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Chirality and Dirac Operator on Noncommutative Sphere

Ursula Carow-Watamura*, **Satoshi Watamura****

Department of Physics, Graduate School of Science, Tohoku University, Aoba-ku, Sendai 980-77, Japan

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Abstract: We give a derivation of the Dirac operator on the noncommutative 2-sphere within the framework of the bosonic fuzzy sphere and define Connes' triple. It turns out that there are two different types of spectra of the Dirac operator and correspondingly there are two classes of quantized algebras. As a result we obtain a new restriction on the Planck constant in Berezin's quantization. The map to the local frame in noncommutative geometry is also discussed.

1. Introduction

The description of spacetime at the order of the Planck scale and the description of the nature of quantum gravity is a long-standing problem. Quite a number of proposals have been made in order to describe consistently gravitational interaction and quantum field theory. However these proposals either do not give a satisfying formulation of the quantum theory of gravity or, in their present form an interpretation as a theory of the geometry is difficult.

Thus, recently the modification of the concept of geometry itself is also discussed by many authors. Of course one may argue that a successful theory of gravitation will naturally exhibit the necessary structure of a suitable theory of geometry and a natural language to describe it. On the other hand it is not very probable that the standard language of ordinary differential geometry is a suitable tool. The noncommutative geometry from the physicist's point of view is a possibility to describe such a geometry.

The noncommutative geometry is discussed in many contexts. The common idea is that one deals with a function algebra over the space one is interested in and the description of its geometry is made free from the concept of a point [11]. In other words, the geometry of a manifold is reformulated in terms of an algebra of functions defined over it, which may be called structure algebra. Once the algebraic description of the geometry is obtained, the structure algebra can be made noncommutative.

* Fellow of the Japan Society for Promotion of Science, E-mail: ursula@tuhep.phys.tohoku.ac.jp.
** E-mail: watamura@tuhep.phys.tohoku.ac.jp.

As an application for physics, recently Connes gave a formulation of the standard model within the framework of noncommutative geometry [10]; the Higgs field in the standard model is interpreted as a gauge field in the noncommutative space given by a product of Minkowski space and a two-point space. This means that, in an appropriate geometry there is a chance to understand all bose particles through the gauge principle [8]. On the other hand we may consider this formulation of the standard model as a kind of Kaluza–Klein theory with the noncommutative space playing the role of an internal space. This idea is also natural when the scale of the internal space is the Planck scale where we expect the breakdown of the classical picture of geometry. (See for example [27] and references therein.) From this point of view it is very interesting to investigate other possibilities where we consider as an internal space a more complicated object such as the noncommutative sphere. With the above problem in mind, the aim of the present paper is to find a proper formulation of the quantum sphere which fits with Connes' idea.

When discussing the noncommutative analogue of the sphere, so far there are two methods available. One is the q-deformation of the sphere [29] and the other one, which is the case considered here, is the Berezin (or Berezin–Toeplitz) quantization of the sphere [3], which is recently also known as the fuzzy sphere [26].

In order to define the differential calculus within Connes' framework one has to construct the triple $(\mathscr{A}, \mathscr{D}, \mathfrak{H})$, where $(\mathfrak{H}, \mathscr{D})$ is a K-cycle over the algebra $\mathscr{A}$ [11]. $\mathfrak{H}$ is a Hilbert space, $\mathscr{A}$ is an algebra of bounded operators acting on this Hilbert space, and $\mathscr{D}$ is a linear operator on $\mathfrak{H}$, the Dirac operator. The Hilbert space is graded (i.e. we consider the even case). The algebra $\mathscr{A}$ is even with respect to the grading and the Dirac operator is odd. Once this K-cycle is given, the construction of the differential calculus is rather straightforward.

The aim of this paper is to define a Dirac operator as well as a chirality operator on the noncommutative sphere such that it can be naturally combined with field theory using Connes' framework.

The Dirac operator on the fuzzy sphere has been also discussed in Refs. [17, 19, 20], where the supersymmetric extension of the fuzzy sphere is constructed using the supersymmetry algebra $OSp(2,1)$; then the Dirac operator is defined on that super fuzzy sphere. On the contrary, our construction of the Dirac operator does not make use of the supersymmetry algebra, i.e., we use the algebra of the bosonic fuzzy sphere. The resulting chirality operator, the Dirac operator and the definition of the spinors are different from the ones obtained by the supersymmetry algebra approach.

Our approach is the following: We define the algebra $\mathscr{A}_N$ of the fuzzy sphere which is generated by noncommutative operators $\mathbf{x}_i$, $(i = 1,2,3)$, where N is an integer relating to the Planck constant in Berezin's quantization. Then we introduce the Lie algebra $\mathscr{L}_N$ defined by the derivations $\mathbf{L}_i$ given by the adjoint action of the generators. Including the derivations we consider the bigger algebra $(\mathscr{A}_N, \mathscr{L}_N)$ generated by $(\mathbf{x}_i, \mathbf{L}_i)$. The chirality operator and the Dirac operator are constructed algebraically in the algebra $(\mathscr{A}_N, \mathscr{L}_N) \otimes M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is the algebra of complex 2×2 matrices.

Then using the result, we investigate its properties such as the spectrum. We reconsider the algebra of the noncommutative sphere and obtain the corresponding triple which we need for applying Connes' approach. In the naive commutative limit, the structure of the spectrum reveals the fine structure of Berezin's quantization and gives the restriction on the quantization parameter N to be an even integer.

Furthermore, we also discuss the Dirac operator in the local patch within the framework of noncommutative geometry.

The organization of this paper is as follows. In Sect. 2, we give a brief review of the operator algebra of the fuzzy sphere and introduce notations. In Sect. 3 we derive the chirality operator and Dirac operator and examine its properties and the Hilbert space structure. Section 4 is devoted to discussions and conclusions.

2. Algebra of Fuzzy Sphere and Derivations

2.1. Brief Summary on Fuzzy Sphere. The noncommutative sphere has been considered by several authors in different contexts such as an example for a general quantization procedure [3, 4] (see also for example [5,6,9,24,7] and references therein), the algebra appearing in a membrane [22], in relation with coherent states [1,28], and recently in connection with noncommutative geometry [26, 19, 20]. The same structure also appears in the context of the quantum Hall effect [21, 16]. The noncommutative sphere is described in various ways, however the resulting algebra is the same. It is easy to understand the idea of the fuzzy sphere from the point of view of approximation.

It is well known that the square integrable functions on a 2-sphere $\mathscr{L}^2(S^2)$ form a Hilbert space the basis of which is given by the spherical harmonics Y_{lm}. By usual multiplication these functions form a closed algebra. Thus we have a basic function algebra $\mathscr{A}_\infty$ over the sphere, and any element of the algebra can be expanded in the basis of spherical harmonics. The idea of the fuzzy sphere may be formulated in brief as we approximate the functions on the sphere by a finite number of spherical harmonics where this number is limited by the maximal angular momentum $\{Y_{lm}; l \leqq N\}$. However with respect to the usual multiplication this set of functions does not form a closed algebra since the product of two spherical harmonics Y_{lm} and $Y_{l'm'}$ contains $Y_{l+l',m}$ due to the product rule. It is a new multiplication rule that cures the above described situation and gives a closed function algebra with a finite number of basis elements. The resulting algebra $\mathscr{A}_N$ is noncommutative.[1] The interpretation of this property is that we obtain a geometry where the concept of a point is "dissolved."[2] Since the number of the independent functions with angular momentum $l \leqq N$ is $(N+1)^2$, as a vector space this has the same dimension as the vector space spanned by the $(N+1) \times (N+1)$ matrices. This is not accidental. We can identify the algebra of the fuzzy sphere with the algebra of complex matrices $M_{N+1}(\mathbb{C})$ and thus we can consider it as a special case of the matrix geometry [12–15], which is the way the fuzzy sphere is introduced in Ref. [26].

In the above description the truncation of the function algebra which leads to the noncommutative algebra is rather ad hoc. However below we shall see that this type of construction of a noncommutative algebra is equivalent to Berezin's quantization and can be generalized to a general Kähler manifold. In the Berezin–Toeplitz quantization one quantizes the algebra using the Poisson structure on the manifold defined by the Kähler form. For this, consider a finite dimensional Hilbert space

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[1] We denote the algebra of fuzzy sphere as $\mathscr{A}_N$ with the suffix N which becomes the eigenvalue of the number operator defined in the next section.

[2] The "fuzziness" of the sphere is removed by taking the limit $N \to \infty$, in which the function algebra becomes commutative.

$\mathscr{F}_N = \Gamma(L^N)$, given by holomorphic sections on L^N, where $L^N = \otimes^N L$ is the N^{th} tensor of the line bundle L. Using the coherent states $|v\rangle$, where $(v, \bar{v})$ parametrize S^2 (see below), one can define an operator acting on the Hilbert space $\mathscr{F}_N$ for any function $f(v, \bar{v}) \in \mathscr{L}^2(S^2)$ by

$$T_f|v\rangle = \mathscr{P} f|v\rangle\,, \tag{1}$$

where $\mathscr{P}$ is the projection operator from the general sections to the subspace $\mathscr{F}_N$,

$$\mathscr{P} : \Gamma(L^{\otimes N}) \to \mathscr{F}_N\,.$$

The resulting operator is defined on $\mathscr{F}_N$. The product of two operators is defined by successive application of the above construction:

$$T_f T_g|v\rangle = \mathscr{P} f \mathscr{P} g|v\rangle\,, \tag{2}$$

We define the algebra $\mathscr{A}_N$ of the operators T_f with this multiplication. T_f is called Toeplitz operator. From the above definition one easily sees that the multiplication of operators T_f is in general noncommutative. Furthermore, their commutator satisfies Berezin's quantization condition [3]

$$\lim_{N\to\infty} \frac{1}{N}[T_f, T_g] = \{f, g\}_{PB}\,, \tag{3}$$

where the r.h.s. is the Poisson bracket defined through the Kähler form and thus the resulting algebra approximates the Poisson algebra of the function [6, 24, 7]. See also [25].

By taking an appropriate basis in the Hilbert space $\mathscr{F}_N$ one can represent the Toeplitz operator by a matrix. For the case of a 2-sphere the dimension of $\mathscr{F}_N$ is $N+1$ and thus we obtain an algebra of $(N+1)\times(N+1)$ matrices which is a representation of the algebra discussed in Ref. [26]. On the other hand, by defining coherent states and representing the Toeplitz operator as an expectation value with respect to these coherent states $\frac{\langle z|T_f|z\rangle}{\langle z|z\rangle}$, called a covariant symbol [2], we obtain Berezin's quantized algebra. (See also [28, 18].) The product of this algebra, i.e. the $*$-product among the covariant symbols is simply defined by rewriting the above operator multiplication in the language of the covariant symbols and thus it is in general noncommutative as is the multiplication of operators. In this way, the algebra of the fuzzy sphere can be understood as a finite approximation of the function algebra. The problem is to clarify within which framework the differential calculus must be introduced.

2.2. Operator Representation of Algebra $\mathscr{A}_N$ and Derivations. As we discussed above, when considering the algebra of the noncommutative sphere, it is natural to start with the algebra of operators acting on the Hilbert space $\mathscr{F}_N$. Since $\mathscr{F}_N$ is a representation space of the rotation group, we introduce in the standard way a pair of creation-annihilation operators $\mathbf{a}^{\dagger b}, \mathbf{a}_b$ $(b = 1,2)$ which transform as a fundamental representation under $SU(2)$, satisfying

$$[\mathbf{a}^a, \mathbf{a}_b^\dagger] = \delta_b^a\,. \tag{4}$$

Define the number operator by

$$\mathbf{N} = \mathbf{a}_b^\dagger \mathbf{a}^b\,, \tag{5}$$

then the set of states satisfying

$$\mathbf{N}|v\rangle = N|v\rangle \tag{6}$$

provides an $N+1$ dimensional irreducible representation space denoted by $\mathscr{F}_N$. The operator algebra $\mathscr{A}_N$ on $\mathscr{F}_N$ is unital and given by the operators $\{\mathcal{O}; [\mathbf{N}, \mathcal{O}] = 0\}$. The generators of the algebra $\mathscr{A}_N$ are defined by

$$\mathbf{x}_i = \frac{1}{2}\alpha\sigma_i{}^a{}_b \mathbf{a}_a^\dagger \mathbf{a}^b , \tag{7}$$

where the normalization factor α is a central element $[\alpha, \mathbf{x}_i] = 0$, introduced for later convenience[3]. The commutation relations among these operators are

$$[\mathbf{x}_i, \mathbf{a}_a^\dagger] = \frac{1}{2}\alpha\sigma_i{}^b{}_a \mathbf{a}_b^\dagger ,$$

$$[\mathbf{x}_i, \mathbf{a}^a] = -\frac{1}{2}\alpha\sigma_i{}^a{}_b \mathbf{a}^b . \tag{8}$$

The algebra of the fuzzy sphere is generated by $\mathbf{x}^i$ and the basic relation is

$$[\mathbf{x}_i, \mathbf{x}_j] = i\alpha\varepsilon_{ijk}\mathbf{x}_k . \tag{9}$$

The normalization α is defined by

$$\mathbf{x}_i\mathbf{x}_i = \frac{\alpha^2}{4}\mathbf{N}(\mathbf{N}+2) = l^2 . \tag{10}$$

This means that $l > 0$ is the radius of the 2-sphere and we get

$$\alpha = \frac{2l}{\sqrt{\mathbf{N}(\mathbf{N}+2)}} . \tag{11}$$

Note that in the formulation on algebra level we use the number operator $\mathbf{N}$ in order to make the independence of the algebra from the representation space $\mathscr{F}_N$ clear. Thus the "Planck constant" α is also an operator. However, when we discuss the commutative limit $N \to \infty$, we take a definite Hilbert space $\mathscr{F}_N$ and thus the number operator is replaced by its eigenvalue N. The Planck constant α becomes also a number $\frac{2l}{\sqrt{N(N+2)}}$. Then, we take the limit $N \to \infty$, i.e., $\alpha \to 0$. Therefore, the commutative limit discussed in the later part of this paper is taken with respect to the eigenvalue N of the number operator on the space $\mathscr{F}_N$.

Proof of Eq. (10). From the Fierz identity we get

$$\mathbf{N}^2 + \frac{4}{\alpha^2}\mathbf{x}_i\mathbf{x}_i = \mathbf{a}_a^\dagger(\sigma_\mu)^a{}_b\mathbf{a}^b\mathbf{a}_d^\dagger(\sigma_\mu)^d{}_c\mathbf{a}^c = 2\mathbf{a}_a^\dagger\mathbf{a}^b\mathbf{a}_d^\dagger\mathbf{a}^c\delta_c^a\delta_b^d$$

$$= 2\mathbf{a}_a^\dagger(\mathbf{N}+2)\mathbf{a}^c\delta_c^a = 2(\mathbf{N}+1)\mathbf{N} , \tag{12}$$

where $\mu = 0, 1, 2, 3$ and $\sigma_0{}^a{}_b = \delta_b^a$. We have also used

$$\mathbf{a}^a\mathbf{a}_a^\dagger = \delta_a^a + \mathbf{a}_a^\dagger\mathbf{a}^a = 2 + \mathbf{N} , \tag{13}$$

$$\mathbf{N}\mathbf{a}^\dagger = \mathbf{a}^\dagger(\mathbf{N}+1) . \tag{14}$$

Thus we get the above relation. □

[3] Formally we have to include N and α as generators of the algebra, however they can be treated as numbers in the following calculations.

Now let us consider derivations of $\mathcal{A}_N$. The derivations are defined by the commutator with any algebra element in $\mathcal{A}_N$, since the adjoint action always defines an inner derivation. We introduce the derivative operator $\mathbf{L}_i$ by the adjoint action of $\mathbf{x}^i$ [26] as

$$\frac{1}{\alpha} ad_{\mathbf{x}_i}\mathbf{x}_k = \frac{1}{\alpha}[\mathbf{x}_i, \mathbf{x}_k] \equiv \mathbf{L}_i\mathbf{x}_k \ . \tag{15}$$

These objects are the noncommutative analogue of the Killing vector fields on the sphere and the algebra of $\mathbf{L}_i$ closes and we obtain a Lie algebra $\mathcal{L}_N \subset \mathrm{Der}(\mathcal{A}_N)$ (see also Ref. [13]).

The action of $\mathbf{L}_i$ generates rotations of the noncommutative sphere, and $\mathbf{L}_i$ and $\mathbf{x}_k$ satisfy the usual commutation relations of $SU(2)$. Thus the algebra $(\mathcal{A}_N, \mathcal{L}_N)$ of the noncommutative sphere together with derivations is defined by the operator algebra given by $\mathbf{x}_i$ and $\mathbf{L}_i$,

$$[\mathbf{L}_i, \mathbf{x}_j] = i\varepsilon_{ijk}\mathbf{x}_k \ , \qquad [\mathbf{L}_i, \mathbf{L}_j] = i\varepsilon_{ijk}\mathbf{L}_k \ , \tag{16}$$

together with (9).

2.3. Integration and Coherent State Representation. The integration on the noncommutative sphere is defined as a trace over the Hilbert space $\mathcal{F}_N$. The simplest way to take the trace is to use the orthogonal basis:

$$|k;N\rangle = \frac{1}{\sqrt{k!(N-k)!}}(\mathbf{a}_1^\dagger)^k(\mathbf{a}_2^\dagger)^{N-k}|0\rangle \ , \tag{17}$$

where $k = 0, \ldots, N$ and $|0\rangle$ is the vacuum. Then

$$\langle \mathcal{O} \rangle_N = \frac{1}{N+1}\mathrm{Tr}\{\mathcal{O}\} = \frac{1}{N+1}\sum_k \langle k;N|\mathcal{O}|k;N\rangle \ . \tag{18}$$

In order to see the relation of the above result with integration in the commutative limit, we define the trace using coherent states [28, 18]. The coherent states are parametrized by a point on the sphere. In order to keep track with the complex structure on the sphere, we take here the parametrization obtained by projecting stereographically to the complex plane and represent the corresponding point by complex coordinates $(z, \bar{z})$. Then the coherent states are defined by

$$|z;N\rangle = \frac{1}{\sqrt{N!}}(z\mathbf{a}_1^\dagger + \mathbf{a}_2^\dagger)^N|0\rangle \ , \tag{19}$$

which satisfy

$$\langle \bar{z};N|z;N\rangle = (1+\bar{z}z)^N \ . \tag{20}$$

Here we have taken the notation $|w\rangle^* = \langle w^*| = \langle \overline{w}|$. In the following we simply denote $|z;N\rangle$ as $|z\rangle$ for the coherent state in $\mathcal{F}_N$.

Using the above definitions we can take the coherent state representation. Then the orthogonal basis of $\mathcal{F}_N$ is given by z:

$$\langle z|k\rangle = f_k(z) = \sqrt{\frac{N!}{k!(N-k)!}}z^k \ . \tag{21}$$

The inner product of two states $|f\rangle$ and $|g\rangle$ is defined by

$$\langle \overline{f}|g\rangle = \int d\mu_N(z)\overline{f(\overline{z})}g(z) \ , \tag{22}$$

where the measure μ_N is defined such that it gives an orthonormal basis:

$$\int d\mu_N(z)\overline{f_l(z)}f_k(z) = \frac{(N+1)}{2\pi i}\int dz \wedge d\bar{z}\frac{f_k(z)\overline{f_l(z)}}{(1+|z|^2)^{N+2}} = \delta_{kl} \,. \tag{23}$$

The reproducing kernel is given by

$$L_N(z,\bar{v}) = \sum f_k(z)\overline{f_k(v)} = \sum \frac{N!}{k!(N-k)!}(z\bar{v})^k = (1+z\bar{v})^N \,. \tag{24}$$

Thus in the coherent state representation, the trace over the Hilbert space $\mathscr{F}_N$ is represented by the integration of the symbol of an operator over S^2,

$$\langle \mathscr{O} \rangle_N = \frac{1}{N+1}\mathrm{Tr}\{\mathscr{O}\} = \frac{1}{N+1}\int d\mu_N \langle \bar{z}|\mathscr{O}|z\rangle \,. \tag{25}$$

3. The Dirac Operator

3.1. Chirality and Dirac Operator. In this section we examine the algebraic relations among the operators with the aim to define the chirality operator and to construct the Dirac operator algebraically on the noncommutative sphere. For this purpose we consider the product algebra of $(\mathscr{A}_N, \mathscr{L}_N)$ and $\mathscr{A}_1$. $\mathscr{A}_1$ is simply the algebra of 2×2 matrices $M_2(\mathbb{C})$ the elements of which are represented by

$$\mathbf{M} = \sum_{\mu=0}^{4} a_\mu \sigma_\mu \,, \tag{26}$$

and transform under rotation as

$$\mathbf{M} \to U\mathbf{M}U^\dagger \,. \tag{27}$$

U is the spin representation matrix of $SU(2)$. Thus we define the chirality operator γ_χ and the Dirac operator $\mathbf{D}$ in the product algebra of $(\mathscr{A}_N, \mathscr{L}_N) \otimes M_2(\mathbb{C})$, i.e. 2×2 matrices the entries of which are polynomials in $(\mathbf{x}_i, \mathbf{L}_j)$.

Our strategy taken here is to define first the chirality operator and, once this is achieved, the Dirac operator is constructed by the requirement that it should anticommute with this chirality operator.

Thus let us first discuss the possibilities for defining the chirality operator. The simplest choice would be to take $\mathbf{1} \otimes \sigma^3$. However, this choice breaks the $SU(2)$ symmetry. It is better for our purpose to keep the $SU(2)$ symmetry and thus we take a rotational invariant operator as the chirality.

On the commutative sphere, as is disscussed in ref. [23] a natural chirality operator is

$$\gamma_\infty = \frac{1}{l}\sum_i x_i \otimes \sigma_i \,, \tag{28}$$

where x_i is the homogeneous coordinate and l is the radius of the sphere ($\sum_i x_i x_i = l^2$). Then $\gamma_\infty^2 = 1$. On the fuzzy sphere the coordinate function is replaced by the operator $\mathbf{x}_i$. However, if we replace x_i by the operator $\mathbf{x}_i$ in the above definition, the square of the resulting operator is not unity due to the noncommutativity of $\mathbf{x}_i$.

In order to investigate this situation in detail (and also for later convenience) let us introduce the following operators in $(\mathscr{A}_N, \mathscr{L}_N) \otimes M_2(\mathbb{C})$:

$$\chi = \sum_i (\mathbf{x}_i \otimes \sigma_i)\,,$$

$$\Lambda = \sum_i (\mathbf{L}_i \otimes \sigma_i)\,,$$

$$\Sigma = -i \sum_{ijk} \varepsilon_{ijk} (\mathbf{x}_i \mathbf{L}_j \otimes \sigma_k)\,. \tag{29}$$

These are all $SU(2)$ invariant operators.

As we discussed above, among the operators given in Eq. (29) χ is a good candidate for the chirality operator. Here, however, the coordinates are not commutative and the square of the operator χ is not unity. Instead, this operator satisfies the relation

$$\chi\chi = (\mathbf{x} \cdot \mathbf{x}) \otimes \mathbf{1} - \alpha\chi\,, \tag{30}$$

where $(\mathbf{A} \cdot \mathbf{B}) = \sum_{i=1}^3 \mathbf{A}_i \mathbf{B}_i$. Therefore we obtain from Eq. (30)

$$\left(\chi + \frac{1}{2}\alpha\right)^2 = \chi^2 + \alpha\chi + \frac{1}{4}\alpha^2 = (\mathbf{x} \cdot \mathbf{x}) + \frac{1}{4}\alpha^2\,. \tag{31}$$

This suggests that the chirality operator is given by

$$\gamma_\chi = \frac{1}{\mathcal{N}_N}\left(\chi + \frac{1}{2}\alpha\right)\,, \tag{32}$$

where the normalization constant is determined by the requirement

$$\gamma_\chi^2 = 1\,. \tag{33}$$

This gives

$$\mathcal{N}_N^2 = (\mathbf{x} \cdot \mathbf{x}) + \frac{1}{4}\alpha^2\,, \tag{34}$$

and thus

$$\mathcal{N}_N = \frac{\alpha}{2}(\mathrm{N} + 1) = l\sqrt{\frac{(\mathrm{N}+1)^2}{\mathrm{N}(\mathrm{N}+2)}}\,, \tag{35}$$

where we imposed relation (10). Thus, we define the chirality operator by Eq. (32) with normalization (35).

Once the chirality operator is chosen, the construction of the Dirac operator is rather straightforward. For this end we have to consider the relations among the above operators. The relevant relations are as follows.

The square of the other operators given in Eq. (29) satisfies

$$\Lambda\Lambda = (\mathbf{L} \cdot \mathbf{L}) \otimes \mathbf{1} - \Lambda\,, \tag{36}$$

$$\Sigma\Sigma = \alpha(\mathbf{x} \cdot \mathbf{L}) - (\mathbf{x} \cdot \mathbf{x})(\mathbf{L} \cdot \mathbf{L}) + (\mathbf{x} \cdot \mathbf{L})(\mathbf{x} \cdot \mathbf{L}) - \alpha\Sigma + \alpha(\mathbf{x} \cdot \mathbf{L})\Lambda - (\mathbf{x} \cdot \mathbf{x})\Lambda\,; \tag{37}$$

and their commutators and anticommutators are given by

$$\{\chi, \Lambda\} = 2[(\mathbf{x} \cdot \mathbf{L}) \otimes \mathbf{1} - \chi]\,, \tag{38}$$

$$[\chi, \Lambda] = 2[\chi - \Sigma]\,, \tag{39}$$

$$\{\Sigma, \chi\} = 2(\mathbf{x} \cdot \mathbf{x}) \otimes \mathbf{1} - \alpha[\chi + \Sigma], \tag{40}$$

$$[\Sigma, \chi] = 2(\mathbf{x} \cdot \mathbf{x})(\Lambda + \mathbf{1}) - 2\alpha(\mathbf{x} \cdot \mathbf{L}) \otimes \mathbf{1} - \{(\mathbf{x} \cdot \mathbf{L}), \chi\}, \tag{41}$$

$$\{\Sigma, \Lambda\} = 2(\mathbf{x} \cdot \mathbf{L}) \otimes \mathbf{1} + 2\Sigma, \tag{42}$$

$$[\Sigma, \Lambda] = 2(\Lambda(\mathbf{x} \cdot \mathbf{L}) - \chi(\mathbf{L} \cdot \mathbf{L})). \tag{43}$$

In order to determine the Dirac operator we make use of the requirement that it must anticommute with the chirality operator. It turns out that this requirement defines the Dirac operator rather uniquely. Combining the above relations we find the following identities:

$$\{\chi - \Sigma, \Lambda\} = 2(\Sigma - \chi), \tag{44}$$

$$\{\chi - \Sigma, \chi\} = \alpha(\Sigma - \chi). \tag{45}$$

From the latter one we obtain

$$\left\{\chi - \Sigma, \chi + \frac{1}{2}\alpha\right\} = 0. \tag{46}$$

Thus there are two independent operators which anticommute with the chirality operator γ_χ

$$(\Sigma - \chi), \tag{47}$$

and

$$\gamma_\chi(\Sigma - \chi). \tag{48}$$

These two are the candidates for the Dirac operator. It turns out that in the limit the second operator coincides with the commutative case and thus we define the Dirac operator as

$$\mathbf{D} = \frac{1}{l}\gamma_\chi(\Sigma - \chi) = \frac{l}{\mathcal{N}_N}\left\{(\Lambda + \mathbf{1}) - \frac{1}{2l^2}\{\chi, (\mathbf{x} \cdot \mathbf{L})\} - \frac{\alpha}{l^2}(\mathbf{x} \cdot \mathbf{L})\right\}, \tag{49}$$

where $\mathcal{N}_N$ is the normalization given in Eq. (35). It satisfies the required condition

$$\{\gamma_\chi, \mathbf{D}\} = 0. \tag{50}$$

Note that the Dirac operator in this paper is dimensionless.

The first term on the r.h.s. in Eq. (49) has the same form as the Dirac operator for the commutative case. The other terms in Eq. (49) can be interpreted as "quantum corrections." Since

$$\lim_{N \to \infty} \mathcal{N}_N = l, \tag{51}$$

in the naive limit, the above Dirac operator has the standard form of the commutative case if the correction terms including $(\mathbf{x} \cdot \mathbf{L})$ vanish in this limit (which we expect since in the commutative case $(\mathbf{x} \cdot \mathbf{L})$ is identically zero).

To understand the structure of the new terms it is necessary to know the properties of the operator $(\mathbf{x} \cdot \mathbf{L})$. Actually we can show that in the algebra $(\mathcal{A}_N, \mathcal{L}_N)$, $(\mathbf{x} \cdot \mathbf{L})$ does not vanish but is of the order α.

Since

$$[\mathbf{L}_i, (\mathbf{x} \cdot \mathbf{L})] = 0 \tag{52}$$

the action of $(\mathbf{x}\cdot\mathbf{L})$ on a polynomial of $\mathbf{x}_i$ is either a constant or proportional to the Casimir operator. An explicit calculation of the action on the highest weight vector shows that

$$(\mathbf{x}\cdot\mathbf{L})\mathbf{x}_\pm^n = \alpha\frac{n(n+1)}{2}\mathbf{x}_\pm^n . \tag{53}$$

Furthermore since

$$[(\mathbf{L}\cdot\mathbf{L}),\mathbf{x}_i] = 2[\mathbf{x}_i - \mathbf{P}_i] , \tag{54}$$

$$[(\mathbf{x}\cdot\mathbf{L}),\mathbf{x}_i] = \alpha[\mathbf{x}_i - \mathbf{P}_i] , \tag{55}$$

$(\mathbf{x}\cdot\mathbf{L})$ has the same commutator with $\mathbf{x}_i$ as $(\mathbf{L}\cdot\mathbf{L})$ and thus, together with relation (52), on an algebra level we can make the following identification:

$$(\mathbf{x}\cdot\mathbf{L}) = \frac{\alpha}{2}(\mathbf{L}\cdot\mathbf{L}) . \tag{56}$$

This confirms our expectation, i.e. $(\mathbf{x}\cdot\mathbf{L})$ vanishes in the commutative limit.

Knowing the pair $(\gamma_\chi, \mathbf{D})$ of chirality operator and Dirac operator in algebraic terms we introduce the spinors on which these operators are acting. In our formulation the Dirac spinor is simply $\mathscr{A}_N \otimes \mathbb{C}^2$,

$$\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} , \tag{57}$$

where $\psi^a \in \mathscr{A}_N$.

As usual the above spinor can be split into two sectors according to the chirality, by the projection operators

$$\mathscr{P}_\pm \equiv \frac{1}{2}(1 \pm \gamma_\chi) , \tag{58}$$

as

$$\Psi_\pm = \mathscr{P}_\pm \Psi . \tag{59}$$

Correspondingly we can split the Dirac operator as

$$\mathbf{D} = \mathbf{D}_+ + \mathbf{D}_- , \tag{60}$$

where

$$\mathbf{D}_\pm = \frac{l}{\mathscr{N}_N}\mathscr{P}_\mp \left\{\Lambda + \mathbf{1} - \frac{\alpha}{2l^2}(\mathbf{x}\cdot\mathbf{L})\right\}\mathscr{P}_\pm . \tag{61}$$

3.2. The Spectrum of the Dirac Operator. To complete the discussion on the Dirac operator we compute its spectrum. For this end we first calculate the square of the Dirac operator which is given by

$$\mathbf{D}^2 = -\frac{1}{l^2}(\Sigma - \chi)^2 . \tag{62}$$

After a tedious but straightforward calculation we obtain

$$\mathbf{D}^2 = [(\mathbf{L}\cdot\mathbf{L}) + 1 + \Lambda] - \frac{1}{l^2}(\mathbf{x}\cdot\mathbf{L})[(\mathbf{x}\cdot\mathbf{L}) + \alpha(\Lambda + 1)] , \tag{63}$$

where we have used the following relation:

$$(\Sigma - \chi)(\Sigma - \chi) = \Sigma\Sigma - \{\Sigma, \chi\} + \chi\chi$$
$$= -l^2[(\mathbf{L}\cdot\mathbf{L}) + 1 + \Lambda] + (\mathbf{x}\cdot\mathbf{L})[\alpha + (\mathbf{x}\cdot\mathbf{L}) + \alpha\Lambda]\,. \quad (64)$$

Using the above formula, the spectrum of the Dirac operator can be represented by using the Casimir operator as in the classical case. For this purpose we introduce the "total angular momentum" as

$$\mathbf{J}_i = \mathbf{L}_i + \frac{1}{2}\sigma_i\,. \quad (65)$$

The states are labeled by the eigenvalues of the Casimir operator as

$$\mathbf{J}^2\Psi_j = j(j+1)\Psi_j\,, \quad (66)$$

where for the case of the algebra $\mathscr{A}_N$, $0 \leqq j \leqq \frac{N+1}{2}$ is an integer or a half integer depending on whether N is odd or even, respectively. Denoting the eigenvalues of $\mathbf{D}^2$ by λ_j^2, i.e.,

$$\mathbf{D}^2\Psi_j = \lambda_j^2\Psi_j\,, \quad (67)$$

their values are given by

$$\lambda_j^2 = \left(j+\frac{1}{2}\right)^2 - \frac{1}{N(N+2)}\left\{\left(j+\frac{1}{2}\right)^4 - \left(j+\frac{1}{2}\right)^2\right\}\,. \quad (68)$$

Proof. Using the relation $(\mathbf{x}\cdot\mathbf{L}) = \frac{\alpha}{2}\mathbf{L}^2$ we obtain

$$\mathbf{D}^2 = \mathbf{L}^2 + 1 + \Lambda - \frac{\alpha^2}{4l^2}\mathbf{L}^2(\mathbf{L}^2 + 2(\Lambda+1))\,. \quad (69)$$

Then substituting the relations

$$\Lambda = \mathbf{J}^2 - \mathbf{L}^2 - \frac{3}{4}\,,$$
$$\mathbf{J}^2 = j(j+1)\,,$$
$$\mathbf{L}^2 = (j+s)(j+s+1)\,, \quad (70)\,.$$

into Eq. (69). Since $s = \pm\frac{1}{2}$, we see that Eq. (69) depends not on s but on $s^2 = \frac{1}{4}$, and we obtain the above result which depends only on j. □

The spectrum of the Dirac operator is then given by the square root of Eq. (68). Taking the case where $\lambda_j > 0$ as

$$\mathbf{D}\Psi_j = \lambda_j\Psi_j\,, \quad (71)$$

since $\mathbf{D}$ anticommutes with the chirality operator γ_χ, we obtain another spinor

$$\mathbf{D}(\gamma_\chi\Psi_j) = -\lambda_j(\gamma_\chi\Psi_j)\,, \quad (72)$$

as expected.

It is interesting to compare the above results with the commutative case, i.e., to take the limit $N \to \infty$. The above expression in Eq. (68) can be rewritten as

$$\lambda_j^2 = \left(j + \frac{1}{2}\right)^2 \left[1 + \frac{1 - (j + \frac{1}{2})^2}{N(N+2)}\right] . \tag{73}$$

We obtain two different types of spectra.

1. N is even integer. Then j is half integer and the spectrum is

$$\lambda_j^2 = 1, \left(4 - \frac{12}{N(N+2)}\right), \ldots, \left(n^2 - \frac{n^4 - n^2}{N(N+2)}\right), \ldots, \tag{74}$$

where n is integer and $1 \leqq n = j + \frac{1}{2} \leqq \frac{N}{2} + 1$.

2. N is odd integer. Then j is integer and the spectrum is

$$\lambda_j^2 = \left(\frac{1}{4} + \frac{3}{16N(N+2)}\right), \left(\frac{9}{4} + \frac{9}{2N(N+2)}\right), \ldots . \tag{75}$$

For $N \to \infty$ the first case has a proper limit which gives the same spectrum as in the commutative case. The appearance of the second case can be understood rather easily. Following Berezin's general formulation, in principle, any dimension of the Hilbert space $\mathscr{F}_N$ can be chosen. However, when we consider the case where N is odd, $\mathscr{F}_N$ is the spin representation of $SU(2)$. On the other hand, for the case where N is an even integer $\mathscr{F}_N$ is a representation space of $SO(3)$. Since in the limit $N \to \infty$, with an appropriate reinterpretation of the measure, $\mathscr{F}_N$ should approximate the Hilbert space of spherical harmonics, it must be a single valued representation. Therefore, the second case does not fit into our scheme and we have to restrict the quantization parameter N to an even integer.

To conclude, although it seems that in the general formulation of the matrix algebra representation of the fuzzy sphere any integer N can be chosen for the $(N+1) \times (N+1)$ matrices, the above results show that N must be an even integer in order to have a proper classical limit.

3.3. Hilbert Space and New Algebra. In the previous section we constructed the Dirac operator by algebraic calculation, then we introduced Dirac spinors as the space onto which the Dirac operator is acting in Eq. (57). Below we shall see that the structure of the algebra has to be reconsidered.

Using the trace norm on the algebra the above spinor space becomes a Hilbert space with norm

$$\|\Psi\|^2 \equiv \frac{1}{N+1} \sum_{a=1,2} \mathrm{Tr}\{\psi^{a*}\psi^a\} . \tag{76}$$

Thus we have the Hilbert space

$$\mathfrak{H}_N \equiv \mathscr{A}_N \otimes \mathbb{C}^2 . \tag{77}$$

In order to complete Connes' K-cycle and thus put the noncommutative sphere into his framework, we have to reconsider the algebra of the fuzzy sphere. The reason for this is that the chirality operator defined above does not commute with the elements of $\mathscr{A}_N$. Instead we have

$$[\gamma_\chi, \mathbf{X}_i] = 0 , \tag{78}$$

where

$$\mathbf{X}_i = \mathbf{x}_i + \frac{\alpha}{2}\sigma_i\,. \tag{79}$$

Since the algebra should be trivial under the grading, we rather have to consider the algebra $\widetilde{\mathscr{A}}_N$ of $\mathbf{X}_i$ instead of the one of $\mathbf{x}_i$. The new generator $\mathbf{X}_i$ satisfies the same commutation relation as $\mathbf{x}_i$, i.e.,

$$[\mathbf{X}_i, \mathbf{X}_j] = i\alpha\varepsilon_{ijk}\mathbf{X}_k\,. \tag{80}$$

Thus the new algebra $\widetilde{\mathscr{A}}_N$ has the same multiplication rule and we may consider it as a different realization of the algebra of the fuzzy sphere.

Now we have completed the construction of the triple $(\widetilde{\mathscr{A}}_N, \mathbf{D}, \mathfrak{H}_N)$, starting from the algebra of the fuzzy sphere. With this K-cycle we may construct the differential calculus and write down the field theory Lagrangian, which we will discuss in a separate paper. In the remaining part of this paper, we want to examine a bit further the properties of the Dirac operator.

3.4. Local Coordinates on Noncommutative Sphere. In Ref. [23], the Schwinger model on S^2 is investigated and the classical Dirac operator on the sphere is discussed in great detail. In this context the globally $SU(2)$ invariant form of the Dirac operator is compared with the Dirac operator in local coordinates. It is shown that the transformation from the covariant to the local frame is given as $\tilde{D}_{\text{local}} = u_c D_{\text{covariant}} u_c^{-1}$, where u_c is a unitary matrix with determinant equal to unity.

It is interesting to examine the similar transformation for the Dirac operator on the noncommutative sphere. This will help us to understand the meaning of a local coordinate patch in noncommutative geometry.

We define the matrix u which gives the map discussed above by the requirement

$$\sigma_3 u = u\gamma_\chi\,. \tag{81}$$

A lengthy but straightforward calculation yields

$$u = \frac{1}{\sqrt{\alpha(N+1)}}\begin{pmatrix} \sqrt{\mathbf{x}_3 + \frac{\alpha}{2}(N+2)} & \frac{1}{\sqrt{(\mathbf{x}_3 + \frac{\alpha}{2}(N+2))}}\mathbf{x}_- \\ \frac{-1}{\sqrt{(\mathbf{x}_3+\frac{\alpha}{2}N)}}\mathbf{x}_+ & \sqrt{\mathbf{x}_3 + \frac{\alpha}{2}N} \end{pmatrix}, \tag{82}$$

where $\mathbf{x}_\pm = \mathbf{x}_1 \pm i\mathbf{x}_2$ and α is given in Eq. (11).

The above matrix becomes singular on the states satisfying $(\mathbf{x}_3 + \frac{\alpha}{2}N)|v\rangle = 0$ and $(\mathbf{x}_3 + \frac{\alpha}{2}(N+2))|v'\rangle = 0$. Taking the basis given in Eq. (17), the eigenvalues of the operator $\mathbf{x}_3$ are limited by $\alpha\frac{N}{2}$, i.e., $\|\mathbf{x}_3\| \leqq \alpha\frac{N}{2}$, thus we have to impose the following restriction on the states in $\mathscr{F}_N$:

$$\left(\mathbf{x}_3 + \frac{\alpha}{2}N\right)|v\rangle \neq 0\,. \tag{83}$$

Therefore the following discussion holds only for the restricted Hilbert space, not for the whole space $\mathscr{F}_N$.

In the limit $N \to \infty$ which corresponds to the commutative case and using Eq. (11) we obtain

$$u_{N\to\infty} = \frac{1}{\sqrt{2(1+\frac{x_3}{l})}} \begin{pmatrix} 1+\frac{x_3}{l} & \frac{x_-}{l} \\ -\frac{x_+}{l} & 1+\frac{x_3}{l} \end{pmatrix}. \tag{84}$$

Rewriting this expression in stereographic coordinates

$$x_1 = 2l^2 z_1 (l^2 + \mathbf{z}^2)^{-1}, \tag{85}$$

$$x_2 = 2l^2 z_2 (l^2 + \mathbf{z}^2)^{-1}, \tag{86}$$

$$x_3 = l(l^2 - \mathbf{z}^2)(l^2 + \mathbf{z}^2)^{-1}, \tag{87}$$

where z_i are the coordinates on the 2-sphere, we get

$$u_{N\to\infty}(z) = \frac{1}{\sqrt{(l^2+z^2)}} \begin{pmatrix} l & z_1 - iz_2 \\ -(z_1+iz_2) & l \end{pmatrix}. \tag{88}$$

Up to a phase factor this is equivalent to the result obtained in [23].[4]

Under this transformation u the coordinates $\mathbf{X}_\pm$, $\mathbf{X}_3$ transform into diagonal form as

$$u(\mathbf{x}_+ + \alpha\sigma_+)u^\dagger = \begin{pmatrix} \sqrt{1+\frac{\alpha}{\mathbf{x}_3+\frac{\alpha}{2}N}}\,\mathbf{x}_+ & 0 \\ 0 & \sqrt{1-\frac{\alpha}{\mathbf{x}_3+\frac{\alpha}{2}N}}\,\mathbf{x}_+ \end{pmatrix}, \tag{90}$$

$$u(\mathbf{x}_- + \alpha\sigma_-)u^\dagger = \begin{pmatrix} \sqrt{1+\frac{\alpha}{\mathbf{x}_3+\frac{\alpha}{2}(N+2)}}\,\mathbf{x}_- & 0 \\ 0 & \sqrt{1-\frac{\alpha}{\mathbf{x}_3+\frac{\alpha}{2}(N+2)}}\,\mathbf{x}_- \end{pmatrix}, \tag{91}$$

and

$$u\left(\mathbf{x}_3 + \frac{\alpha}{2}\sigma_3\right)u^\dagger = \begin{pmatrix} \mathbf{x}_3 + \frac{\alpha}{2} & 0 \\ 0 & \mathbf{x}_3 - \frac{\alpha}{2} \end{pmatrix}. \tag{92}$$

This is consistent since the chirality operator is now σ_3 and the elements which commute with the chirality operator are the diagonal matrices. In this parametrization the Dirac operator has the form

$$\mathbf{D}' = \begin{pmatrix} 0 & \mathbf{D}'_- \\ \mathbf{D}'_+ & 0 \end{pmatrix}, \tag{93}$$

where $\mathbf{D}'_\pm$ is defined by transforming the operators in Eq. (61).

As already mentioned, when we perform the above transformation we have to restrict the states by Eq. (83). The interpretation of the restriction (83) on the states

[4] The phase factor is

$$f = \frac{1}{\sqrt{2}} \begin{pmatrix} (1+i) & 0 \\ 0 & (1-i) \end{pmatrix}. \tag{89}$$

can be considered as an analogy to the idea of a local patch in the noncommutative geometry.

To complete the discussion, consider another transformation defined by

$$\sigma_3 u' = -u'\gamma_\chi\,, \tag{94}$$

which differs from condition (81) only by a sign. The matrix u' of this transformation

$$u' = \frac{1}{\sqrt{\alpha(N+1)}}\begin{pmatrix} \sqrt{\mathbf{x}_3 - \frac{\alpha}{2}N} & \frac{1}{\sqrt{(\mathbf{x}_3-\frac{\alpha}{2}N)}}\mathbf{x}_- \\ \frac{-1}{\sqrt{(\mathbf{x}_3-\frac{\alpha}{2}(N+2))}}\mathbf{x}_+ & \sqrt{\mathbf{x}_3 - \frac{\alpha}{2}(N+2)} \end{pmatrix}. \tag{95}$$

This transformation matrix has also a singularity and correspondingly we remove the state satisfying

$$\left(\mathbf{x}_3 - \frac{\alpha}{2}N\right)|v'\rangle = 0\,. \tag{96}$$

from the Hilbert space $\mathscr{F}_N$.

Also in this case, we obtain a chirality operator given by σ_3 after the transformation and thus the algebra $\tilde{\mathscr{A}}$ is transformed into diagonal matrices and the Dirac operator has nonzero elements only in the off-diagonal components.

We can consider the transformations u and u' as maps from symmetric coordinate to local coordinate systems. The restriction on the states can be understood as the analogue of the definition of the regions of local patches in the commutative case.

In the coherent state representation this restriction defines an open set on the sphere, since the complex value v parametrizing the state $|v\rangle$ corresponds to a point on S^2. Consider the normalized state $\frac{1}{(1+\bar{z}z)^{N/2}}|z\rangle$, then the state which has to be removed corresponds to the point $z=\infty$, $(z=0)$ for transformation u (u', respectively). Therefore, the transformation u gives a "coordinate patch," an open set of states over the noncommutative sphere where the "south pole" is excluded. Correspondingly u' gives a "coordinate patch" where the "north pole" is excluded.

4. Conclusions and Discussions

In this paper we have defined the Dirac operator on the noncommutative sphere and determined the triple $(\tilde{\mathscr{A}}_N, \mathbf{D}, \mathfrak{H}_N)$ with grading operator γ_χ, which defines Connes' K-cycle $(\mathbf{D}, \mathfrak{H}_N)$ over the algebra $\tilde{\mathscr{A}}_N$. We have started with the well known algebra of the fuzzy sphere $\mathscr{A}_N$, and then have considered the bigger algebra $(\mathscr{A}_N, \mathscr{L}_N)$ by including the derivations $\mathbf{L}_i$. The derivations are defined by the adjoint action of the coordinate function $\mathbf{x}_i$. Then, we defined the chirality operator and the Dirac operator in the algebra $(\mathscr{A}_N, \mathscr{L}_N)\otimes M_2(\mathbb{C})$. The Dirac operator has been determined by requiring that it anticommutes with the chirality operator. The construction is performed purely algebraically.

In order to determine the chirality operator γ_χ we have required a "correspondence principle," i.e. the condition that in the naive limit $N\to\infty$ this operator corresponds to the one of the commutative case. We also used this correspondence principle to choose the Dirac operator between two possibilities. In this way we singled out a pair $(\mathbf{D}, \gamma_\chi)$ in the algebra of $(\mathscr{A}_N, \mathscr{L}_N)\otimes M_2(\mathbb{C})$.

The spinors are introduced as Hilbert space vectors on which these operators are acting, thus they are vectors in the space $\mathscr{A}_N \otimes \mathbb{C}^2$. Then, we have calculated the spectrum of the Dirac operator. As a result, we have found that there are two different sequences of quantized algebras depending on whether the quantization parameter N is an even or an odd integer, i.e., $\{\mathscr{A}_{2k}\}$ and $\{\mathscr{A}_{2k+1}\}$ with integer k. We have seen that a restriction of N to even integers gives the desired classical limit. This requirement guarantees that the Hilbert space $\mathscr{F}_N$ appearing in Berezin's quantization is a single valued representation of the rotation group of S^2.

We also defined the Connes' triple $(\widetilde{\mathscr{A}_N}, \mathbf{D}, \mathfrak{H}_N)$. The algebra of the noncommutative sphere $\widetilde{\mathscr{A}_N}$ is obtained by modifying the original algebra $\mathscr{A}_N$. This modification is necessary since the algebra $\mathscr{A}_N$ does not commute with the chirality operator and the new algebra $\widetilde{\mathscr{A}_N}$ is defined as a subalgebra of $\mathscr{A}_N \otimes M_2(\mathbb{C})$ by the requirement that it commutes with the chirality operator.

Finally we considered the transformation of the chirality operator γ_χ into the standard form, i.e., $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We found two unitary matrices u and u' defined by Eqs. (81) and (94). However, the unitary matrices u and u' are not defined on the whole set of states in the Hilbert space $\mathscr{F}_N$ since they have singularities depending on the eigenvalues of the operator $\mathbf{x}_3$. We meet the similar situation in the commutative case when we transform the coordinate system from symmetric coordinates to local ones. To avoid these singularities, we restrict the vectors in the Hilbert space $\mathscr{F}_N$ in analogy to the commutative case. In the coherent state representation, this restriction corresponds to considering the states labeled by points in a certain open set on S^2. Actually, under the maps given by u and u' the algebra $\tilde{A}_N$ is diagonalized, and the set of coherent states given by the open set of parameters restricted to the region around the north pole and south pole, respectively, can be interpreted as the analogue of a local patch in the noncommutative case.

The starting point of our construction is near to the original idea of Ref. [17], where the fermion fields are also considered as $\mathscr{A} \otimes \mathbb{C}^2$ with the association to the classical case in Ref. [23]. However unlike in the papers [20] we do not make use of the supersymmetry algebra in order to define the Dirac operator. As a consequence our results are different from the ones obtained in [20].

Especially we have found that in the noncommutative case there is a contribution from an operator $(\mathbf{x} \cdot \mathbf{L})$ which is zero in the commutative limit. Without this contribution of $(\mathbf{x} \cdot \mathbf{L})$, the Dirac operator does not anticommute with the chirality operator in noncommutative space. As a result, the spectrum of the Dirac operator has a correction term compared to the commutative case.

With the results obtained here we can construct the differential calculus à la Connes. This is now in preparation.

Finally we want to make a comment on the property of the Planck constant in our formulation. As we mentioned, in the method described here the formulation of the function algebra is independent of the dimension of the Hilbert space. The $\mathbf{N}$, and thus the Planck constant α, appears as an operator in the algebra. The dependence on the dimension of the Hilbert space enters when computing the expectation value with respect to a certain coherent state or the trace in Hilbert space.

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