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Zeta function for the Laplace operator acting on forms in a ball with gauge boundary conditions

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Abstract: The Laplace operator acting on antisymmetric tensor fields in a D-dimensional Euclidean ball is studied. Gauge-invariant local boundary conditions (absolute and relative ones, in the language of Gilkey) are considered. The eigenfuctions of the operator are found explicitly for all values of D. Using in a row a number of basic techniques, as Mellin transforms, deformation and shifting of the complex integration contour and pole compensation, the zeta function of the operator is obtained. From its expression, in particular, $\zeta(0)$ and $\zeta'(0)$ are evaluated exactly. A table is given in the paper for $D = 3, 4, \ldots, 8$. The functional determinants and Casimir energies are obtained for $D = 3, 4, \ldots, 6$.

1. Introduction

In this paper we obtain the zeta function of the Laplace operator acting on antisymmetric tensor fields defined in a D-dimensional ball with gauge-invariant boundary conditions. Mathematically this computation is quite an imposing challenge, as is proven by the number of erroneous results reported in the literature on this and related computations (details will be given later). The physical motivations for such a study are to be found in quantum cosmology, where the ζ function of the Laplacian describes the contribution of antisymmetric tensor fields and ghosts to the pre-factor of the wave function of the universe (see e.g. [1]). An intriguing problem in this context is the non-compensation of the boundary contributions to the one–loop divergences between different members of the supergravity supermultiplet [2]. Another motivation is to provide the numerical material needed to extend previous analysis of the heat kernel asymptotics [3] to the case of mixed boundary conditions.

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There are two admissible sets of gauge-invariant local boundary conditions – which have been called by Gilkey, respectively, absolute and relative boundary conditions [4]. These sets are dual to each other and are becoming highly interesting in connection with recent developments in string theory. Hence one can study p-forms with $p \leq \left[\frac{D+1}{2}\right]$ for both types of boundary conditions. Due to duality, the determinant of the Laplacian for p-forms with absolute boundary conditions is the same as the one for $(D-p)$-forms with relative boundary conditions. Furthermore, owing to gauge-invariance we can restrict ourselves to transversal p-forms. The complete result will just be a sum of the contributions corresponding to transversal p- and $(p-1)$-forms, provided zero modes are properly taken into account. To find the spectrum of the Laplace operator on transversal p-forms we use the method proposed in [5, 6]. To obtain the zeta function we use the powerful procedures developed in [7, 8] (see also [9, 10]). They involve integral representations of the spectral sums, Mellin transformations, non-trivial commutation of series and integrals and skillful analytic continuation of zeta functions on the complex plane. Here we will focus again on a class of situations for which the eigenvalues of the differential operator, A, are not known explicitly but where, nevertheless, the exact calculation of $\ln \det A$ is possible. The method is applicable whenever an implicit equation satisfied by the eigenvalues is at hand and some asymptoticity properties of the equation are known too.

More specifically, we will find here explicit solutions of the D-dimensional transversality condition in terms of p- and $(p-1)$-forms obeying a $(D-1)$-dimensional transversality condition. These forms will satisfy now pure Dirichlet or Robin boundary conditions, instead of mixed absolute or relative boundary conditions. Such a clever procedure will enable us to find exact eigenfunctions and to express the eigenvalues of the Laplace operator in terms of Bessel functions and their derivatives. After this, we will be able to perform explicitly the evaluation of the zeta function at the origin, and that of the determinant of the Laplacian as well. We will also calculate the Casimir energy. A table of results will be given in the paper for $D = 3, \ldots, 8$, which cover the situations that appear in the usual supersymmetric theories. However, our final expressions are actually valid (and can be used) for any dimension D and yield explicit, exact values in a reasonable amount of algebraic computation time. Usual methods for the acceleration of the series convergence improve performance considerably.

In connection with previous results, we should point out that for some scattered values of $D = 4$ and $p = 1, 2$, several first heat-kernel coefficients have been calculated in [5, 6, 11]. These results agree with the analytical formulas in [12] once the corrections that were found in [5] are taken into account (see also [2]). For $D = 4$, $p = 1$, the one-loop effective action has been evaluated in [13] for a specific choice of gauge and of boundary conditions.

The paper is organized as follows. In Sect. 2 we use the Hodge-de Rham decomposition of p-forms in order to simplify the structure of the spectrum of the Laplacian operator in the D-dimensional ball and, subsequently, of the corresponding determinant. After writing the absolute and relative boundary conditions as Dirichlet and Robin ones, a convenient analytical continuation of the associated zeta function is performed in Sect. 3, in some detail, what leaves us in a position wherefrom we can find the heat-kernel coefficients, the determinant, Casimir energies, and so on. The calculation of the zeta function at the origin is undertaken in Sect. 4, that of the determinant in Sect. 5, and the Casimir energy is obtained in Sect. 6. Finally, in an Appendix we give an exhaustive list of all the determinants explicitly calculated, both for the case of absolute and relative boundary conditions.

2. Spectrum of the Laplace Operator in a Ball

Consider the $D = d + 1$ dimensional unit disk with the metric

$$ds^2 = dr^2 + r^2 d\Omega^2, \quad 0 \leq r \leq 1, \tag{1}$$

where $d\Omega^2$ is the metric on the unit sphere S^d. Throughout this paper we shall use the notations $\{x_\mu\} = \{x_0, x_i\}$, $x^0 = r$, $\mu = 0, 1, \ldots, d$. The $(d+1)$-dimensional Laplace operator, $\Delta = \nabla^\mu \nabla_\mu$, acting on a p-form, B, can be written as

$$(\Delta B)_{i_1 \ldots i_{p-1} 0} = \left(\partial_0^2 + \frac{d - 2p + 2}{r} \partial_0 + \frac{p^2 - dp - 1}{r^2} + {}^{(d)}\Delta \right) B_{i_1 \ldots i_{p-1} 0} - \\ - \frac{2}{r} {}^{(d)}\nabla^k B_{i_1 \ldots i_{p-1} k}, \tag{2}$$

$$(\Delta B)_{i_1 \ldots i_p} = \left(\partial_0^2 + \frac{d - 2p}{r} \partial_0 + \frac{p^2 - dp}{r^2} + {}^{(d)}\Delta \right) B_{i_1 \ldots i_p} + \\ + \frac{2}{r} \sum_{a=1}^{p} {}^{(d)}\nabla_{i_a} B_{i_1 \ldots i_{a-1} 0 i_{a+1} \ldots i_p}, \tag{3}$$

where ${}^{(d)}\nabla$ and ${}^{(d)}\Delta$ are the covariant derivative and Laplace operator corresponding to the d-dimensional metric g_{ik}.

Any p-form B^p admits the Hodge-de Rham decomposition:

$$B^p = B^{p\perp} + dB^{(p-1)\perp}, \tag{4}$$

where $B^{p\perp}$ denotes a transversal p-form. The decomposition (4) commutes with the Laplace operator. Thus, in order to define the spectrum of the Laplacian on the space of all antisymmetric forms, it is enough to study the case of transversal forms only.

There are two sets of local boundary conditions consistent with the decomposition (4). They are the so-called absolute and relative boundary conditions [4]. In the coordinate system (1) the absolute boundary conditions read

$$\partial_0 B_{i_1, \ldots, i_p} |_{\partial M} = 0, \quad B_{0, i_1, \ldots, i_{p-1}} |_{\partial M} = 0, \tag{5}$$

while the relative boundary conditions have the form

$$B_{i_1, \ldots, i_p} |_{\partial M} = 0, \quad \left(\partial_0 + \frac{d - 2p + 2}{r} \right) B_{0, i_1, \ldots, i_{p-1}} |_{\partial M} = 0. \tag{6}$$

Consider the $(d+1)$-dimensional transversality condition

$$\nabla^\mu B_{\mu\nu \ldots \rho} = 0. \tag{7}$$

On a disk it can be written as

$$(\nabla B)_{i_1 \ldots i_{p-2} 0} = {}^{(d)}\nabla^i B_{i i_1 \ldots i_{p-2} 0} = 0, \tag{8}$$

$$(\nabla B)_{i_1 \ldots i_{p-1}} = (\partial_0 + \frac{d - 2p + 2}{r}) B_{0 i_1 \ldots i_{p-1}} + {}^{(d)}\nabla^i B_{i i_1 \ldots i_{p-1}} = 0. \tag{9}$$

According to the general method developed in the papers [5, 6], the solutions of Eqs. (8) and (9) can be expressed in terms of d-dimensional transversal forms:

$$B^{p\perp} = B^{pT} + B^{p\perp}(\psi^{(p-1)T}), \tag{10}$$

where such d-dimensional transversal forms satisfy the equations:

$$^{(d)}\nabla^i A^{pT}_{i,i_1,\ldots,i_{p-1}} = 0, \qquad A^{pT}_{0,i_1,\ldots,i_{p-1}} = 0. \tag{11}$$

Here A^T denotes either B^T or ψ^T. The second term in (10) has the form:

$$\begin{aligned} B^{\perp}_{0i_1\ldots i_{p-1}}(\psi^T) &= (-^{(d)}\Delta + \frac{(p-1)d-(p-1)^2}{r^2})r\psi^T_{i_1\ldots i_{p-1}}, \\ B^{\perp}_{i_1\ldots i_p}(\psi^T) &= (\partial_0 + \frac{d-2p}{r})r\,{}^{(d)}\nabla_{[i_1}\psi^T_{i_2\ldots i_p]}, \\ \nabla_{[i_1}\psi_{i_2\ldots i_p]} &:= \sum_{n=1}^{p}(-1)^{n+1}\nabla_{i_n}\psi_{i_1\ldots i_{n-1}i_{n+1}\ldots i_p}. \end{aligned} \tag{12}$$

One can prove that the Laplace operator (2), (3), commutes with the decomposition (10):

$$\Delta B^{p\perp} = \Delta B^{pT} + B^{p\perp}(\Delta\psi^{(p-1)T}). \tag{13}$$

The determinant of the Laplace operator on the space of $(d+1)$-dimensional transversal p forms can be represented as a product of two determinants, taken over d-dimensional transversal p- and $(p-1)$-forms:

$$\det(-\Delta)_{p\perp} = \det(-\Delta)_{pT} \times \det(-\Delta)_{(p-1)T}. \tag{14}$$

Moreover, the fields B^T and ψ^T satisfy pure boundary conditions. The boundary conditions for B^T are defined by the first equations in (5) and (6), for absolute and relative boundary conditions, respectively. For absolute boundary conditions on the field $B^{p\perp}$, the form $\psi^{T(p-1)}$ satisfies Dirichlet boundary conditions,

$$\psi^T|_{\partial M} = 0. \tag{15}$$

For relative boundary conditions we have, for the $(p-1)$-form ψ^T,

$$\left(\partial_0 + \frac{d-2p+1}{r}\right)\psi^T|_{\partial M} = 0, \tag{16}$$

that is, Robin boundary conditions. We thus see that the initial eigenvalue problem for the $(d+1)$-dimensional transversal p-forms with mixed boundary conditions is reduced to two eigenvalue problems, for d-dimensional transversal p- and $(p-1)$-forms with pure boundary conditions (Dirichlet and Robin).

In the particular cases when $p = 1, 2$, the boundary conditions (15) and (16) agree with the corresponding expressions in [5, 6]. Note that the $l = 0$ scalar mode generates a zero mode of the mapping $\psi \to B^{1\perp}$. Hence this mode should be excluded when one considers the path integral over transversal 1-forms and from the second determinant on the r.h.s. of Eq. (14).

Let us introduce the set of d-dimensional spherical harmonics, $Y^{(l)p}_{i_1,\ldots,i_p}(x_j)$, corresponding to transversal p-forms on S^d. They are eigenmodes of the d-dimensional Laplacian. The associated eigenvalues and degeneracies D^p_l are found to be [14, 15]

$$
\begin{aligned}
{}^{(d)}\Delta Y^{(l)p}_{i_1,\ldots,i_p}(x_j) &= \frac{1}{r^2}[-l(l+d-1)+p]Y^{(l)p}_{i_1,\ldots,i_p}(x_j), \\
D^p_l &= \frac{(2l+d-1)\,(l+d-1)!}{p!\,(d-p-1)!\,(l-1)!\,(l+p)(l+d-p-1)}.
\end{aligned}
\tag{17}
$$

We can represent the eigenfunctions of the complete ($D = d + 1$)-dimensional Laplace operator as a Fourier series in the harmonics (17):

$$
B^{pT}_{i_1,\ldots,i_p}(r, x_j) = \sum_{(l)} Y^{(l)p}_{i_1,\ldots,i_p}(x_j) f(l)(r). \tag{18}
$$

Here we need to sum over (l), what means summation over the index l from 1 to ∞ and over another index, from 1 to D^p_l, which describes the different harmonics with degenerate eigenvalues of the Laplacian ${}^{(d)}\Delta$. This last summation is not shown explicitly.

We can now substitute the decomposition (18) in the eigenvalue equation

$$
\Delta B^{pT}_{\mu_1,\mu_2,\ldots,\mu_p} = -\lambda^2 B^{pT}_{\mu_1,\mu_2,\ldots,\mu_p} \tag{19}
$$

for the $(d + 1)$-dimensional Laplacian (2), (3). Let us recall the fact that for the fields B^{pT} the zeroth components vanish identically: $B^{pT}_{0i_1\ldots i_{p-1}} = 0$. The equation for the components (2) reduces to the trivial identity $0 = 0$. The other components (3) lead to an equation of Bessel type for $f^{(l)}(r)$. After a rather lengthy algebra, one finds that the eigenfunctions of the Laplace operator (3) have the following form:

$$
r^{(1-d)/2+p} J_{(d-1)/2+l}(\lambda_l r) Y^{(l)p}_{i_1,\ldots,i_p}(x_j), \tag{20}
$$

where J_n denote Bessel functions. The eigenvalues λ^2_l are defined by boundary conditions and their degeneracies D^p_l are given by (17).

From the preceding expressions, we are able to evaluate the determinant of the Laplace operator on the space of transversal p-forms. We obtain

$$
\det(-\Delta)_{p\perp} = \prod_{l=1}^{\infty} \lambda_l^{2D^p_l} \prod_{k=1}^{\infty} \kappa_k^{2D^{p-1}_k}, \tag{21}
$$

where for absolute boundary conditions the eigenvalues λ and κ are defined by (5) and (15), namely

$$
\partial_0 r^{(1-d)/2+p} J_{(d-1)/2+l}(\lambda_l r)|_{\partial M} = 0, \qquad J_{(d-1)/2+k}(\kappa_k r)|_{\partial M} = 0. \tag{22}
$$

For relative boundary conditions we have, from (6) and (16),

$$
J_{(d-1)/2+l}(\lambda_l r)|_{\partial M} = 0, \qquad (\partial_0 + d - 2p + 1) r^{(1-d)/2+p-1} J_{(d-1)/2+k}(\kappa_k r)|_{\partial M} = 0. \tag{23}
$$

One can easily check that the eigenfunctions defined in this section satisfy all necessary orthogonality properties.

3. Analytical Continuation of the Zeta Function

Both the absolute and the relative boundary conditions can be written as Dirichlet and Robin boundary conditions, in the form

$$J_\nu(\lambda_l r)|_{\partial M} = 0, \tag{24}$$

$$u(d,p)J_\nu(\lambda_l r) + \lambda_l J'_\nu(\lambda_l r)|_{\partial M} = 0. \tag{25}$$

For the absolute boundary conditions we have $u(d,p) = (1-d)/2 + p$, while for the relative boundary condition, $u(d,p) = (1+d)/2 - p$. The boundary ∂M is here described by $r = a$. The zeta function is

$$\zeta^d_{pT}(s) = \sum_{l=1}^{\infty} D^p_l(d)\lambda^{-2s}. \tag{26}$$

The decomposition described in the last section will, at the level of zeta functions, manifest itself as a sum of the different zeta functions belonging to each term of the decomposition. Thus we can write

$$\zeta^d_{p\perp}(s) = \zeta^d_{pT}(s) + \zeta^d_{(p-1)T}(s). \tag{27}$$

The zeta function is in general convergent for $s > \frac{d+1}{2}$ only, but it can be analytically continued in the complex plane to all values of s, in particular to the vicinity of $s = 0$. Several authors have considered zeta functions corresponding to operators whose eigenvalues are not given explicitly. In particular, they have investigated in detail the case when they are given under the form of roots of equations involving Bessel functions (see, for example [7]). In [7, 8] it has been shown explicitly how this analytical continuation can be carried out for zeta functions of this kind, and we will follow this path closely. The reader may resort to those papers for all particularities skipped in the present calculation.

Writing the boundary conditions (24) and (25) symbolically as $\Psi_\nu(\lambda_l r)|_{\partial M} = 0$, the first idea is to express the zeta function as a contour integral along a path γ enclosing all positive solutions of the boundary condition equation, namely

$$\zeta^d_{p\perp}(s) = \sum_{l=1}^{\infty} D_l(p,d) \int_\gamma \frac{dk}{2\pi}(k^2+m^2)^{-s}\frac{\partial}{\partial k}\ln\Psi_\nu(ak). \tag{28}$$

Here we have introduced the constant m in order to simplify the treatment of the problem. It is, however, not essential in order to obtain the final result and we shall let this constant vanish later in the calculation. To start, we expand the degeneracy as

$$D_l(p,d) = \sum_{j=0}^{d-1} e_j(d,p)\left(l + \frac{d-1}{2}\right)^j. \tag{29}$$

The zeta function reads then

$$\begin{aligned}\zeta^d_{p\perp}(s) &= \sum_{j=0}^{d-1} e_j(d,p)\sum_{l=1}^{\infty}\left(l+\frac{d-1}{2}\right)^j \int_\gamma \frac{dk}{2\pi}(k^2+m^2)^{-s}\frac{\partial}{\partial k}\ln\Psi_{l+\frac{d-1}{2}}(ak)\\ &= \sum_{j=0}^{d-1} e_j(d,p)\sum_{l=0}^{\infty}\left(l+\frac{d+1}{2}\right)^j \int_\gamma \frac{dk}{2\pi}(k^2+m^2)^{-s}\frac{\partial}{\partial k}\ln\Psi_{l+\frac{d+1}{2}}(ak).\end{aligned} \tag{30}$$

Now we are already in the position of performing the analytic continuation. This involves subtraction and addition of the leading asymptotic terms of the uniform expansion of the Bessel function $I_\nu(k)$ and its derivative. For $\nu \to \infty$ and $z = k/\nu$ being fixed, these terms are [16]

$$I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\mu}}{(1+z^2)^{1/4}} \left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right] \tag{31}$$

and

$$I'_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\mu}(1+z^2)^{1/4}}{z} \left[1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k}\right], \tag{32}$$

respectively. Here u_k and v_k are functions obtained in a recursive way in [16], while $t = 1/\sqrt{1+z^2}$ and $\mu = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$. Furthermore, we define the coefficients $D_n(t)$ and $M_n(t)$ by

$$\ln\left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right] \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n} \tag{33}$$

and

$$\ln\left[1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k} + \frac{u(p,d)}{\nu} t \left(1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right)\right] \sim \sum_{n=1}^{\infty} \frac{M_n(p,d)(t)}{\nu^n}. \tag{34}$$

Then, by adding and subtracting the first N terms of these last two expansions, we can write the zeta function for Dirichlet boundary conditions as

$$\zeta^d_{p\perp}(s) = \sum_{i=-1}^{N} A_i(s) + Z_N(s), \tag{35}$$

where, with $\nu = l + \frac{d+1}{2}$ and $m = 0$, we have

$$A_{-1}(s) = \frac{a^{2s}\Gamma(s-\frac{1}{2})}{4\sqrt{\pi}\Gamma(s+1)} \sum_{j=0}^{d-1} e_j(d,p)\zeta_H(2s-1-j, \frac{d+1}{2}), \tag{36}$$

$$A_0(s) = -\frac{a^{2s}}{4} \sum_{j=0}^{d-1} e_j(d,p)\zeta_H(2s-j, \frac{d+1}{2}), \tag{37}$$

$$A_i(s) = -\frac{a^{2s}}{2\Gamma(s)} \sum_{j=0}^{d-1} e_j(d,p)\zeta_H(2s+i-j, \frac{d+1}{2}) \sum_{k=0}^{i} x_{k,i} \frac{(i+2k)\Gamma(s+k+\frac{i}{2})}{\Gamma(1+k+\frac{i}{2})}, \tag{38}$$

and

$$Z_N(s) = 2sa^{2s}\frac{\sin(\pi s)}{\pi} \sum_{j=0}^{d-1} e_j(d,p) \sum_{l=0}^{\infty} \nu^{j-2s}$$
$$\times \int_0^\infty dz z^{-2s-1} \left\{ \ln I_\nu(\nu z) - \ln\left[\frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\mu}}{(1+z^2)^{1/4}}\right] - \sum_{n=1}^{N} \frac{D_n(t)}{\nu^n} \right\}. \tag{39}$$

The coefficients $x_{k,i}$ in Eq. (38) are obtained from the polynomial expansion of $D_i(t)$:

$$D_i(t) = \sum_{k=0}^{i} x_{k,i} t^{i+2k}. \tag{40}$$

Similarly, denoting by $z_{i,k}$ the coefficients in the expansion of $M_i(t)$,

$$M_i(p,d)(t) = \sum_{k=0}^{2i} z_{i,k}(p,d) t^{i+k}, \tag{41}$$

we can write the zeta function for Robin boundary conditions as

$$\zeta_p^d(s) = \sum_{i=-1}^{N} A_i^R(s) + Z_N^R(s), \tag{42}$$

where

$$A_{-1}^R(s) = A_{-1}(s), \tag{43}$$

$$A_0^R(s) = -A_0(s), \tag{44}$$

$$A_i^R(s) = -\frac{a^{2s}}{2\Gamma(s)} \sum_{j=0}^{d-1} e_j(d,p) \zeta_H(2s+i-j, \frac{d+1}{2}) \sum_{k=0}^{2i} z_{k,i}(p,d) \frac{(i+k)\Gamma(s+\frac{i+k}{2})}{\Gamma(1+\frac{i+k}{2})}, \tag{45}$$

and

$$\begin{aligned} Z_N^R(s) = {} & 2sa^{2s} \frac{\sin(\pi s)}{\pi} \sum_{j=0}^{d-1} e_j(d,p) \sum_{l=0}^{\infty} \nu^{j-2s} \\ & \times \int_0^\infty dz z^{-2s-1} \left\{ \ln\left[u(p,d) I_\nu(\nu z) + z\nu I'_\nu(\nu z)\right] \right. \\ & \left. - \ln\left[\frac{\nu}{\sqrt{2\pi\nu}} e^{\nu\mu}(1+z^2)^{1/4}\right] - \sum_{n=1}^{N} \frac{M_n(t)}{\nu^n} \right\}. \end{aligned} \tag{46}$$

It can be shown that both zeta functions, (35) and (42), are well defined for $\frac{d-1-N}{2} < s$. Both $Z_N(s)$ and $Z_N^R(s)$ are analytic here, so that all the poles are contained in the A's. The analytical continuation can therefore reach the desired range of s, by just changing the value of N. We are thus in a position where we can find the heat kernel coefficients, the zeta function determinant and also the Casimir energy.

In the following we shall work in the unit sphere. Since the only change we have to do on this zeta function in order to include an arbitrary radius is to multiply by a^{2s}, the results that we will obtain for the unit sphere can be easily converted into corresponding ones for the general case.

4. Calculation of the Zeta Function at $s = 0$

From the zeta functions we have defined above, we can now calculate the heat kernel coefficients, using the relations that exist between them. From a physical (and maybe also from a mathematical) point of view the most interesting coefficient is the one of $\zeta(0)$, namely

$$\zeta(0) = \frac{B_{\frac{d+1}{2}}}{4\pi^{\frac{d+1}{2}}}, \tag{47}$$

where the numerator $B_{\frac{d+1}{2}}$ is the corresponding coefficient that comes from the short time expansion of the integrated heat kernel:

$$K(t) \sim (4\pi t)^{\frac{d+1}{2}} \sum_{m=0}^{\infty} B_{\frac{m}{2}} t^{\frac{m}{2}}. \tag{48}$$

Note that $Z_d(0) = 0$ and $Z_d^R(0) = 0$, since the sum over l and integral over z is convergent here for $N = d$. Therefore, we need only consider $A_i(0)$, $i = -1, 0, \ldots d$. Using the expansions

$$\Gamma(-n+\varepsilon) \simeq \frac{(-1)^n}{n!\varepsilon} + \cdots \tag{49}$$

and

$$\zeta(1+\varepsilon, \nu) \simeq \frac{1}{\varepsilon} - \Psi(\nu), \tag{50}$$

we find the expressions

$$A_{-1}(0) = \frac{\Gamma(-\frac{1}{2})}{4\sqrt{\pi}} \sum_{j=0}^{d-1} e_j(d,p) \zeta_H(-1-j, \frac{d+1}{2}), \tag{51}$$

$$A_0(0) = -\frac{1}{4} \sum_{j=0}^{d-1} e_j(d,p) \zeta_H(-j, \frac{d+1}{2}), \tag{52}$$

$$A_i(0) = -\frac{1}{4} e_{i-1} \sum_{k=0}^{i} x_{k,i} \frac{(i+2k)\Gamma(k+\frac{i}{2})}{\Gamma(1+k+\frac{i}{2})}, \tag{53}$$

and

$$A_i^R(0) = -\frac{1}{4} e_{i-1} \sum_{k=0}^{2i} z_{k,i}(p,d) \frac{(i+k)\Gamma(\frac{i+k}{2})}{\Gamma(1+\frac{i+k}{2})}. \tag{54}$$

The numerical values obtained from these expressions are given in Tables 1 and 2. Only those values that are independent have been given (the rest are obtained using duality). For absolute boundary conditions and $p = 1$, the scalar field $B^{0\perp}$ has a zero mode satisfying the boundary condition $\partial_0 B^{0\perp} = 0$. But this mode does not contribute in the Hodge-de Rham decomposition. The scalar field is treated by constructing the zeta function for Neumann boundary conditions. From the definition of the integrated heat kernel

$$K(t) = \sum_n e^{-\lambda_n t}, \tag{55}$$

Table 1. Values of $\zeta(0)$ for absolute boundary conditions, transversal p-forms

$p\backslash d$	7	6	5	4	3	2
4	$-\frac{3559}{9072}$					
3	$\frac{36583}{45360}$	$\frac{2929}{4608}$	$\frac{358}{945}$			
2	$-\frac{20467}{25200}$	$-\frac{1624993}{1935360}$	$-\frac{1531}{1890}$	$-\frac{81}{128}$	$-\frac{7}{20}$	
1	$\frac{185449}{226800}$	$\frac{785567}{967680}$	$\frac{6199}{7560}$	$\frac{2429}{2880}$	$\frac{49}{60}$	$\frac{5}{8}$
0	$-\frac{3629089}{3628800}$	$-\frac{1934993}{1935360}$	$-\frac{6379}{7560}$	$-\frac{9647}{11520}$	$-\frac{151}{180}$	$-\frac{41}{48}$

Table 2. Values of $\zeta(0)$ for relative boundary conditions, transversal p-forms

$p\backslash d$	7	6	5	4	3	2
3	$-\frac{3559}{9072}$	$\frac{2929}{4608}$				
2	$\frac{4283}{25200}$	$-\frac{416417}{1935360}$	$\frac{358}{945}$	$-\frac{81}{128}$		
1	$-\frac{33521}{226800}$	$\frac{143263}{967680}$	$-\frac{1109}{7560}$	$\frac{541}{2880}$	$-\frac{7}{20}$	$\frac{5}{8}$
0	$-\frac{289}{3628800}$	$-\frac{367}{1935360}$	$\frac{1}{1512}$	$\frac{17}{11520}$	$-\frac{1}{180}$	$-\frac{1}{48}$

we see that omission of the zero mode corresponds to subtracting 1 from this sum. Since the relationship (47) is still valid – also when the zero mode is projected out – we conclude that in order to get the values of $\zeta(0)$ without the zero mode we actually need to consider $\zeta_{incl}(0) - 1$. In this way we have obtained the values also for $p = 0$, by extracting the values of $\zeta_{incl}(0)$ from [7]. For absolute boundary conditions and $d = 3$, all our values are in agreement with those calculated in [2]. For relative boundary conditions and $d = 3$, $p = 1$, we have found the value given in [5]. In a subject where discrepancies have been so common, this serves as a check of consistency of our whole tables.

5. Calculation of the Determinants

We shall employ the zeta function definition of the determinant of an operator, A, namely

$$\ln \det(A) = -\frac{d}{ds}\, \zeta^A(s)\big|_{s=0}\,. \tag{56}$$

The determinant of our Laplacian is thus

$$-\ln \det{}^d_{p\perp}(\Delta) = \zeta'^d_{pT}(0) + \zeta'^d_{(p-1)T}(0). \tag{57}$$

The determinant in the case of the sphere of radius a is obtained by adding the terms coming over from the derivation of a^{2s}:

$$-\ln \det{}^d_{p\perp}(\Delta)(a) = 2\ln a \left[\zeta^d_{pT}(0) + \zeta^d_{(p-1)T}(0)\right] + \zeta'^d_{pT}(0) + \zeta'^d_{(p-1)T}(0). \tag{58}$$

Differentiation can be carried out without difficulty. Following the same steps as in [8], one obtains the formulas

$$
\begin{aligned}
Z_d'(0,x) = \sum_{j=0}^{d-1} e_j(d,p) \Bigg\{ & \int_0^\infty dt t^x \frac{e^{-t(\frac{d+1}{2})}}{1-e^{-t}} \frac{d^j}{dt^j}\left(\frac{t^{-1}}{e^t-1}\right) \\
& + \sum_{n=j+1}^{d} \frac{D_n(1)\Gamma(x+n-j)}{\Gamma(n-j)} \zeta_H(x+n-j, \frac{d+1}{2}) \\
& + \frac{(-1)^j j!}{2} \zeta_H(x-j, \frac{d+1}{2}) \Gamma(x-j) \\
& - (-1)^j (j+1)! \zeta_H(x-j-1, \frac{d+1}{2}) \Gamma(x-j-1) \Bigg\}, \qquad (59)
\end{aligned}
$$

and

$$
\begin{aligned}
Z_d^{R\prime}(0,x) = Z_d'(0,x) + \sum_{j=0}^{d-1} e_j(d,p) \Bigg\{ & (-1)^{j+1} \frac{u(p,d)^j + 1}{j+1} \Psi(\frac{d+1}{2}) \\
& + \sum_{n=j+2}^{d} \frac{u(p,d)^n}{n} (-1)^{n+1} \zeta_H(n-j, \frac{d+1}{2}) \\
& - (-1)^j j \int_0^u dx x^{j-1} \ln \Gamma(\frac{d+1}{2} + x) \\
& + (-1)^j u(p,d) \ln \Gamma(\frac{d+1}{2} + u(p,d)) \Bigg\}. \qquad (60)
\end{aligned}
$$

The parameter x is introduced in order to allow for the individual terms to be finite. In the final answer this parameter will disappear. The determinants obtained in this way are listed in the Appendix. We have also included the determinants for transversal $p = 0$ forms given in [8]. For Neumann boundary conditions the zero mode must be treated specially, what yields an answer we will be able to use directly later on.

6. The Casimir Energy

As is well known, the Casimir energy (or vacuum energy) density can be written as a (usually formal) sum over the eigenvalues of the energy equation, that is $\frac{1}{2}\sum_k \omega_k$. The energy density difference gives rise to the Casimir force. However, this sum is usually divergent and has to be regularized. A very simple and elegant way of performing the regularization is to use the zeta function method (see [9, 10] for extensive and updated expositions of this procedure). But, sometimes, it happens that even after analytical continuation the zeta function at the desired value still diverges. The normal procedure consists then in resorting to the principal part prescription [17, 18]. In [18] the physical meaning of this prescription has been investigated in depth. A finite part of the vacuum energy is found by separating off the pole. Obviously, from our zeta function, the vacuum energy is obtained by computing its value at $s = -1/2$. Writing the zeta function around $s = -1/2$,

$$
\zeta(s) = \frac{1}{R}\left[\frac{c}{s+\frac{1}{2}} + \phi + \mathcal{O}(s+\frac{1}{2})\right], \qquad (61)
$$

we see that the vacuum energy is given by

$$E_C = \frac{1}{2R}\phi. \tag{62}$$

At $s = -1/2$ one observes that the poles come from the gamma function in A_{-1},

Table 3. Values of ϕ for $(d+1)$-dimensional transversal p-forms

d	p	absolute boundary conditions	relative boundary conditions
2	0	$0.008891 + \frac{2\ \ln(a)}{315\ \pi}$	$0.02806 + \frac{2\ \ln(a)}{45\ \pi}$
	1	$0.1678 + \frac{16\ \ln(a)}{315\ \pi}$	$0.1678 + \frac{16\ \ln(a)}{315\ \pi}$
3	0	-0.001793	$-0.03537 - \frac{2213\ \ln(a)}{65536}$
	1	$0.3462 + \frac{1339\ \ln(a)}{32769}$	$-0.04881 - \frac{1631\ \ln(a)}{65536}$
	2	$-0.04881 - \frac{1631\ \ln(a)}{65536}$	
4	0	$-0.000945 - \frac{38\ \ln(a)}{45045\ \pi}$	$0.03054 + \frac{2344\ \ln(a)}{15015\ \pi}$
	1	$0.4677 + \frac{11048\ \ln(a)}{45045\ \pi}$	$0.01881 + \frac{6632\ \ln(a)}{45045\ \pi}$
	2	$-0.1749 - \frac{212\ \ln(a)}{3465\ \pi}$	$-0.1749 - \frac{212\ \ln(a)}{3465\ \pi}$
5	0	0.0002050	$-0.02312 - \frac{3118613\ \ln(a)}{50331648}$
	1	$0.5249 + \frac{871339\ \ln(a)}{8388608}$	$-0.02027 - \frac{1052991\ \ln(a)}{16777216}$
	2	$-0.3459 - \frac{1063379\ \ln(a)}{25165824}$	$0.05573 + \frac{31697\ \ln(a)}{1048576}$
	3	$0.05573 + \frac{31697\ \ln(a)}{1048576}$	

Table 4. Values of ϕ for p-forms on the unit sphere with absolute boundary conditions

d/p	5	4	3	2	1
2				0.1959	0.1767
3			−0.08417	0.2974	0.3444
4		0.04935	−0.1561	0.2928	0.4668
5	−0.04340	0.03546	−0.2902	0.1790	0.5251

from the gamma function in A_i, for $b = 0$ and $i = 1$, and from the zeta function in A_i, when $i = m + 2$. We perform a Laurent expansion around these poles and isolate the corresponding finite parts. The rest of the functions will only contribute to the finite part. The values $Z_{d+1}(-\frac{1}{2})$ and $Z^R_{d+1}(-\frac{1}{2})$ have to be computed numerically. These contributions are generally quite small compared with the finite part which comes from the sum $\sum_{i=-1}^{N} A_i$. By adjusting N, the values of $Z^{(R)}_{d+1}(-\frac{1}{2})$ can be further improved, allowing us to obtain the same accuracy with much less effort. For some values of d and p the sum over l converges very slowly. Use of Richardson extrapolation leads to a dramatical improvement of the convergence speed. This extrapolation is a general procedure of numerical analysis. It is here valid because the partial sum has the following asymptotic behavior:

$$\sum_{l=0}^{n} \nu^{j+1} \int_0^\infty \{\ldots\} \sim Q_0 + Q_1 n^{-1} + Q_2 n^{-2} + Q_3 n^{-3} + \ldots, n \to \infty. \tag{63}$$

The finite contributions for the $(d+1)$-transversal forms are listed in Table 3. We have included the scalar field, $p = 0$. When the argument of the zeta function is negative, the constant term for Neumann boundary conditions does not contribute. Special care of this term need therefore not be taken here. The coefficients belonging to $\ln(a)$ equal the heat-kernel coefficient $-\frac{B_{\frac{d}{2}+1}}{(4\pi)^{\frac{d+1}{2}}\sqrt{\pi}}$. Table 4 gives the vacuum energy for the unit sphere for all p-forms, for absolute boundary condition. For $p = 1$ we see that the energy increases with increasing d. For $p = 2$ there is a maximum at $d = 3$, while the energy for $p = 3$ and $p = 4$ decreases with d. For constant d there are actually less systematic trends.

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A. The Zeta Function Determinants

A.1 Absolute boundary conditions In this case, we have obtained

$$\begin{aligned}
-\ln \det{}^5_{3\perp}(\Delta) &= \frac{55120073}{64864800} - \frac{4}{3}\int_0^1 dy\, y^2\, \Psi(3+y) + \frac{1}{3}\int_0^1 dy\, y^4\, \Psi(3+y) \\
&\quad - \frac{29\ln 2}{3780} - \frac{215\ln 3}{2} + \frac{\ln 4}{8} + \frac{\zeta_R'(-5)}{6} - \frac{\zeta_R'(-4)}{12} \\
&\quad - \zeta_R'(-3) + \frac{7\zeta_R'(-2)}{12} + \frac{4\zeta_R'(-1)}{3} - \zeta_R'(0) \\
-\ln \det{}^5_{2\perp}(\Delta) &= \frac{-38814043}{64864800} + \frac{5179\ln 2}{3780} - \frac{3\ln 4}{8} + \frac{\zeta_R'(-5)}{6} \\
&\quad + \frac{\zeta_R'(-4)}{12} - \zeta_R'(-3) - \frac{7\zeta_R'(-2)}{12} + \frac{4\zeta_R'(-1)}{3} + \zeta_R'(0), \\
-\ln \det{}^5_{1\perp}(\Delta) &= \frac{75711793}{64864800} + \frac{8}{3}\int_0^{-1} dy\, y \ln\Gamma(3+y) - \frac{4}{3}\int_0^{-1} dy\, y^3 \ln\Gamma(3+y) \\
&\quad - \frac{3\ln(\frac{1}{2})}{2} - \frac{11869\ln 2}{3780} - + \frac{\zeta_R'(-5)}{12} + \frac{\zeta_R'(-4)}{8} \\
&\quad - \frac{\zeta_R'(-3)}{3} - \frac{5\zeta_R'(-2)}{8} - \frac{\zeta_R'(-1)}{4}, \\
-\ln \det{}^5_{0\perp}(\Delta) &= -\frac{7087979}{32432400} - \frac{1181\ln 2}{3780} \\
&\quad + \ln 3 + \frac{\zeta_R'(-5)}{60} + \frac{\zeta_R'(-4)}{24} - \frac{\zeta_R'(-2)}{24} - \frac{\zeta_R'(-1)}{60}
\end{aligned}$$

$$-\frac{1}{6}\int_0^2 dy\ (y-2)\ln\Gamma(y)+\frac{1}{3}\int_0^2 dy\ (y-2)^3\ln\Gamma(y),$$

$$-\ln\det{}^4_{2\perp}(\Delta)=-\frac{3411}{2560}+\frac{9}{4}\int_0^{\frac{1}{2}} dy\, y\Psi(\frac{5}{2}+y)-\int_0^{\frac{1}{2}} dy\, y^3\,\Psi(\frac{5}{2}+y)-\frac{45\ln 2}{64}$$
$$-\frac{15\,\zeta_R{}'(-4)}{32}+\frac{21\,\zeta_R{}'(-2)}{16},$$

$$-\ln\det{}^4_{1\perp}(\Delta)=\frac{10633}{11520}-\frac{9}{4}\int_0^{-\frac{1}{2}} dy\ \ln\Gamma(\frac{5}{2}+y)+3\int_0^{-\frac{1}{2}} dy\, y^2\ \ln\Gamma(\frac{5}{2}+y)+\ln(\frac{2}{3})$$
$$+\frac{581\ln 2}{2880}-\frac{5\,\zeta_R{}'(-4)}{16}-\frac{7\,\zeta_R{}'(-3)}{24}+\frac{5\,\zeta_R{}'(-2)}{8}+\frac{13\,\zeta_R{}'(-1)}{24},$$

$$-\ln\det{}^4_{0\perp}(\Delta)=-\frac{19261}{46080}-\frac{713\ln 2}{720}+\ln 5-\frac{5\,\zeta_R{}'(-4)}{64}$$
$$-\frac{7\,\zeta_R{}'(-3)}{48}-\frac{\zeta_R{}'(-2)}{32}+\frac{\zeta_R{}'(-1)}{48}$$
$$+\frac{1}{12}\int_0^{3/2} dy\ \ln\Gamma(y)-\int_0^{3/2} dy\ \left(y-\frac{3}{2}\right)^2\ln\Gamma(y),$$

$$-\ln\det{}^3_{2\perp}(\Delta)=-\frac{2081}{3360}+\int_0^1 dy\, y^2\,\Psi(2+y)-\frac{\ln 2}{20}-\frac{\ln 4}{8}+\zeta_R{}'(-3)-\frac{\zeta_R{}'(-2)}{2}$$
$$-\frac{3\,\zeta_R{}'(-1)}{2}+\zeta_R{}'(0),$$

$$-\ln\det{}^3_{1\perp}(\Delta)=\frac{5989}{10080}-\frac{83\ln 2}{60}+\frac{3\ln 4}{8}+\zeta_R{}'(-3)$$
$$+\frac{\zeta_R{}'(-2)}{2}-\frac{3\,\zeta_R{}'(-1)}{2}-\zeta_R{}'(0),$$

$$-\ln\det{}^3_{0\perp}(\Delta)=-\frac{493}{4320}+\frac{61\ln 2}{90}+\frac{\zeta_R'(-3)}{3}+\frac{\zeta_R'(-2)}{2}+\frac{\zeta_R'(-1)}{6}$$
$$+2\int_0^1 dy\ (y-1)\ln\Gamma(y),$$

$$-\ln\det{}^2_{1\perp}(\Delta)=\frac{19}{16}+2\int_0^{\frac{1}{2}} dy\ \ln\Gamma(\frac{3}{2}+y)+\frac{3}{4}\ln 2-\frac{3}{2}\zeta_R{}'(-2),$$

$$-\ln\det{}^2_{0\perp}(\Delta)=-\frac{7}{32}-\frac{7\ln 2}{6}+\ln 3-\frac{3\,\zeta_R{}'(-2)}{4}-\frac{\zeta_R{}'(-1)}{2}$$
$$-2\int_0^{1/2} dy\ \ln\Gamma(y). \tag{64}$$

A.2 Relative boundary conditions In this case, the results are

$$
\begin{aligned}
-\ln \det{}^5_{2\perp}(\Delta) = {} & \frac{55120073}{64864800} - \frac{4}{3}\int_0^1 dy\, y^2\, \Psi(3+y) + \frac{1}{3}\int_0^1 dy\, y^4\, \Psi(3+y) \\
& - \frac{29\ln 2}{3780} - \frac{215\ln 3}{2} + \frac{\ln 4}{8} + \frac{\zeta_R{}'(-5)}{6} - \frac{\zeta_R{}'(-4)}{12} - \zeta_R{}'(-3) \\
& + \frac{7\zeta_R{}'(-2)}{12} + \frac{4\zeta_R{}'(-1)}{3} - \zeta_R{}'(0), \\
-\ln \det{}^5_{1\perp}(\Delta) = {} & \frac{-11173163}{64864800} - \frac{1}{12}\int_0^2 dy\, y^2\, \Psi(3+y) + \\
& + \frac{1}{12}\int_0^2 dy\, y^4\, \Psi(3+y) - \frac{3\ln(\frac{1}{2})}{2} \\
& - \frac{8341\ln 2}{3780} + \frac{\zeta_R{}'(-5)}{12} - \frac{\zeta_R{}'(-4)}{8} - \frac{\zeta_R{}'(-3)}{3} \\
& + \frac{5\zeta_R{}'(-2)}{8} - \frac{\zeta_R{}'(-1)}{4}, \\
-\ln \det{}^5_{0\perp}(\Delta) = {} & -\frac{4027}{6486480} - \frac{1}{756}\ln 2 + \frac{1}{60}\zeta_R'(-5) \\
& - \frac{1}{24}\zeta_R'(-4) + \frac{1}{24}\zeta_R'(-2) - \frac{1}{60}\zeta_R'(-1), \\
-\ln \det{}^4_{2\perp}(\Delta) = {} & -\frac{3411}{2560} + \frac{9}{4}\int_0^{\frac{1}{2}} dy\, y\Psi(\frac{5}{2}+y) - \int_0^{\frac{1}{2}} dy\, y^3\, \Psi(\frac{5}{2}+y) - \frac{45\ln 2}{64} \\
& - \frac{15\zeta_R{}'(-4)}{32} + \frac{21\zeta_R{}'(-2)}{16}, \\
-\ln \det{}^4_{1\perp}(\Delta) = {} & \frac{17021}{11520} + \frac{1}{12}\int_0^{\frac{3}{2}} dy\, y\Psi(\frac{5}{2}+y) - \frac{1}{3}\int_0^{\frac{3}{2}} dy\, y^3\, \Psi(\frac{5}{2}+y) - \frac{2561\ln 2}{2880} \\
& - \frac{5\zeta_R{}'(-4)}{16} + \frac{7\zeta_R{}'(-3)}{24} + \frac{5\zeta_R{}'(-2)}{8} - \frac{13\zeta_R{}'(-1)}{24}, \\
-\ln \det{}^4_{0\perp}(\Delta) = {} & \frac{47}{9216} + \frac{17}{2880}\ln 2 + \frac{5}{64}\zeta_R'(-4) + \frac{7}{48}\zeta_R'(-3) \\
& - \frac{1}{32}\zeta_R'(-2) - \frac{1}{,}48\zeta_R'(-1), \\
-\ln \det{}^3_{1\perp}(\Delta) = {} & -\frac{2081}{3360} + \int_0^1 dy\, y^2\, \Psi(2+y) - \frac{\ln 2}{20} - \frac{\ln 4}{8} + \zeta_R{}'(-3) \\
& - \frac{\zeta_R{}'(-2)}{2} - \frac{3\zeta_R{}'(-1)}{2} + \zeta_R{}'(0), \\
-\ln \det{}^3_{0\perp}(\Delta) = {} & \frac{173}{30240} + \frac{1}{90}\ln 2 + \frac{1}{3}\zeta_R'(-3), \\
-\ln \det{}^2_{1\perp}(\Delta) = {} & \frac{19}{16} + 2\int_0^{\frac{1}{2}} dy\, \ln\Gamma(\frac{3}{2}+y) + \frac{3}{4}\ln 2 - \frac{3}{2}\zeta_R{}'(-2), \\
-\ln \det{}^2_{0\perp}(\Delta) = {} & -\frac{3}{32} - \frac{1}{12}\ln 2 - \frac{3}{4}\zeta_R'(-2) + \frac{1}{2}\zeta_R'(-1).
\end{aligned} \tag{65}
$$

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