom{q}{l}\int_{\mathbb{R}}(1+x^2)^{2q}f^{(q-l)}(\gamma\psi)^{(l+1)}(\phi\psi)^{(q)}dx$$

$$+\sum_{k=0}^{q-1}\int_{\mathbb{R}}(1+x^2)^{2q}(f(\gamma\psi)')^{(k)}(\phi\psi)^{(k)}dx$$

$$-\sum_{k=0}^{q}\int_{\mathbb{R}}(1+x^2)^{2q}(f_1\gamma\psi)^{(k)}(\phi\psi)^{(k)}dx$$

$$=-\int_{\mathbb{R}}(1+x^2)^{2q}f(\gamma\psi)^{(q)}(\phi\psi)^{(q+1)}dx+O(\|\gamma\|_q\|\phi\|_q).$$

Since $\|\gamma\|_q\geq\|(\gamma\psi)^{(j)}\|_{L_q}$, (A.3) follows by the Cauchy-Schwartz inequality.

Finally, if $\gamma=\phi$, we can add the two identities to obtain

$$\sum_{k=0}^{q}\int_{\mathbb{R}}(1+x^2)^{2q}(f\phi'\psi)^{(k)}(\phi\psi)^{(k)}dx=O(\|\phi\|_q^2).$$

∎

Write $F_1=F_{11}+F_{12}$ and $F_2=F_{21}+F_{22}$, where

$$F_{11}\phi=\frac{1}{2}a\phi''+b\phi',\qquad F_{21}\phi=\alpha\phi'.\tag{A.4}$$

For $\nu\in\mathcal{M}(\mathbb{R})$, let $\rho_\nu=\int_{\mathbb{R}}\psi(x)^{-1}\nu(dx)$.

LEMMA A.2. *For $v\in\Phi_{-q}$ such that $\phi=\theta_q v\in\Phi_{q+2}$,*

$$2\langle v,F_{11}^*v\rangle_{-q}=-\int_{\mathbb{R}}a(x,\nu)(1+x^2)^{2q}|(\phi\psi)^{(q+1)}(x)|^2dx+O(\|v\|_{-q}^2),$$

where $|O(\|v\|_{-q}^2)|\leq c_{20}\|v\|_{-q}^2$ with c_{20} independent of ν.
Proof: By definition,

$$\langle v,F_{11}^*v\rangle_{-q}=\langle\phi,F_{11}^*v\rangle=\langle F_{11}\phi,v\rangle=\langle F_{11}\phi,\phi\rangle_q$$

$$=\sum_{k=0}^{q}\int_{\mathbb{R}}(1+x^2)^{2q}(\phi\psi)^{(k)}(\frac{1}{2}a\phi''\psi)^{(k)}dx$$

$$+\sum_{k=0}^{q}\int_{\mathbb{R}}(1+x^2)^{2q}(\phi\psi)^{(k)}(b\phi'\psi)^{(k)}dx,\tag{A.5}$$

where the third equality follows from the fact that $F_{11}\phi\in\Phi_q$. By Lemma A.1, the term involving b is bounded by a constant times $\|\phi\|_q^2\equiv\|v\|_{-q}^2$.

For the term involving a, let $a_1 = \frac{a\psi''}{\psi}$ and $a_2 = \frac{2a\psi'}{\psi}$. Then

$$\sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (\phi\psi)^{(k)} (a\phi''\psi)^{(k)} dx$$

$$= \sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (\phi\psi)^{(k)} (a(\phi\psi)^{(2)} - a_1(\psi\phi) - a_2(\psi\phi'))^{(k)} dx$$

$$= -\sum_{k=0}^{q} \int_{\mathbb{R}} (1+x^2)^{2q} (\phi\psi)^{(k+1)} a(\phi\psi)^{(k+1)} dx + O(\|\phi\|_q^2)$$

$$= -\int_{\mathbb{R}} a(1+x^2)^{2q} |(\phi\psi)^{(q+1)}|^2 dx + O(\|\phi\|_q^2),$$

where the second equality follows by integrating by parts and applying Lemma A.1.
∎

LEMMA A.3. *There exists a constant c_{21} such that for $\epsilon > 0$, $v \in \Phi_{-\kappa}$ such that $\phi = \theta_q v \in \Phi_{q+2}$, and $\nu \in \mathcal{M}(\mathbb{R})$,*

$$\int_U \|F_{21}^*(\nu, u)v\|_{-q}^2 \mu(du)$$

$$\leq (1+\epsilon) \int_{\mathbb{R}} (1+x^2)^{2q} \int_U |\alpha(\cdot, \nu, u)|^2 \mu(du) |(\phi\psi)^{(q+1)}|^2 dx + c_{21}(1+\epsilon^{-1}) \|v\|_{-q}^2.$$

Proof: Noting that

$$\|F_{21}^*(\nu, u)v\|_{-q} = \sup_{\gamma \in \Phi_q} \frac{\langle F_{21}^* v, \gamma \rangle}{\|\gamma\|_q} = \sup_{\gamma \in \Phi_q} \frac{\langle \phi, F_{21}\gamma \rangle_q}{\|\gamma\|_q},$$

by (A.3),

$$\|F_{21}^*(\nu, u)v\|_{-q}^2 \leq \left(\sqrt{\int_{\mathbb{R}} (1+x^2)^{2q} |\alpha(x, \nu, u)(\phi\psi)^{(q+1)}|^2 dx} \right.$$

$$\left. + c_{19} \sum_{1 \leq i, j \leq q} \sqrt{\int_{\mathbb{R}} (1+x^2)^{2q} |\alpha^{(i)}(x, \nu, u)(\phi\psi)^{(j)}|^2 dx} \right)^2.$$

Consequently, there exists a constant c_{22} such that

$$\|F_{21}^*(\nu, u)v\|_{-q}^2 \leq (1+\epsilon) \int_{\mathbb{R}} (1+x^2)^{2q} |\alpha(x, \nu, u)(\phi\psi)^{(q+1)}|^2 dx$$

$$+ c_{22}(1+\epsilon^{-1}) \sum_{1 \leq i, j \leq q} \int_{\mathbb{R}} (1+x^2)^{2q} |\alpha^{(i)}(x, \nu, u)(\phi\psi)^{(j)}|^2 dx^2,$$

and integrating with respect to μ, the boundedness of $\alpha^{(i)}$ in $L^2(U, \mu)$ implies the existence of c_{21}. ∎

LEMMA A.4. *For $v \in \Phi_{-\kappa}$,*

$$2\langle v, F_{12}^* v \rangle_{-q} \leq c_{23} \rho_\nu \|v\|_{-q}^2.$$

Proof: Assume that $\phi = \theta_q v \in \Phi_{q+2}$. As in Lemma A.2,

$$\langle v, F_{12}^* v \rangle_{-q} = \langle F_{12}\phi, \phi \rangle_q.$$

Write

$$\langle F_{12}(\nu)\phi, \phi \rangle_q = \langle d(\cdot, \nu)\phi, \phi \rangle_q + \langle G_1(\nu)\phi, \phi \rangle_q \tag{A.6}$$

where

$$G_1(\nu)\phi = \int_{\mathbb{R}} (\partial d(x,\nu,\cdot)\phi(x) + \frac{1}{2}\partial a(x,\nu,\cdot)\phi''(x) + \partial b(x,\nu,\cdot)\phi'(x))\nu(dx).$$

The boundedness of the derivatives of d implies that the first term on the right of (A.6) is $O(\|\phi\|_q)$ uniformly in ν. Note that

$$|\phi(x)\psi(x)|^2 \le \int_{\mathbb{R}} |(\phi(y)\psi(y))'|^2 (1+y^2)^2 dy \int_{\mathbb{R}} (1+y^2)^{-2} dy,$$

so

$$|\phi(x)\psi(x)| \le O(\|\phi\|_1).$$

Similarly, letting $K_1 = \sup_z \frac{|\psi'(z)|}{\psi(z)}$ and $K_2 = \sup_z \frac{|\psi''(z)|}{\psi(z)}$,

$$|\phi'(x)\psi(x)| \le |(\phi(x)\psi(x))'| + K_1|\phi(x)\psi(x)| \le O(\|\phi\|_2)$$

and

$$|\phi''(x)\psi(x)| \le |(\phi(x)\psi(x))''| + 2K_1|\phi'(x)\psi(x)| + K_2|\phi(x)\psi(x)| \le O(\|\phi\|_3).$$

Consequently,

$$\|G_1(\nu)\phi\|_q = \int_{\mathbb{R}} (\|\partial d(x,\nu,\cdot)\|_q|\phi(x)\psi(x)| + \frac{1}{2}\|\partial a(x,\nu,\cdot)\|_q|\phi''(x)\psi(x)|$$
$$+ \|\partial b(x,\nu,\cdot)\|_q|\phi'(x)\psi(x)|)\psi^{-1}(x)\nu(dx)$$
$$\le c_{31}\rho_\nu\|\phi\|_3 \le c_{31}\rho_\nu\|\phi\|_q = c_{31}\rho_\nu\|v\|_{-q}.$$

Combining this inequality with the estimate on the first term gives the result. ∎

LEMMA A.5. *There exists c_{24} such that for $v \in \Phi_{-\kappa}$,*

$$\int_U \|F_{22}^*(\nu,u)v\|_{-q}^2 \mu(du) \le c_{24}\rho_\nu^2\|v\|_{-q}^2. \tag{A.7}$$

Proof: To bound $\|F_{22}^* v\|_{-q}$, note that

$$\|F_{22}^* v\|_{-q} = \sup_{\gamma \in \Phi_{q+2}} \frac{\langle \gamma, F_{22}^* v \rangle}{\|\gamma\|_q} = \sup_{\gamma \in \Phi_{q+2}} \frac{\langle F_{22}\gamma, \phi \rangle_q}{\|\gamma\|_q} \le \sup_{\gamma \in \Phi_{q+2}} \frac{\|F_{22}\gamma\|_q}{\|\gamma\|_q}\|\phi\|_q.$$

Then $\|F_{22}\gamma\|_q \le c_{25}(u)\rho_\nu\|\gamma\|_q$ by the same argument used to estimate G_1 in Lemma A.4, and c_{25} can be selected to be integrable with respect to μ. ∎

Combining the previous lemmas we have

LEMMA A.6. *For* $v \in \Phi_{-\kappa}$,

$$2\langle v, F_1^*(\nu)v\rangle_{-q} + \int_U \|F_2^*(\nu,u)v\|_{-q}^2 \mu(du) \leq c_{18}\rho_\nu^2 \|v\|_{-q}^2. \qquad (A.8)$$

Proof: Selecting $\epsilon > 0$ so that $(1+\epsilon)^2 \leq (1+\delta)$ for δ in (S_4),

$$2\langle v, F_1^*(\nu)v\rangle_{-q} + \int_U \|F_2^*(\nu,u)v\|_{-q}^2 \mu(du)$$

$$\leq 2\langle v, F_{11}^*(\nu)v\rangle_{-q} + (1+\epsilon)\int_U \|F_{21}^*(\nu,u)v\|_{-q}^2 \mu(du)$$

$$+ 2\langle v, F_{12}^*(\nu)v\rangle_{-q} + (1+\epsilon^{-1})\int_U \|F_{22}^*(\nu,u)v\|_{-q}^2 \mu(du)$$

$$\leq -\int_{\mathbb{R}}\left(a(x,\nu) - (1+\epsilon)^2\int_U |\alpha(x,\nu,u)|^2\mu(du)\right)(1+x^2)^{2q}|(\phi\psi)^{(q+1)}(x)|^2 dx$$

$$+ (c_{20} + c_{21}(1+\epsilon)(1+\epsilon^{-1}) + c_{23}\rho_\nu + (1+\epsilon^{-1})c_{24}\rho_\nu^2)\|v\|_{-q}^2$$

$$\leq c_{18}\rho_\nu^2\|v\|_{-q}^2,$$

since the first term in the third expression is less than or equal to zero by (S4). ■

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