Commun. Math. Phys. 183, 423–461 (1997)

Communications in Mathematical Physics

Extensions of the Matrix Gelfand–Dickey Hierarchy from Generalized Drinfeld–Sokolov Reduction

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Received: 10 April 1995 / Accepted: 11 July 1996

Abstract: The $p \times p$ matrix version of the r-KdV hierarchy has been recently treated as the reduced system arising in a Drinfeld–Sokolov type Hamiltonian symmetry reduction applied to a Poisson submanifold in the dual of the Lie algebra $\widehat{gl}_{pr} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. Here a series of extensions of this matrix Gelfand–Dickey system is derived by means of a generalized Drinfeld–Sokolov reduction defined for the Lie algebra $\widehat{gl}_{pr+s} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ using the natural embedding $gl_{pr} \subset gl_{pr+s}$ for s any positive integer. The hierarchies obtained admit a description in terms of a $p \times p$ matrix pseudo-differential operator comprising an r-KdV type positive part and a non-trivial negative part. This system has been investigated previously in the $p = 1$ case as a constrained KP system. In this paper the previous results are considerably extended and a systematic study is presented on the basis of the Drinfeld–Sokolov approach that has the advantage that it leads to local Poisson brackets and makes clear the conformal ($\mathcal{W}$-algebra) structures related to the KdV type hierarchies.

0. Introduction

This paper is a continuation of [1], where it was shown how the matrix Gelfand–Dickey hierarchy [2, 3] fits into the Drinfeld–Sokolov approach [4] (see also [5–9]) to generalized KdV hierarchies.

The phase space of the matrix Gelfand–Dickey hierarchy is the space of $p \times p$ matrix Lax operators

$$L_{p,r} = P\partial^r + u_1\partial^{r-1} + \cdots + u_{r-1}\partial + u_r, \quad u_i \in C^\infty(S^1, gl_p), \tag{0.1}$$

where P is a diagonal constant matrix with distinct, non-zero entries. This phase space has two compatible Poisson brackets: the linear and quadratic matrix Gelfand–Dickey Poisson brackets. The Hamiltonians generating a commuting hierarchy of bihamiltonian flows are obtained from the residues of the componentwise fractional

★ On leave from Theoretical Physics Department of Szeged University, H-6720 Szeged, Hungary powers of the $p \times p$ diagonal matrix pseudo-differential operator $\hat{L}_{p,r}$ determined by diagonalizing $L_{p,r}$ in the algebra of matrix pseudo-differential operators. This system arises from a Drinfeld–Sokolov type Hamiltonian symmetry reduction applied to a Poisson submanifold in the dual of the Lie algebra $\widehat{gl}_{pr} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ – where $\widehat{gl}_{pr}$ is the central extension of the loop algebra $\widetilde{gl}_{pr} = C^\infty(S^1, gl_{pr})$ – endowed with the family of compatible Poisson brackets and commuting Hamiltonians provided by the r-matrix (AKS) construction (see e.g. [10]). The corresponding reduced phase space is identified with the set of first order matrix differential operators $\mathscr{L}_{p,r}$ of the form

$$\mathscr{L}_{p,r} = \mathbf{1}_{pr}\partial + j_{p,r} + \Lambda_{p,r}\,, \tag{0.2}$$

where $j_{p,r} \in C^\infty(S^1, gl_{pr})$ and $\Lambda_{p,r} \in gl_{pr} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ are written as $r \times r$ matrices with $p \times p$ matrix entries as follows:

$$j_{p,r} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ v_r & \cdots & \cdots & v_1 \end{pmatrix}, \qquad \Lambda_{p,r} = \begin{pmatrix} 0 & \Gamma & 0 & \cdots & 0 \\ \vdots & 0 & \Gamma & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & \Gamma \\ \lambda\Gamma & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{0.3}$$

where Γ is a $p \times p$ diagonal constant matrix for which $P = (-\Gamma^{-1})^r$; $\mathbf{1}_{pr}$ is the $pr \times pr$ identity matrix. The correspondence between $L_{p,r}$ in (0.1) and $\mathscr{L}_{p,r}$ in (0.2) is established through the relation

$$\mathscr{L}_{p,r}(\psi_1^t, \psi_2^t, \ldots, \psi_r^t)^t = 0 \Leftrightarrow L_{p,r}\psi_1 = \lambda\psi_1\,, \tag{0.4}$$

where the ψ_i are p-component column vectors, yielding $u_i = \Delta v_i \Delta^{r-i}$ with $\Delta := -\Gamma^{-1}$.

The purpose of the present paper is to derive a series of extensions of the above system using the natural embedding of the Lie algebra gl_{pr} into gl_{pr+s} for any positive integer s. This embedding is given by writing the general element $m \in gl_{pr+s}$ in the block form

$$m = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{0.5}$$

where $A \in gl_{pr}$ is written as an $r \times r$ matrix with $p \times p$ matrix entries, D is an $s \times s$ matrix and B (respectively C) is an r-component column (row) vector with $p \times s$ ($s \times p$) matrix entries. In particular, the image of $\Lambda_{p,r}$ under this embedding is $\Lambda_{p,r,s} \in gl_{pr+s} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$,

$$\Lambda_{p,r,s} := \begin{pmatrix} \Lambda_{p,r} & 0 \\ 0 & \mathbf{0}_s \end{pmatrix}. \tag{0.6}$$

A generalized Drinfeld–Sokolov reduction based on the Lie algebra $\widehat{gl}_{pr+s} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ will be defined in such a way that the matrix Gelfand–Dickey system is recovered when setting all fields outside the $gl_{pr} \subset gl_{pr+s}$ block to zero. The corresponding reduced phase space will turn out to be the set of first order matrix differential operators $\mathscr{L}_{p,r,s}$ of the form

$$\mathscr{L}_{p,r,s} = \mathbf{1}_{pr+s}\partial + j_{p,r,s} + \Lambda_{p,r,s}\,, \tag{0.7}$$

where $j_{p,r,s}$ reads

$$j_{p,r,s} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ v_r & \cdots & \cdots & v_1 & \zeta_+ \\ \zeta_- & 0 & \cdots & 0 & w \end{pmatrix} . \tag{0.8}$$

The dynamical variables encoded in $j_{p,r,s}$ reduce to those in $j_{p,r}$ (0.3) upon setting the fields $\zeta_\pm, w$ to zero. The phase space whose points are the operators $\mathscr{L}_{p,r,s}$ carries two compatible local Poisson brackets, which are naturally induced by the reduction procedure, and an infinite family of commuting local Hamiltonians defined by the local monodromy invariants of $\mathscr{L}_{p,r,s}$. The "second" Poisson bracket can be identified as a classical $\mathscr{W}$-algebra (see e.g. [11]). This phase space can be mapped into the space of $p \times p$ matrix pseudo-differential operators with the aid of the usual elimination procedure:

$$\mathscr{L}_{p,r,s}(\psi_1^t, \psi_2^t, \ldots, \psi_r^t, \phi^t)^t = 0 \Leftrightarrow L_{p,r,s}\psi_1 = \lambda\psi_1 , \tag{0.9}$$

which yields

$$L_{p,r,s} = P\partial^r + u_1\partial^{r-1} + \cdots + u_{r-1}\partial + u_r + z_+(\mathbf{1}_s\partial + w)^{-1}z_- , \tag{0.10}$$

where u_i is related to v_i as in (0.4) and $z_+ = \Gamma^{-1}\zeta_+, z_- = \zeta_-$. In contrast to the standard case, the operator $L_{p,r,s}$ attached to $\mathscr{L}_{p,r,s}$ is now not a differential operator but contains a non-trivial negative part, and the mapping $\mathscr{L}_{p,r,s} \mapsto L_{p,r,s}$ is not a one-to-one mapping. In fact, this mapping corresponds to factoring out a residual $\widehat{gl}_s$ symmetry of the hierarchy resulting from the generalized Drinfeld–Sokokov reduction, which is generated by the current $w \in C^\infty(S^1, gl_s)$ through the second Poisson bracket.

Our main results are the following:

First, we shall prove that the set of operators $L_{p,r,s}$ is a Poisson submanifold in the space of $p \times p$ matrix pseudo-differential operators with respect to the compatible Gelfand–Dickey Poisson brackets, and that the mapping $\mathscr{L}_{p,r,s} \mapsto L_{p,r,s}$ is a Poisson mapping from the bihamiltonian manifold obtained as the reduced phase space in the Drinfeld–Sokolov reduction onto this Poisson submanifold. We shall also present the explicit form of the Poisson brackets on the reduced phase space.

Second, we shall show that the mapping $\mathscr{L}_{p,r,s} \mapsto L_{p,r,s}$ converts the commuting Hamiltonians determined by the local monodromy invariants of $\mathscr{L}_{p,r,s}$ into the Hamiltonians generated by the residues of the componentwise fractional powers of the diagonalized form $\hat{L}_{p,r,s}$ of $L_{p,r,s}$.

Third, we shall derive a "modified" version of the generalized KdV hierarchy carried by the manifold of operators $\mathscr{L}_{p,r,s}$, which via the mapping $\mathscr{L}_{p,r,s} \mapsto L_{p,r,s}$ corresponds to an interesting factorization of $L_{p,r,s}$ (given in Eq. (4.18)). One of the factors in this factorization (the factor ΔK in (4.18)) arises also independently in the $r = 1$ case of our construction, when there is no Drinfeld–Sokolov reduction and we are dealing with a generalized AKNS hierarchy.

In the $p = 1$, $s = 1$ special case, the system based on the Lax operator $L_{p,r,s}$ (0.10) has been considered in several recent papers (see [12–14] and references therein) from the point of view of constrained KP hierarchies. Specifically, the system considered in the literature may be obtained from (0.10) by setting $w = 0$.

Setting $w = 0$ is consistent with the flows of the hierarchy, but has the inconvenient feature that the resulting Dirac brackets turn out to be *non-local* for the second Hamiltonian structure. It is interesting that one can in this way recover the second Hamiltonian structure *postulated* by Oevel and Strampp [13], for $p = s = 1$, as a non-local reduction of the local second Hamiltonian structure that automatically results from the Hamiltonian reduction in the Drinfeld–Sokolov approach. The Drinfeld–Sokolov approach to these systems that we shall present leads to a systematic understanding and for this reason we think it is of interest in its own right. This approach has clear advantages in that it leads to local Poisson brackets and it incorporates the construction of Miura maps. The construction also makes clear the conformal, $\mathscr{W}$-algebra structures related to these hierarchies (see e.g. [9, 11]). It is worth noting that in the $p = 1$ case the quantum mechanical versions of these $\mathscr{W}$-algebras have been recently found to have interesting applications in conformal field theory [15].

The paper is organized as follows. Section 1 is a brief review of the version of the Drinfeld–Sokolov approach that will be used. In Sect. 2 the generalized Drinfeld–Sokolov reduction relevant in our case is defined and the resulting reduced system is analyzed in terms of convenient gauge slices. In Sect. 3 the residual symmetries of the reduced system are pointed out. Section 4 is devoted to describing the mapping of the reduced system into the space of matrix pseudo-differential operators and to deriving the form of the Poisson brackets in terms of the reduced space variables. This mapping will be considered both in the "Drinfeld–Sokolov gauge" (0.8) and in a "block-diagonal gauge" which gives rise to a factorization of the Lax operator $L_{p,r,s}$ and to a modified version of the generalized KdV hierarchy. Section 5 contains the pseudo-differential operator description of the local monodromy invariants that in the first order matrix differential operator setting generate the natural family of commuting Hamiltonians. There are two appendices, Appendix A and Appendix B, containing the technical details of certain proofs.

Those readers who are more interested in the concrete description of the reduced system than in its derivation by means of the Hamiltonian reduction could directly turn to look at Theorem 4.4 in Sect. 4 and Theorem 5.2 together with Corollary 5.3 in Sect. 5, which give the form of the reduced Poisson brackets and the commuting Hamiltonians together with the compatible evolution equations.

Notational conventions. Throughout the paper, $\widetilde{N} = C^\infty(S^1, N)$ will denote the space of smooth loops in N for N a Lie group, a Lie algebra, a vector space, or $\mathrm{mat}(m \times n)$: the algebra of $m \times n$ matrices over the field of complex numbers $\mathbb{C}$. All algebraic operations such as addition, multiplication, Lie bracket, are extended to $\widetilde{N}$ in the standard pointwise fashion. For a finite dimensional vector space V_i (or for V a finite or infinite direct sum of such vector spaces) if we write $f_i : S^1 \to V_i$ it is understood that $f_i \in \widetilde{V}_i$ (or so for the components of $f : S^1 \to V$). The symbol e_{ik} will stand for the element of $\mathrm{mat}(m \times n)$ containing a single non-zero entry 1 at the ik position; $\mathbf{0}_n$ and $\mathbf{1}_n \in gl_n = \mathrm{mat}(n \times n)$ will denote the zero and identity matrices, respectively. The (smooth) dual of the vector space $\widetilde{\mathrm{mat}}(m \times n)$ will be identified with $\widetilde{\mathrm{mat}}(n \times m)$ by means of the pairing $\langle\ ,\ \rangle : \widetilde{\mathrm{mat}}(n \times m) \times \widetilde{\mathrm{mat}}(m \times n) \to \mathbb{C}$ given by

$$\langle \alpha, v \rangle := \int_{S^1} \mathrm{tr}\,(\alpha v) \quad \text{for } \alpha \in \widetilde{\mathrm{mat}}(n \times m), \quad v \in \widetilde{\mathrm{mat}}(m \times n)\,. \tag{0.11}$$

For a (suitably smooth) complex function F on $\widetilde{\mathrm{mat}}(m \times n)$, the functional derivative $\frac{\delta F}{\delta u} \in \widetilde{\mathrm{mat}}(n \times m)$ at $u \in \widetilde{\mathrm{mat}}(m \times n)$ is defined by

$$\frac{d}{dt}F(u+tv)\Big|_{t=0} = \left\langle \frac{\delta F}{\delta u}, v \right\rangle = \int_0^{2\pi} dx\, \mathrm{tr}\left(\frac{\delta F}{\delta u(x)} v(x)\right) \quad \forall v \in \widetilde{\mathrm{mat}}(m \times n)\,. \tag{0.12}$$

For F depending on several arguments we shall use partial functional derivatives defined by extending the above formula in a natural manner. For instance, if F depends on $u \in \widetilde{gl}_n$ for $n = n_1 + n_2$ and $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ with $u_{ij} \in \widetilde{\mathrm{mat}}(n_i \times n_j)$, then we have $\frac{\delta F}{\delta u} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ in terms of the partial functional derivatives $\frac{\delta F}{\delta u_{ij}} = \alpha_{ji}$.

Finally, for a Lie algebra $\mathscr{G}$, we let $\ell(\mathscr{G}) := \mathscr{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ denote the space of Laurent polynomials in the spectral parameter λ with coefficients in $\mathscr{G}$. The Lie bracket is extended to $\ell(\mathscr{G})$ in the standard way. The reason why $\ell(\mathscr{G})$ has to be carefully distinguished from $\widetilde{\mathscr{G}} = C^\infty(S^1, \mathscr{G})$ is that $x \in [0, 2\pi]$ parametrizing S^1 has the rôle of the physical space variable (we adopt the periodic boundary condition for definiteness), while λ will appear essentially as a bookkeeping device.

1. Hamiltonian Reduction Approach to KdV Type Hierarchies

To make the paper self-contained, we here present a review of the Drinfeld–Sokolov approach to KdV type hierarchies concentrating on the special case that will be used. From our viewpoint, the main idea of this approach is to combine Hamiltonian symmetry reduction with the Adler–Kostant–Symes approach to constructing commuting Hamiltonian flows. Since in general one can define many reductions once the symmetries of a system are understood, the Hamiltonian reduction approach to constructing KdV systems is in principle more flexible than the algebraic approach described in [6–8] (see also [9]) and it may eventually prove more general. Although the concrete systems studied later in this paper are in fact special cases of the class of systems defined in [6–8], we found it more convenient to base our analysis of them on the Hamiltonian reduction viewpoint. The reader may consult [16] for a review where the full class of KdV systems of [6–8] is interpreted in the framework of Hamiltonian reduction.

Let $\mathscr{G}$ be a finite dimensional Lie algebra with an invariant scalar product "tr." (For most of this paper it may be assumed that $\mathscr{G} = gl_n$ for some n, in which case tr really is the standard matrix trace.) For $C_- \in \mathscr{G}$ arbitrarily fixed, consider the manifold

$$\mathscr{M} := \{\mathscr{L} = \partial + J + \lambda C_- \,|\, J \in \widetilde{\mathscr{G}}\} \tag{1.1}$$

of first order differential operators for which λ is a free parameter, the so-called spectral parameter. As is well known (see e.g. [10]), $\mathscr{M}$ can be identified with a subspace of the dual of the Lie algebra $\widehat{\mathscr{G}} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$, where $\widehat{\mathscr{G}}$ is the central extension of the loop algebra $\widetilde{\mathscr{G}} = C^\infty(S^1, \mathscr{G})$, and there are two compatible Poisson brackets (PBs) on $\mathscr{M}$ defined by the following formulae. The current algebra PB, or affine Kac–Moody algebra PB, is given by

$$\{f,h\}_2(J) = \int_{S^1} \mathrm{tr}\left(J\left[\frac{\delta f}{\delta J}, \frac{\delta h}{\delta J}\right] - \frac{\delta f}{\delta J}\left(\frac{\delta h}{\delta J}\right)'\right). \tag{1.2}$$

This PB (or its reduction) is often referred to as the *second* PB. The *first* PB reads

$$\{f,h\}_1(J) = -\int_{S^1} \mathrm{tr}\, C_- \left[\frac{\delta f}{\delta J}, \frac{\delta h}{\delta J}\right]. \tag{1.3}$$

Notice that the first PB $\{\ ,\ \}_1$ is minus the Lie derivative of the second PB $\{\ ,\ \}_2$ along translations of J in the direction of C_-. The Hamiltonians of special interest here are provided by evaluation of the invariants ("eigenvalues") of the monodromy matrix $T(J,\lambda)$ of $\mathscr{L}$, given by the path ordered exponential

$$T(J,\lambda) = \mathscr{P}\exp\left(-\int_0^{2\pi} dx(J(x) + \lambda C_-)\right), \tag{1.4}$$

where $x \in [0, 2\pi]$ parametrizes the space S^1. The corresponding Hamiltonian flows commute as a special case of the Adler–Kostant–Symes (r-matrix) construction (see e.g. [10]). We call the hierarchy of these bihamiltonian flows on $\mathscr{M}$ the "AKS hierarchy." The AKS hierarchy on $\mathscr{M}$ is *non-local* in general since the invariants of the monodromy matrix (1.4) are non-local functionals of J. However, in some circumstances it is possible to perform a Hamiltonian reduction of the AKS hierarchy to obtain a *local* hierarchy. The locality refers both to the commuting Hamiltonians and the reduced PBs.

The standard method [4–8] to obtain commuting local Hamiltonians relies on a perturbative procedure that uses some *graded, semisimple* element Λ of the Lie algebra $\ell(\mathscr{G}) = \mathscr{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. The grading in which Λ is supposed to be homogeneous with a non-zero grade, say grade $k > 0$, is defined by the eigenspaces of a linear operator $d_{N,H} : \ell(\mathscr{G}) \to \ell(\mathscr{G})$,

$$d_{N,H} = N\lambda\frac{d}{d\lambda} + \mathrm{ad}\, H\,, \tag{1.5}$$

where N is a non-zero integer and $H \in \mathscr{G}$ is diagonalizable with (usually) integer eigenvalues in the adjoint representation. Note that $\mathrm{ad}\, H$ defines a grading of $\mathscr{G}$,

$$\mathscr{G} = \bigoplus_i \mathscr{G}_i, \qquad [\mathscr{G}_i, \mathscr{G}_l] \subset \mathscr{G}_{i+l}\,. \tag{1.6}$$

In the cases of our interest the grade of Λ is small (usually 1) and hence Λ takes the form

$$\Lambda = (C_+ + \lambda C_-) \quad \text{with some } C_\pm \in \mathscr{G}\,. \tag{1.7}$$

The requirement that Λ is *semisimple* means that it defines a direct sum decomposition

$$\ell(\mathscr{G}) = \mathrm{Ker}(\mathrm{ad}\,\Lambda) + \mathrm{Im}(\mathrm{ad}\,\Lambda), \qquad \mathrm{Ker}(\mathrm{ad}\,\Lambda) \cap \mathrm{Im}(\mathrm{ad}\,\Lambda) = \{0\}\,, \tag{1.8}$$

where the *centralizer* $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$ of Λ is a subalgebra of $\ell(\mathscr{G})$. Having chosen Λ of the form given by (1.7), thereby defining C_- in (1.1); and having chosen also a compatible grading operator $d_{N,H}$, the construction then involves imposing constraints on $\mathscr{M}$ so that the constrained manifold $\mathscr{M}_c \subset \mathscr{M}$ consists of operators $\mathscr{L}$ of the form

$$\mathscr{L} = \partial + (j + C_+) + \lambda C_- = \partial + j + \Lambda \quad \text{with } j : S^1 \to \sum_{i<k} \mathscr{G}_i\,. \tag{1.9}$$

That is to say, in addition to the semisimple leading term $\Lambda, \mathcal{L}$ contains only terms of strictly smaller grade than the grade of Λ. In concrete applications there might be further constraints on $\mathcal{L} \in \mathcal{M}_c$, but the above grading and semisimplicity assumptions are already sufficient to obtain local Hamiltonians by the subsequent procedure. The crucial step is to transform $\mathcal{L}$ in (1.9) as follows:

$$(\partial + j + \Lambda) \mapsto e^{\mathrm{ad}\, \Xi}(\partial + j + \Lambda) := (\partial + h + \Lambda)\,, \tag{1.10a}$$

where $\Xi = \sum_{i<0} \Xi_i$ and $h = \sum_{i<k} h_k$ are (formal) infinite series consisting of terms that take their values in homogeneous subspaces in the decomposition (1.8) according to

$$\Xi : S^1 \to (\mathrm{Im}(\mathrm{ad}\, \Lambda))_{<0}\,, \qquad h : S^1 \to (\mathrm{Ker}(\mathrm{ad}\, \Lambda))_{<k}\,, \tag{1.10b}$$

with the subscripts referring to the grading (1.5). The above grading and semisimplicity assumptions ensure that (1.10) can be solved recursively, grade by grade, for both $\Xi = \Xi(j)$ and $h = h(j)$ and the solution is uniquely given by *differential polynomials* in the components of j. The Hamiltonians of interest are associated to the grade larger than $-k$ subspace in the *centre* of the subalgebra $\mathrm{Ker}(\mathrm{ad}\, \Lambda) \subset \ell(\mathcal{G})$ as follows. Suppose that $\{X_i\}$ is a basis of this linear space. The corresponding Hamiltonians are defined by

$$H_{X_i}(j) := \int_0^{2\pi} dx (X_i, h(j(x)))\,, \tag{1.11}$$

where we use the canonical scalar product

$$(X, Y) := \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} \mathrm{tr}(X(\lambda)Y(\lambda)) \tag{1.12}$$

for any $X, Y \in \ell(\mathcal{G})$. As we shall see in examples, the local functionals $H_{X_i}(j)$ can be interpreted as the coefficients in an asymptotic expansion of eigenvalues of the monodromy matrix of $\mathcal{L}$ (1.9) for $\lambda \approx \infty$. They will inherit the property of the monodromy invariants that they commute among themselves with respect to the PBs (1.2–3) on $\mathcal{M}$ if the restriction to $\mathcal{M}_c \subset \mathcal{M}$ is implemented by means of an appropriate Hamiltonian symmetry reduction.

Let G be a finite dimensional Lie group corresponding to $\mathcal{G}$ and let $\mathrm{Stab}(C_-) \subset G$ be the subgroup which stabilizes the element $C_- \in \mathcal{G}$ appearing in the definition (1.1) of $\mathcal{M}$. Let $\widetilde{\mathrm{Stab}}(C_-)$ be the loop group based on $\mathrm{Stab}(C_-)$. Consider the action of $\widetilde{\mathrm{Stab}}(C_-)$ on $\mathcal{M}$ given by

$$(\partial + J + \lambda C_-) \mapsto g(\partial + J + \lambda C_-)g^{-1} = g(\partial + J)g^{-1} + \lambda C_-, \quad \forall g \in \widetilde{\mathrm{Stab}}(C_-)\,. \tag{1.13}$$

The action defined by (1.13) is a symmetry of the AKS hierarchy on $\mathcal{M}$. That is it leaves invariant the compatible PBs (1.2–3) and the monodromy invariants. If $\{T^i\} \subset \mathcal{G}$ is a basis of the centralizer of C_- in $\mathcal{G}$, $[T^i, C_-] = 0$, then the current components $J^i = \mathrm{tr}\,(T^i J)$ serve as the generators of this symmetry (components of the appropriate momentum map) with respect to the second PB (1.2). The same current components are Casimir functions with respect to the first PB (1.3).

The possibility to apply Hamiltonian reduction to the AKS hierarchy on $\mathcal{M}$ rests on the action (1.13) of the symmetry group $\widetilde{\mathrm{Stab}}(C_-)$. The aim is to perform a symmetry reduction using an appropriate *subgroup* of this group in such a way to ensure the *locality* of the reduced AKS hierarchy. In a Hamiltonian symmetry reduction the first step is to introduce constraints (e.g. by restricting to the inverse image of a value of the momentum map). To get local Hamiltonians, the constrained manifold should consist of operators satisfying the conditions in (1.9). Typically, the constraints also bring a gauge freedom into the system, which is to be factored out to obtain the reduced system. Another requirement on the constraints is that the reduced PBs inherited from (1.2) and (1.3) should be given by local, differential polynomial formulae. This is automatically ensured if i) the reduced PBs are given by the original PBs of the gauge invariant differential polynomials on $\mathcal{M}_c$ and ii) these invariants form a *freely generated differential ring*. In practice, property ii) is satisfied if the gauge orbits admit a global cross section for which the components of the gauge fixed current (regarded as a function of j in (1.9)) define a free generating set of the gauge invariant differential polynomials on $\mathcal{M}_c$. The existence of such gauges is a very strong condition on the constraints and the gauge group. Gauge slices of this type have been used by Drinfeld and Sokolov [4], and we refer to such gauges as *DS gauges*. A detailed description of the notion of DS gauges can be found in [11].

We end this overview by recalling that in the original Drinfeld–Sokolov case Λ was chosen to be a grade 1 *regular* semisimple element from the principal Heisenberg subalgebra of $\ell(\mathcal{G})$ for $\mathcal{G}$ a complex simple Lie algebra [4, 17]. Requiring Λ to be a *regular* semisimple element means by definition that the centralizer $\mathrm{Ker}(\mathrm{ad}\,\Lambda) \subset \ell(\mathcal{G})$ appearing in the decomposition (1.8) is an *abelian* subalgebra. In the original case this centralizer is the principal Heisenberg subalgebra (disregarding the central extension). For $\mathcal{G}$ a complex simple Lie algebra or gl_n, the graded regular semisimple elements taken from the other Heisenberg subalgebras (graded, maximal abelian subalgebras) of $\ell(\mathcal{G})$ [18] can be classified using the results of Springer [19] on the regular conjugacy classes of the Weyl group of $\mathcal{G}$ (see also [20]). One may associate constraints to any graded regular Λ in such a way that DS gauges are available [6–9]. More generally, if one implements the generalized Drinfeld–Sokolov reduction procedure proposed in [6, 7] using a graded semisimple but non-regular element Λ, then the existence of a DS gauge must be separately imposed as a condition on Λ.

2. A Generalized Drinfeld–Sokolov Reduction

We wish to apply the above formalism to $\mathcal{G} := gl_n$. The graded Heisenberg subalgebras of $\ell(gl_n)$ are classified by the partitions of n [18]. We shall choose Λ to be a graded semisimple element of minimal positive grade from a Heisenberg subalgebra of $\ell(gl_n)$ associated to a partition of n into "equal blocks plus singlets"

$$n = pr + s = \overbrace{r + \cdots + r}^{p\ \text{times}} + \overbrace{1 + \cdots + 1}^{s\ \text{times}} \quad \text{for some } p \geqq 1,\ r > 1,\ s \geqq 1\,. \tag{2.1}$$

It was explained in [1] that a graded *regular* semisimple element only exists for the partitions into equal blocks $n = pr$ or equal blocks plus *one* singlet $n = pr + 1$. When continuing our previous study of the equal blocks case [1] with the equal blocks plus singlet case, it was realized that DS gauge fixing is possible also when the partition contains an arbitrary number of singlets (1's). This is the motivation for the study of the more general case $s \geqq 1$ here; and actually the analysis will be the same for any $s \geqq 1$.

Next we introduce the necessary notation. Adapted to the partition in (2.1), an element $m \in gl_n$ will often be presented in the following $(r+1) \times (r+1)$ block form:

$$m = \sum_{i,j=1}^{r} e_{i,j} \otimes A_{i,j} + \sum_{i=1}^{r} e_{i,r+1} \otimes B_i + \sum_{i=1}^{r} e_{r+1,i} \otimes C_i + e_{r+1,r+1} \otimes D\,, \tag{2.2}$$

where the $e_{i,j} \in gl_{r+1}$ are the usual elementary matrices, $A_{i,j} \in gl_p$, $D \in gl_s$, $B_i \in \mathrm{mat}(p \times s)$ and $C_i \in \mathrm{mat}(s \times p)$. Alternatively, we may write

$$m = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.3}$$

Introduce the $r \times r$ "DS matrix" $\Lambda_r \in \ell(gl_r)$ given by

$$\Lambda_r := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ \lambda & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{2.4}$$

Recall that the Heisenberg subalgebra, $\mathscr{H}_{p,r,s} \subset \ell(gl_n)$, associated to the partition (2.1) is the linear span of the elements

$$\begin{pmatrix} (\Lambda_r)^l \otimes Y_p & 0 \\ 0 & \mathbf{0}_s \end{pmatrix}, \quad \forall l \in \mathbb{Z}, \qquad \forall Y_p = \mathrm{diag}(y_1, \ldots, y_p), \quad (y_i \in \mathbb{C},\ i = 1, \ldots, p)\,, \tag{2.5}$$

together with the elements

$$\begin{pmatrix} \mathbf{0}_p & 0 \\ 0 & Y_s \lambda^l \end{pmatrix}, \quad \forall l \in \mathbb{Z}, \qquad \forall Y_s = \mathrm{diag}(y_1, \ldots, y_s), \quad (y_k \in \mathbb{C},\ k = 1, \ldots, s)\,. \tag{2.6}$$

$\mathscr{H}_{p,r,s}$ is a graded maximal abelian subalgebra of $\ell(gl_n)$ if we choose the grading defined by

$$d := d_{r,H} = r\lambda \frac{d}{d\lambda} + \mathrm{ad}\, H \tag{2.7a}$$

(see (1.5) and (1.6)), where we take

$$H := \begin{cases} \mathrm{diag}\,(m\mathbf{1}_p, (m-1)\mathbf{1}_p, \ldots, -m\mathbf{1}_p, \mathbf{0}_s), & \text{if } r = (2m+1) \text{ odd}\,; \\ \mathrm{diag}\,(m\mathbf{1}_p, (m-1)\mathbf{1}_p, \ldots, -(m-1)\mathbf{1}_p, \mathbf{0}_s) & \text{if } r = 2m \text{ even}\,. \end{cases} \tag{2.7b}$$

We choose a grade 1 element $\Lambda := \Lambda_{p,r,s}$ (0.6) from the Heisenberg subalgebra,

$$\Lambda = \begin{pmatrix} \Lambda_r \otimes \Gamma & 0 \\ 0 & \mathbf{0}_s \end{pmatrix} \quad \text{with } \Gamma := \mathrm{diag}\,(d_1, \ldots, d_p), \ (d_i)^r \neq (d_j)^r, \quad d_i \neq 0\,. \tag{2.8}$$

For generic λ, all of the eigenvalues of Λ except for the eigenvalue 0 are distinct. The eigenvalue 0 has multiplicity s. It follows that Λ is a *regular* element of $\ell(gl_n)$

if and only if $s = 1$. In addition to the linear span of the elements given in (2.5), the *centralizer* $\mathrm{Ker}(\mathrm{ad}\,\Lambda) \subset \ell(gl_n)$ of Λ contains the algebra $\ell(gl_s) \subset \ell(gl_n)$ spanned by the elements

$$\begin{pmatrix} \mathbf{0}_p & 0 \\ 0 & D\lambda^l \end{pmatrix}, \quad \forall l \in \mathbb{Z},\ \forall D \in gl_s\,. \tag{2.9}$$

Hence the *centre of the centralizer* of Λ is spanned by the elements given in (2.5) together with the set of elements of the form

$$\begin{pmatrix} \mathbf{0}_p & 0 \\ 0 & \mathbf{1}_s\lambda^l \end{pmatrix}, \quad \forall l \in \mathbb{Z}\,. \tag{2.10}$$

Now we consider the generalized Drinfeld–Sokolov reduction of the AKS hierarchy on the manifold $\mathcal{M}$ defined in (1.1), with C_- and C_+ given by writing Λ as $\Lambda = \lambda C_- + C_+$, i.e., from now on

$$C_- = e_{r,1} \otimes \Gamma \quad \text{and} \quad C_+ = \sum_{i=1}^{r-1} e_{i,i+1} \otimes \Gamma\,. \tag{2.11}$$

Using the grading of the Lie algebra $\mathcal{G} = gl_n$ defined by the eigenvalues of $\mathrm{ad}\,H$,

$$\mathcal{G} = \mathcal{G}_{-(r-1)} + \cdots + \mathcal{G}_{-1} + \mathcal{G}_0 + \mathcal{G}_1 + \cdots + \mathcal{G}_{r-1} = \mathcal{G}_{<0} + \mathcal{G}_0 + \mathcal{G}_{>0}\,, \tag{2.12}$$

we first write the current J as

$$J = J_{<0} + J_0 + J_{>0}\,, \tag{2.13}$$

and impose the constraint

$$J_{>0} = C_+\,, \tag{2.14}$$

which restricts the system to the submanifold $\mathcal{M}_c \subset \mathcal{M}$ given by

$$\mathcal{M}_c := \{\mathcal{L} = \partial + j + \Lambda \,|\, j \in C^\infty(S^1, \mathcal{G}_{\leqq 0})\}\,. \tag{2.15}$$

Then we factorize the constrained manifold $\mathcal{M}_c$ by the group $\mathcal{N}$ of *gauge transformations* e^f acting according to

$$e^f : \mathcal{L} \mapsto e^f \mathcal{L} e^{-f}, \quad f \in C^\infty(S^1, \mathcal{G}_{<0})\,. \tag{2.16}$$

The Hamiltonian interpretation of this reduction procedure is the same as in the case of the standard Drinfeld–Sokolov reduction for gl_n. Briefly, from the point of view of the second PB (1.2), it is a Marsden–Weinstein type reduction with $\mathcal{N}$ being the symmetry group and C_+ a character of $\mathcal{N}$ (which means that the constraints defining $\mathcal{M}_c \subset \mathcal{M}$ are *first class*). From the point of view of the first PB (1.3), the reduction amounts to fixing the values of Casimir functions and subsequently factoring by the group of Poisson maps $\mathcal{N}$. It follows that the compatible PBs on $\mathcal{M}$ induce compatible PBs on the space of the gauge invariant functions on $\mathcal{M}_c$, identified as the space of functions on the reduced space $\mathcal{M}_{\mathrm{red}} := \mathcal{M}_c/\mathcal{N}$. The reduced first PB is the Lie derivative of the reduced second PB with respect to the one parameter group action on the reduced space induced by the following one parameter group action on $\mathcal{M}_c$:

$$\mathcal{L} \mapsto \mathcal{L} - \tau C_-, \quad \forall \tau \in \mathbb{R}\,. \tag{2.17}$$

This follows by combining the fact that these translations commute with the action of $\mathcal{N}$ on $\mathcal{M}_c$ with the fact that the first PB (1.3) is the Lie derivative of the second

PB (1.2) with respect to the respective one parameter group action on $\mathcal{M}$. It is clear from (2.16) that the invariants (eigenvalues) of the monodromy matrix of $\mathcal{L} \in \mathcal{M}_c$ define gauge invariant functions on $\mathcal{M}_c$. In conclusion, we have a local hierarchy of bihamiltonian flows on the reduced phase space $\mathcal{M}_{\text{red}}$, which is generated by the Hamiltonians provided by the local monodromy invariants of $\mathcal{L}$ defined by the procedure in (1.10) and (1.11).

In order to describe the reduced system in more detail, one reverts to gauge slices. We wish to mention two important gauges. The first is the DS gauge whose gauge section is the manifold $\mathcal{M}_{\text{DS}} \subset \mathcal{M}_c$ given by

$$\mathcal{M}_{\text{DS}} := \Big\{ \mathcal{L} = \partial + j_{\text{DS}} + \Lambda \,|$$

$$j_{\text{DS}} = \sum_{i=1}^{r} e_{r,i} \otimes v_{r-i+1} + e_{r,r+1} \otimes \zeta_+ + e_{r+1,1} \otimes \zeta_- + e_{r+1,r+1} \otimes w$$

$$\text{with } v_i \in \widetilde{gl}_p,\ w \in \widetilde{gl}_s,\ \zeta_+ \in \widetilde{\text{mat}}(p \times s),\ \zeta_- \in \widetilde{\text{mat}}(s \times p) \Big\} . \tag{2.18}$$

In explicit matrix notation $j_{\text{DS}} := j_{p,r,s}$ is given in (0.8). The space $\mathcal{M}_{\text{DS}}$ is a *one-to-one* model of $\mathcal{M}_c/\mathcal{N}$ with the property that, when regarded as functions on $\mathcal{M}_c$, the components of the gauge fixed current $j_{\text{DS}} = j_{\text{DS}}(j)$ provide a basis of the gauge invariant differential polynomials on $\mathcal{M}_c$. This follows from a standard argument on DS gauge fixing (see e.g. [21]), which relies on the grading and the non-degeneracy condition

$$\text{Ker}(\text{ad}\, C_+) \cap \mathcal{G}_{<0} = \{0\} , \tag{2.19}$$

which is satisfied in our case. The reduced AKS hierarchy on the phase space $\mathcal{M}_{\text{red}} \simeq \mathcal{M}_{\text{DS}}$ is a generalization of the well-known r-KdV hierarchy, as we shall see in Sects. 4 and 5.

The other important gauge is what we call the "Θ-gauge" (also called block-diagonal gauge), which is defined by the following submanifold $\Theta \subset \mathcal{M}_c$,

$$\Theta := \{ \mathcal{L} = \partial + j_0 + \Lambda \,|\, j_0 \in C^\infty(S^1, \mathcal{G}_0) \} . \tag{2.20}$$

Correspondingly to the grading operator H in (2.7), we parametrize j_0 as

$$j_0 = \text{diag}(\theta_m, \ldots, \theta_1, a, \theta_{-1}, \ldots, \theta_{-[\frac{r-1}{2}]}, d) + e_{m+1,r+1} \otimes b + e_{r+1,m+1} \otimes c , \tag{2.21a}$$

where $[\frac{r-1}{2}]$ is the integral part of $\frac{r-1}{2}$, $m = [\frac{r}{2}]$. The variables on Θ are $\theta_i \in \widetilde{gl}_p$ for $i \neq 0$ and

$$a \in \widetilde{gl}_p, \qquad b \in \widetilde{\text{mat}}(p \times s), \qquad c \in \widetilde{\text{mat}}(s \times p), \qquad d \in \widetilde{gl}_s , \tag{2.21b}$$

which we collect into the matrix

$$\theta_0 := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{gl}_{p+s} . \tag{2.22}$$

In the Θ-gauge, in terms of the variables θ_i, the reduced second PB becomes just the direct sum of free current algebra PBs given by

$$\{\bar{f}, \bar{h}\}(\theta) = \sum_{i=-[\frac{r-1}{2}]}^{m} \int_{S^1} \text{tr} \left(\theta_i \left[\frac{\delta \bar{f}}{\delta \theta_i}, \frac{\delta \bar{h}}{\delta \theta_i} \right] - \frac{\delta \bar{f}}{\delta \theta_i} \left(\frac{\delta \bar{h}}{\delta \theta_i} \right)' \right) \tag{2.23}$$

for $\bar{f}$, $\bar{h}$ smooth functions on Θ. In fact, the restriction of the second PB of arbitrary gauge invariant functions f, h on $\mathcal{M}_c$ to Θ has the form (2.23) with $\bar{f} = f|_\Theta$, $\bar{h} = h|_\Theta$. This can be proved in the same way as Lemma 2.1 in [1], which is of course essentially the same as the proof found in [4] in the scalar r-KdV case. Using DS gauge fixing we obtain a local, differential polynomial mapping $\mu : \Theta \to \mathcal{M}_{\mathrm{DS}}$ yielding a generalized Miura transformation from the modified KdV type hierarchy on Θ to the KdV type hierarchy on $\mathcal{M}_{\mathrm{DS}}$. As is expected from a Miura map, the inverse of μ is non-local and is not single valued. In other words, Θ cannot be reached by a local gauge fixing procedure and the intersection of $\Theta \subset \mathcal{M}_c$ with a gauge orbit of $\mathcal{N}$ in $\mathcal{M}_c$ is not unique.

Remark 2.1. The construction presented in the above is a straightforward generalization of the construction of the n-KdV hierarchy due to Drinfeld and Sokolov [4]. In particular, the DS gauge in (2.18) and the Θ-gauge in (2.20) that parametrize the KdV and modified KdV type systems are quite similar to respective gauges in [4]. Following further the spirit of [4], our aim in Sects. 4 and 5 will be to find pseudo-differential operator models of these systems. This question was not addressed in [6–8] where an algebraic generalization of the Drinfeld–Sokolov construction of KdV and modified KdV type systems was given (for a review, see e.g. [16]). The systems that are defined rather abstractly in these papers include our systems as special cases[1], but our presentation is different in that we proceed consistently from the viewpoint of Hamiltonian reduction, since this proves advantageous in developing concrete models of the systems obtained.

Remark 2.2. In the case $r = 1$ it is natural to define the element Λ to be the diagonal matrix $\Lambda := \Lambda_{p,r=1,s}$ given by

$$\Lambda_{p,r=1,s} := \lambda \begin{pmatrix} \Gamma & 0 \\ 0 & \mathbf{0}_s \end{pmatrix} = \lambda\, \mathrm{diag}(d_1, \ldots, d_p, 0, \ldots, 0)\,, \tag{2.24}$$

which contains s zeros and distinct, non-zero $d_i \in \mathbb{C}$ for $i = 1, \ldots, p$. This is a grade 1 semisimple element of $\ell(gl_n)$, $n = p + s$, with respect to the homogeneous grading. In this case the Drinfeld–Sokolov reduction becomes trivial, i.e., $\mathcal{M} = \mathcal{M}_c = \mathcal{M}_{\mathrm{DS}} = \Theta$. For this reason the assumption has been made so far that $r > 1$. All considerations in the rest of this paper apply to the $r = 1$ case too and all of the results follow through. There are interesting consequences for $r = 1$ as well as for $r > 1$.

3. Residual Symmetries

We have performed a Drinfeld–Sokolov type reduction on the system $\mathcal{M}$ (1.1) using the subgroup $\mathcal{N}$ (2.16) of the symmetry group $\widetilde{\mathrm{Stab}}(C_-)$ defined by the stabilizer of the element C_- in (2.11). Here we wish to point out a residual symmetry of the reduced system so obtained. Consider the following transformations on $\mathcal{M}$:

$$\mathcal{L} \mapsto \exp\left(\begin{pmatrix} \mathbf{1}_r \otimes \alpha & 0 \\ 0 & \mathbf{0}_s \end{pmatrix}\right) \mathcal{L} \exp\left(-\begin{pmatrix} \mathbf{1}_r \otimes \alpha & 0 \\ 0 & \mathbf{0}_s \end{pmatrix}\right), \quad \alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_p)\,, \tag{3.1a}$$

[1] In the notation of [7] the DS gauge (2.18) represents the phase space of a "generalized KdV hierarchy" while the Θ-gauge (2.20) corresponds precisely to "generalized modified KdV."

with $\alpha_i \in C^\infty(S^1, \mathbf{C})$ for $i = 1, \ldots, p$, and

$$\mathscr{L} \mapsto \exp\left(\begin{pmatrix} \mathbf{0}_{pr} & 0 \\ 0 & D \end{pmatrix}\right) \mathscr{L} \exp\left(-\begin{pmatrix} \mathbf{0}_{pr} & 0 \\ 0 & D \end{pmatrix}\right), \quad \text{with } D \in C^\infty(S^1, gl_s)\,. \tag{3.1b}$$

The group generated by these transformations is a $\widetilde{GL}_1 \times \cdots \times \widetilde{GL}_1 \times \widetilde{GL}_s$ subgroup of the symmetry group $\widetilde{\mathrm{Stab}}(C_-)$ acting on $\mathscr{M}$ according to (1.13). We call it the *group of residual symmetries* and denote it by G_R. It is easily verified that these transformations map the constrained manifold $\mathscr{M}_c \subset \mathscr{M}$ to itself. For grading reasons, $G_R \subset \widetilde{\mathrm{Stab}}(C_-)$ normalizes the gauge group $\mathscr{N} \subset \widetilde{\mathrm{Stab}}(C_-)$, which implies that the transformations in (3.1) induce a corresponding action of G_R on the space of gauge orbits $\mathscr{M}_c/\mathscr{N}$. By construction, this induced action leaves invariant the commuting Hamiltonians as well as the compatible PBs of the hierarchy on $\mathscr{M}_{\mathrm{red}} = \mathscr{M}_c/\mathscr{N}$.

Let us write the current $J \in \widetilde{gl}_{pr+s}$ defining $\mathscr{L} = (\partial + J + \lambda C_-) \in \mathscr{M}$ in the block form

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \tag{3.2}$$

where $J_{11} \in \widetilde{gl}_{pr}$, $J_{22} \in \widetilde{gl}_s$, etc., similarly to the matrix m in (2.2),(2.3). With respect to the current algebra PB (1.2), the infinitesimal generators (the momentum map) of the transformations in (3.1a) and (3.1b) on $\mathscr{M}$ are provided by the current components Φ_i and w given by

$$\Phi_i(J) := \mathrm{tr}(J_{11}(\mathbf{1}_r \otimes e_{ii})), \quad i = 1, \ldots, p\,, \tag{3.3a}$$

where $e_{ii} \in gl_p$ is the usual elementary matrix, and

$$w(J) := J_{22}\,, \tag{3.3b}$$

respectively. The restrictions of these current components to $\mathscr{M}_c$ (by setting $J = (j + C_+)$ as in (2.15)) are gauge invariant. These gauge invariant current components generate the induced action of the group of residual symmetries G_R on $\mathscr{M}_{\mathrm{red}}$ with respect to the reduced second PB. Since the Hamiltonians and the compatible PBs of the hierarchy on $\mathscr{M}_{\mathrm{red}}$ are invariant under this group, it follows that the current components Φ_i and w are constants along the flows of the hierarchy. Of course this also follows from the fact that Φ_i and w are Casimir functions with respect to the first PB (1.3). Incidentally, these Casimir functions are examples of the centres of the first Poisson bracket given by Proposition 1 in [9].

The residual symmetry (3.1) may be used to perform further reductions on the hierarchy obtained from the generalized Drinfeld–Sokolov reduction. In fact, the mapping to pseudo-differential operators studied in the next section corresponds to such a reduction.

4. The Poisson Brackets on the Reduced Phase Space

In this section we shall find a Poisson mapping from the reduced phase space $\mathscr{M}_c/\mathscr{N}$, endowed with the compatible PBs induced by the Drinfeld–Sokolov reduction, to the space of pseudo-differential operators (PDOs) with gl_p valued coefficients endowed with the usual Gelfand–Dickey Poisson brackets [2, 3, 22, 23]. The image includes the phase space of the matrix r-KdV hierarchy. The mapping will be defined by means of the elimination procedure similarly to the r-KdV case [4, 1]. In the present case the mapping will not be one-to-one, but we shall be able to present the explicit form of the reduced PBs on $\mathcal{M}_c/\mathcal{N}$ nonetheless.

Let $\mathcal{A}$ be the space of pseudo-differential operators with $p \times p$ matrix coefficients:

$$\mathcal{A} = \left\{ L = \sum_{s=-\infty}^{N} L_s \partial^s \mid L_s \in \widetilde{gl}_p,\ N \in \mathbb{Z} \right\}. \tag{4.1}$$

Multiplication of matrix pseudo-differential operators is defined in the usual way, i.e., the product rule is given by matrix multiplication together with the formulae

$$\partial\partial^{-1} = 1 \quad \text{and} \quad \partial F = F\partial + F' \quad \text{for } F \in \widetilde{gl_p}\,, \tag{4.2a}$$

which engender the formula

$$\partial^{-1} F = \sum_{i=0}^{\infty} (-1)^i F^{(i)} \partial^{-i-1} \quad \text{for } F \in \widetilde{gl_p}\,. \tag{4.2b}$$

The Adler trace [22], $\mathrm{Tr}: \mathcal{A} \to \mathbb{C}$, is given by

$$\mathrm{Tr}\, L := \int_{S^1} \mathrm{tr}\ \mathrm{res}(L) = \int_{S^1} \mathrm{tr}\ L_{-1}\,, \tag{4.3}$$

where tr is the ordinary matrix trace and $\mathrm{res}(L) = L_{-1}$ is the coefficient of ∂^{-1}. Let $P_\pm$ be the projectors on $\mathcal{A}$ onto the subalgebras

$$\mathcal{A}_+ := \left\{ L = \sum_{s=0}^{N} L_s \partial^s \right\}, \qquad \mathcal{A}_- := \left\{ L = \sum_{s=-\infty}^{-1} L_s \partial^s \right\}, \tag{4.4}$$

respectively. Put $L_\pm := P_\pm(L)$. The space $\mathcal{A}$ is a bihamiltonian manifold. For f, h smooth functions on $\mathcal{A}$, the quadratic (second) Gelfand–Dickey PB is given by

$$\{f, h\}^{(2)}(L) = \mathrm{Tr} \left(\frac{\delta h}{\delta L} L \left(\frac{\delta f}{\delta L} L \right)_+ - L \frac{\delta h}{\delta L} \left(L \frac{\delta f}{\delta L} \right)_+ \right), \tag{4.5}$$

where the gradient $\frac{\delta f}{\delta L} \in \mathcal{A}$ of f at $L \in \mathcal{A}$ is defined by

$$\left. \frac{d}{dt} f(L + tA) \right|_{t=0} = \mathrm{Tr} \left(A \frac{\delta f}{\delta L} \right), \quad \forall A \in \mathcal{A}\,. \tag{4.6}$$

The Lie derivative of the quadratic bracket (4.5) with respect to the one parameter group of translations

$$L \mapsto (L + \tau \mathbf{1}_p), \quad \forall \tau \in \mathbb{R}\,, \tag{4.7}$$

is the linear (first) Gelfand–Dickey PB:

$$\{f, h\}^{(1)}(L) = \mathrm{Tr} \left(L \left[\left(\frac{\delta f}{\delta L} \right)_+, \left(\frac{\delta h}{\delta L} \right)_+ \right] - L \left[\left(\frac{\delta f}{\delta L} \right)_-, \left(\frac{\delta h}{\delta L} \right)_- \right] \right), \tag{4.8}$$

which is compatible (coordinated) with the quadratic PB.

In order to relate the Drinfeld–Sokolov reduction to the above formalism, consider the linear problem for $\mathscr{L} \in \mathscr{M}_c$ (2.15):

$$\mathscr{L}\psi = 0, \qquad \psi = (\psi_1^t, \psi_2^t, \ldots, \psi_r^t, \phi^t)^t\,, \tag{4.9}$$

where ψ is a $(pr+s)$-component column vector consisting of the p-component column vectors ψ_i $(i = 1, \ldots, r)$ and the s-component column vector ϕ. This system of equations is covariant under the gauge transformation (2.16) accompanied by the transformation

$$\psi \mapsto e^f \psi, \quad f \in C^\infty(S^1, \mathscr{G}_{<0})\,. \tag{4.10}$$

Observe that the component ψ_1 is *invariant* under (4.10). This implies that if we derive from (4.9) an equation on ψ_1, then the operator entering that equation will be a gauge invariant object. The process of obtaining an equation on ψ_1 from (4.9) is what we refer to as the *elimination procedure*. This is particularly simple in the DS gauge given by (2.18) and we obtain

$$L\psi_1 = \lambda\psi_1\,, \tag{4.11}$$

where

$$L = \Delta^r\partial^r + u_1\partial^{r-1} + u_2\partial^{r-2} + \cdots + u_{r-1}\partial + u_r + z_+(\mathbf{1}_s\partial + w)^{-1}z_-\,, \tag{4.12}$$

with $\Delta = -\Gamma^{-1}$ for Γ defined in (2.8) and the variables being related to those in (2.18) by

$$u_i = \Delta v_i \Delta^{r-i}, \qquad z_+ = -\Delta\zeta_+, \qquad z_- = \zeta_-\,. \tag{4.13}$$

Let $M \subset \mathscr{A}$ denote the manifold of "Lax operators" L of the form (4.12). As discussed above, the elimination procedure gives rise to a mapping

$$\pi : \mathscr{M}_c \to M, \quad \pi(\mathscr{L}) = L\,, \tag{4.14a}$$

which is constant along the gauge orbits in $\mathscr{M}_c$. Thus we have a corresponding induced mapping

$$\bar{\pi} : \mathscr{M}_c/\mathscr{N} \to M\,. \tag{4.14b}$$

Observe that L in (4.12) contains only quadratic combinations of the fields $z_\pm$ that parametrize the manifold $\mathscr{M}_{\mathrm{DS}} \simeq \mathscr{M}_c/\mathscr{N}$ (2.18), but does not contain these fields in a linear manner. This shows that (unlike in the usual r-KdV case) the mapping $\bar{\pi}$ is *not* one-to-one. This can be explained by the fact that the action of the group $\widetilde{GL}_s$ on $\mathscr{M}_c/\mathscr{N}$ defined by (3.1b) is a symmetry of the mapping $\bar{\pi}$, i.e., every $\widetilde{GL}_s$ orbit in $\mathscr{M}_c/\mathscr{N}$ is mapped to a single point. To understand this, observe that the original linear problem (4.9) is covariant not only with respect to the gauge group $\mathscr{N}$ but also with respect to the group of residual symmetries G_R acting according to the formulae in (3.1a) and (3.1b) complemented with the formulae

$$\psi \mapsto \exp\left(\begin{pmatrix} \mathbf{1}_r \otimes \alpha & 0 \\ 0 & \mathbf{0}_s \end{pmatrix}\right)\psi \tag{4.15a}$$

and

$$\psi \mapsto \exp\left(\begin{pmatrix} \mathbf{0}_{pr} & 0 \\ 0 & D \end{pmatrix}\right)\psi\,. \tag{4.15b}$$

Since the component ψ_1 is *invariant* under the $\widetilde{GL}_s$ action (4.15b), the operator $L = \pi(\mathscr{L})$ entering (4.11) must be also invariant with respect to the $\widetilde{GL}_s$ action (3.1b). Correspondingly, for the infinitesimal generator w (3.3b) of the $\widetilde{GL}_s$ symmetry and for any function $\mathscr{F}$ of L, we have

$$\{w, \mathscr{F}\}_2 = 0 . \tag{4.16}$$

In particular, the expansion of L in powers of ∂ contains only such $\mathcal{N}$-invariant differential polynomials in the components of $\mathscr{L} \in \mathcal{M}_c$ which commute with w under the second PB.

The elimination procedure may be performed on the linear problem (4.9) in the Θ-gauge (2.20) analogously as was done above in the DS gauge (2.18). We obtain

$$L_\Theta \psi_1 = \lambda \psi_1 \tag{4.17}$$

with the factorized Lax operator

$$L_\Theta = (\Delta(\partial + \theta_{-[\frac{r-1}{2}]})) \cdots (\Delta(\partial + \theta_{-1}))(\Delta K)(\Delta(\partial + \theta_1)) \cdots (\Delta(\partial + \theta_m)) , \tag{4.18}$$

where $\Delta = -\Gamma^{-1}$ and the operator K is given by

$$K = (\mathbf{1}_p \partial + a) - b(\mathbf{1}_s \partial + d)^{-1} c . \tag{4.19}$$

Here θ_i, a, b, c, d are the fields parametrizing the Θ-gauge according to (2.21). Since ψ_1 is gauge invariant, the operators L and L_Θ attached to such points of $\mathcal{M}_{\mathrm{DS}}$ and Θ that lie on the same gauge orbit are equal: That is L_Θ (4.18) is a factorized form of L (4.12). In particular,

$$M = M_\Theta , \tag{4.20}$$

where $M_\Theta := \{L_\Theta\}$ is the set of operators L_Θ (4.18). Note that if $r = 2m$ is even, there appear $(m-1)$ factors before (ΔK) and m factors after (ΔK) in (4.18). The factorization (4.18) (but not the results below) was derived previously in the case $p = s = 1$ in [24].

We now consider the relationship between the set M_K of operators K of the form (4.19) and the space $\Theta_0 := \widetilde{gl}_{p+s}$. Parametrizing the general element $\theta_0 \in \Theta_0$ as $\theta_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ like in (2.22), we have the mapping

$$\eta : \Theta_0 \to M_K, \qquad \eta(\theta_0) := \mathbf{1}_p \partial + a - b(\mathbf{1}_s \partial + d)^{-1} c . \tag{4.21}$$

The natural PB on the space Θ_0 is given by the appropriate term in (2.23),

$$\{\bar{f}, \bar{h}\}(\theta_0) = \int_{S^1} \mathrm{tr} \left(\theta_0 \left[\frac{\delta \bar{f}}{\delta \theta_0}, \frac{\delta \bar{h}}{\delta \theta_0} \right] - \frac{\delta \bar{f}}{\delta \theta_0} \left(\frac{\delta \bar{h}}{\delta \theta_0} \right)' \right) \tag{4.22}$$

for $\bar{f}$, $\bar{h}$ smooth functions on Θ_0. We have the following result.

Proposition 4.1. *The set $M_K \subset \mathscr{A}$ of operators K of the form* (4.19) *is a Poisson submanifold of $\mathscr{A}$ with respect to the quadratic Gelfand–Dickey* PB (4.5). *The mapping η* (4.21) *is a Poisson mapping with respect to the free current algebra* PB (4.22) *on Θ_0 and the quadratic Gelfand–Dickey* PB *on M_K.*

Proof. Let f and h be arbitrary (smooth) functions on $\mathscr{A}$. The statement of the proposition is equivalent to the equality

$$\{f,h\}^{(2)} \circ \eta = \{f \circ \eta, h \circ \eta\}\,, \tag{4.23}$$

where the l.h.s. is determined by the quadratic Gelfand–Dickey PB (4.5) and the r.h.s. is determined by the free current algebra PB (4.22). This equality can be verified by a straightforward computation. Since the computation is rather long, we have relegated it to Appendix A. □

Thanks to Proposition 4.1, we are now ready to describe the relationship between the PBs on $\mathscr{M}_c/\mathscr{N}$ induced from the PBs (1.2–3) on $\mathscr{M}$ by the Drinfeld–Sokolov reduction and the Gelfand–Dickey PBs on $M \subset \mathscr{A}$.

Theorem 4.2. *The set $M \subset \mathscr{A}$ of operators L* (4.12) *is a Poisson submanifold with respect to the quadratic Gelfand–Dickey* PB (4.5)*. The mapping $\bar{\pi} : \mathscr{M}_c/\mathscr{N} \to M$* (4.14b) *defined by the elimination procedure is a Poisson mapping, where M is endowed with the quadratic Gelfand–Dickey* PB *and $\mathscr{M}_c/\mathscr{N}$ is endowed with the reduced second* PB *resulting from the current algebra* PB (1.2) *on $\mathscr{M}$ by means of the Drinfeld–Sokolov reduction.*

Proof. We have seen that $M_K \subset \mathscr{A}$ is a Poisson submanifold with respect to the PB (4.5). It is well-known (see [1] Sect. 2.2) that the other factors $\{\Delta(\partial + \theta_i) \mid \theta_i \in \widetilde{gl}_p\} \subset \mathscr{A}$ appearing in L_Θ (4.18) are also Poisson submanifolds with respect to the PB (4.5), which coincides with the free current algebra on these submanifolds:

$$\{\bar{f},\bar{h}\}^{(2)}(\theta_i) = \int_{S^1} \operatorname{tr}\left(\theta_i\left[\frac{\delta\bar{f}}{\delta\theta_i},\frac{\delta\bar{h}}{\delta\theta_i}\right] - \frac{\delta\bar{f}}{\delta\theta_i}\left(\frac{\delta\bar{h}}{\delta\theta_i}\right)'\right), \tag{4.24}$$

for $\bar{f},\bar{h}$ smooth functions of θ_i. Recall the "product property" of the quadratic bracket according to which the product of Poisson submanifolds is also a Poisson submanifold. The statement of the theorem follows from this on account of (4.20) and the fact that in the Θ-gauge (2.20) the reduced second PB is given by the current algebra (2.23). □

So far we have dealt with the reduced second PB on the bihamiltonian manifold $\mathscr{M}_c/\mathscr{N}$. Remember that the reduced first PB on $\mathscr{M}_c/\mathscr{N}$, which results from the PB (1.3) on $\mathscr{M}$, is the Lie derivative of the reduced second PB with respect to the infinitesimal generator of the one parameter group action on $\mathscr{M}_c/\mathscr{N}$ induced by the one parameter group action (2.17) on $\mathscr{M}_c$. The manifold M of Lax operators (4.12) is also a bihamiltonian manifold since the linear Gelfand–Dickey PB $\{\cdot,\cdot\}^{(1)}$ (4.8) on $\mathscr{A}$ can be restricted to $M \subset \mathscr{A}$. To see this recall that the linear PB $\{\cdot,\cdot\}^{(1)}$ (4.8) on $\mathscr{A}$ is the Lie derivative of the quadratic PB $\{\cdot,\cdot\}^{(2)}$ (4.5) on $\mathscr{A}$ with respect to the infinitesimal generator of the one parameter group action (4.7) on $\mathscr{A}$ and notice that this group maps $M \subset \mathscr{A}$ to itself. This together with the first statement of Theorem 4.2 implies that $M \subset \mathscr{A}$ is in fact a Poisson submanifold also with respect to the linear PB (4.8). Theorem 4.2 and Theorem 4.3 below state that the mapping $\bar{\pi}$ (4.14b) is a Poisson mapping of bihamiltonian manifolds.

Theorem 4.3. *The manifold $M \subset \mathscr{A}$ is a Poisson submanifold with respect to the linear Gelfand–Dickey* PB (4.8) *on $\mathscr{A}$ and the mapping $\bar{\pi} : \mathscr{M}_c/\mathscr{N} \to M$* (4.14b) *is a Poisson mapping with respect to the reduced first* PB (1.3) *on $\mathscr{M}_c/\mathscr{N}$ and the linear Gelfand–Dickey* PB (4.8) *on M.*

Proof. Using the identification $\mathscr{M}_c/\mathscr{N} \simeq \mathscr{M}_{\mathrm{DS}}$, it is enough to show that the mapping $\bar{\pi} : \mathscr{M}_{\mathrm{DS}} \to M$ (4.14) intertwines the one parameter group action (2.17) on $\mathscr{M}_{\mathrm{DS}}$ and the one parameter group action (4.7) on M. By (4.13), this follows from the elimination procedure that converts $\mathscr{L} \in \mathscr{M}_{\mathrm{DS}}$ appearing in (4.9) into $L \in M$ in (4.11). □

The above results may be used to determine the reduced first and second PBs between such functions on $\mathscr{M}_{\mathrm{red}} = \mathscr{M}_c/\mathscr{N}$ which are of the form $\mathscr{F} \circ \bar{\pi}$, $\mathscr{H} \circ \bar{\pi}$ with functions $\mathscr{F}, \mathscr{H}$ on M. We now wish to present the explicit formulae for the PBs of arbitrary functions on $\mathscr{M}_{\mathrm{red}}$. We can parametrize $\mathscr{M}_{\mathrm{red}} \simeq \mathscr{M}_{\mathrm{DS}}$ by the variables $u_i, z_\pm, w$ or equivalently by the variables $\ell, z_\pm, w$, where ℓ is the positive part of L,

$$\ell = \Delta^r \partial^r + \sum_{i=1}^{r} u_i \partial^{r-i}, \qquad L = \ell + z_+ (\mathbf{1}_s \partial + w)^{-1} z_- \,. \tag{4.25}$$

With the aid of the usual functional derivatives, the arbitrary variation δH of a function H on $\mathscr{M}_{\mathrm{red}}$ may be written as

$$\delta H = \int_{S^1} \mathrm{tr} \left(\sum_{i=1}^{r} \frac{\delta H}{\delta u_i} \delta u_i + \frac{\delta H}{\delta z_+} \delta z_+ + \frac{\delta H}{\delta z_-} \delta z_- + \frac{\delta H}{\delta w} \delta w \right) , \tag{4.26a}$$

or equivalently as

$$\delta H = \mathrm{Tr} \left(\frac{\delta H}{\delta \ell} \delta \ell \right) + \int_{S^1} \mathrm{tr} \left(\frac{\delta H}{\delta z_+} \delta z_+ + \frac{\delta H}{\delta z_-} \delta z_- + \frac{\delta H}{\delta w} \delta w \right) , \tag{4.26b}$$

using the Adler trace (4.3) and the definition

$$\frac{\delta H}{\delta \ell} = \sum_{i=1}^{r} \partial^{i-r-1} \frac{\delta H}{\delta u_i} \,. \tag{4.26c}$$

We can write the reduced first and second PBs, denoted by $\{F,H\}_i^*$ ($i = 1,2$), of the arbitrary functions F, H as follows:

$$\begin{aligned} \{F,H\}_i^* &= \mathbf{X}_H^i(F) \\ &= \mathrm{Tr} \left(\frac{\delta F}{\delta \ell} \mathbf{X}_H^i(\ell) \right) + \int_{S^1} \mathrm{tr} \left(\frac{\delta F}{\delta z_+} \mathbf{X}_H^i(z_+) + \frac{\delta F}{\delta z_-} \mathbf{X}_H^i(z_-) + \frac{\delta F}{\delta w} \mathbf{X}_H^i(w) \right) , \end{aligned} \tag{4.27}$$

where $\mathbf{X}_H^i$ is the corresponding Hamiltonian vector field associated to the function H and $\mathbf{X}_H^i(G) = \langle \delta G, \mathbf{X}_H^i \rangle$ is the derivative of the function G with respect to $\mathbf{X}_H^i$. For an arbitrary PDO A, we define

$$P_0(A) := \mathrm{res} \left(A \partial^{-1} \right) \quad \text{and} \quad P_0^\dagger(A) := \mathrm{res} \left(\partial^{-1} A \right) . \tag{4.28}$$

Expanding A in the right form, $A := \sum_k A_k \partial^k$, or in the left form, $A := \sum_k \partial^k \tilde{A}_k$, we have $P_0(A) = A_0$ and $P_0^\dagger(A) = \tilde{A}_0$. It will be also convenient to rewrite the PDO $(\partial + w)$ as

$$(\partial + w) = W^{-1}\partial W \quad \text{with } w = W^{-1}W' , \tag{4.29}$$

where the GL_s valued function W on $\mathbb{R}$ is uniquely associated to w by (4.29) and the condition $W(0) = \mathbf{1}_s$. In terms of these notations we can now write down $\mathbf{X}_H^i$.

Theorem 4.4. *The Hamiltonian vector field* $\mathbf{X}_H^2$ *associated to a function* H *on* $\mathcal{M}_c/\mathcal{N}$ *by means of the reduced second* PB *is given by*

$$\begin{aligned}
\mathbf{X}_H^2(\ell) &= \left(\ell\frac{\delta H}{\delta \ell}\right)_+ \ell - \ell\left(\frac{\delta H}{\delta \ell}\ell\right)_+ + \left(\ell\frac{\delta H}{\delta z_-}(\partial + w)^{-1} z_-\right)_+ \\
&\quad - \left(z_+(\partial + w)^{-1}\frac{\delta H}{\delta z_+}\ell\right)_+ , \\
\mathbf{X}_H^2(z_+) &= P_0\left(\ell\left(\frac{\delta H}{\delta \ell}z_+ + \frac{\delta H}{\delta z_-}\right)W^{-1}\right)W - z_+\frac{\delta H}{\delta w} , \\
\mathbf{X}_H^2(z_-) &= -W^{-1}P_0^\dagger\left(W\left(z_-\frac{\delta H}{\delta \ell} + \frac{\delta H}{\delta z_+}\right)\ell\right) + \frac{\delta H}{\delta w}z_- , \\
\mathbf{X}_H^2(w) &= \frac{\delta H}{\delta z_+}z_+ - z_-\frac{\delta H}{\delta z_-} + \left[\frac{\delta H}{\delta w}, w\right] - \left(\frac{\delta H}{\delta w}\right)' .
\end{aligned} \tag{4.30}$$

The Hamiltonian vector field $\mathbf{X}_H^1$ *corresponding to the reduced first* PB *reads*

$$\mathbf{X}_H^1(\ell) = \left[\ell, \frac{\delta H}{\delta \ell}\right]_+ , \qquad \mathbf{X}_H^1(z_\pm) = \pm\frac{\delta H}{\delta z_\mp}, \quad \mathbf{X}_H^1(w) = 0 . \tag{4.31}$$

Proof. On account of Theorem 4.2, in order to verify (4.30) it is enough to compute the Hamiltonian vector fields separately for functions on $\mathcal{M}_c/\mathcal{N}$ that have the special form $\int_{S^1} \mathrm{tr}(f_\pm z_\pm)$ or $\int_{S^1} \mathrm{tr}(\alpha w)$ with some matrix valued test functions $f_\pm$, α. This computation is presented in Appendix B. After writing down the formula of the reduced second PB from (4.30), it is easy to compute its Lie derivative with respect to the vector field V on $\mathcal{M}_{\mathrm{red}}$ given by

$$V(\ell) = \mathbf{1}_p, \qquad V(z_\pm) = V(w) = 0 . \tag{4.32}$$

We know from (2.17) that this gives the formula of the reduced first PB and we find

$$\{F, H\}_1^* = -\mathrm{Tr}\left(\ell\left[\frac{\delta F}{\delta \ell}, \frac{\delta H}{\delta \ell}\right]\right) + \int_{S^1} \mathrm{tr}\left(\frac{\delta F}{\delta z_+}\frac{\delta H}{\delta z_-} - \frac{\delta F}{\delta z_-}\frac{\delta H}{\delta z_+}\right) , \tag{4.33}$$

which is equivalent to (4.31). □

Note that the introduction of the "integrating factor" W in the above is only a notational trick which we used to get compact formulae. For instance,

$$\begin{aligned} P_0\left(\ell\frac{\delta H}{\delta z_-}W^{-1}\right)W &= (-\Gamma)^{-r}\tilde{\mathscr{D}}^r\left(\frac{\delta H}{\delta z_-}\right)+\sum_{k=1}^{r}u_k\tilde{\mathscr{D}}^{r-k}\left(\frac{\delta H}{\delta z_-}\right), \\ W^{-1}P_0^{\dagger}\left(W\frac{\delta H}{\delta z_+}\ell\right) &= \mathscr{D}^r\left(\frac{\delta H}{\delta z_+}\Gamma^{-r}\right)+\sum_{k=1}^{r}(-1)^{r-k}\mathscr{D}^{r-k}\left(\frac{\delta H}{\delta z_+}u_k\right), \end{aligned} \tag{4.34a}$$

where for arbitrary $s\times p$ and $p\times s$ matrix valued functions β and $\tilde{\beta}$ on S^1 we define their covariant derivatives

$$\mathscr{D}(\beta):=(\beta'+w\beta),\qquad \tilde{\mathscr{D}}(\tilde{\beta}):=(\tilde{\beta}'-\tilde{\beta}w)\,. \tag{4.34b}$$

All other terms containing W can be rewritten in terms of w in an analogous fashion.

Let us now consider a function H that depends on $\ell, z_\pm, w$ only through the Lax operator L in (4.25),

$$H(\ell,z_+,z_-,w)=\mathscr{H}(L)\,, \tag{4.35}$$

i.e., $H=\mathscr{H}\circ\bar{\pi}$ for some function $\mathscr{H}$ on M. Naturally, in this case we have the equality

$$\delta H=\mathrm{Tr}\left(\frac{\delta\mathscr{H}}{\delta L}\delta L\right)\,. \tag{4.36}$$

Comparing (4.26b) with (4.36) using (4.25) leads to the relations

$$\left(\frac{\delta\mathscr{H}}{\delta L}\right)_-=\frac{\delta H}{\delta\ell}\,, \tag{4.37a}$$

$$\frac{\delta H}{\delta z_+}=W^{-1}P_0^{\dagger}\left(Wz_-\left(\frac{\delta\mathscr{H}}{\delta L}\right)_+\right),\qquad \frac{\delta H}{\delta z_-}=P_0\left(\left(\frac{\delta\mathscr{H}}{\delta L}\right)_+z_+W^{-1}\right)W\,, \tag{4.37b}$$

$$\frac{\delta H}{\delta w}=-\mathrm{res}\left(W^{-1}\partial^{-1}Wz_-\left(\frac{\delta\mathscr{H}}{\delta L}\right)_+z_+W^{-1}\partial^{-1}W\right)\,. \tag{4.37c}$$

The Hamiltonian vectors fields $\mathbf{X}_H^i$ of Theorem 4.4 can then be simplified to give

$$\begin{aligned} \mathbf{X}_H^2(L) &= \left(L\frac{\delta\mathscr{H}}{\delta L}\right)_+L-L\left(\frac{\delta\mathscr{H}}{\delta L}L\right)_+\,, \\ \mathbf{X}_H^2(z_+) &= P_0\left(L\frac{\delta\mathscr{H}}{\delta L}z_+W^{-1}\right)W\,, \\ \mathbf{X}_H^2(z_-) &= -W^{-1}P_0^{\dagger}\left(Wz_-\frac{\delta\mathscr{H}}{\delta L}L\right)\,, \\ \mathbf{X}_H^2(w) &= 0\,, \end{aligned} \tag{4.38}$$

and

$$\begin{aligned}
\mathbf{X}^1_H(L) &= \left[L, \left(\frac{\delta \mathscr{H}}{\delta L}\right)_-\right]_+ - \left[L, \left(\frac{\delta \mathscr{H}}{\delta L}\right)_+\right]_- , \\
\mathbf{X}^1_H(z_+) &= P_0\left(\frac{\delta \mathscr{H}}{\delta L} z_+ W^{-1}\right) W , \\
\mathbf{X}^1_H(z_-) &= -W^{-1} P_0^\dagger\left(W z_- \frac{\delta \mathscr{H}}{\delta L}\right) , \\
\mathbf{X}^1_H(w) &= 0 .
\end{aligned} \tag{4.39}$$

These formulae are consistent with the claims of Theorems 4.2 and 4.3. They will be used at the end of Sect. 5 to determine the evolution equations of the KdV type hierarchy that results from the Drinfeld–Sokolov reduction.

Remark 4.1. The compact presentation of the formulae for the PBs in (4.38), (4.39), (4.30) and (4.31) was suggested by the formulae in [13]. It may be verified that our PBs reduce to those in [13] in the scalar case $p = s = 1$ upon constraining to $w = 0$.

Remark 4.2. Notice that for $r = p = s = 1$ the AKS hierarchy on $\mathscr{M}$ is the gl_2 version of the well-known AKNS hierarchy. For $r = 1$ and arbitrary p, s (see Remark 2.2.), it is reasonable to call the system on $\mathscr{M}$ a generalized AKNS hierarchy. For the generalized AKNS hierarchy the pseudo-differential Lax operator associated to $\mathscr{L} \in \mathscr{M}$ becomes just the operator ΔK (4.19). For this reason we can call ΔK in the factorization (4.18) the "AKNS factor." (It is an easy exercise to directly verify the equivalence between the respective formulae (1.2–3) and (4.30–31) for $r = 1$.) In the simplest case $r = p = s = 1$ the connection between the AKNS hierarchy on $\mathscr{M}$ and the constrained KP hierarchy on M was observed in [12, 13, 25] too. The nonlinear Schrödinger (NLS) hierarchy results from constraining the AKNS hierarchy, and it has many generalizations [26]. The connection between generalized AKNS and NLS systems and constrained (matrix) KP systems given by the results in Sects. 4 and 5 can be extended to more general cases than those treated in this paper.

5. Local Monodromy Invariants and Residues of Fractional Powers

In Sect. 4 we established a relationship between the Poisson brackets on the reduced phase space $\mathscr{M}_{\mathrm{red}} \simeq \mathscr{M}_c/\mathscr{N}$ and the Gelfand–Dickey Poisson brackets on $M \subset \mathscr{A}$. Our next task is to characterize the Hamiltonians generated by the *local* monodromy invariants of $\mathscr{L} \in \mathscr{M}_c$. These Hamiltonians, which define the commuting hierarchy of evolution equations on $\mathscr{M}_{\mathrm{red}}$, turn out to admit a description purely in terms of the Lax operator $L \in M$ attached to $\mathscr{L}$ by the elimination procedure, $L = \pi(\mathscr{L})$. Namely, the Hamiltonians defined by the local monodromy invariants of $\mathscr{L}$ can be identified in terms of integrals of componentwise residues of fractional powers of the diagonal PDO $\hat{L}$ obtained by diagonalizing L in the PDO algebra $\mathscr{A}$. This identification results from computing the local monodromy invariants of $\mathscr{L} \in \mathscr{M}_{\mathrm{DS}}$ in two alternative ways: first using the procedure of (1.10–11) outlined in Sect. 1 and second using the diagonalization of L combined with a reverse of the elimination procedure. The same method was used in [1], but the presence of the singlets in the partition (2.1) gives rise to complications requiring a non-trivial refinement of the argument.

5.1. Local monodromy invariants and solutions of exponential type. We wish to compute the local invariants of the monodromy matrix T associated to the linear problem

$$\mathscr{L}\Psi = 0 \;\Leftrightarrow\; (\partial_x + j(x) + \Lambda)\Psi(x) = 0\,, \tag{5.1}$$

where $j(x+2\pi) = j(x)$, since $\mathscr{L} = (\partial + j + \Lambda) \in \mathscr{M}_c$. If $\Psi : \mathbb{R} \to GL_n$ is a solution of (5.1), which means that the columns of the matrix Ψ are a complete set of solutions, then the monodromy matrix is given by

$$T = \Psi(2\pi)\Psi^{-1}(0)\,. \tag{5.2}$$

Following the procedure outlined in Sect. 1, perform the transformation

$$\mathscr{L} \mapsto \tilde{\mathscr{L}} = e^{\Xi}\mathscr{L}e^{-\Xi} = (\partial + h + \Lambda), \quad \tilde{\Psi} = e^{\Xi}\Psi\,, \tag{5.3}$$

where

$$\Xi \in (\mathrm{Im}(\mathrm{ad}\,\Lambda))_{<0}, \qquad h \in (\mathrm{Ker}(\mathrm{ad}\,\Lambda))_{<1}\,, \tag{5.4}$$

and the subscripts refer to the grading d in (2.7). The fact that Ξ and h are uniquely determined differential polynomial expressions in the components of j implies that $\Xi(j(x))$ and $h(j(x))$ are periodic functions of $x \in \mathbb{R}$. It follows that the invariants of the monodromy matrix T that we are interested in are the same as the invariants of the transformed monodromy matrix

$$\tilde{T} := \tilde{\Psi}(2\pi)\tilde{\Psi}^{-1}(0) = G^{-1}(0)TG(0)\,, \tag{5.5}$$

with the definition

$$G(x) := \exp(-\Xi(j(x)))\,. \tag{5.6}$$

Using the notation (2.2–3), h in (5.3) may be written as

$$h(j) = \begin{pmatrix} A(j) & 0 \\ 0 & D(j) \end{pmatrix}, \tag{5.7a}$$

where $A(j)$ and $D(j)$ are uniquely determined series of the form

$$A(j) = \sum_{k=0}^{\infty}\sum_{i=1}^{p} h_{k,i}(j)\,\Lambda_r^{-k} \otimes e_{i,i}, \qquad D(j) = \sum_{k=0}^{\infty} \lambda^{-k} e_{r+1,r+1} \otimes D_k(j)\,, \tag{5.7b}$$

with $h_{k,i}(j(x)) \in \mathbb{C}$ and $D_k(j(x)) \in gl_s$. A basis of the centre of $\mathrm{Ker}(\mathrm{ad}\,\Lambda) \subset \ell(gl_n)$ is given in (2.5) and (2.10). The Hamiltonians defined by the procedure in (1.10–11) are then

$$H_{k,i}(j) := \int_0^{2\pi} dx\, h_{k,i}(j(x)), \quad i = 1,\ldots,p, \;\; k = 0,1,2,\ldots \tag{5.8}$$

and

$$E_k(j) := \int_0^{2\pi} dx\, \mathrm{tr}\, D_k(j(x))\,. \tag{5.9}$$

Taking the trace of Eq. (5.3) using the identity $\Lambda_r^{-kr} = \lambda^{-k}\mathbf{1}_r$, we obtain the equality

$$E_k(j) = \delta_{k,0}\int_0^{2\pi} dx\,\mathrm{tr}\,j(x) - r\sum_{i=1}^{p} H_{kr,i}(j)\,. \tag{5.10}$$

Since $j \mapsto \int_0^{2\pi} dx\,\mathrm{tr}\,j(x)$ defines a Casimir function with respect to both Poisson brackets on $\mathcal{M}_{\mathrm{red}}$, this equality means that the complete set of independent Hamiltonians associated to the centre of $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$ is given by the $H_{k,i}(j)$ above. General arguments that go back to the r-matrix (AKS) construction (see e.g. [10]) guarantee that the Hamiltonians $H_{k,i}$ are in involution (commute among themselves) since they can be interpreted as particular monodromy invariants. To explain this interpretation, notice that the transformed linear problem $\tilde{\mathscr{L}}\tilde{\Psi} = 0$ has the solution

$$\tilde{\Psi}(x) = \begin{pmatrix} \tilde{\Psi}_{11}(x) & 0 \\ 0 & \tilde{\Psi}_{22}(x) \end{pmatrix}, \tag{5.11}$$

where

$$\tilde{\Psi}_{11}(x) = \exp\left(-x\Lambda_r \otimes \Gamma - \sum_{k=0}^{\infty}\sum_{i=1}^{p}\int_0^x d\xi\, h_{k,i}(j(\xi))\Lambda_r^{-k} \otimes e_{i,i}\right), \tag{5.12}$$

and

$$\tilde{\Psi}_{22}(x) = \mathscr{P}\exp\left(-\int_0^x d\xi \sum_{k=0}^{\infty} \lambda^{-k} D_k(j(\xi))\right). \tag{5.13}$$

The corresponding monodromy matrix is

$$\tilde{T} = \begin{pmatrix} \tilde{\Psi}_{11}(2\pi) & 0 \\ 0 & \tilde{\Psi}_{22}(2\pi) \end{pmatrix}. \tag{5.14}$$

To diagonalize $\tilde{T}$, let ζ be any r^{th} root of λ, $\zeta^r = \lambda$, and define

$$\zeta_a := \zeta\omega^a \quad \text{with } \omega := \exp(2i\pi/r)\,. \tag{5.15}$$

The matrix Λ_r is conjugate to

$$\tilde{\Lambda}_r := \mathrm{diag}(\zeta_1, \zeta_2, \ldots, \zeta_r)\,. \tag{5.16}$$

In fact, we have

$$\Lambda_r = S\tilde{\Lambda}_r S^{-1}, \tag{5.17a}$$

with

$$S_{ab} = \frac{1}{\sqrt{r}}(\zeta_b)^{a-1}, \qquad (S^{-1})_{ab} = \frac{1}{\sqrt{r}}(\zeta_a)^{1-b}\,. \tag{5.17b}$$

Using this conjugation the upper block $\tilde{\Psi}_{11}(2\pi)$ of $\tilde{T}$ becomes diagonal,

$$\tilde{\Psi}_{11}(2\pi) = (S \otimes \mathbf{1}_p)\exp\left(-2\pi\tilde{\Lambda}_r \otimes \Gamma - \sum_{k=0}^{\infty}\sum_{i=1}^{p} H_{k,i}(j)\,\tilde{\Lambda}_r^{-k} \otimes e_{i,i}\right)(S^{-1} \otimes \mathbf{1}_p)\,. \tag{5.18}$$

Hence, up to the constant $-2\pi\tilde{\Lambda}_r \otimes \Gamma$, the Hamiltonians $H_{k,i}(j)$ can be identified as expansion coefficients defining the expansions of logarithms of certain eigenvalues of the monodromy matrix around $\zeta \approx \infty$. As usual, this expansion has to be interpreted as an asymptotic – or formal – series. It is clear from (5.13) that the spectral invariants determined by the "lower block" $\tilde{\Psi}_{22}(2\pi)$ of the monodromy matrix are in general, except the functionals $E_k(j)$ given above, *non-local* functionals of j.

We have seen that the local monodromy invariants $H_{k,i}(j)$ associated to the centre of $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$ are determined purely in terms of the upper block $\tilde{\Psi}_{11}(2\pi)$ of the transformed monodromy matrix $\tilde{T}$. To see what this means in terms of the original linear problem (5.1) consider the solution Ψ given by

$$\Psi(x) := G(x)\tilde{\Psi}(x)\begin{pmatrix} S\otimes \mathbf{1}_p & 0 \\ 0 & \mathbf{1}_s \end{pmatrix}. \tag{5.19}$$

Using a block notation similar to (2.2–3),

$$\Psi(x) = \begin{pmatrix} \Psi_{11}(x) & \Psi_{12}(x) \\ \Psi_{21}(x) & \Psi_{22}(x) \end{pmatrix}, \qquad G(x) = e^{-\Xi(j(x))} = \begin{pmatrix} G_{11}(x) & G_{12}(x) \\ G_{21}(x) & G_{22}(x) \end{pmatrix}, \tag{5.20}$$

we have

$$\begin{pmatrix} \Psi_{11} \\ \Psi_{21} \end{pmatrix} = \begin{pmatrix} G_{11}\tilde{\Psi}_{11} S\otimes \mathbf{1}_p \\ G_{21}\tilde{\Psi}_{11} S\otimes \mathbf{1}_p \end{pmatrix} \tag{5.21a}$$

and

$$\begin{pmatrix} \Psi_{12} \\ \Psi_{22} \end{pmatrix} = \begin{pmatrix} G_{12}\tilde{\Psi}_{22} \\ G_{22}\tilde{\Psi}_{22} \end{pmatrix}. \tag{5.21b}$$

It follows from (5.4) that the infinite series $\Xi(j(x))$ contains only *non-positive* powers of λ and therefore the entries of the matrix G are given by similar series. This together with (5.12–13), (5.17) implies that the column vector solutions of the linear problem $\mathscr{L}\psi = 0$ comprising the matrices in (5.21a) and in (5.21b) are qualitatively different. They are different in the sense that – apart from the constant $S(\zeta)\otimes \mathbf{1}_p$ that we included in the definition of Ψ for later convenience – the columns of the matrix in (5.21a) have the form of descending series in non-positive powers of ζ multiplied by a leading term of the type $e^{-x\zeta_b\Gamma}$ while the columns of the matrix in (5.21b) do not contain such a leading term only a descending series in non-positive powers of ζ. We refer to the series solutions containing a leading term $e^{-x\zeta_b\Gamma}$ as "solutions of exponential type." The solutions of exponential type *contain all information about the local monodromy invariants* since the matrix $\tilde{\Psi}_{11}(2\pi)$ in (5.18), whose eigenvalues generate the Hamiltonians $H_{k,i}(j)$, is conjugate to the matrix

$$\Psi_{11}(2\pi)(\Psi_{11}(0))^{-1} = G_{11}(0)\tilde{\Psi}_{11}(2\pi)(G_{11}(0))^{-1}. \tag{5.22}$$

It is convenient to write the $(rp+s)\times(rp)$ matrix in (5.21a) in the following detailed form:

$$\begin{pmatrix} \Psi_{11} \\ \Psi_{21} \end{pmatrix} = \begin{pmatrix} \psi_1^1 & \cdots & \psi_1^r \\ \psi_2^1 & \cdots & \psi_2^r \\ \vdots & & \vdots \\ \psi_r^1 & \cdots & \psi_r^r \\ \phi^1 & \cdots & \phi^r \end{pmatrix}, \quad (\psi_a^b \in gl_p,\ \phi^b \in \mathrm{mat}(s\times p)). \tag{5.23}$$

Putting the above formulae together we have

$$\begin{aligned}\psi_a^b(x,\zeta) &= \frac{1}{\sqrt{r}}\sum_{c=1}^{r} G_{11}^{ac}(x,\lambda)(\zeta_b)^{c-1}\exp(-\mathscr{D}(x,\zeta_b) - x\zeta_b\Gamma)\,,\\ \phi^b(x,\zeta) &= \frac{1}{\sqrt{r}}\sum_{c=1}^{r} G_{21}^{c}(x,\lambda)(\zeta_b)^{c-1}\exp(-\mathscr{D}(x,\zeta_b) - x\zeta_b\Gamma)\,,\end{aligned} \tag{5.24a}$$

where G_{11}^{ac} and G_{21}^{c} are $p \times p$ and $s \times p$ matrices, respectively, and $\mathscr{D}(x,\zeta)$ is the $p \times p$ diagonal matrix series

$$\mathscr{D}(x,\zeta) = \sum_{k=0}^{\infty} \zeta^{-k} \int_0^x d\xi\, \mathrm{diag}(h_{k,1}(j(\xi)),\dots,h_{k,p}(j(\xi)))\,. \tag{5.24b}$$

Observe that all the gauge invariant components $\psi_1^b(x,\zeta)$ can be obtained from $\psi_1^r(x,\zeta)$ simply replacing the argument $\zeta = \zeta_r$ by ζ_b. Since the other components of the solution in (5.23) are determined by the ψ_1^b by means of (4.9), this means that the complete set of solutions of exponential type can be recovered from $\psi_1^r(x,\zeta)$. In particular, the local monodromy invariants $H_{k,i}$ in (5.8) can be read off from the relation

$$\psi_1^r(x+2\pi,\zeta) = \psi_1^r(x,\zeta)\exp\left(-2\pi\zeta\Gamma - \sum_{k=0}^{\infty}\zeta^{-k}\mathrm{diag}(H_{k,1},\dots,H_{k,p})\right), \tag{5.25}$$

which is a consequence of (5.24). This relation will play a crucial rôle in finding the link between the local monodromy invariants and the residues of fractional powers, which is the ultimate aim of the present section.

In this subsection we obtained the matrix solution (5.23) of exponential type to the linear problem (5.1) by transforming $\mathscr{L}$ to $\tilde{\mathscr{L}}$ and imposing on the solution $\tilde{\Psi}$ (5.11) of $\tilde{\mathscr{L}}\tilde{\Psi} = 0$ the condition $\tilde{\Psi} = \mathbf{1}_n$ at $x = 0$, see (5.12–13). We noticed that the matrix solution in (5.23) is determined completely by the block ψ_1^r. We then observed that the block $\psi_1^r(x,\zeta)$ directly encodes the Hamiltonians of our interest, the Hamiltonians $H_{k,i}$ given in (5.8), through the relation (5.25). In the next subsection we will consider the consequence of looking at ψ_1^r from a different point of view, namely as a solution of Eq. (4.11). The result will be an explicit connection between the family $\{H_{k,i}\}$ of local monodromy invariants and the components of the residues of the fractional powers of the diagonalized form of L.

5.2. Solutions of exponential type and residues of fractional powers. Consider the $p \times p$ matrix PDO

$$L = \Delta^r\partial^r + u_1\partial^{r-1} + u_2\partial^{r-2} + \cdots + u_{r-1}\partial + u_r + z_+(\mathbf{1}_s\partial + w)^{-1}z_-, \quad \Delta = -\Gamma^{-1}\,, \tag{5.26}$$

attached to $\mathscr{L} \in \mathscr{M}_c$ by the elimination procedure. The strategy of this subsection will be to determine ψ_1^r in (5.24) as a solution of the linear problem

$$L\psi_1 = \lambda\psi_1 \tag{5.27a}$$

given by a (asymptotic or formal) series of the form

$$\psi_1(x,\zeta) = \left(\sum_{k=0}^{\infty} \chi_k(x)\zeta^{-k}\right) e^{-x\zeta\Gamma} \quad \text{with } \chi_k \in C^\infty(\mathbb{R}, gl_p),\ \det(\chi_0(x)) \neq 0\,. \tag{5.27b}$$

The elimination procedure implies that $\psi_1^r(x,\zeta)$ in (5.24) is a solution of (5.27a). We shall see below that $\psi_1^r(x,\zeta)$ can be expanded in the form given in (5.27b) and that the solution of (5.27a), (5.27b) is essentially unique.

That ψ_1^r in (5.24) can be expanded in the form given in (5.27b) can be seen by inspection. The key step is to check that the series Ξ defining $G = e^{-\Xi} = \mathbf{1}_n - \Xi + \frac{1}{2}\Xi^2 \cdots$ in (5.20) contains only negative powers of λ in its first row due to the grading condition (5.4), which implies that the leading term of ψ_1^r comes from the unit matrix contained in the $c = 1$ contribution in the sum in (5.24a). Computing the first term of the "abelianised current" $h(j)$ in (5.7), one obtains

$$h_{0,i}(j) = \frac{1}{r}((-\Gamma)^r u_1)_{ii}\,, \tag{5.28a}$$

where u_1 is the gauge invariant component of j entering the Lax operator L (5.26) attached to $\mathcal{L} = (\partial + j + \Lambda)$. It follows that when rewritten as a series of the form (5.27b), the leading term χ_0 of ψ_1^r in (5.24) is given by

$$\chi_0(x) = \exp\left(-\frac{1}{r}(-\Gamma)^r \int_0^x d\xi\,(u_1(\xi))_{\mathrm{diag}}\right), \tag{5.28b}$$

and indeed has non-zero determinant. Incidentally, the constant factor $S(\zeta)$ was inserted in the definition (5.19) to set the leading power of ζ in (5.27b), which multiplies the product of χ_0 (5.28b) and $e^{-x\zeta\Gamma}$, to be ζ^0.

Below our aim is to determine $\psi_1(x,\zeta)$ from Eqs. (5.27a) and (5.27b). To make precise the meaning of Eq. (5.27a), which has been derived by formally applying the elimination procedure, we note that an arbitrary $p \times p$ matrix PDO $\alpha = \sum_i \alpha_i(x)\partial^i$ acts on a series of the form $\psi_1(x,\zeta)$ in (5.27b) as follows. Defining the action of ∂^i for *any* integer i on $e^{-x\zeta\Gamma}$ by $(\partial^i e^{-x\zeta\Gamma}) := (-\zeta\Gamma)^i e^{-x\zeta\Gamma}$ one first associates the PDO χ to ψ_1 (5.27b) by writing

$$\psi_1(x,\zeta) = (\chi e^{-x\zeta\Gamma}), \quad \text{i.e. } \chi(x,\partial) = \sum_{k=0}^{\infty} \chi_k(x)\Delta^{-k}\partial^{-k}\,. \tag{5.29a}$$

Then $(\alpha\psi_1)(x,\zeta) := (\beta e^{-x\zeta\Gamma})$ with $\beta = \alpha\chi$ being defined by the composition rule of PDOs (as in (4.2)). To avoid confusion, we stress that here the coefficients of the PDOs χ, α, β are not required to be periodic functions on $\mathbb{R}$. When Eq. (5.27a) is understood in this sense it is easily seen to be equivalent to the "dressing equation"

$$\chi^{-1}L\chi = \Delta^r\partial^r\,. \tag{5.29b}$$

This reformulation of (5.27) is well-known, see [27] and references therein. Using the reformulation (5.29) and the fact that Δ^r has distinct, non-zero eigenvalues, it is not hard to verify[2] that (5.27a) uniquely determines the series solution $\psi_1(x,\zeta)$

[2] A proof can be found in [28] for the case when $(u_1)_{\mathrm{diag}} = 0$ and $\chi_0 = \mathbf{1}_p$, but these assumptions can be dropped.

of the form (5.27b) up to multiplication on the right by a constant (x-independent) series $c(\zeta)$ of the form

$$c(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^{-k} \quad \text{with } \det(c_0) \neq 0\,, \tag{5.30}$$

where all the c_k are $p \times p$ *diagonal* matrices.

Thanks to the above uniqueness property, the following procedure may be used to determine the series $\psi_1(x,\zeta)$. First we determine a $p \times p$ matrix PDO g of the form

$$g = \mathbf{1}_p + \sum_{i=1}^{\infty} g_i \partial^{-i}\,, \tag{5.31}$$

with periodic coefficients, $g_i(x+2\pi) = g_i(x)$, such that

$$L = g\hat{L}g^{-1}\,, \tag{5.32}$$

where $\hat{L}$ is *diagonal*, i.e.,

$$\hat{L} = \Delta^r \partial^r + \sum_{i=1}^{\infty} a_i \partial^{r-i}, \qquad a_i : \text{all diagonal}\,. \tag{5.33}$$

Since Δ^r is a diagonal matrix with distinct, non-zero entries, then if we require the g_i's to be off-diagonal matrices we can recursively determine both the g_i's and the a_i's by comparing the two sides of $Lg = g\hat{L}$ term-by-term, according to powers of ∂. The solution is given by unique differential polynomial expressions in the coefficients defining the expansion of L in ∂. For instance, we have

$$a_1 = (u_1)_{\mathrm{diag}}\,. \tag{5.34}$$

Then we consider a $p \times p$ matrix series

$$\hat{\psi}_1(x,\zeta) = \left(\sum_{k=0}^{\infty} \zeta^{-k} \hat{\chi}_k(x)\right) e^{-x\zeta\Gamma} \quad \text{with } \det(\hat{\chi}_0(x)) \neq 0\,, \tag{5.35}$$

which satisfies the equation

$$(\hat{L}\hat{\psi}_1)(x,\zeta) = \lambda\hat{\psi}_1(x,\zeta), \quad (\lambda = \zeta^r)\,. \tag{5.36}$$

At the end of the procedure, we find the desired solution of (5.27) from

$$\psi_1(x,\zeta) := (g\hat{\psi}_1)(x,\zeta)\,. \tag{5.37}$$

We know that up to a diagonal constant matrix $c(\zeta)$ of the form (5.30) the series solution of (5.27) determined by this procedure must coincide with $\psi_1^r(x,\zeta)$ given by Eq. (5.24). To proceed further we need the following identity.

Proposition 5.1. *The operator $\mathbf{1}_p\partial$ can be expressed as*

$$\mathbf{1}_p\partial = -\Gamma\hat{L}^{1/r} + \sum_{k=0}^{\infty} \mathscr{F}_k \Delta^k (\hat{L}^{1/r})^{-k}, \tag{5.38}$$

with $\hat{L}^{1/r}$ being defined by

$$(\hat{L}^{1/r})^r = \hat{L}, \qquad \hat{L}^{1/r} = \Delta\partial + \sum_{i=0}^{\infty} b_i \partial^{-i} \quad \textit{for some } b_i, \tag{5.39}$$

and uniquely determined $p \times p$ diagonal matrix valued functions $\mathscr{F}_k$, which satisfy

$$\mathscr{F}_0 = -\frac{1}{r}(-\Gamma)^r a_1 \quad \textit{and} \quad \int_0^{2\pi} dx\,(k\mathscr{F}_k + (-\Gamma)^k \operatorname{res}(\hat{L}^{k/r}))(x) = 0 \quad \textit{for } k > 0\,. \tag{5.40}$$

The above proposition, which was crucial in [1] for obtaining results analogous to those under consideration here, is taken from [4] and is originally due to Cherednik [29] (see also [27, 30]). Since $\hat{\psi}_1$ is uniquely determined by (5.35–36) up to multiplication by a diagonal constant matrix $c(\zeta)$ of the form given in (5.30), Eq. (5.36) implies

$$(\hat{L}^{1/r}\hat{\psi}_1)(x,\zeta) = \zeta\hat{\psi}_1(x,\zeta)\,. \tag{5.41}$$

This together with (5.38) leads to

$$\begin{aligned} \hat{\psi}_1'(x,\zeta) &= \left(-\zeta\Gamma + \sum_{k=0}^{\infty} \mathscr{F}_k(x)\Delta^k\zeta^{-k}\right)\hat{\psi}_1(x,\zeta) \\ &\Rightarrow \hat{\psi}_1(x,\zeta) = \exp\left(-x\zeta\Gamma + \sum_{k=0}^{\infty} \Delta^k\zeta^{-k}\int_0^x d\xi\,\mathscr{F}_k(\xi)\right)\hat{\psi}_1(0,\zeta)\,, \end{aligned} \tag{5.42}$$

where $\hat{\psi}_1(0,\zeta)$ is an arbitrary diagonal matrix series of the form given in (5.30). Combining this with (5.34) and (5.40) results in the crucial relation

$$\hat{\psi}_1(x+2\pi,\zeta) = \hat{\psi}_1(x,\zeta)\tau(\zeta) \tag{5.43a}$$

with

$$\tau(\zeta) = \exp\left(-2\pi\zeta\Gamma - \frac{(-\Gamma)^r}{r}\int_0^{2\pi} dx\,(u_1(x))_{\text{diag}} - \sum_{k=1}^{\infty}\frac{\zeta^{-k}}{k}\int_0^{2\pi} dx \operatorname{res}(\hat{L}^{k/r})(x)\right). \tag{5.43b}$$

Observe now that $\psi_1(x,\zeta)$ defined in (5.37) satisfies

$$\psi_1(x+2\pi,\zeta) = \psi_1(x,\zeta)\tau(\zeta) \tag{5.44}$$

with the same $\tau(\zeta)$, because the coefficients $g_i(x)$ defining the PDO g are periodic functions of x. This immediately leads to the following result.

Theorem 5.2. *The set of commuting Hamiltonians provided by the local monodromy invariants $H_{k,i}(j)$ given in (5.8) is exhausted by the Hamiltonians $H_{0,i}(j)$ together with the Hamiltonians defined by the residues of componentwise fractional powers of the* PDO $\hat{L}$ (5.33) *obtained by diagonalizing the Lax operator* L (5.26) *attached to* $\mathscr{L} = (\partial + j + \Lambda)$ *by the elimination procedure. More precisely,*

$$\begin{aligned}
\operatorname{diag}(H_{0,1},\ldots,H_{0,p}) &= \frac{(-\Gamma)^r}{r}\int_0^{2\pi} dx\,(u_1(x))_{\mathrm{diag}}\,,\\
\operatorname{diag}(H_{k,1},\ldots,H_{k,p}) &= \frac{1}{k}\int_0^{2\pi} dx\,\operatorname{res}(\hat{L}^{k/r})(x), \quad \textit{for } k > 0\,.
\end{aligned} \tag{5.45}$$

Proof. The statement follows by comparing (5.25) with (5.44) taking into account that $\psi_1(x,\zeta) = \psi_1^r(x,\zeta)c(\zeta)$, where $c(\zeta)$ is a diagonal constant matrix of the form (5.30). □

Finally, we shall write down the evolution equations generated by the above Hamiltonians on $\mathscr{M}_{\mathrm{red}} = \mathscr{M}_c/\mathscr{N}$. For this it is convenient to consider an arbitrary $p \times p$ diagonal constant matrix Q and associate to it the Hamiltonian

$$H_k^Q := r\sum_{i=1}^{p} Q_{ii}H_{k,i}\,. \tag{5.46}$$

Observe that H_k^Q has the form (4.35) since

$$H_k^Q(\ell, z_+, z_-, w) = \mathscr{H}_k^Q(L) \tag{5.47a}$$

with

$$\begin{aligned}
\mathscr{H}_0^Q(L) &= \int_0^{2\pi} dx\,\operatorname{tr}((-\Gamma)^r Q u_1(x))\,,\\
\mathscr{H}_k^Q(L) &= \frac{r}{k}\operatorname{Tr}(Q\hat{L}^{k/r}), \quad k \geqq 1\,.
\end{aligned} \tag{5.47b}$$

Using (4.36) the gradients of these functions are found to be

$$\frac{\delta\mathscr{H}_0^Q}{\delta L} = (-\Gamma)^r Q\partial^{-r}, \qquad \frac{\delta\mathscr{H}_k^Q}{\delta L} = gQg^{-1}L^{(k-r)/r}, \quad k \geqq 1\,, \tag{5.48}$$

where g is the PDO (5.31) which diagonalizes L and

$$L^{l/r} := g\hat{L}^{l/r}g^{-1}, \quad \forall l\,. \tag{5.49}$$

Note that $L^{l/r}$ commutes with gQg^{-1}. We let $\mathbf{X}_{k,Q}^i$ denote the Hamiltonian vector field associated to H_k^Q by means of the first and second PBs on $\mathscr{M}_{\mathrm{red}}$ given in Theorem 4.4. Inserting the gradients (5.48) into formulae (4.38) and (4.39) we obtain the following result.

Corollary 5.3. *The Hamiltonian vector fields* $\mathbf{X}^i_{k,Q}$ *defining the local hierarchy of compatible evolution equations on* $\mathcal{M}_c/\mathcal{N}$ *take the form*

$$\begin{aligned}
\mathbf{X}^2_{k,Q}(L) &= \mathbf{X}^1_{k+r,Q}(L) = [(gQg^{-1}L^{k/r})_+, L]\,,\\
\mathbf{X}^2_{k,Q}(z_+) &= \mathbf{X}^1_{k+r,Q}(z_+) = P_0(gQg^{-1}L^{k/r}z_+W^{-1})W\,,\\
\mathbf{X}^2_{k,Q}(z_-) &= \mathbf{X}^1_{k+r,Q}(z_-) = -W^{-1}P_0^\dagger(Wz_-gQg^{-1}L^{k/r})\,,\\
\mathbf{X}^2_{k,Q}(w) &= \mathbf{X}^1_{k+r,Q}(w) = 0, \quad \forall k = 0, 1, \dots\,.
\end{aligned} \tag{5.50}$$

In particular, the flows are bihamiltonian. In terms of the notation

$$\left(gQg^{-1}L^{k/r}\right)_+ := \sum_{i=0}^{k} A_{k,i}\partial^i \tag{5.51}$$

and the covariant derivatives defined in (4.34b) *the second and third equations in* (5.50) *can be rewritten as*

$$\begin{aligned}
\mathbf{X}^2_{k,Q}(z_+) = \mathbf{X}^1_{k+r,Q}(z_+) &= \sum_{i=0}^{k} A_{k,i}\tilde{\mathscr{D}}^i(z_+)\,,\\
\mathbf{X}^2_{k,Q}(z_-) = \mathbf{X}^1_{k+r,Q}(z_-) &= -\sum_{i=0}^{k}(-1)^i\mathscr{D}^i(z_-A_{k,i})\,.
\end{aligned} \tag{5.52}$$

This completes our general analysis of the KdV type hierarchy resulting from the generalized Drinfeld–Sokolov reduction defined in Sect. 2. The evolution equations associated to the vector fields in (5.50) define natural "covariantized matrix generalizations" of the constrained KP hierarchy considered previously in the literature (see [12–14] and references therein) in the scalar case $p = s = 1$ with the constraint $w = 0$. Notice that w and the diagonal components of u_1 (the subleading term of L in (5.26)) do not evolve with respect to the flows determined by the vector fields in (5.50). The reason for this is the fact that $(u_1)_{ii}$ for $i = 1, \dots, p$ and the components of w are Casimir functions with respect to the reduced first PB, and generators of residual symmetries with respect to the reduced second PB. Indeed, evaluating the gauge invariant current component Φ_i given by (3.3a) in the DS gauge (2.18), we see that Φ_i is proportional with $(u_1)_{ii}$.

6. Discussion

In this paper we derived a Gelfand–Dickey type PDO model of an integrable hierarchy resulting from generalized Drinfeld–Sokolov reduction. The reduction was based on the grade 1 semisimple element $\Lambda_{p,r,s}$ in (0.6), which belongs to the Heisenberg subalgebra of $\ell(gl_n)$ associated with the partion of $n = pr + s$ in (2.1). The reduced phase space $\mathcal{M}_{\rm red} = \mathcal{M}_c/\mathcal{N}$ turned out to be the space of quadruplets (ℓ, z_+, z_-, w), where ℓ is a $p \times p$ matrix r-KdV type operator which is coupled to the fields $z_\pm, w$. The compatible PBs and the commuting Hamiltonians of the hierarchy on $\mathcal{M}_{\rm red}$ are given by Theorems 4.4 and 5.2 with the corresponding evolution equations being determined in Corollary 5.3. These results extend the results of [1] on the matrix r-KdV system for which $s = 0$.

We wish to note that the above systems possess "discrete symmetries" given by certain involutive Poisson maps $\sigma_{m,q}$ on $\mathcal{M}_{\mathrm{red}}$ according to

$$\sigma_{m,q}: \begin{pmatrix} \ell \\ z_+ \\ z_- \\ w \end{pmatrix} \mapsto \begin{pmatrix} m\ell^\dagger m^{-1} \\ -mz_-^t q^{-1} \\ qz_+^t m^{-1} \\ -qw^t q^{-1} \end{pmatrix}, \tag{6.1}$$

where $\ell^\dagger = (-1)^r \Delta^r \partial^r + \sum_{i=1}^r (-1)^{r-i} \partial^{r-i} u_i^t$ for $\ell = \Delta^r \partial^r + \sum_{i=1}^r u_i \partial^{r-i}$, and the matrices $m \in GL_p$, $q \in GL_s$ are constants determined by the requirements that $\sigma_{m,q}$ must map $\mathcal{M}_{\mathrm{red}}$ to itself and $\sigma_{m,q}^2 = \mathrm{id}$ must hold. Given an involutive symmetry $\sigma := \sigma_{m,q}$, the commuting Hamiltonians in Theorem 5.2 admit a basis consisting of invariant and anti-invariant (that change sign) linear combinations with respect to σ. Hence a "discrete reduced hierarchy" may be defined by restricting the flows generated on $\mathcal{M}_{\mathrm{red}}$ by the σ-invariant Hamiltonians to the fixed point set $\mathcal{M}_{\mathrm{red}}^\sigma \subset \mathcal{M}_{\mathrm{red}}$ of σ. These flows are bihamiltonian with respect to the restricted Hamiltonians and the bihamiltonian structure on $\mathcal{M}_{\mathrm{red}}^\sigma$, defined by restricting the original PBs of functions of σ-invariant linear combinations of the components of ℓ, z_+, z_-, w – which may be regarded as coordinates on $\mathcal{M}_{\mathrm{red}}^\sigma$ – to $\mathcal{M}_{\mathrm{red}}^\sigma$.

As explained in particular cases in [20], the above discrete reductions are actually induced by the reductions of gl_n to a simple Lie algebra of B, C or D type. Correspondingly, many generalized KdV hierarchies that may be associated with graded semisimple elements of $\ell(\mathcal{G})$ for $\mathcal{G}$ a classical simple Lie algebra by Drinfeld–Sokolov reduction can be also obtained from discrete reduction applied to the matrix r-KdV hierarchy or its extended version associated with gl_n.

We also dealt with the modified KdV type system related to the factorization of the Lax operator $L = \ell + z_+(\mathbf{1}_s\partial + w)^{-1} z_-$ in (4.18). Alternative factorizations of L leading to isomorphic modified systems can be defined by simply changing the position of the AKNS factor K in (4.18). A more interesting possibility is to obtain a new modification by further factorizing K as

$$K = \left(\mathbf{1}_p\partial + a - b(\mathbf{1}_s\partial + d)^{-1}c\right) = (\mathbf{1}_p\partial + \vartheta)(\mathbf{1}_p - \Delta\gamma(\mathbf{1}_s\partial + \chi + \xi\Delta\gamma)^{-1}\xi) \tag{6.2}$$

with $\xi \in \widetilde{\mathrm{mat}}(s \times p)$, $\gamma \in \widetilde{\mathrm{mat}}(p \times s)$, $\chi \in \widetilde{gl}_s$ and $\vartheta \in \widetilde{gl}_p$. All these modified KdV systems fit into the Drinfeld–Sokolov framework [4–8] in which modifications of KdV systems correspond to special gauge fixings if one also considers versions of the underlying Hamiltonian reduction. These can be defined by replacing H in the grading operator in (2.7) by $H_k := I_0 + \frac{k}{2}\,\mathrm{diag}\,(\mathbf{1}_{pr}, \mathbf{0}_s)$ with any $k = 0, \pm 1, \pm 2, \ldots, \pm(r-1)$, where $2I_0 = \mathrm{diag}\,((r-1)\mathbf{1}_p, (r-3)\mathbf{1}_p, \ldots, -(r-1)\mathbf{1}_p, \mathbf{0}_s)$. The element $\Lambda_{p,r,s}$ in (0.6) has grade 1 with respect to d_{r,H_k} for any k. Using any of these gradings in the definition of the reduction like in Sect. 2, the reduced phase space would be the same. In fact, $\mathcal{M}_{\mathrm{DS}}$ in (2.18) would still be a global gauge slice. However, the elimination procedure applied in the analogues of the Θ-gauge (2.20) would give rise to alternative factorizations of the Lax operator L.

The discrete reductions and modifications of generalized KdV and AKNS systems just mentioned will be further discussed elsewhere.

As a final remark, recall that there exists a $\mathcal{W}$-algebra in correspondence with each sl_2 subalgebra of a simple Lie algebra or gl_n. It follows immediately from the definition of these $\mathcal{W}$-algebras [31, 21] that the second Poisson bracket of our extended matrix r-KdV hierarchy is identical with the $\mathcal{W}$-algebra corresponding to the sl_2 subalgebra of gl_n that contains I_0 as its semisimple generator (see also [9, 16]).

Note added. The papers in [32–35] deal with particular cases and further reductions of the constrained KP hierarchies that we derived systematically from the Drinfeld–Sokolov approach. These papers came to our attention after the completion of our work, and in them one can find further references on the subject of constrained KP systems. A more detailed analysis of discrete reductions, examples and the modified system given by (6.2) can be found in the original version of the present paper in hep-th/9503217. The classical Wakimoto realizations of current algebras described in [36] may be used to define a family of "second modifications" of the matrix r-KdV hierarchies, their extended versions and discrete reductions. Many, if not all, of these second modifications can be interpreted in terms of the generalized Drinfeld–Sokolov approach, and the system given by (6.2) is a special case.

Acknowledgements. Reference [19] was pointed out by a referee of an earlier version of [20]. We wish to thank I.R. McIntosh for making available ref. [6] and we are grateful to F. Delduc, L.A. Dickey, L. Gallot and F. Toppan for discussions, encouragement and comments on the manuscript. Part of the work for this paper was carried out whilst LF was a visitor at ENSLAPP in Lyon. We wish to thank the CNRS for supporting the visit of IM to ENSLAPP and especially to thank P. Sorba and the members of ENSLAPP for their kind hospitality.

Appendix A: The Poisson Submanifold $M_K \subset \mathscr{A}$

In this appendix we prove Eq. (4.23) of Proposition 4.1.

Let f and h be arbitrary linear functions on $\mathscr{A}$. In other words, let $X, Y \in \mathscr{A}$ and let f, h be given by

$$f(L) = \operatorname{Tr} LX, \qquad h(L) = \operatorname{Tr} LY \quad \forall L \in \mathscr{A}\,. \tag{A.1}$$

It follows by a standard argument for Poisson vector spaces that if (4.23) is valid for arbitrary linear functions f, h on $\mathscr{A}$, then it is valid for any smooth functions f, h on $\mathscr{A}$. Let

$$\theta_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A.2}$$

denote an arbitrary element of the space $\Theta_0 = \widetilde{gl}_{p+s}$ as in (2.21b), (2.22). Consider

$$\begin{aligned} f \circ \eta &\begin{pmatrix} a + t\alpha & b + t\beta \\ c + t\gamma & d + t\delta \end{pmatrix} \\ &= \operatorname{Tr}([\partial + a + t\alpha - (b + t\beta)(\partial + d + t\delta)^{-1}(c + t\gamma)]X) \\ &= f(\eta(\theta_0)) + t\operatorname{Tr}([\alpha - \beta(\partial + d)^{-1}c + b(\partial + d)^{-1} \\ &\quad \times \delta(\partial + d)^{-1}c - b(\partial + d)^{-1}\gamma]X) + O(t^2) \end{aligned} \tag{A.3}$$

which gives

$$\frac{\delta(f \circ \eta)}{\delta\theta_0} = \operatorname{res}\begin{pmatrix} X & -Xb(\partial + d)^{-1} \\ -(\partial + d)^{-1}cX & (\partial + d)^{-1}cXb(\partial + d)^{-1} \end{pmatrix}. \tag{A.4}$$

We have

$$\{f\circ\eta,h\circ\eta\}(\theta_0)=\left\langle\theta_0,\left[\frac{\delta f\circ\eta}{\delta\theta_0},\frac{\delta h\circ\eta}{\delta\theta_0}\right]\right\rangle-\left\langle\frac{\delta f\circ\eta}{\delta\theta_0},\left(\frac{\delta h\circ\eta}{\delta\theta_0}\right)'\right\rangle . \tag{A.5}$$

The following lemma is easy to prove.

Lemma. *Let* $L\in \mathrm{PDO}(n\times n)$ *and let* $\theta\in \widetilde{gl}_n\subset \mathrm{PDO}(n\times n)$*. Then*

$$\begin{aligned}&\text{(i)} && [\partial,\operatorname{res}L]=\operatorname{res}[\partial,L]\,,\\ &\text{(ii)} && [\theta,\operatorname{res}L]=\operatorname{res}[\theta,L]\,.\end{aligned} \tag{A.6}$$

Using the lemma and the ad-invariance of the inner product $\langle\,,\rangle$ we get

$$\{f\circ\eta,h\circ\eta\}(\theta_0)=-\left\langle\mathbf{X}_{f\circ\eta}(\theta_0),\frac{\delta h\circ\eta}{\delta\theta_0}\right\rangle , \tag{A.7}$$

where the Hamiltonian vector field $\mathbf{X}_{f\circ\eta}$ is given by

$$-\mathbf{X}_{f\circ\eta}(\theta_0)=\operatorname{res}\left[\begin{pmatrix}\partial+a & b\\ c & \partial+d\end{pmatrix},\begin{pmatrix}X & -Xb(\partial+d)^{-1}\\ -(\partial+d)^{-1}cX & (\partial+d)^{-1}cXb(\partial+d)^{-1}\end{pmatrix}\right] . \tag{A.8}$$

From this we obtain

$$-\mathbf{X}_{f\circ\eta}(\theta_0)=\operatorname{res}\begin{pmatrix}[K,X] & -KXb(\partial+d)^{-1}\\ (\partial+d)^{-1}cXK & 0\end{pmatrix}\quad\text{with } K=\eta(\theta_0)\,. \tag{A.9}$$

Hence we can write

$$\begin{aligned}\{f\circ\eta,h\circ\eta\}(\theta_0)&=\langle\operatorname{res}[K,X],\operatorname{res}Y\rangle-\langle\operatorname{res}(\partial+d)^{-1}cXK,\operatorname{res}Yb(\partial+d)^{-1}\rangle\\ &\quad+\langle\operatorname{res}KXb(\partial+d)^{-1},\operatorname{res}(\partial+d)^{-1}cY\rangle\\ &=\operatorname{Tr}([K,X]\operatorname{res}Y)-\operatorname{Tr}((\partial+d)^{-1}cXK\operatorname{res}(Yb(\partial+d)^{-1}))\\ &\quad+\operatorname{Tr}(KXb(\partial+d)^{-1}\operatorname{res}(\partial+d)^{-1}cY)\,,\end{aligned} \tag{A.10}$$

where in each of the above terms, $\langle x,y\rangle$ means $\int \operatorname{tr}xy$ for the appropriate size of matrices, and we have used the identity $\langle\operatorname{res}A,B\rangle=\operatorname{Tr}AB$, for $A\in\mathscr{A}$, $B\in\widetilde{gl}_n$. Using the integrating factor, $(\partial+d)=W^{-1}\partial W$ with $d=W^{-1}W'$, we simplify each term as follows:

$$\begin{aligned}\operatorname{Tr}([K,X]\operatorname{res}Y)&=\operatorname{Tr}(KX(\partial Y_-)_+-XK(Y_-\partial)_+)\\ &=\operatorname{Tr}(KX(KY_-)_+-XK(Y_-K)_+)\\ &=\operatorname{Tr}(KX(KY)_+-XK(YK)_+-KX(KY_+)_++XK(Y_+K)_+)\,.\end{aligned} \tag{A.11}$$

$$\begin{aligned}\operatorname{Tr}(KXb(\partial+d)^{-1}\operatorname{res}[(\partial+d)^{-1}cY]) &= \operatorname{Tr}(KXb(\partial+d)^{-1}\operatorname{res}[W^{-1}\partial^{-1}WcY_+])\\ &= \operatorname{Tr}(KXb(\partial+d)^{-1}W^{-1}\partial[\partial^{-1}WcY_+])\\ &= \operatorname{Tr}([KXbW^{-1}]_+\partial^{-1}WcY_+)\\ &= -\operatorname{Tr}((KX)_+KY_+)\,. \qquad (A.12)\end{aligned}$$

$$\begin{aligned}\operatorname{Tr}((\partial+d)^{-1}cXK\operatorname{res}[Yb(\partial+d)^{-1}]) &= \operatorname{Tr}(cXK[Y_+bW^{-1}\partial^{-1}]_-\partial W(\partial+d)^{-1})\\ &= \operatorname{Tr}(cXK[Y_+bW^{-1}\partial^{-1}]_-W^{-1})\\ &= \operatorname{Tr}((XK_+Y_+b(\partial+d)^{-1}c)\\ &= -\operatorname{Tr}((XK)_+Y_+K)\,. \qquad (A.13)\end{aligned}$$

Collecting terms we obtain the desired result.

Appendix B: The Formula of the Reduced PB in the DS Gauge

The purpose of this appendix is to present the computation leading to formula (4.30) of Sect. 4 that describes the reduced current algebra PB on $\mathscr{M}_c/\widetilde{\mathscr{N}} \simeq \mathscr{M}_{\mathrm{DS}}$. We recall that the constrained current $J \in \widetilde{gl}_{pr+s}$ defining a generic point of $\mathscr{M}_{\mathrm{DS}}$ has the form

$$J = j_{p,r,s} + C_+ = \sum_{i=1}^{r} e_{r,i}\otimes v_{r-i+1} + e_{r,r+1}\otimes\zeta_+ + e_{r+1,1}\otimes\zeta_- + e_{r+1,r+1}\otimes w + C_+\,, \qquad (B.1a)$$

where the explicit matrix form of $j_{p,r,s}$ is given by Eq. (0.8) with the variables

$$v_i \in \widetilde{gl}_p,\quad w\in\widetilde{gl}_s,\quad \zeta_+\in\widetilde{\mathrm{mat}}(p\times s),\quad \zeta_-\in\widetilde{\mathrm{mat}}(s\times p)\,, \qquad (B.1b)$$

and $C_+ = \sum_{k=1}^{r-1} e_{k,k+1}\otimes\Gamma$ is the constant matrix appearing in $\Lambda_{p,r,s} = C_+ + \lambda C_-$ in (0.6). We know already by Theorem 4.2 that the PBs of functions of the components $v_1,\ldots,v_r$ are given by the standard quadratic Gelfand–Dickey PB on the space of operators ℓ,

$$\ell = L_+ = \Delta^r\partial^r + \sum_{i=1}^{r} u_i\partial^{r-i},\quad u_i = \Delta v_i\Delta^{r-i},\ \Delta = -\Gamma^{-1}\,. \qquad (B.2)$$

We need then only to compute the other PB relations. We choose to do this by computing the Hamiltonian vector field $\mathbf{X}_H := \mathbf{X}_H^2$ for $H = Q, P, R$ respectively where

$$Q(J) = \int_0^{2\pi}\operatorname{tr}(f\zeta_+),\qquad P(J) = \int_0^{2\pi}\operatorname{tr}(\varphi\zeta_-),\qquad R(J) = \int_0^{2\pi}\operatorname{tr}(\alpha w) \qquad (B.3)$$

with f,φ,α being matrix valued test functions. It is not hard to see, for instance from the theory of reduction by constraints, that $\mathbf{X}_H$ takes the form

$$\mathbf{X}_H(J) = [K_H,J] - K_H',\qquad K_H = \frac{\delta H}{\delta J} + B_H\,, \qquad (B.4)$$

where $[B_H, J] - B_H'$ is a linear combination of the Hamiltonian vector fields associated to the ("second class") constraints that define the above special form of J. The polynomial nature of the DS gauge ensures (see e.g. [21]) that once J and $\frac{\delta H}{\delta J}$ are given one can uniquely solve (B.4) (where the form of $\mathbf{X}_H(J)$ must be consistent with that of J in (B.1)) for B_H, $\mathbf{X}_H$ in terms of differential polynomial expressions in the components of J and $\frac{\delta H}{\delta J}$.

Let us determine in turn $\mathbf{X}_Q$, $\mathbf{X}_P$, $\mathbf{X}_R$. Inspecting the simplest examples leads us to search for K_Q in the form

$$K_Q = \sum_{i=1}^{r} e_{r+1,r+1-i} \otimes f_i = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ f_r & \cdots & f_1 & 0 \end{pmatrix}, \quad \text{with } f_1 = f . \tag{B.5}$$

Substituting from (B.5) in (B.4) we get

$$\mathbf{X}_Q(w) = f\zeta_+, \qquad \mathbf{X}_Q(\zeta_+) = 0, \qquad \mathbf{X}_Q(\zeta_-) = fv_r - \mathscr{D}(f_r), \qquad \mathbf{X}_Q(v_i) = -\zeta_+ f_i , \tag{B.6}$$

and the recursion relation

$$f_{i+1} = f v_i \Delta - \mathscr{D}(f_i)\Delta , \tag{B.7}$$

where $\mathscr{D}(\beta)$ is given by $\mathscr{D}(\beta) = \beta' + w\beta$ for any $\beta \in \widetilde{\text{mat}}(s \times p)$. The solution of this recursion relation is found to be

$$f_i = (-1)^{i-1}\mathscr{D}^{i-1}(f)\Delta^{i-1} - \sum_{k=1}^{i-1} (-1)^{i-k}\mathscr{D}^{i-1-k}(fv_k)\Delta^{i-k} . \tag{B.8}$$

Substitution from (B.8) in (B.6) gives $\mathbf{X}_Q$ explicitly.

To do the analogous computation for $H = P$, it is advantageous to change variables by transforming to a different gauge section. In fact, there exists a unique gauge transformation

$$J \mapsto \tilde{J} = gJg^{-1} - g'g^{-1}, \quad g = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{1}_s \end{pmatrix} , \tag{B.9}$$

where $\mathbf{A}$ is a block lower triangular matrix with $p \times p$ unit matrices along the diagonal, for which $\tilde{J}$ takes the form

$$\tilde{J} = \sum_{i=1}^{r} e_{i,1} \otimes \tilde{v}_i + e_{r,r+1} \otimes \zeta_+ + e_{r+1,1} \otimes \zeta_- + e_{r+1,r+1} \otimes w + C_+ . \tag{B.10}$$

The $\tilde{v}_i$'s in (B.10) are unique differential polynomials in the v_i's in (B.1), which may be determined from the equality

$$\Delta^r \partial^r + \sum_{i=1}^{r} u_i \partial^{r-i} + z_+(\partial + w)^{-1} z_- = L = \Delta^r \partial^r + \sum_{i=1}^{r} \partial^{r-i} \tilde{u}_i + z_+(\partial + w)^{-1} z_- \tag{B.11}$$

with

$$z_+ = -\Delta\zeta_+, \qquad z_- = \zeta_-, \qquad u_i = \Delta v_i \Delta^{r-i}, \qquad \tilde{u}_i = \Delta^{r+1-i}\tilde{v}_i . \tag{B.12}$$

This equality results from the elimination procedure performed in the respective gauges (B.1) and (B.10). In the latter gauge we have a formula for $\mathbf{X}_P$ analogous to (B.4),

$$\mathbf{X}_P(\tilde{J}) = [\tilde{K}_P,\tilde{J}] - \tilde{K}_P', \qquad \tilde{K}_P = \frac{\delta P}{\delta \tilde{J}} + \tilde{B}_P\,, \tag{B.13}$$

and $\tilde{K}_P$ turns out to have the form

$$\tilde{K}_P = \sum_{i=1}^{r} e_{i,r+1} \otimes \varphi_i = \begin{pmatrix} 0 & \cdots & 0 & \varphi_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \varphi_r \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \text{with } \varphi_1 = \varphi\,. \tag{B.14}$$

In fact, substituting from (B.14) in (B.13) leads to

$$\mathbf{X}_P(w) = -\zeta_-\varphi, \qquad \mathbf{X}_P(\zeta_+) = -\tilde{\mathscr{D}}(\varphi_r) - \tilde{v}_r\varphi, \qquad \mathbf{X}_P(\zeta_-) = 0, \qquad \mathbf{X}_P(\tilde{v}_i) = \varphi_i\zeta_- \tag{B.15}$$

with the recursion relation

$$\varphi_{i+1} = \Delta(\tilde{\mathscr{D}}(\varphi_i) + \tilde{v}_i\varphi)\,, \tag{B.16}$$

where $\tilde{\mathscr{D}}(\tilde{\beta})$ is given by $\tilde{\mathscr{D}}(\tilde{\beta}) = \tilde{\beta}' - \tilde{\beta}w$ for any $\tilde{\beta} \in \widetilde{\mathrm{mat}}(p \times s)$. This yields

$$\varphi_i = \Delta^{i-1}\tilde{\mathscr{D}}^{i-1}(\varphi) + \sum_{k=1}^{i-1} \Delta^{i-k}\tilde{\mathscr{D}}^{i-k-1}(\tilde{v}_k\varphi)\,. \tag{B.17}$$

Plugging this back into (B.15) gives $\mathbf{X}_P$ explicitly.

In the case of $H = R$ we find that K_R equals $\frac{\delta R}{\delta J}$, i.e., $K_R = e_{r+1,r+1} \otimes \alpha$. Therefore

$$\mathbf{X}_R(w) = [\alpha, w] - \alpha', \qquad \mathbf{X}_R(\zeta_+) = -\zeta_+\alpha, \qquad \mathbf{X}_R(\zeta_-) = \alpha\zeta_-, \qquad \mathbf{X}_R(v_i) = 0\,. \tag{B.18}$$

The remaining task is to find a neater form of the above formulae. First we rewrite them in terms of the operator ℓ in (B.2) as follows.

Claim B1: Formula (B.6) with (B.8) is equivalent to

$$\mathbf{X}_Q(\ell) = (\Delta\zeta_+(\partial + w)^{-1} f\Gamma\ell)_+, \qquad \mathbf{X}_Q(\zeta_+) = 0,$$

$$\mathbf{X}_Q(\zeta_-) = -\mathscr{D}(f_r) + f v_r, \qquad \mathbf{X}_Q(w) = f\zeta_+\,. \tag{B.19}$$

Claim B2: Formula (B.15) with (B.17) is equivalent to

$$\mathbf{X}_P(\ell) = (\ell\varphi(\partial + w)^{-1}\zeta_-)_+, \qquad \mathbf{X}_P(\zeta_+) = -\tilde{\mathscr{D}}(\varphi_r) - \tilde{v}_r\varphi,$$

$$\mathbf{X}_P(\zeta_-) = 0, \qquad \mathbf{X}_P(w) = -\zeta_-\varphi\,. \tag{B.20}$$

Claim B3: Formula (B.18) is equivalent to

$$\mathbf{X}_R(\ell) = 0, \qquad \mathbf{X}_R(\zeta_+) = -\zeta_+\alpha, \qquad \mathbf{X}_R(\zeta_-) = \alpha\zeta_-, \qquad \mathbf{X}_R(w) = [\alpha, w] - \alpha' . \tag{B.21}$$

Claim B3 is obvious from (B.18). To verify Claim B1, we use (4.29) to get

$$- \Delta\zeta_+(\partial + w)^{-1} f\Gamma\ell = \Delta\zeta_+ W^{-1}\partial^{-1} W f \Delta^{r-1}\partial^r - \Delta\sum_{i=0}^{r-1} \zeta_+ W^{-1}\partial^{-1} W f \Gamma u_{r-i}\partial^i . \tag{B.22}$$

Using $\mathscr{D}(F) = W^{-1}(WF)'$ and $\partial^{-1}F = \sum_{i=0}^{\infty}(-1)^i F\partial^{-i-1}$ we then write the contribution to non-negative powers of ∂ in the right-hand side of (B.22) in the form

$$\Delta\zeta_+ \sum_{k=0}^{r-1} (-1)^k \mathscr{D}^k(f)\Delta^{r-1}\partial^{r-k-1} - \Delta\zeta_+ \sum_{i=1}^{r-1}\sum_{k=0}^{i-1} (-1)^k \mathscr{D}^k(f\Gamma u_{r-i})\partial^{i-k-1} , \tag{B.23}$$

which simplifies to

$$- (\Delta\zeta_+(\partial + w)^{-1} f\Gamma\ell)_+ = \sum_{k=1}^{r} \Delta\zeta_+ f_k \Delta^{r-k}\partial^{r-k} = -\sum_{k=1}^{r} \mathbf{X}_Q(u_k)\partial^{r-k} , \tag{B.24}$$

as required by Claim B1. Similarly, one may check Claim B2 using the identity $\tilde{\mathscr{D}}(\tilde{F}) = (\tilde{F}W^{-1})'W$ and the expression of $\ell = L_+$ in the variables $\tilde{u}_i$ provided by (B.11) and (B.12).

Let us now consider the Hamiltonian $H = H_1 + Q + P + R$ given by

$$H(J) = \int_0^{2\pi} \operatorname{tr}\operatorname{res}(\ell\xi) + \int_0^{2\pi} \operatorname{tr}(\zeta_+ f) + \int_0^{2\pi} \operatorname{tr}(\zeta_-\varphi) + \int_0^{2\pi} \operatorname{tr}(w\alpha) , \tag{B.25}$$

where ξ is a $p \times p$ matrix PDO of the form $\xi = \sum_{i=1}^{r} \xi_i\partial^{i-r-1}$ with arbitrarily chosen $\xi_i \in \widetilde{gl_p}$. We wish to present the Hamiltonian vector field $\mathbf{X}_H$ associated to H by means of the reduced current algebra PB in terms of the variables ℓ, w, and $z_\pm$ in (B.12). To make contact with formula (4.30), we now substitute

$$f = -\frac{\delta H}{\delta z_+}\Delta, \qquad \varphi = \frac{\delta H}{\delta z_-}, \qquad \alpha = \frac{\delta H}{\delta w}, \qquad \xi = \frac{\delta H}{\delta \ell} . \tag{B.26}$$

Combining the above claims with Theorem 4.2 of Sect. 4 implies the following formula:

$$\mathbf{X}_H(\ell) = \left(\ell\frac{\delta H}{\delta \ell}\right)_+ \ell - \ell\left(\frac{\delta H}{\delta \ell}\ell\right)_+ + \left(\ell\frac{\delta H}{\delta z_-}(\partial + w)^{-1}z_-\right)_+ - \left(z_+(\partial + w)^{-1}\frac{\delta H}{\delta z_+}\ell\right)_+ ,$$

$$\mathbf{X}_H(z_+) = P_0\left(\ell\frac{\delta H}{\delta \ell}z_+W^{-1}\right)W + \Delta^r\tilde{\mathscr{D}}^r\left(\frac{\delta H}{\delta z_-}\right) + \sum_{k=1}^{r}\tilde{\mathscr{D}}^{r-k}\left(\tilde{u}_k\frac{\delta H}{\delta z_-}\right) - z_+\frac{\delta H}{\delta w} ,$$

$$\mathbf{X}_H(z_-) = -W^{-1}P_0^\dagger\left(Wz_-\frac{\delta H}{\delta \ell}\ell\right) - (-1)^r\mathscr{D}^r\left(\frac{\delta H}{\delta z_+}\right)\Delta^r$$

$$-\sum_{k=1}^{r}(-1)^{r-k}\mathscr{D}^{r-k}\left(\frac{\delta H}{\delta z_+}u_k\right) + \frac{\delta H}{\delta w}z_- ,$$

$$\mathbf{X}_H(w) = \frac{\delta H}{\delta z_+}z_+ - z_-\frac{\delta H}{\delta z_-} + \left[\frac{\delta H}{\delta w}, w\right] - \left(\frac{\delta H}{\delta w}\right)' . \tag{B.27}$$

Here the first term in $\mathbf{X}_H(z_\pm)$ has been found using the skew-symmetry property of the PB, and the notations P_0 and $P_0^\dagger$ have been defined in (4.28). It is now easy to convert (B.27) into formula (4.30), which completes the derivation.

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Communicated by M. Jimbo