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Supersymmetry and Localization

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Abstract: We study conditions under which an odd symmetry of the integrand leads to localization of the corresponding integral over a (super)manifold. We also show that in many cases these conditions guarantee exactness of the stationary phase approximation of such integrals.

1. Introduction

Localization formulae express certain integrals over (super)manifolds as sums of contributions of some subsets of these manifolds. Such formulae were studied in various contexts both in mathematics and physics. Some important examples of localization formulae are based on the theory of equivariant cohomology (see [3] for a review of the theory and its applications). One famous particular case is the Duistermaat–Heckman integration formula [5] which became a powerful (though not completely rigorous) tool in Quantum Field Theory (QFT). In the context of QFT, the Duistermaat–Heckman theorem gives sufficient conditions for exactness of semi-classical approximation of field theoretical models (see [4, 11] and references therein for a review of applications of Duistermaat–Heckman and some other localization formulae to QFT).

The aim of the present paper is to derive very general localization formulae in the framework of supergeometry. Namely, we consider an integral over a finite dimensional (super)manifold M, where the integrand is invariant under the action of an odd vector field Q. We formulate sufficient conditions on M and Q under which the integral localizes onto the zero locus of the number part of Q. It is important to stress that without the conditions below, the localization formula can be wrong. (Physicists often used the localization formulae without rigorous justification and without mentioning the conditions of applicability of these formulae).

One of the possible ways to apply the theorems of the present paper is based on the use of the Batalin–Vilkovisky [1] formalism where the calculation of physical

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quantities reduces to the calculation of integrals of functions invariant with respect to an odd vector field. However the conditions of our theorems are not always satisfied in this situation.

In the present paper we do not consider concrete examples of the application of our results. Let us mention only that they can be used, for instance, to calculate integrals of $OSp(n\,|\,m)$-invariant functions and obtain the statement about dimensional reduction in the Parisi–Sourlas model [7] and in similar situations [8].

To conclude the introduction let us define a technically convenient notion of compact vector field which will be used throughout the paper: we say that a vector field A on a (super)manifold M is a compact vector field if it generates an action of a one-parameter subgroup of some compact group G of transformations of M. In other words we assume that there exists a homomorphism φ_* of the Lie algebra $\mathscr{G}$ of the compact Lie group G into the Lie algebra Vect(M) of vector fields on M such that $A \in \operatorname{Im} \varphi_*$.

One can say also that the vector field A on M is compact if the closure G_A of the one-parameter group generated by A in Diff(M) is compact. Here Diff(M) denotes the group of diffeomorphisms of M equipped with the compact open topology (see [6]). It is easy to see that G_A is a commutative connected compact Lie group. Therefore it is isomorphic to a torus. The space of compact vector fields on M will be denoted by $\mathscr{K}(M)$.

Several definitions we use are explained in the Appendix. All manifolds, maps, functions considered in this paper are assumed to be smooth.

2. The Duistermaat–Heckman Formula in the Language of Supergeometry

First let us recall the conventional formulation of the Duistermaat–Heckman integration formula (see [3] or [4] for a review).

Let (W^{2n}, Ω) be a compact $2n$-dimensional symplectic manifold with a symplectic form $\Omega = \Omega_{ij}(x)dx^i dx^j$. Let $X \in \mathscr{K}(W)$ be a compact Hamiltonian vector field. Let us denote the corresponding Hamiltonian by H. The Duistermaat–Heckman theorem states that

$$\int_W \Omega^n e^{iH} = \text{sum of contributions of the zero locus of the vector field } X\,. \quad (1)$$

If in particular the zero locus R of X is a finite subset of W and all zeros of X are non-degenerate then

$$\int_W \Omega^n e^{iH} = i^n \sum_{p \in R} e^{(i\frac{\pi}{4}\operatorname{sgn} H(p))} \frac{e^{iH(p)}}{\sqrt{\det \operatorname{Hess} H(p)}}\,, \quad (2)$$

where $\operatorname{Hess} H(p)$ stands for the Hessian of H at the point p and $\operatorname{sgn} H(p)$ is the signature of $\operatorname{Hess} H(p)$.

To show the relation of the Duistermaat–Heckman theorem to supergeometry we notice, following [11], that the left-hand side of (1) can be rewritten as an integral over a supermanifold. Namely, it is easy to check that

$$\int_W \Omega^n e^{iH} = i^{-n} \int_{\Pi TW} \prod_{k=1}^{2n} dx^k d\xi^k e^{i(H(x)+\Omega_{ab}(x)\xi^a\xi^b)}\,, \quad (3)$$

where ΠTW denotes the total space of the tangent bundle over W with reversed parity of the fibers. In other words if $(x^1,\ldots,x^{2n})$ is a local coordinate system in W, then a local coordinate system in ΠTW consists of even coordinates $(x^1,\ldots,x^{2n})$ and odd coordinates $(\xi^1,\ldots,\xi^{2n})$; the coordinates $(\xi^1,\ldots,\xi^{2n})$ transform as a vector by the change of local coordinates $(x^1,\ldots,x^{2n})$.

A function on ΠTW can be identified with a differential form on W. Using this remark we can identify the exponential $S(x,\xi)\equiv H(x)+\Omega_{ij}(x)\xi^i\xi^j$ with the (inhomogeneous) differential form $H+\Omega$. Let us consider the vector field $Q=\xi^i\frac{\partial}{\partial x^i}+X^i(x)\frac{\partial}{\partial \xi^i}$. As an operator acting on forms on M, Q coincides with the "equivariant differential" $d+i_X$, where i_X denotes the contraction with the vector field X. It is easy to check that the function $S(x,\xi)$ on ΠTW satisfies $QS=0$. (The differential form $H+\Omega$ obeys $(d+i_X)(H+\Omega)=0$ because X is a Hamiltonian vector field with Hamiltonian H.)

We conclude from this that the Duistermaat–Heckman theorem can be reformulated in the following way:

$$\int_{\Pi TW} dVe^S = \text{sum of contributions of the zero locus of the vector field } Q\,. \quad (4)$$

Notice that Q is an odd vector field on ΠTW such that $Q^2\in\mathcal{K}(\Pi TW)$. Let us explain this fact. There exists a natural homomorphism of the algebra of vector fields on W into the algebra of vector fields on ΠTW. (An infinitesimal diffeomorphism of W induces an infinitesimal diffeomorphism of ΠTW.) This homomorphism transforms a vector field X into the Lie derivative L_X, considered as a vector field on ΠTW (recall that the functions on ΠTW can be considered as forms on W; therefore the Lie derivative can be regarded as a first order differential operator acting on functions on ΠTW). It is clear that when a vector field A on W generates an action of a subgroup of a compact Lie group, the corresponding vector field L_A also generates an action of a subgroup of the compact Lie group on ΠTW. We identified Q with $d+i_X$, therefore Q^2 is identified with $di_X+i_Xd=L_X$. Thus we conclude that $Q^2\in\mathcal{K}(\Pi TW)$.

In (4), dV stands for $\prod_{i=1}^{2n}dx^id\xi^i$, which is the canonical volume element on ΠTW. Note that this volume element is Q-invariant; i.e., the divergence of Q with respect to dV vanishes: $\mathrm{div}_{dV}Q=0$. (The notion of divergence is naturally generalized to the case of vector fields on an arbitrary supermanifold M by the formula $\int_M dV(Q\cdot f)=-\int_M dV(\mathrm{div}_{dV}Q)f$).

Equation (4) is equivalent to Eq. (1) by virtue of (3) and the one-to-one correspondence between sets of zeros of Q and zeros of X.

It is natural to conjecture that Eq. (4) remains correct if ΠTW is replaced with an arbitrary supermanifold M, Q is an odd vector field on M, dV is any Q-invariant volume form on M, and S stands for any Q-invariant function. We will see that this conjecture is essentially correct if $Q^2\in\mathcal{K}(M)$; i.e., it is compact.

3. Localization of Integrals over Supermanifolds

In this section we will formulate and prove several statements giving sufficient conditions under which an integral over a supermanifold M is localized on a certain subset of M.

In what follows M is a compact supermanifold with $\dim M = (n_+, n_-)$. For any vector field F on M we denote by $R_F \subset M$ the zero locus of its number part $m(F)$.

Theorem 1. *Let M be a compact supermanifold with a volume form dV. Let Q be an odd vector field on M which satisfies the following conditions:*

$$\text{(i)} \quad \operatorname{div}_{dV} Q = 0\,,$$

$$\text{(ii)} \quad Q^2 \in \mathscr{K}(M)\,.$$

Then for any neighborhood $U(R_Q)$ of R_Q in M there exists an even Q-invariant function g_0 which is equal to 1 on some neighborhood $O(R_Q) \subset U(R_Q)$ of R_Q and vanishes outside of $U(R_Q)$. For every Q-invariant function $h \in C(M)$ and every g_0 obeying the above conditions we have

$$\int_M dV h = \int_M dV g_0 \cdot h\,. \tag{5}$$

The proof of Theorem 1 can be deduced from the following.

Lemma 1. *There exists an odd Q^2-invariant function σ on M which satisfies $m(Q\sigma)(x) \neq 0$ if $x \notin R_Q$.*

Proof of Lemma 1. Let us begin with preliminary remarks concerning the structure of the supermanifolds. Consider an ordinary manifold N and a vector bundle α over N. Then we obtain a supermanifold from the total space of α by reversing the parity of fibers. This manifold will be denoted by $\Pi\alpha N$ (a particular case of this construction was used in Sect. 2). It is well known that every supermanifold can be obtained by means of this construction [2].

If we begin with a supermanifold M, then the construction of a bundle α over $N = m(M)$ with $M = \Pi\alpha(N)$ can be described in the following way: let us fix an atlas of the supermanifold M. Even local coordinates will be denoted by Latin letters, odd local coordinates will be denoted by Greek letters. Let us represent the transition functions from local coordinates $(x^1, \ldots, x^{n_+}; \xi^1, \ldots, \xi^{n_-})$ to local coordinates $(\tilde{x}^1, \ldots, \tilde{x}^{n_+}; \tilde{\xi}^1, \ldots, \tilde{\xi}^{n_-})$ as follows:

$$\tilde{x}^i = f^i(x) + \cdots\,, \tag{6}$$

$$\tilde{\xi}^\alpha = \phi^\alpha_\beta(x)\xi^\beta + \cdots\,, \tag{7}$$

where the omitted terms are of higher order with respect to the ξ's. The set of functions f^i can be considered as the set of transition functions between local coordinate systems on the body N of M. The functions ϕ^α_β are transition functions of a bundle α over N. One can prove that M is diffeomorphic to $\Pi\alpha(N)$. There exists also an invariant construction of the bundle α (which is often referred to as the conormal bundle, see [2]).

There is a natural homomorphism of the group $\operatorname{Diff}(M)$ of diffeomorphisms of M into the group of automorphisms of the bundle α (the existence of such a homomorphism follows immediately from the existence of an invariant construction of α). Even vector fields on M can be considered as infinitesimal diffeomorphisms of M. Therefore they generate infinitesimal automorphisms of the vector bundle α. In other words there exists a natural homomorphism of the Lie algebra of vector fields on M into the Lie algebra of infinitesimal automorphisms of the vector bundle α; the infinitesimal automorphism corresponding to the vector field A

will be denoted by $\bar{A}$. In local coordinates $(z) = (x^i, \xi^\alpha)$ on M,

$$Q = \sum_{i=1}^{n_+} a^i_\alpha(z)\xi^\alpha \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{n_-} b^\alpha(z)\frac{\partial}{\partial \xi^\alpha}, \tag{8}$$

where $a^i_\alpha(z) = a^i_\alpha(x) + \cdots$, $b^\alpha(z) = b^\alpha(x) + \cdots$. Here and below we denote the higher order terms in ξ's by $\cdots$. Also, $Q^2 = \sum_{i=1}^{n_+} k^i(z)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{n_-} l^\alpha_\beta(z)\xi^\beta \frac{\partial}{\partial \xi^\alpha}$, where $k^i(z) = k^i(x) + \cdots, l^\alpha_\beta(z) = l^\alpha_\beta(x) + \cdots$. The coefficients $k^i(z), l^\alpha_\beta(z)$ can be easily expressed in terms of $a^i_\alpha(z), b^\alpha(z)$, but we will not need explicit formulae. It follows from $[Q, Q^2] = 0$ that

$$k^i(x)\frac{\partial b^\alpha(x)}{\partial x^i} - l^\alpha_\beta(x)b^\beta(x) = 0\,. \tag{9}$$

It is easy to see that $b^\alpha(x)$ defines a section of the bundle α; by definition this section coincides with the number part of Q. The relation (9) shows that this section is invariant with respect to the infinitesimal automorphism $\overline{Q^2}$ of the bundle α. In the conditions of Theorem 1 we can assume without loss of generality that the group generated by Q^2 is dense in some compact group G of transformations of M. We conclude from this fact and from $\overline{Q^2}$-invariance of the section b^α that this section is also invariant with respect to the natural action of the group G on the bundle α.

Consider now any odd function σ on M. In local coordinates $\sigma(z) = \sigma_\alpha(x)\xi^\alpha +$ higher order terms in ξ's. It is clear therefore that σ determines a section of the vector bundle α^* dual to α. Let $g_{\alpha\beta}(x)$ be a G-invariant non-degenerate scalar product on the fibers of $\alpha(m(M))$, establishing an isomorphism $\alpha(m(M)) \cong \alpha^*(m(M))$. Such a scalar product always exists, since G is compact. Let us take an odd function σ on M such that

$$\sigma(z) = g_{\alpha\beta}(x)b^\alpha(x)\xi^\beta + \cdots\,.$$

One can assume that σ is G-invariant. (If σ is not G-invariant take its average with respect to G. As b^α and $g_{\alpha\beta}$ are G-invariant this operation does not change the terms linear in ξ's.) Therefore $Q^2\sigma = 0$ and $m(Q\sigma)(x) = g_{\alpha\beta}(x)b^\alpha(x)b^\beta(x) \neq 0$, whenever $x \notin R_Q$. This completes the proof of the lemma.

The lemma admits a simple corollary.

Let us introduce the notation

$$\beta = \frac{\sigma}{Q\sigma}, \tag{10}$$

where σ is the function constructed above. The odd function β is defined on the complement of R_Q in M and satisfies there the condition $Q\beta = 1$. (To check the last fact note that $(Q\sigma)\beta = \sigma$ and apply Q to both sides of this equation.)

Consider now an arbitrary neighborhood $U(R_Q)$ of R_Q. Using the function β one can construct a partition of unity on $M, \sum_{n\in J} g_n = 1$, which satisfies the following conditions:

i) $\operatorname{supp}(g_0) \subset U(R_Q), 0 \in J$,

ii) There exists a neighborhood $O(R_Q)$ of R_Q such that (11)

$O(R_Q) \subset U(R_Q)$ and $g_0 |_{O(R_Q)} = 1$,

iii) $Qg_n = 0, n \in J, g_n = Q\rho_n, n \neq 0$,

where the ρ_n's are some odd functions on M.

Proof. One can choose a finite atlas $\{U_n, n \in I\}$ of M such that $R_Q \subset \bigcup_{n\in I' \subset I} U_n \subset U(R_Q)$ and $O(R_Q) \cap (\bigcup_{n\in I\setminus I'} U_n) = \emptyset$. Now take a partition of unity on M, $\sum_{n\in I} f_n = 1$, such that $\text{supp}(f_n) \subset U_n, n \in I$. Without loss of generality we can consider this partition of unity to be G-invariant (otherwise take its average with respect to the action of group G). Therefore our partition of unity is also Q^2-invariant. The partition of unity having the desired properties is given by $\{g_n, n \in J\}$, where $J = (I\setminus I') \cup \{0\}, g_n = Q(\beta f_n), n \in I\setminus I', g_0 = 1 - \sum_{n\in I\setminus I'} g_n$. To prove this note that by construction $\sum_{n\in I\setminus I'} f_n(p) = 0$ for $p \in O(R_Q)$ and $\sum_{n\in I\setminus I'} f_n(p) = 1$ for $p \notin \bigcup_{n\in I'} U_n$. Then $\sum_{n\in I\setminus I'} Q(\beta f_n)(p) = 1$ for $p \notin \bigcup_{n\in I'} U_n$ and $\sum_{n\in I\setminus I'} Q(\beta f_n)(p) = 0$ for $p \in R$. Also, $Q(Q(\beta f_n)) = 0$, as $Q^2 f_n = 0, n \in I$.

Thus the existence of the Q-invariant partition of unity satisfying (11) is proved.

Now we are in position to complete the proof of Theorem 1. Namely, consider an integral $Z = \int_M dVh$, where h is any Q-invariant function on M. Using the Q-invariant partition of unity satisfying (11) one can rewrite an expression for Z in the following way:

$$Z = \sum_{n\in I\setminus I'} \int_M dVQ(\rho_n h) + \int_M g_0 h .$$

But $\text{div}_{dV} Q = 0$, therefore $\int_M dVQ(\rho_n h) = 0$ for all $n \in I\setminus I'$. So we conclude that

$$Z = \int_M dV g_0 h , \tag{12}$$

$$\text{supp}\,(g_0) \subset U(R_Q) \subset M, Qg_0 = 0, g_0 \,|_{O(R_Q)} = 1 . \tag{13}$$

The last thing to be proved is that the function g_0 entering the partition of unity can be replaced by any function obeying (13) without changing the value of the integral (12). Suppose there is an even function $\tilde{g}_0$ on M which obeys (x) with $O(R_Q)$ replaced by $\tilde{O}(R_Q), R_Q \subset \tilde{O}(R_Q) \subset U(R_Q)$. But then $(g_0 - \tilde{g}_0)\,|_{O(R_Q)\cap\tilde{O}(R_Q)} \equiv 0$, therefore $g_0 - \tilde{g}_0 = Q(\beta(g_0 - \tilde{g}_0))$, where β is defined by the formula (x). We arrive at the desired result by means of the following simple calculation:

$$\int_M dV g_0 h - \int_M dV \tilde{g}_0 h = \int_M dVQ(\beta(g_0 - \tilde{g}_0))h = \int_M dVQ(\beta h(g_0 - \tilde{g}_0)) = 0 .$$

This completes the proof of Theorem 1.

One can weaken the conditions on Q^2 in Theorem 1 in the following way. Notice that the condition $Q^2 \in \mathscr{K}(M)$ means that there exists a compact group $G \subset \text{Diff}(M)$ such that Q^2 can be represented in the form

$$Q^2 = \sum_{i=1}^{\dim G} p_i e_i, \qquad p_i \in \mathbf{R} , \tag{14}$$

where $\{e_i\}_{i=1}^{\dim G}$ is a basis of Lie algebra $\mathscr{G}$ of G. It is always possible to choose $G = G_{Q^2}$ (see the Introduction). Then the one parameter subgroup generated by Q^2 will be dense in G.

One can generalize Theorem 1, assuming that the coefficients p_i in (14) are arbitrary even functions on M. In the proof of Theorem 1 we used the fact that the p_i's are constants only once – when we proved that from $\overline{Q^2}$-invariance of the section b^α we can derive the invariance of this section with respect to the natural action of the group G on the bundle α. However the G-invariance of the section b^α can be proved if we know just that the vector field Q is G-invariant.

From this remark it becomes clear that one can prove the following statement:

Theorem 2. *Suppose that all conditions of Theorem* 1 *are satisfied except* (ii). *Impose instead the following conditions on* Q:

$$\text{a)}\quad Q^2 = \sum_{i=1}^{\dim G} p_i e_i; \text{ where } p_i \text{ are even functions on } M\,.$$

b) *The vector field* Q *on* M *is* G*-invariant.*

Then the conclusion of Theorem 1 *is true.*

It follows from Theorems 1 and 2 that the values of the function h on the complement to an arbitrary neighborhood of R_Q are irrelevant in the calculation of $\int_M dVh$. One can express this statement by saying that this integral is localized on R_Q.

Now let us discuss the localization of the integrals of functions invariant with respect to several anticommuting odd symmetries. We will prove the following.

Theorem 3. *Let* $\{Q_i\}_{i=1}^N$ *be odd anticommuting volume preserving vector fields on* M *(i.e.,* $\{Q_i, Q_j\} = 0$ *for* $i \neq j, \operatorname{div}_{dV} Q_i = 0, 1 \leqq i,j \leqq N$). *Let us assume that* $\{Q_i, Q_i\} \in \mathscr{K}(M)$ *and* $R_{Q_i} = R_{Q_i^2}, 1 \leqq i \leqq N$.

Then for every function h *on* M, *obeying* $Q_1 h = Q_2 h = \cdots = Q_N h = 0$, *the integral of* h *over* M *is localized on* $R_{Q_1} \cap R_{Q_2} \cap \cdots \cap R_{Q_N}$.

The precise meaning of the word "localized" is explained above.

To prove Theorem 3 let us notice that it follows from $\{Q_i, Q_j\} = 0$ that $[Q_i^2, Q_j^2] = 0$. Consider the group $G_i = [e^{tQ_i^2}]$ defined as the closure of the one-parameter subgroup $\{e^{tQ_i^2}\} \subset \operatorname{Diff}(M)$ in the compact open topology on $\operatorname{Diff}(M)$ (see Introduction). Taking into account that Q_i^2 commutes with Q_j^2 we see that the subgroups G_i and G_j of $\operatorname{Diff}(M)$ commute (we use the fact that the one-parameter subgroup $\{e^{tQ_i^2}\}$ is dense in the corresponding group G_i).

This means that the group $G = G_1 \times G_2 \times \cdots \times G_N$ acts in a natural way on M:

$$G \times M \to M$$
$$((g_1, \ldots, g_N), x) \mapsto g_1 \circ \cdots \circ g_N(x)\,.$$

Consider the odd vector field $Q = \sum_{i=1}^N c_i Q_i$, where $\{c_i\}_{i=1}^N$ is a set of real numbers.

It is easy to check that $Qh = 0, \operatorname{div}_{dV} Q = 0$ and $Q^2 = \sum_{i=1}^N c_i^2 Q_i^2$. Therefore, $Q^2 \in \mathscr{L}ie\ G$ and we conclude that $Q^2 \in \mathscr{K}(\mathscr{M})$.

One can choose $\{c_i\}_{i=1}^N$ in such a way that the one-parameter subgroup generated by Q^2 is dense in G. Then $R_{Q^2} = \bigcap_{i=1}^N R_{Q_i^2} = \bigcap_{i=1}^N R_{Q_i}$. Taking into account that $R_Q \subset R_{Q^2}$, we see that $R_Q \subset \bigcap_{i=1}^N R_{Q_i}$. But Q is a linear combination of Q_i's, therefore $\bigcap_{i=1}^N R_{Q_i} \subset R_Q$. Thus R_Q coincides with $\bigcup_{i=1}^N R_{Q_i}$. Using Theorem 1 we see that the integral $\int_M dVh$ is localized to $R_Q = \bigcap_{i=1}^N R_{Q_i}$. This proves Theorem 3.

To conclude the present section let us make the following remark. Theorems 1 and 2 state the localization of integrals of Q-invariant functions on the zero locus R_Q of the number part of Q. In the next section we will apply Theorem 1 to give conditions of exactness of the stationary phase approximation. We will see that under these conditions one can make the stronger statement that the integrals at hand are localized on the zero locus K_Q of the vector field Q.

4. Exactness of Stationary Phase Approximation

In the previous section we formulated sufficient conditions for the localization of integrals over supermanifolds. Here we will consider the problem of calculation of such integrals. Namely, we are going to describe a class of examples in which the integral can be *exactly* computed by means of the stationary phase method.

Throughout this section we consider the integral

$$Z = \int_M dV \cdot h ,$$

where h is a Q-invariant function on M and M, dV, Q satisfy conditions of Theorem 1. This means that our integral is localized on R_Q. Denote the zero locus of Q by K_Q. We restrict ourselves to the situations when K_Q is either a finite subset of M or a compact submanifold of M. Moreover, we will assume that the odd codimension of $K_Q \subset M$ is equal to its even codimension. In the case when K_Q is a finite subset of M the last restriction means that the even dimension of M is equal to its odd dimension.

Let us begin with one important remark which will be exploited throughout the rest of the paper. Suppose one can find an odd function σ on M such that $Q^2\sigma = 0$. Then $Q(e^{i\lambda Q\sigma}) = 0$ and therefore

$$\frac{d}{d\lambda}\int_M dV h e^{i\lambda Q\sigma} = \int_M dV Q(h e^{i\lambda Q\sigma}) = 0 .$$

(To conclude that the last integral is zero we used the fact that the volume element dV is Q-invariant; i.e., $\mathrm{div}_{dV} Q = 0$.) We see that $\int_M dV h e^{i\lambda Q\sigma}$ does not depend on λ. Therefore,

$$Z = \lim_{\lambda\to\infty}\int_M dV h e^{i\lambda Q\sigma} . \tag{15}$$

In what follows we will use (15) with the function σ which was constructed in the proof of Lemma 1.

First we will compute Z for the case when $K_Q \subset M$ is finite. As was mentioned above we assume in this situation that the even dimension n_+ of M is equal to its odd dimension n_-. It follows from the condition of finiteness that actually $K_Q \subset m(M)$. (To prove this, notice that if the point $p \in M$ having local coordinates $(x(p), \xi(p))$ belongs to K_Q, then any point with coordinates $(x(p), c\xi(p))$, where c is a real number, also does.)

Let $\{z = (x, \xi)\}$ be a local coordinate system on M centered at a fixed point p such that $Q(p) = 0$. In such coordinates

$$Q = (b^\alpha(x) + b^\alpha_{\beta\gamma}(x)\xi^\beta\xi^\gamma + \cdots)\frac{\partial}{\partial\xi^\alpha} + (c^i_\alpha(x)\xi^\alpha + \cdots)\frac{\partial}{\partial x^i}\,, \tag{16}$$

$$b^\alpha(0) = 0\,,$$

where "..." denotes as always higher order terms in ξ's. We call this zero non-degenerate if $\det(\frac{\partial b^\alpha}{\partial x^i}(0)) \neq 0$, $\det(c^i_\alpha(0)) \neq 0$ (recall that $n_+ = n_-$).

Consider the odd function σ constructed in the proof of Lemma 1:

$$\sigma(x, \xi) = g_{\alpha\beta}(x)b^\alpha(x)\xi^\beta + \cdots\,. \tag{17}$$

Notice that $Q\sigma(x, \xi) = g_{\alpha\beta}(x)b^\alpha(x)b^\beta(x) + d_{\alpha\beta}(x)\xi^\alpha\xi^\beta + \cdots$, where $d_{\alpha\beta}(x)$ is some matrix. The condition $Q(Q\sigma) = 0$ leads to the following relation between $d_{\alpha\beta}(0)$ and $b_{ij}(0) \equiv \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}(g_{\alpha\beta}b^\alpha b^\beta)(0)$:

$$\frac{\partial b^\alpha}{\partial x^j}(0)d_{\alpha\beta}(0) + c^i_\beta(0)b_{ij}(0) = 0\,. \tag{18}$$

From (18), the nondegeneracy of the zero of the vector field Q at p and the nondegeneracy of scalar product g it follows that the matrix $d_{\alpha\beta}(0)$ is also non-degenerate. Therefore the point $p \in K_Q$ is a non-degenerate isolated critical point of the function $Q\sigma$. Consequently, there exists a neighborhood $U(p)$ of the point p such that p is the only critical point of $Q\sigma$ restricted to $U(p)$. It also follows from (18) that

$$\text{sdet}(\text{Hess}Q\sigma(p)) = \text{sdet}(-IQ'(p))\,, \tag{19}$$

where $I = \text{antidiag}(1, 1, \ldots, 1)$ is $2n \times 2n$ matrix, $Q'(p)$ is the (super)matrix of first derivatives of coefficient functions of operator Q at the point p. Let us note that the condition of non-degeneracy given above can be reformulated in terms of Q': a point p is a non-degenerate zero of Q if $IQ'(0)$ is non-degenerate as a supermatrix. Finally, the answer for Z can be obtained by means of the following calculation:

$$\begin{aligned} Z &\equiv \int_M dVh \overset{\text{Thm.1}}{=} \int_M dVg_0h \overset{(15)}{=} \lim_{\lambda\to\infty} \int_M dVg_0he^{i\lambda Q\sigma} \\ &= \sum_{p\in K_Q} \frac{\rho(p)h(p)}{\sqrt{\text{sdet}(\text{Hess}(Q\sigma)(p))}} \overset{(19)}{=} \sum_{p\in K_Q} \frac{\rho(p)h(p)}{\sqrt{\text{sdet}(-IQ'(p))}}\,, \end{aligned} \tag{20}$$

where $\text{supp}(g_0) \subset \bigcup_{p\in K_Q} U(p)$, $g_0\,|_{K_Q} = 1$ and $U(p)$ are chosen in such a way that the intersection of the critical set of σ with $\bigcup_{p\in K_Q} U(p)$ is just K_Q; $\rho(p)$ is the volume density at p. The third equality in (20) is due to the formula generalizing the stationary phase approximation to the supercase (see e.g. [9] for details).

Let us make a few remarks about (20). First, as it should be, the final answer for Z doesn't depend on the choice of the non-degenerate inner product $g_{\alpha\beta}$ in $\alpha(m(M))$. Second, notice that (20) states that Z is represented as a sum of contributions of the zeros of Q; this is a stronger statement than the one given by the general theorems of the previous section.

Finally, consider the integrand of Z in the form $h = e^{iS}$, where S is an even Q-invariant function. It is easy to check that K_Q is contained in the critical set of

S. (This fact will be proved below in a more general situation.) Suppose also that Hess(S) is non-degenerate at every $p \in K_Q$. Then one can rewrite (20) as follows:

$$Z = \sum_{p \in K_Q} \frac{\rho(p) e^{iS(p)}}{\sqrt{\operatorname{sdet}(\operatorname{Hess}(S)(p))}} . \tag{21}$$

The stationary phase approximation leads to precisely such an expression for Z, but with p running through the critical set K_S of S. Formula (21) means that the stationary approximation is exact and the points of $K_S \backslash K_Q$ do not contribute to Z.

Now we will discuss more general conditions on h and Q leading to the exactness of the stationary phase approximation for Z.

Our considerations are based on the following statement:

Lemma 2. *Let h be a Q-invariant function on M such that $h|_{K_Q} = 0$. Let us assume that K_Q is a compact submanifold of M and suppose Q is non-degenerate on K_Q; i.e., the supermatrix $\partial_\perp Q(p)$ of transversal derivatives of Q at every $p \in K_Q$ is invertible. Then*

$$\int_M dVh = 0 . \tag{22}$$

Let us introduce the following system of local coordinates in a neighborhood of K_Q: let $\{x^i; \xi^\alpha\} = \{x^{i'}, x^{i''}; \xi^{\alpha'}, \xi^{\alpha''}\}$, $1 \leqq i \leqq n_+$, $1 \leqq \alpha \leqq n_-$, $1 \leqq i' \leqq n'_+$, $\leqq \alpha' \leqq n'_-$, $1 \leqq i'' \leqq n''_+$, $1 \leqq \alpha'' \leqq n''_-$, $n_+ = n'_+ + n''_+, n_- = n'_- + n''_-$, be such that K_Q is singled out by the equations $x' = 0$, $\xi' = 0$. In other words the indices labeled by $'$ are related to transversal directions and tangent indices are labeled by $''$. Note that by our assumption $n'_+ = n'_-$. Let us also introduce cumulative notations for even and odd local coordinates in the vicinity of $K_Q : \{z^{I'}\} = \{x^{i'}, \xi^{\alpha'}\}, \{z^{I''}\} = \{x^{i''}, \xi^{\alpha''}\}$. In this notation K_Q is singled out by the equation $z' = 0$.

We begin the proof with the following remark. Consider any Q-invariant function S on M which is locally constant on K_Q (i.e., it is constant on each connected component of K_Q). Then $K_Q \subset K_S$, where K_S is the critical set of S. To see this it is sufficient to present S in the vicinity of K_Q as a power series in (z')'s and impose the condition $QS = 0$. As a consequence of the nondegeneracy of Q on K_Q we will obtain $\frac{\partial S}{\partial z^{I'}}|_{K_Q} = 0$. Also $\frac{\partial S}{\partial z^{I''}}|_{K_Q} = 0$ as S is locally constant on K_Q. Therefore the inclusion $K_Q \subset K_S$ is proved.

Consider now the odd function σ constructed in Lemma 1. By construction $Q\sigma|_{K_Q} = 0$, $Q(Q\sigma) = 0$. Therefore in the vicinity of K_Q we have

$$Q\sigma(z'', z') = S_{I',J'}(z'') z^{I'} z^{J'} + \text{higher order terms in } (z')\text{'s} . \tag{23}$$

To deduce the statement of Lemma 2 we notice that

$$\int_M dV \cdot h \overset{\text{Thm.1}}{=} \int_M dV \cdot g_0 h \overset{(15)}{=} \lim_{\lambda \to \infty} \int_M dV \cdot g_0 h e^{i\lambda Q\sigma} . \tag{24}$$

The last expression in (24) is equal to zero; we obtain this result from the standard formula for the stationary phase approximation. It is important to stress that such a formula can be applied because (as we are going to prove below) the Hessian of $\lambda Q\sigma$ in the directions transverse to K_Q is non-degenerate. We don't need an exact formula for stationary phase approximation; we use only that the leading term in this approximation is of order λ^0 (because $n'_+ = n'_-$) and that the answer is proportional to $h|_{K_Q} = 0$.

So, let us prove the nondegeneracy of the Hessian of $\lambda Q\sigma$ in the directions transversal to K_Q. At the point of K_Q with coordinates z'' this Hessian coincides with the supermatrix $S_{I',J'}(z'')$ entering (23). First we will prove that the even-even and odd-odd parts of $S_{I',J'}$ are degenerate or non-degenerate simultaneously. Then we will show that the even-even part of $S_{I',J'}$ is non-degenerate.

In the vicinity of K_Q we have

$$Q = a^{I'}_{J'}(z'',z')z^{J'}\frac{\partial}{\partial z^{I'}} + b^{I''}_{J'}(z'',z')z^{J'}\frac{\partial}{\partial z^{I''}}\,. \tag{25}$$

The nondegeneracy of Q in a neighborhood of K_Q means that the matrices $a^{i'}_{\lambda'}(z'',0)$, $a^{\lambda'}_{i'}(z'',0)$ are invertible. The condition $Q(Q\sigma) = 0$ leads in particular to the following condition on $S_{I',J'}$:

$$\begin{aligned} &a^{i'}_{j'}(z'',0)S_{i',\gamma}(z'') + a^{\delta'}_{\gamma'}(z'',0)S_{j',\delta'}(z'') \\ &+a^{\delta'}_{j'}(z'',0)S_{\delta',\gamma'}(z'') + a^{i'}_{\gamma'}(z'',0)S_{i',j'}(z'') = 0\,. \end{aligned} \tag{26}$$

We will need a corollary of (26) for the number parts of matrices which enter it. The odd matrices $a^{i'}_{j'}(z'',0)$ and $a^{\delta'}_{\gamma'}(z'',0)$ have zero number parts, therefore it follows from (26) that

$$m(a^{\delta'}_{j'}(z'',0))m(S_{\delta',\gamma'}(z'')) + m(a^{i'}_{\gamma'}(z'',0))m(S_{i',j'}(z'')) = 0\,. \tag{27}$$

From the invertibility of $a^{i'}_{\lambda'}(z'',0)$, $a^{\lambda'}_{i'}(z,0)$, it follows that $\det(m(a^{i'}_{\lambda'}(z'',0))) \neq 0$, $\det(m(a^{\lambda'}_{i'}(z'',0))) \neq 0$. Therefore, in view of (27), the matrices $m(S_{\delta',\gamma'}(z''))$, $m(S_{i',j'}(z''))$ are singular or non-singular simultaneously. We will prove that $\det(m(S_{i',j'}(z''))) \neq 0$. Then by (27) $\det(m(S_{\delta',\gamma'}(z''))) \neq 0$. Invertibility of the number parts of matrices depending on odd variables leads to the invertibility of matrices themselves; therefore we conclude from the above considerations that the matrices $S_{i'j'}(z'')$, $S_{\delta'\gamma'}(z'')$ are invertible.

So it remains to prove invertibility of the number part $m(S_{i',j'}(z''))$ of the even-even block $S_{i'j'}(z'')$ for all z'''s. By construction, $m(Q\sigma(x,\xi)) = g_{\alpha\beta}(x)b^\alpha(x)b^\beta(x)$, where g is a $\overline{Q^2}$-invariant scalar product in $\alpha(m(M))$. Therefore

$$m(S_{i',j'}(z'')) = g_{\alpha\beta}(x'')b^\alpha_{i'}(x'')b^\beta_{j'}(z''), b^\alpha_{i'}(x'') = \frac{\partial}{\partial x^{i'}}b^\alpha(x)\bigg|_{x'=0}\,. \tag{28}$$

It follows from $L_{\overline{Q^2}}g = 0$ that

$$(L_{\overline{Q^2}}g)_{\alpha\beta}(x) = \left(k^i\frac{\partial}{\partial x^i}g_{\alpha\beta} + g_{\gamma\beta}l^\gamma_\alpha + g_{\gamma\alpha}l^\gamma_\beta\right)(x) = 0\,, \tag{29}$$

where k^i, l^β_α are the coefficients of Q^2 introduced in the text following (8). Consider (29) in such coordinates in the vicinity of the point $(z'',0)$ that $g_{\alpha\beta}(z'',0) = \delta_{\alpha\beta}$.

Then (29) written at the point $(z'',0) \in K_Q$ gives

$$a_\delta^{i'} \frac{\partial}{\partial x^{i'}} b^\gamma + a_\gamma^{i'} \frac{\partial}{\partial x^{i'}} b^\delta = 0 .$$

Noticing that $Q|_{K_Q} = 0$, we get $a_{\delta''}^{i'}(z'',0) = 0$ for all tangent indices δ''. Then it follows from the last equation that $a_{\gamma'}^{i'} b_{i'}^{\delta''}|_{K_Q} = 0$. But $\det(a_{\gamma'}^{i'})|_{K_Q}(x) \neq 0$ by nondegeneracy of Q in the vicinity of K_Q. Therefore in the given coordinates $b_{i'}^{\delta''}|_{K_Q} = 0$. Therefore by (28) $m((Q\sigma)_{i',j'})|_{K_Q} = (b_{i'}^{k'} \cdot b_{j'}^{k'})|_{K_Q}$, which is non-singular (recall that nondegeneracy of Q means that $b_{i'}^{k'}$ is non-singular). Therefore Lemma 2 is proved.

Lemma 2 admits a simple corollary which is important for our further considerations:

Lemma 3. *Let S be an even Q-invariant function, where Q satisfies all conditions of Lemma 2. Suppose $S|_{K_Q} = 0$. Then $\int_M dVe^{i\beta S}$ does not depend on the parameter β and can be calculated therefore according to the following formula*:

$$\int_M dV = \lim_{\beta\to\infty} \int_M dVe^{i\beta S} . \tag{30}$$

Really, since $S|_{K_Q} = 0, \frac{d}{d\beta} \int_M dVe^{i\beta S} = \int_M dVSe^{i\beta S} = 0$ in accordance with Lemma 2 applied to the integral Z with the integrand $h = Se^{i\beta S}$.

Before proceeding further let us explain what we mean by saying that the stationary phase approximation for the integral $\int_M dVe^{i\beta S}$ is exact. Consider an asymptotical expansion in powers of β^{-1} for such an integral. Let us present the critical set of S as a union of level sets of S on K_S; $K_S = K_S^{(1)} \cup K_S^{(2)} \cup \cdots \cup K_S^N$; i.e., $S_i \equiv S|_{K_S^i}$ is constant and $S_i \neq S_j$ if $i \neq j$ (each set $K_S^{(i)}$ is a union of connected components). Then the asymptotical expansion of the integral at hand takes the form

$$\int_M dVe^{i\beta S} = \sum_{i=1}^N e^{i\beta S_i}(c_{-\Delta}^i\beta^\Delta + c_{-\Delta+1}\beta^{-\Delta+1} + \cdots + c_0^i + c_1^i\beta^{-1} + \cdots) , \tag{31}$$

where Δ is some number equal to the difference between the odd and even codimensions of K_S in the case when the latter is a submanifold of M; c's are some constants. We say that the stationary phase approximation for the integral is exact if the equation $\int_M dVe^{i\beta S} = \sum_{j=1}^N e^{i\beta S_j} c_0^j$ is satisfied (in particular this means that $c_k^i = 0$ for $1 \leqq i \leqq N$ and $k \neq 0$).

Based on Lemma 3 one can easily prove the following theorem:

Theorem 4. *Let S be an even Q-invariant function on M which is locally constant on K_Q. Suppose Q is non-degenerate in a neighborhood of K_Q. Then the stationary phase approximation of the integral $\int_M dVe^{i\beta S}$ is exact.*

We proved already that under the conditions of Theorem 4, $K_Q \subset K_S$. Let us decompose K_Q into a union of level sets $K_Q = \bigcup_{j=1}^L K_Q^j, L \leqq N$. In other words $K_Q^j = K_S^j \cap K_Q$.

Using Theorem 1 let us present the integral under consideration in the form

$$\int_M dVe^{i\beta S} = \sum_{j=1}^{L} e^{i\beta S_j} \int_M dVg_0^{(j)} e^{i\beta(S-S_j)}\,, \tag{32}$$

where $S_j \equiv S\,|_{K_Q^j}$ and $g_0^{(j)}$ is an even Q-invariant function from Theorem 1 having support in some neighborhood of K_Q^j. By virtue of Lemma 3, the integrals in the right-hand side of (32) do not depend on β, since $S - S_j$ vanishes on K_Q^j.

Therefore the stationary phase approximation for the integral $\int_M dVe^{i\beta S}$ is exact. This proves Theorem 4.

We proved even a stronger result: the coefficients c_k^i in the decomposition (31) vanish if $K_S^j \cap K_Q$ is empty. This means that the asymptotical expansion for $\int_M dVe^{\beta S}$ receives a contribution only from part of the critical set K_S, namely from the set K_Q.

Let us also notice that all statements in the present section can be generalized to the case when K_Q is not a submanifold of M, but is a union of compact submanifolds, not necessarily of the same dimension.

5. Appendix

In this paper we utilize more or less the standard terminology of supergeometry (see for example [2, 10]). We will use the definition of an $(m\,|\,n)$-dimensional supermanifold as an object obtained from domains in $(m\,|\,n)$-dimensional superspace $R^{(m\,|\,n)}$ pasted together by means of invertible maps. (See for example [10] for a more precise definition.) The body $m(M)$ of the supermanifold M can be identified with the submanifold of M singled out by the equations $\xi^1 = \xi^2 = \cdots = \xi^n = 0$, where $\xi^1, \xi^2, \ldots, \xi^n$ are odd coordinates. This condition is independent of the choice of coordinate system because we do not consider families of supermanifolds and therefore the transition functions between different local coordinate systems do not depend on external odd parameters.

With every $(m\,|\,n)$-dimensional supermanifold M one can associate an n-dimensional vector bundle over $m(M)$, the so-called conormal bundle (see [2] for an invariant definition and Sect. 3 for the coordinate construction). For every function F on a supermanifold M we can consider its number part $m(F)$ as a restriction of F to the body $m(M) \subset M$. If A is an even vector field then one can define its number part as a vector field on $m(M)$. If in local coordinates the vector field A corresponds to a first order differential operator

$$A = A^i(x,\xi)\frac{\partial}{\partial x^i} + A^\alpha(x,\xi)\frac{\partial}{\partial \xi^\alpha}\,,$$

then its number part corresponds to an operator

$$m(A) = A^i(x,0)\frac{\partial}{\partial x^i}\,.$$

The number part of an odd vector field Q is defined as a section of the conormal bundle. If in local coordinates

$$Q = \kappa^i(x,\xi)\frac{\partial}{\partial x^i} + q^\alpha(x,\xi)\frac{\partial}{\partial \xi^\alpha},$$

then this section is specified by means of functions $q^1(x,0), q^2(x,0), \ldots, q^n(x,0)$.

Let us finally mention that we denote by sdet M the superdeterminant (Berezinian) of a supermatrix M. The supermatrix M is invertible iff the even-even and odd-odd blocks of M are non-singular in the usual sense; then sdet M exists and its number part does not vanish.

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