Index Theory, Gerbes, and Hamiltonian Quantization

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Abstract: We give an Atiyah-Patodi-Singer index theory construction of the bundle of fermionic Fock spaces parametrized by vector potentials in odd space dimensions and prove that this leads in a simple manner to the known Schwinger terms (Faddeev-Mickelsson cocycle) for the gauge group action. We relate the APS construction to the bundle gerbe approach discussed recently by Carey and Murray, including an explicit computation of the Dixmier-Douady class. An advantage of our method is that it can be applied whenever one has a form of the APS theorem at hand, as in the case of fermions in an external gravitational field.

1. Introduction

There are subtleties in defining the fermionic Fock spaces in the case of chiral (Weyl) fermions in external vector potentials. The difficulty is related to the fact that the splitting of the one particle fermionic Hilbert space H into positive and negative energies is not continuous as a function of the external field. One can easily construct paths in the space of external fields such that at some point on the path a positive energy state dives into the negative energy space (or vice versa). These points are obviously discontinuities in the definition of the space of negative energy states and therefore the fermionic vacua do not form a smooth vector bundle over the space of external fields. This problem does not arise if we have massive fermions in the temporal gauge $A_0 = 0$. In that case there is a mass gap $[-m, m]$ in the spectrum of the Dirac hamiltonians and the polarization to positive and negative energy subspaces is indeed continuous.

If λ is a real number not in the spectrum of the hamiltonian then one can define a bundle of fermionic Fock spaces $\mathcal{F}_{A,\lambda}$ over the set U_λ of external fields A, $\lambda \notin Spec(D_A)$. It turns out that the Fock spaces $\mathcal{F}_{A,\lambda}$ and $\mathcal{F}_{A,\lambda'}$ are naturally isomorphic up to a phase. The phase is related to the arbitrariness in filling the Dirac sea between vacuum levels λ, λ'. In order to compensate this ambiguity one defines a tensor product

$\mathcal{F}'_{A,\lambda} = \mathcal{F}_{A,\lambda} \otimes DET_{A,\lambda}$, where the second factor is a complex line bundle over U_λ. By a suitable choice of the determinant bundle the tensor product becomes independent of λ and one has a well-defined bundle $\mathcal{F}'$ of Fock spaces over all of $\mathcal{A}$.

Next one can ask what is the action of the gauge group on $\mathcal{F}'$. The gauge action in U_λ lifts naturally to $\mathcal{F}$. Thus the only problem is to construct a lift of the action on the base to the total space of DET_λ. Note that the determinant bundle here is a bundle over external fields in *odd dimension,* and therefore one would expect that it is trivial (curvature equal to zero) on the basis of the families index theorem. However, it turns out that the relevant determinant bundle actually comes from a determinant bundle in even dimensions. Instead of single vector potentials we must study paths in $\mathcal{A}$, thus the extra dimension. The relevant index theorem is then the Atiyah Patodi Singer (APS) index theorem for even dimensional manifolds with a boundary [AtPaSi]; physically, the boundary can be interpreted as the union of the space at the present time and in the infinite past.

The gauge action in the bundle $\mathcal{F}$ leads to Schwinger terms in the Lie algebra commutation relations of the gauge currents. These commutator anomalies have been discussed before in the literature from different points of view. However, whereas in even space-time dimensions Atiyah and Singer [AtSi] gave a definitive mathematical treatment of anomalies, the odd dimensional case (that is hamiltonian anomalies) has been the subject of various ad hoc approaches.

In this paper we present a simple derivation using the families index theorem, in the spirit of [AtSi], giving a Fock space formulation for the descent equations leading from the space-time anomalies to hamiltonian anomalies. We also resolve a puzzle in our earlier work by explaining in a direct way the relation between the Schwinger terms and the Dixmier-Douady class (which is a certain closed 3-form on the moduli space of gauge connections introduced in [CaMu1]) in de Rham cohomology.

2. The Odd Determinant Bundles

Let M be a smooth compact manifold without boundary equipped with a spin structure. We assume that the dimension of M is odd and equal to $2n + 1$. Let S be the spin bundle over M, with fiber isomorphic to $\mathbb{C}^{2^n}$. Let H be the space of square integrable sections of the complex vector bundle $S \otimes V$, where V is a trivial vector bundle over M with fiber to be denoted by the same symbol V. The measure is defined by a fixed metric on M and V. We assume that a unitary representation ρ of a compact group G is given in the fiber. The set of smooth vector potentials on M with values in the Lie algebra $\mathbf{g}$ of G is denoted by $\mathcal{A}$ or $\mathcal{A}_{2n+1}$, depending on whether there is a chance of confusion.

For each $A \in \mathcal{A}$ there is a massless hermitian Dirac operator D_A. Fix a potential A_0 such that D_A does not have the zero as an eigenvalue and let H_+ be the closed subspace spanned by eigenvectors belonging to positive eigenvalues of D_{A_0} and H_- its orthogonal complement. More generally for any potential A and any real λ not belonging to the spectrum of D_A we define the spectral decomposition $H = H_+(A, \lambda) \oplus H_-(A, \lambda)$ with respect to the operator $D_A - \lambda$. Let $\mathcal{A}_0$ denote the set of all pairs (A, λ) as above and let $U_\lambda = \{A \in \mathcal{A} | (A, \lambda) \in \mathcal{A}_0\}$.

Over the set $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$ there is a canonical complex line bundle, to be denoted by $DET_{\lambda\lambda'}$. Its fiber at $A \in U_{\lambda\lambda'}$ is the top exterior power

$$DET_{\lambda\lambda'}(A) = \wedge^{\text{top}}(H_+(A, \lambda) \cap H_-(A, \lambda')), \qquad (2.1)$$

where we have assumed $\lambda < \lambda'$. For completeness we put $DET_{\lambda\lambda'} = DET_{\lambda'\lambda}^{-1}$. Since M is compact, the spectral subspace corresponding to the interval $[\lambda, \lambda']$ in the spectrum is finite-dimensional and the complex line above is well-defined.

It is known [Mi1, CaMu1] that there exists a complex line bundle DET_λ over each of the sets U_λ such that

$$DET_{\lambda'} = DET_\lambda \otimes DET_{\lambda\lambda'}, \qquad (2.2)$$

over the set $U_{\lambda\lambda'}$. In [CaMu, CaMu1] the structure of these line bundles was studied with the help of bundle gerbes. In particular, there is an obstruction for passing to the quotient by the group $\mathcal{G}$ of gauge transformations which is given by the Dixmier-Douady class of the bundle gerbe. (In [Mi1] the structure of the bundles and their relation to anomalies was found by using certain embeddings to infinite-dimensional Grassmannians.)

In this paper we shall compute the curvature of the (odd dimensional) determinant bundles from Atiyah-Patodi-Singer index theory and we obtain the Schwinger terms in the Fock bundle directly from the local part of the index density.

To each (A, λ) in $\mathcal{A}_0$ we associate a euclidean Dirac operator on the $2n + 2$ dimensional manifold $M \times [0, 1]$ with the obvious metric and spin structure. This Dirac operator is

$$D_{A(t)}^{(2n+2)} = \frac{\partial}{\partial t} + D_{A(t)}, \qquad (2.3)$$

where the time dependent potential is $A(t) = f(t)A + (1 - f(t))A_0$. Here f is a fixed smooth real valued function on the interval $[0, 1]$ such that $f(0) = 0, f(1) = 1$, and the function is constant near the end points. It turns out that the choice of f does not influence our results as we show at the end of the section.

We fix the boundary conditions for $D_{A(t)}^{(2n+2)}$ such that at the boundary component $t = 0$ the spinor fields should belong to H_- whereas at $t = 1$ the spinor field is in $H_+(A, \lambda)$. This type of boundary condition was used in [AtPaSi] in the proof of index theorems (in even dimensions) when the manifold has a boundary. The Dirac operator is nonhermitian, it is really a map between two different spaces, namely the space of left handed spinors S_- and right handed spinors S_+. The kernel and cokernel of $D_{A(t)}^{(2n+2)}$ are finite dimensional vector spaces.

The tensor product of the top exterior powers of the dual of the kernel and the cokernel of $D_{A(t)}^{(2n+2)}$ defines a complex line $DET_\lambda(A)$. Together these lines define a complex line bundle DET_λ over U_λ, the set of potentials not having λ as an eigenvalue. The bundle does not extend to all of $\mathcal{A}$ since the boundary conditions change abruptly at points in the parameter space such that the corresponding boundary Dirac operator has zero modes.

There is an important alternative description of the determinant line bundle. Let $\{\psi_n\}$ be a basis of eigenvectors at the boundary component $t = 1$ corresponding to eigenvalues $\lambda_n > \lambda$,

$$D_A \psi_n = \lambda_n \psi_n.$$

The nonhermitian time evolution

$$i\partial_t \phi = -iD_{A(t)}\phi \qquad (2.4)$$

defines for each n a unique solution ϕ_n on $M \times [0, 1]$ such that at $t = 1$ $\phi_n(x, 1) = \psi_n(x)$. The vectors $\phi_n(x, 0)$ span an infinite dimensional plane $W = W(A, \lambda)$ in H. Let π_+ be the projection from W to H_+. The kernel of this projection can be identified as the kernel of $D_{A(t)}^{(2n+2)}$ through restriction to the boundary $t = 0$. Similarly, the cokernel of $D_{A(t)}^{(2n+2)}$ is identified as the cokernel of π_+. This is because the boundary conditions for the

adjoint operator $(\partial_t + D_{A(t)})^* = -\partial_t + D_{A(t)}$ are orthogonal to the boundary conditions of $D_{A(t)}^{(2n+2)}$, [AtPaSi]. At $t = 0$ the vectors in the domain of the adjoint belong to H_+ whereas at $t = 1$ they belong to $H_-(A, \lambda)$. On the other hand, coker $\pi_+ = W^\perp \cap H_+$ and a zero mode of the adjoint is orthogonal to a zero mode of $D_{A(t)}^{(2n+2)}$ at $t = 0$. Thus

$$DET_\lambda(A) = \wedge^{\text{top}}(\ker\pi_+)^* \otimes \wedge^{\text{top}}(\text{coker}\pi_+). \tag{2.5}$$

We choose an orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ such that the vectors with a nonnegative n belong to H_+ and those with a negative n belong to H_-. Since π_+ is a Fredholm operator, of index $k = \dim \ker\pi_+ - \dim \text{coker}\pi_+$ say, the projection $\pi_{+,k}$ from W to the plane H_k spanned by the vectors $\{e_n\}_{n \geq -k}$ is almost invertible, i.e., there is a linear map $q : H_k \to W$ such that $q\pi_+$ and $\pi_+ q$ differ from the identity operator by a finite rank operator. The pseudo-inverse q is fixed by a choice of basis $\{u_1, \ldots, u_r\}$ in $\ker\pi_+$ and a basis $\{v_1, \ldots, v_{r-k}\}$ in $\text{coker}\pi_+$: The map π_+ gives an isomorphism between $\pi_+(W) \subset H_+$ and $(\ker\pi_+)^\perp \cap W$. This isomorphism is complemented to an isomorphism between W and H_k by adjoining to u_i the vector v_i for $i = 1, 2, \ldots, r - k$ and $u_i \mapsto e_{i-r}$ for $i = r - k + 1, \ldots, r$, when k is nonnegative. When $k < 0$ we define H_k as the space spanned by e_i with $i \geq -k$ and proceed as before.

The image $\{w_{-k}, w_{-k+1}, \ldots\}$ of the basis of H_k under q is *an admissible basis* of W, [PrSe]. By definition, any admissible bases of W is a basis obtained from $\{w_i\}$ by a unitary rotation by an operator $1 + R$, where R is trace-class. The operators $1 + R$ have a well-defined determinant. Over $W(A, \lambda)$ (that is, over $A \in U_\lambda$) there is a complex line defined as the set of all admissible basis of W modulo basis transformations by operators with unit determinant. As we saw above, the ambiguity in the construction of an admissible basis is the same as the freedom of choosing the basis in $\ker\pi_+$ and $\text{coker}\pi_+$. It follows that the determinant line is naturally identified as the complex line in the Pressley-Segal construction.

Any choice $\{f_n\}$ of a basis of eigenvectors of D_A corresponding to eigenvalues in the interval $[\lambda, \mu]$ gives now an isomorphism between the determinant lines $DET_\lambda(A)$ and $DET_\mu(A)$. Namely, an admissible basis $\{w_n\}$ of $W(A, \mu)$ can be completed to an admissible basis of $W(A, \lambda)$ by adding the time evolved vectors obtained from $\{f_n\}$ by the euclidean time evolution backwards in time from $t = 1$ to $t = 0$. Clearly, a rotation R of the basis $\{f_n\}$ induces a rotation of the determinant line $DET_\lambda(A)$ by a phase equal to det R. On the other hand, a choice of the basis $\{f_n\}$ modulo unitary transformations R with det $R = 1$ is equivalent to choosing an element in the complex line $DET_{\lambda\mu}(A)$. This shows that we can identify

$$DET_\mu(A) = DET_{\lambda\mu}(A) \otimes DET_\lambda(A),$$

as already stated in (2.2).

An alternative proof of this result can be given which uses the APS index theorem as follows. Denote $W_+(A, \lambda) = W(A, \lambda)$ and $W_-(A, \lambda) = W(A, \lambda)^\perp$. Define

$$K(A, \lambda) = W_+(A, \lambda) \cap H_- \quad \text{and} \quad K(A, \lambda') = W_+(A, \lambda') \cap H_-$$

and

$$C(A, \lambda) = H_+ \cap W_-(A, \lambda) \quad \text{and} \quad C(A, \lambda') = H_+ \cap W_-(A, \lambda').$$

These are the kernels and co-kernels of the even dimensional Dirac operators formed out of A_0 and A with projection at $t = 1$ onto the eigenspaces greater than λ' and λ respectively. So we have

$$DET_\lambda = \wedge^{top}(K(A,\lambda)^* \oplus C(A,\lambda)) \quad \text{and} \quad DET_{\lambda'} = \wedge^{top}(K(A,\lambda')^* \oplus C(A,\lambda')).$$

Recall that

$$H = W_-(A,\lambda) \oplus (W_+(A,\lambda) \cap W_-(A,\lambda')) \oplus W_+(A,\lambda')$$

so that we have

$$W_-(A,\lambda') = W_-(A,\lambda) \oplus W_+(A,\lambda) \cap W_-(A,\lambda')$$

and

$$W_+(A,\lambda) = W_+(A,\lambda') \oplus W_+(A,\lambda) \cap W_-(A,\lambda').$$

So orthogonal projection defines a map

$$K(A,\lambda)/K(A,\lambda') \to W_+(A,\lambda) \cap W_-(A,\lambda')$$

and similarly

$$C(A,\lambda')/C(A,\lambda) \to W_+(A,\lambda) \cap W_-(A,\lambda').$$

Adding these gives a map

$$K(A,\lambda)/K(A,\lambda') \oplus C(A,\lambda')/C(A,\lambda) \to W_+(A,\lambda) \cap W_-(A,\lambda').$$

If we can prove that this final map is an isomorphism, then by wedging to the top power on either side we will have constructed an isomorphism

$$DET_{\lambda'}(A) \otimes DET_\lambda(A)^* = DET_{\lambda\lambda'}(A),$$

which gives the desired result in Eq. (2.2). It is easy to prove that this map is injective because the images of the two factors are, in fact, orthogonal. It remains to do surjectivity and this comes from a dimension count which follows from the APS index theorem. It suffices to show that

$$\dim(K_\lambda) - \dim(K_{\lambda'}) + \dim(C_{\lambda'}) - \dim(C_\lambda) = \dim(W_+(A,\lambda) \cap W_-(A,\lambda')).$$

Given $(A_0,0)$ and (A,λ) let $D[(A_0,0),(A,\lambda)]$ be the four dimensional Dirac operator as above. We need to prove then that

$$\text{index}(D[(A_0,0),(A,\lambda')] - \text{index}(D[(A_0,0),(A,\lambda)] = \dim(W_+(A,\lambda') \cap W_-(A,\lambda)).$$

It is easy to show that

$$\dim(W_+(A,\lambda) \cap W_-(A,\lambda')) = \dim(H_+(A,\lambda) \cap H_-(A,\lambda'))$$
$$= \text{index}(D[(A,\lambda'),(A,\lambda)]),$$

so the result follows from the fact that the index is additive. That is

$$\text{index}(D[(A,\lambda),(B,\lambda')] + \text{index}(D[(B,\lambda'),(C,\lambda'')] = \text{index}(D[(A,\lambda),(C,\lambda'')]).$$

This additivity of the index is a direct consequence of the index theorem itself. The index is a sum of two terms. The first is an integral of a *local* differential polynomial of the vector potential and therefore it is manifestly additive in time. The second term is also additive because it is equal to $\frac{1}{2}(\eta(t=1) - \eta(t=0))$. On the common

boundary the eta invariants (for the boundary operator B) in index$(D[(A, \lambda), (B, \lambda')]+$ index$(D[(B, \lambda'), (C, \lambda'')])$ cancel.

The real parameters $\lambda, \lambda', \lambda''$ do not change the discussion since we can always consider operators such as

$$D_{A(t)} - f(t)\lambda - (1 - f(t))\lambda'$$

instead of $D_{A(t)}$. For the shifted Dirac operators we can use the 'vacuum level' value 0.

Finally, the geometry of the determinant bundles DET_λ is described by the families index theorem. Normally, the determinant bundle over $\mathcal{A}$ in even dimensions is trivial whereas the bundle over the moduli space of gauge orbits $\mathcal{A}/\mathcal{G}_0$ is nontrivial. Here $\mathcal{G}_0$ is the group of based gauge transformations, $g(p) = 1$, where $p \in M$ is some fixed point. However, in the present case we are studying potentials on a manifold with boundary and the boundary conditions depend globally on the potential A not having λ as a zero mode. The parameter space is not affine, the determinant bundle is nontrivial. (We stress again that the determinant bundle DET_λ does *not* extend to the space of all vector potentials; there are discontinuities at the points A for which λ is an eigenvalue.)

We use the same APS boundary conditions for the operators $D_{A(t)}^{(2n+2)}$ as before. Then according to [AtPaSi],

$$\text{index} D_{A(t)}^{(2n+2)} = \int Ch(A(t)) - \frac{1}{2}(\eta(t = 1) - \eta(t = 0)), \qquad (2.6)$$

assuming that the boundary operators do not have zero modes. Here Ch is a characteristic class depending in general on the vector potential and the metric. This term is the same as in the case of a manifold without a boundary. The eta invariant $\eta(D_A)$ for a hermitian operator is defined through analytic continuation of

$$\eta_s(A) = \sum_i \frac{\lambda_i}{|\lambda_i|^s},$$

which is well-defined for $s \gg 0$, to the point $s = 1$, where the λ_i's are the eigenvalues of D_A. The η-invariant term in (2.6) depends only on data on the boundary.

The Chern class of the determinant bundle DET over this class of Dirac operators is completely determined by integrating the corresponding de Rham form over two dimensional cycles $S^2 \mapsto$ set of Dirac operators.

We recall some facts about lifting a group action on the base space X of a complex line bundle to the total space E. Let ω be the curvature 2-form of the line bundle. It is integral in the sense that $\int \omega$ over any cycle is $2\pi \times$ an integer. Let G be a group acting smoothly on X. Then there is an extension $\hat{G}$ which acts on E and covers the G action on X. The fiber of $\hat{G} \to G$ is equal to $Map(X, S^1)$. As a vector space, the Lie algebra of the extension is $\mathbf{g} \oplus \text{Map}(X, i\mathbb{R})$. The commutators are defined as

$$[(a, \alpha), (b, \beta)] = ([a, b], \omega(a, b) + \mathcal{L}_a\beta - \mathcal{L}_b\alpha), \qquad (2.7)$$

where $a, b \in \mathbf{g}$ and $\alpha, \beta : X \to i\mathbb{R}$. The vector fields generated by the G action on X are denoted by the same symbols as the Lie algebra elements a, b; thus $\omega(a, b)$ is the function on X obtained by evaluating the 2-form ω along the vector fields a, b. The Jacobi identity

$$\omega([a, b], c) + \mathcal{L}_a\omega(b, c) + \text{cyclic permutations} = 0$$

for the Lie algebra extension $\hat{\mathfrak{g}}$ follows from $d\omega = 0$.

For the computation of the Schwinger term we need only the curvature along gauge directions for the boundary operator $D_{A(t=1)}^{(2n+1)}$. According to the general theory of determinant bundles: the integral of the first Chern class over a S^2 in the parameter space of Dirac operators = the index of the family of Dirac operators. That means: one has to choose (any) connection on $B = S^2 \times [0, 1] \times M$ such that along $[0, 1] \times M$ it is equal to the potential $f(t)A(x, z) + (1 - f(t))A_0$ (here $z \in S^2$ parametrizes the family of operators) and satisfies the appropriate boundary conditions. The appropriate Dirac operator is then the operator D_B on B related to this connection.

Consider a family of gauge transformed potentials $A(x, z) = gAg^{-1} + d_x g g^{-1}$, where $x \mapsto g(x, z)$ is a family of gauge transformations parametrized by points $z \in S^2$. To this family of potentials we associate a Dirac operator D_B on B. Formally

$$D_B = D(S^2) + D_{A(t)}^{(2n+2)} + f(t)\rho(z)^{-1}\gamma^z \cdot \partial_z \rho(z), \qquad (2.8)$$

where the first term is the Dirac operator on S^2 determined by a metric and fixed spin structure; γ^z stand for a pair of gamma matrices to the S^2 directions. The boundary conditions at $t = 1$ are: the spinor field should be in the positive energy plane of the boundary operator, that is, in the gauge transform of the positive energy plane for the operator determined by $g = 1$. We assume that at the "initial point" $g = 1$ there are no zero modes. It follows that the operator D_B does not have zero modes on the boundary $t = 1$. (Otherwise we could modify $D(S^2)$ by adding a small positive constant.) The boundary conditions at $t = 0$ are the usual ones, i.e., the spinor field should be in the negative energy plane of the "free" Dirac hamiltonian.

The index formula (2.6) on manifolds with boundary contains two pieces. The first is an integral of a local differential form in the interior of the manifold. The η-invariant term is a nonlocal expression involving the boundary Dirac operator. Because it is expressed in terms of the eigenvalues of the (hermitian) Dirac operator it is invariant under gauge transformations.

For this reason, when computing the index for the family of operators given by the different gauge configurations, the only part contributing is the local part. If M is a sphere the relevant characteristic class is the Chern class c_{n+2} on B. The Chern class c_k is the coefficient of λ^k in the expansion of $\det(1 + \frac{\lambda}{2\pi i}F)$, where F is the curvature form. In the case of $G = SU(N)$, $\text{tr}F = 0$ and the lowest terms are

$$c_2 = \frac{1}{8\pi^2}\text{tr}F^2, c_3 = \frac{i}{24\pi^3}\text{tr}F^3, c_4 = \frac{1}{2^6\pi^4}(\text{tr}F^4 - \frac{1}{2}(\text{tr}F^2)^2).$$

The Chern classes c_n are normalized such that their integrals over closed submanifolds of the corresponding dimension are integers.

The vector potential is globally defined and therefore the integral of the Chern classes is given by a boundary integral of a Chern-Simons form $CS_i(A)$ in $i = 2n + 3$ dimensions, $d(CS_i) = c_{n+2}$. At the boundary component $t = 0$ the form vanishes. So the only contribution is

$$\int_{S^2 \times M} CS_i(A(1, x, z)). \qquad (2.9)$$

Performing only the M integration gives a closed 2-form on S^2. For example, when $\dim M = 1$ the CS form is $\frac{1}{8\pi^2}\text{tr}(AdA + \frac{2}{3}A^3)$, and we get

$$\omega_A(X, Y) = \frac{1}{4\pi}\int_{S^1}\text{tr}\, A_\phi[X, Y],$$

the curvature at the point A in the directions of infinitesimal gauge transformations X, Y. (Note the normalization factor 2π relating the Chern class to the curvature formula.) This is not quite the central term of an affine Kac-Moody algebra, but it is equivalent to it (in the cohomology with coefficients in Map$(\mathcal{A}, \mathbb{C})$). In other words, there is a 1-form θ along gauge orbits in $\mathcal{A}$ such that $d\theta = \omega - c$, where

$$c(X, Y) = \frac{i}{2\pi} \int \mathrm{tr} X \partial_\phi Y$$

is the central term of the Kac-Moody algebra, considered as a closed constant coefficient 2-form on the gauge orbits. There is a simple explicit expression for θ,

$$\theta_A(X) = \frac{i}{4\pi} \int \mathrm{tr} AX.$$

When $\dim M = 3$ the curvature (or equivalently, the Schwinger term) is obtained from the five dimensional Chern-Simons form

$$CS_5(A) = \frac{i}{24\pi^3} \mathrm{tr}(A(dA)^2 + \frac{3}{2} A^3 dA + \frac{3}{5} A^5).$$

By the same procedure as in the one dimensional case we obtain

$$\omega_A(X, Y) = \frac{i}{4\pi^2} \int \mathrm{tr}\left((AdA + dA\,A + A^3)[X, Y] + XdA\,YA - YdA\,XA\right).$$

This differs from the FM cocycle, [FaSh, Mi],

$$\omega'_A(X, Y) = \frac{i}{24\pi^2} \int \mathrm{tr} A(dXdY - dYdX)$$

by the coboundary of

$$\frac{-i}{24\pi^2} \int \mathrm{tr}(AdA + dA\,A + A^3)X.$$

The use of index theory for describing hamiltonian anomalies was suggested by Nelson and Alvarez-Gaumè in [NeAl]. However, in that paper the appearance of Schwinger terms was not made clear.

3. Bundle Gerbes

The eventual aim of the discussion in Sect. 4 is to obtain formulae for the Dixmier-Douady (D-D) class of the bundle gerbe associated with the determinant bundles described in the previous section. These formulae express the D-D class in terms of de Rham forms on subsets of the space of connections. In this section we review the definition of bundle gerbe and then deal with two technical issues.

Let $\pi: Y \to M$ be a submersion. Recall that the fibre product of Y with itself is the set

$$Y^{[2]} = Y \times_f Y = \bigcup_{m \in M} Y_m \times Y_m,$$

where $Y_m = \pi^{-1}(m)$ is the fibre of Y over $m \in M$. Then in [Mu] a bundle gerbe over a manifold M is defined to be a pair (J, Y), where $\pi: Y \to M$ is a submersion and J is

a line bundle over $Y^{[2]}$. These have to satisfy the condition that there is a product on J, that is a linear isomorphism

$$J_{(y_1,y_2)} \otimes J_{(y_2,y_3)} \to J_{(y_1,y_3)}$$

for any (y_1, y_2), (y_2, y_3) in $Y^{[2]}$. This linear isomorphism is supposed to vary smoothly as y_1, y_2, y_3 vary, the details of how to formulate that condition precisely are given in [Mu]. The product is required to be associative.

As an aside let us note that in [Br] the definition of a gerbe as a sheaf of categories is given. From the simpler notion of a bundle gerbe one can construct a gerbe as follows. Let $U \subset M$ be an open set which is contractible and over which $Y \to M$ admits sections. Define the objects of a category associated to U to be all sections of Y over U. Given two objects s and t we can define a section (s, t) of $Y^{[2]}$ over U by $x \mapsto (s(x), t(x))$. Then we can pull-back J to a line bundle $(s, t)^*(J)$ over U. Because U is contractible this is trivial. Take as the morphisms between s and t the set of all non-vanishing sections of $(s, t)^*(J)$. The bundle gerbe product induces a composition of morphisms. To define a sheaf of categories we need to associate to any open set a category. To do this we have to "sheafify" this construction in a suitable manner; however we omit the details as they are not relevant to our discussion.

There is a natural notion of isomorphism for bundle gerbes where the submersion and the line bundle are isomorphic and the isomorphisms intertwine the product. If $L \to Y$ is a line bundle where $Y \to M$ is submersion then we can define a bundle gerbe $\delta(L) \to Y^{[2]}$ by

$$\delta(L)_{(y_1,y_2)} = L_{y_1} \otimes L_{y_2}^*.$$

A product is easily defined on $\delta(L)$ by the obvious contraction

$$(L_{y_1} \otimes L_{y_2}^*) \otimes (L_{y_2} \otimes L_{y_3}^*) \to L_{y_1} \otimes L_{y_3}^*.$$

We call any bundle gerbe isomorphic to a bundle gerbe of the form $\delta(L)$ *trivial*. The D-D class of a bundle gerbe vanishes precisely when the bundle gerbe is trivial.

If (L, Z) and (J, Y) are two bundle gerbes over M then we can define their product. First we define the fibre product $Z \times_f Y$ of the submersions Z and Y. Then we define $L \otimes J$ over $(Z \times_f Y)^{[2]}$ by

$$(L \otimes J)_{((z_1,y_1),(z_2,y_2))} = L_{(z_1,z_2)} \otimes J_{(y_1,y_2)}.$$

If (J, Y) is a bundle gerbe then we can define its dual (J^*, Y) by $(J^*)_{(y_1,y_2)} = (J_{(y_2,y_1)})^*$.

If $dd(J)$ denotes the D-D class of a bundle gerbe (J, Y) then we have $dd(J \otimes L) = dd(J) + dd(L)$ and $dd(J) = -dd(J^*)$.

We say two bundle gerbes J and L are *stably isomorphic* if there are trivial bundle gerbes T and S such that $J \otimes S$ is isomorphic to $L \otimes T$. Two bundle gerbes are stably isomorphic if and only if they have the same D-D class. As an aside let us note that this definition of stable isomorphism is the same idea used in K-theory to define stable isomorphism of vector bundles, i.e. we say two vector bundles E and F are stably isomorphic if one can find two trivial bundles $\mathbb{R}^n$ and $\mathbb{R}^m$ such that $E \oplus \mathbb{R}^n$ and $F \oplus \mathbb{R}^m$ are isomorphic.

An example of stably isomorphic bundle gerbes we use below is the following. If we have a bundle gerbe $Q \to Z^{[2]}$, where $Z \to X$ is a submersion and another submersion $Y \to X$ and a fibre map $f: Y \to Z$, we get an induced map $f^{[2]}: Y^{[2]} \to Z^{[2]}$ and the line bundle Q pulls back to define a bundle gerbe $(f^{[2]})^*(Q)$ on X. The bundle gerbes Q and $(f^{[2]})^*(Q)$ are stably isomorphic.

We now need to consider two technical results necessary for the later sections. First let (J, Y) be a bundle gerbe over M and assume that M has an open cover $\{U_\alpha\}_{\alpha \in I}$ such that there are sections $s_\alpha: U_\alpha \to Y$ for each α. Consider the disjoint union $\coprod_{\alpha \in I} U_\alpha$. It is useful to think of this as a subset of $M \times I$. Then the projection map to M restricts to define a submersion. The sections s_α define a map

$$s: \coprod_{\alpha \in I} U_\alpha \to Y$$

by $s(x, \alpha) = s_\alpha(x)$. This commutes with the projections and hence we can pull-back the bundle gerbe J to a bundle gerbe on the disjoint union. From the discussion above this is stably isomorphic to the original bundle gerbe.

We need this result in Section 4 where we consider $\mathcal{A}_0$ as part of a bundle gerbe over $\mathcal{A}$. It is the submersion $\mathcal{A}_0 \to \mathcal{A}$. For any $\lambda \in \mathbb{R}$ we have a section $s_\lambda: U_\lambda \to \mathcal{A}_0$ defined by $s_\lambda(A) = (A, \lambda)$. So we can apply the discussion above to obtain the disjoint union

$$Y = \coprod U_\lambda \subset \mathcal{A} \times \mathbb{R}$$

as the set of all (A, λ) such that $A \in U_\lambda$. If we follow the discussion we topologize Y by giving $\mathbb{R}$ the discrete topology. Notice that as a set Y is just $\mathcal{A}_0$ but the topology is different. The identity map $Y \to \mathcal{A}_0$ is continuous. So from the result above we deduce that bundle gerbes over $\mathcal{A}_0$ with either topology on $\mathcal{A}_0$ are stably isomorphic so we can work with either picture. An advantage of the open cover picture is that the map δ introduced in [Mu] is then just the coboundary map in the Céch de-Rham double complex. In the next section $\mathcal{A}_0$ can be interpreted in either sense.

For technical reasons explained below it is worth noting that we may work with a denumerable cover from the very beginning. If we restrict λ to be rational then the sets U_λ form a denumerable cover. It follows that the intersections $U_{\lambda \lambda'} = U_\lambda \cap U_{\lambda'}$ also form a denumerable open cover. Similarly, we have an open cover by sets $V_{\lambda \lambda'} = \pi(U_{\lambda \lambda'})$ on the quotient $X = \mathcal{A}/\mathcal{G}_e$, where $\mathcal{G}_e$ is the group of *based* gauge transformations g, $g(p) = e = $ the identity at some fixed base point $p \in M$. Here $\pi : \mathcal{A} \to X$ is the canonical projection.

The second technical point is the question of existence of partitions of unity. This is one of the major technical difficulties with working with manifolds modelled on infinite dimensional vector spaces which are not Hilbert spaces. We digress here to indicate how this problem is solved for the case we are presently interested in. The main result in this theory appears to be the theorem of [Mil].

Theorem 1. *If M is a Lindelöf, regular manifold modelled on a topological vector space with enough smooth functions then any open cover of M has a refinement which admits a partition of unity.*

Before trying to prove this let us give some definitions. Lindelöf means any open cover has a countable subcover. Regular means any closed set and a point not in it can be separated by disjoint open sets. A topological vector space V has enough smooth functions if the collection of sets of the form $U_f = \{x \in V \mid f(x) > 0\}$, where f runs over all smooth functions, is a basis for the topology of V. Another way of saying this is that for every point $x \in V$ and open set U containing x there is a smooth function f with $x \in U_f \subset U$.

The reason to worry about not having enough smooth functions is that the obvious method of constructing them, by taking a semi-norm ρ and composing it with a bump

function on $\mathbb{R}$, may not work as the semi-norm may not be smooth. However [Mil, Be] show that the set of smooth functions from a manifold into a Hilbert space with the smooth, Fréchet topology has enough smooth functions. The point is that we can realize this topology by semi-norms ρ_k which are inner products defined by summing the L^2 norms of the first k derivatives. Then each of these is smooth because the inner products are bilinear and hence smooth. That this gives rise to the same topology as the uniform norms on derivatives is a consequence of the Sobolev inequalities.

The proof of the theorem above from [Mil] goes as follows. Let U be an element of a given open cover and let $x \in U$. By the assumption, there is a smooth real valued function f on M such that $x \in U_f \subset U$. We can assume that f is nonnegative. Namely, for a given f we can form another smooth function $\tilde{f} = h \circ f$, where $h : \mathbb{R} \to \mathbb{R}$ is the smooth function defined by $h(x) = 0$ for $x \leq 0$ and $h(x) = \exp(-1/x^2)$ for $x > 0$. Clearly $U_{\tilde{f}} = U_f$ and $\tilde{f}$ is nonnegative. Choosing nonnegative functions has the advantage that

$$U_f \cap U_g = U_{fg}. \tag{3.1}$$

By the above discussion we can refine the given open cover to an open cover consisting of U_f's. Then we can take a countable open subcover $U_i = U_{f_i}$ for some smooth, non-negative functions f_i. Consider now the sets

$$V_n = \{x \mid f_1(x) < 1/n\} \cap \{x \mid f_2(x) < 1/n\} \cap \cdots \{x \mid f_{n-1}(x) < 1/n\} \cap U_n.$$

These are a locally finite cover.

First let us prove they are a cover. Note that the U_n cover M so there is some f_m such that $f_m(x) > 0$. Let $x \in M$ and assume that f_k is the first function which doesn't vanish at x. Then $x \in V_k$.

To see that this cover is locally finite pick a point $x \in M$ and some f_m such that $f_m(x) > 0$. But then the only possible V_n that can contain x are those where $n < 1/f_m(x)$. Clearly we can find an open set around x where $f_m(x)$ stays positive and bounded so a similar result holds for all the points in that open set so the cover V_n has to be locally finite. Each of the sets $\{x | f_k(x) < i/n\}$ can be written as U_f for some suitable nonnegative function f (essentially the same argument as before (3.1)). From (3.1) it follows that each V_n is of the form U_{g_n} for some smooth functions g_n. The partitions of unity are obtained by scaling the g_n by their sum.

In the case at hand we can see directly that the open cover we are using is of the form required by the preceding construction of the partition of unity. This is because we can define smooth functions $f_{\lambda\lambda'}$ on X as $f_{\lambda\lambda'}(A) = \exp(-1/d)$, where d is the distance of the spectrum of the operator D_A to the set $\{\lambda, \lambda'\}$. This distance is always positive for $A \in U_{\lambda\lambda'}$, because the spectrum does not have accumulation points on a compact manifold M. When $A \notin U_{\lambda\lambda'}$ we set $f_{\lambda\lambda'}(A) = 0$. Finally $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}_e$ are metric spaces as they are Frechet manifolds modelled on a space with topology given by the (countably many) Sobolev space inner products and hence are regular. We can use the set of functions $f_{\lambda\lambda'}$ in the proof above to show the existence of a locally finite cover and corresponding partition of unity.

4. Calculating the Dixmier-Douady Class

Our starting point is to describe, in the notation of this paper, the bundle gerbe J over $\mathcal{A}$ defined in [CaMu]. This is a line bundle over the fibre product $\mathcal{A}_0^{[2]}$. This fibre product can be identified with all triples (A, λ, λ'), where neither λ nor λ' are in the spectrum

of D_A. The fibre of J over (A, λ, λ') is just $DET_{\lambda\lambda'}$. For this to be a bundle gerbe we need a product which in this case is a linear isomorphism

$$DET_{\lambda\lambda'} \otimes DET_{\lambda'\lambda''} = DET_{\lambda\lambda''}. \tag{4.1}$$

But such a linear isomorphism is a simple consequence of the definition of $DET_{\lambda\lambda'}$ and the fact that taking highest exterior powers is multiplicative for direct sums.

Let $\pi : \mathcal{A}_0 \to \mathcal{A}$ be the projection. Let $p : \mathcal{A} \to \mathcal{A}/\mathcal{G}_e$ be the quotient by the gauge action. We saw in [CaMu1] that the line bundle DET on $\mathcal{A}_0$ satisfies $J = \delta(DET)$. Here $\delta(DET) = \pi_1^*(DET)^* \otimes \pi_2^*(DET)$ where $\pi_i : \mathcal{A}_0^{[2]} \to \mathcal{A}_0$ are the projections,

$$\pi_1((A, \lambda, \lambda')) = (A, \lambda) \quad \text{and} \quad \pi_2((A, \lambda, \lambda')) = (A, \lambda')$$

In other words $J = \delta(DET)$ is equivalent to

$$DET_{\lambda\lambda'} = DET_\lambda^* \otimes DET_{\lambda'}$$

which is equivalent to Eq. (2.2). Note that we also used δ to denote a similar operation on differential forms discussed earlier.

The fibering $\mathcal{A}_0 \to \mathcal{A}$ has, over each open set U_λ a canonical section $A \mapsto (A, \lambda)$. These enable us to suppress the geometry of the fibration and the bundle gerbe J becomes the line bundle $DET_{\lambda\lambda'}$ over the intersection $U_{\lambda\lambda'}$ and its triviality amounts to the fact that we have the line bundle DET_λ over U_λ and over intersections we have the identifications

$$DET_{\lambda\lambda'} = DET_\lambda^* \otimes DET_{\lambda'}.$$

We denote the Chern class of $DET_{\lambda\lambda'}$ by $\theta_2^{[2]}$. Note that these bundles descend to bundles over $V_{\lambda\lambda'} = \pi(U_{\lambda\lambda'}) \subset \mathcal{A}/\mathcal{G}_e$. Therefore, the forms $\theta_2^{\lambda\lambda'} = \theta_2^\lambda - \theta_2^{\lambda'}$ on $U_{\lambda\lambda'}$ (where θ_2^λ is the 2-form giving the curvature of DET_λ) are equivalent (in cohomology) to forms which descend to closed 2-forms $\phi_2^{\lambda\lambda'}$ on $V_{\lambda\lambda'}$.

Our aim in this section is twofold. We show first that the collection of Chern classes θ_2^λ gives rise to the Dixmier-Douady class of the bundle gerbe $J/\mathcal{G}_e$ and second that using the results of the preceding section, we can obtain formulae for this class using standard methods.

To begin, let us choose a bundle gerbe connection on J/G_e. This a connection that preserves the product in Eq. (4.1). In the general setting the existence of such connections follows from a partition of unity argument [Mu]. However in this case it is possible to construct one by orthogonal projection. Call it ∇ and its curvature F_∇. Then we can pull ∇ back to $p^*(\nabla)$ on J with curvature $p^*(F_\nabla)$. Similarly choose a connection ∇_{DET} on the line bundle DET. This induces a connection $\delta(\nabla_{DET})$ on J. The difference of these two connections is a one form a on $\mathcal{A}_0^{[2]}$ and, in fact, $\delta(a) = 0$ so that $a = \delta(\psi)$.

Note that ψ is not unique and we do not have a constructive method of finding it (but if we did then we could construct explicit formulae). Pressing on however if F_{DET} is the curvature (which has class equal to the Chern class $\{\theta_2^\lambda\}$) of ∇_{DET}, then

$$p^*(F_\nabla) = \delta(F_{DET} + d\psi).$$

Now assume that down on $\mathcal{A}/\mathcal{G}_e$ we have solved

$$F_\nabla = \delta(f).$$

We remark that this is a central point. It is not obvious that there is a solution. However by the previous subsection there is a locally finite partition of unity $\{s_\lambda\}$ subordinate to the open cover $\{V_\lambda\}$. The curvature of the bundle gerbe consists of closed 2-forms $\phi_2^{\lambda\lambda'}$ on the intersections $V_{\lambda\lambda'}$ satisfying the cocycle condition

$$\phi_2^{\lambda\lambda'} + \phi_2^{\lambda'\lambda''} = \phi_2^{\lambda\lambda''}$$

on the domains of definition. One can then define

$$\phi_2^\lambda = \sum s_{\lambda'}\phi_2^{\lambda\lambda'},$$

which gives

$$\phi_2^{\lambda\lambda'} = \phi_2^\lambda - \phi_2^{\lambda'}$$

on $V_{\lambda\lambda'}$. The collection of forms ϕ_2^λ defines the form f on $\mathcal{A}_0/\mathcal{G}_e$.

Now, continuing our argument, we have

$$\delta(p^*(f)) = \delta(F_{DET} + d\psi)$$

so that

$$p^*(f) = F_{DET} + d\psi + \pi^*(\rho)$$

as $\pi^*(\rho) = \delta(\rho)$.

By definition the Dixmier-Douady class is the 3-form ω on $\mathcal{A}/\mathcal{G}_e$ defined by

$$df = \pi^*(\omega).$$

If we transgress ω then we want to solve

$$p^*(\omega) = d\mu$$

for some μ. Hence we have

$$p^*(\pi^*(\omega)) = d\pi^*(\mu).$$

But from the above we have

$$p^*\pi^*(\omega) = dp^*(f) = dF_{DET} + dd\psi + d\pi^*(\rho) = \pi^*(d\rho).$$

Hence

$$p^*(\omega) = d\rho.$$

So $[\rho] = [\mu]$. To calculate the Lie algebra cocycle we need to apply ρ to two vectors ξ, η in $\mathcal{A}$ generated by the group action. As the group also acts on $\mathcal{A}_0$ it is equivalent to apply $\pi^*(\rho)$ to two such vectors which we shall denote by the same symbols. Then, noting that $p^*(f)$ is zero on any vectors generated by the gauge group action (because p is the projection $A \to \mathcal{A}/\mathcal{G}_e$) we have

$$\pi^*(\rho)(\xi, \eta) = -F_{DET}(\xi, \eta) - d\psi(\xi, \eta).$$

Hence the Faddeev-Mickelsson cocycle on the Lie algebra of the gauge group is cohomologous to the negative of that defined by the curvature F_{DET} of the line bundle DET.

To obtain the Dixmier-Douady class as a characteristic class we recall that in the case of even dimensional manifolds, Atiyah and Singer [AtSi] gave a construction of 'anomalies' in terms of characteristic classes. In the present case of odd dimensional

manifolds we now demonstrate that a similar procedure yields the Dixmier-Douady class.

We begin with the observation that given a closed integral form Ω_p of degree p on a product manifold $M \times X$ (dim$M = d$ and dim$S = k$) we obtain a closed integral form on S, of degree $p - d$, as

$$\Omega_X = \int_M \Omega.$$

If now A is any Lie algebra valued connection on the product $M \times X$ and F is the corresponding curvature we can construct the Chern form $c_{2n} = c_{2n}(F)$ as a polynomial in F. Apply this to the connection A defined by Atiyah and Singer, [AtSi, DoKr, p. 196], in the case when $X = \mathcal{A}/\mathcal{G}_e$.

First pull back the forms to $M \times \mathcal{A}$. The Atiyah-Singer connection on $M \times X$ becomes a globally defined Lie algebra valued 1-form $\hat{A}$ on $M \times \mathcal{A}$. Along directions u on M it is defined as

$$\hat{A}_{x,a}(u) = u_\mu a_\mu(x), \text{ with } x \in M \text{ and } a \in \mathcal{A},$$

and along a tangent vector $b \in T_a\mathcal{A}$ the value is

$$\hat{A}_{x,a}(b) = -(G_a d_a^* b)(x),$$

where $d_a = d + [a, \cdot]$ is the covariant exterior differentiation acting on functions with values in the adjoint representation of $\mathbf{g}$ and $G_a = (d_a^* d_a)^{-1}$ is the Green's operator. Let $\hat{F}$ be the curvature form determined by $\hat{A}$. A tangent vector b at $a \in \mathcal{A}$ is said to be in the *background gauge* if $d_a^* b = \partial_\mu b_\mu + [a_\mu, b_\mu] = 0$. Any tangent vector b at a point $\pi(a) \in X$ is represented by a unique potential b in the background gauge. For this reason we need to evaluate the curvature $\hat{F}$ only along background gauge directions.

Along tangent vectors u, v at $x \in M$ the curvature is $\mathcal{F}_{x,a}(u, v) = f_x(u, v),$, where $f = da + \frac{1}{2}[a, a]$. Along directions b, b' in the background gauge at $a \in \mathcal{A}$ the value of $\hat{F}$ is $G_a[b_\mu, b'_\mu]$ and finally along mixed directions $\hat{F}_{x,a}(u, b) = u_\mu b_\mu(x)$, [AtSi]. For example, when dim$M = 3$ and $p = 6$ the 3-form Ω_X becomes now, evaluated at $a \in \mathcal{A}$,

$$\Omega_X(b, b', b'') = \frac{-i}{8\pi^3} \int_M \text{tr} b[b', b''] +$$
$$\frac{i}{8\pi^3} \int_M \text{tr} f(b'' G_a[b, b'] + G_a[b, b']b'' + \text{cycl. combin. }),$$

when b, b', b'' are in the background gauge.

The integral of Ω_X over a sphere $S^3 \subset X$ can be evaluated without computing the nonlocal Green's operators in the above formula. The pull-back of S^3 becomes a disk D^3 on $\mathcal{A}$ with boundary points identified through gauge transformations. We can therefore write

$$\int_{S^3} \Omega_X = \int_{M \times D^3} c_{2n}(\hat{F}).$$

But the integral of the Chern form over a manifold with a boundary (when the potential is globally defined) is equal to the integral

$$\int_{M \times \partial D^3} CS_{2n-1}(\hat{A}).$$

Along gauge directions the form $\hat{A}$ is particularly simple: $\hat{A}_{a,x}(b) = Z(x)$, where $x \in M, a \in \mathcal{A}$, and $b = -d_a Z = [Z, a] - dZ$ is a tangent vector along a gauge orbit at a. For example, when $M = S^1$ and $2n = 4$ we get (here $S^2 = \partial D^3$)

$$\int_{S^3} \Omega_X = \int CS_3(\hat{A}) = \frac{1}{8\pi^2} \int_{S^1 \times S^2} \text{tr}(a^g da^g + \frac{2}{3}(a^g)^3) = \frac{1}{24\pi^2} \int_{S^1 \times S^2} \text{tr}(dgg^{-1})^3,$$

where $g = g(x, z)$ is a family of gauge transformations parametrized by $z \in S^2$. Similar results hold in higher dimensions: The exponent 3 on the right is replaced by $\dim M + 2 = 2n + 3$ and the normalization factor is $-(\frac{i}{2\pi})^{n+2}((n + 2)! \cdot (2n + 3))^{-1}$.

Now we can prove that Ω_X represents the Dixmier-Douady class of the bundle gerbe. The integral of the DD form ω over a closed 3-cycle $S \subset \mathcal{A}/\mathcal{G}_e$ (which can be assumed to be a sphere S^3) is evaluated, using the pull-back form $df = \pi^*(\omega)$, on $\mathcal{A}_0/\mathcal{G}_0$ and further pulling back this by p^* to $\mathcal{A}_0$. But the cover of S^3 in the latter space is a disk D^3 such that the boundary points are gauge related. Because the spectrum of the Dirac operator is gauge invariant we can choose a single label λ such that $\partial D^3 \subset U_\lambda$. Since $p^*(df) = d(p^*f)$ the integral over D^3 can be evaluated by Stokes theorem over the boundary ∂D^3. The form p^*f on U_λ is equal to $\theta_2{}^\lambda$, the Chern class of the determinant bundle over U_λ. But the integral of $\theta_2{}^\lambda$ over the gauge orbit ∂D^3 is given by the integral of the Chern-Simons form (2.9), thus giving exactly the same result as the integration of Ω_X above. We conclude that the de Rham cohomology classes $[\omega]$ and $[\Omega_X]$ are the same.

References

[AtSi]   Atiyah, M.F. and Singer, I.M.: Dirac operators coupled to vector potentials. Natl. Acad. Sci. (USA) **81**, 2597 (1984)

[AtPaSi] Atiyah, M.F., Patodi, V.K. and Singer, I. M.: Spectral asymmetry and Riemannian Geometry, I-III. Math. Proc. Camb. Phil. Soc. **77**,43 (1975); **78**, 405 (1975); **79**, 71 (1976)

[Be]   Beggs, Edwin J.: *The de Rham complex on infinite dimensional manifolds.* Quart. J. Math. Oxford (2), **38**, 131-154 (1987)

[Br]   Brylinski, J.-L.: *Loop Spaces, Characteristic Classes and Geometric Quantization.* Boston-Basel-Berlin: Birkhäuser (1993)

[CaMu]   Carey, A.L. and Murray, M.K.: In: "Confronting the Infinite", Proceedings of a Conference in Celebration of the 70th Years of H.S. Green and C.A. Hurst. Singapore: World Scientific, 1995 hep-th/9408141

[CaMu1]   Carey, A.L. and Murray, M.K.: Faddeev's anomaly and bundle gerbes. Lett. Math. Phys. 1995. **37**, 29–36 (1996)

[CaMuWa] Carey, A.L., Murray, M.K. and Wang, B.: Higher bundle gerbes, descent equations and 3-Cocycles. To appear in J. Geom. and Phys. hep-th/9511169

[DiDo]   Dixmier, J. and Douady, A.: Champs continus d'espaces hilbertiens et de C^* algebres. Bull. Soc. Math. Fr. **91**, 227 (1963)

[DoKr]   Donaldson, S.K. and Kronheimer, P.B.: *The Geometry of Four-Manifolds.* Oxford: Clarendon Press (1990)

[FaSh]   Faddeev, L. and Shatasvili, S.: Algebraic and hamiltonian methods in the theory of nonabelian anomalies. Theoret. Math. Phys. **60**, 770 (1985)

[Mi]   Mickelsson, J.: Chiral anomalies in even and odd dimensions. Commun. Math. Phys. **97**, 361 (1985)

[Mi1]   Mickelsson, J.: On the hamiltonian approach to commutator anomalies in 3 + 1 dimensions. Phys. Lett. **B 241**, 70 (1990)

[Mil]   Milnor, J.: *On infinite dimensional Lie groups.* Preprint.

[Mu]   Murray. M.K.: *Bundle gerbes.* J. of the Lond. Math. Soc. (2) **54**, 403–416 (1996)

[NeAl]   Nelson, P. and Alvarez-Gaumè, L.: Hamiltonian interpretation of anomalies. Commun. Math. Phys. **99**, 103 (1985)

[PrSe]   Pressley, A. and Segal, G.: *Loop Groups*. Oxford: Clarendon Press (1986)
[Se]   Segal, G.: Faddeev's anomaly in Gauss's law. Preprint

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