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C*-Algebras Associated With One Dimensional Almost Periodic Tilings

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Abstract: For each irrational number, $0 < \alpha < 1$, we consider the space of one dimensional almost periodic tilings obtained by the projection method using a line of slope α. On this space we put the relation generated by translation and the identification of the "singular pairs." We represent this as a topological space X_α with an equivalence relation R_α. On R_α there is a natural locally Hausdorff topology from which we obtain a topological groupoid with a Haar system. We then construct the C*-algebra of this groupoid and show that it is the irrational rotation C*-algebra, A_α.

Given a topological space X and an equivalence relation R on X, one can form the quotient space X/R and give it the quotient topology. It frequently happens however that the quotient topology has very few open sets. For example let X be the unit circle, which we shall write as [0,1] with the endpoints identified and the group law given by addition modulo 1. Fix α, irrational, $0 < \alpha < 1$, and let $R = \{(x,y) \mid x - y \in \mathbb{Z} + \alpha\mathbb{Z}\}$. Since each equivalence class of R is dense in X, the only open sets in X/R are $\emptyset$ and X/R.

However the equivalence relation R has the structure of a groupoid and if we can put a topology on R, (usually not the product topology of $X \times X$), so that R becomes a topological groupoid:

(i) $R \ni (x,y) \mapsto (y,x) \in R$ is continuous, and
(ii) $R^2 \ni ((x,y),(y,z)) \mapsto (x,z) \in R$ is continuous,

and we can find a compatible family $\{\mu^x\}$ of measures (μ^x is a measure on $R^x = \{(x,y) \mid x \sim y\}$), called a Haar system (see Renault [7, Definition I.2.2]), one can construct a C*-algebra, $C^*(R,\mu)$, by completing $C_{oo}(R)$, the continuous functions on R with compact support in a suitable norm.

In the example above of the relation R on the unit circle S^1, suppose $(x,y) \in R$, so there is $n \in \mathbb{Z}$ such that $(x + n\alpha) - y \in \mathbb{Z}$ and let $\mathcal{U} \subseteq S^1$ be a neighbourhood

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of x, then a basic neighbourhood of (x,y) in R is given by $\{(a,a+n\alpha)\,|\,a\in\mathscr{U}\}$. On $R^x=\{(x,x+n\alpha)\,|\,n\in\mathbb{Z}\}$ we put the counting measure. With this information one can construct the C*-algebra of this topological groupoid by completing $C_{oo}(R)$ in a C*-norm; see Renault [7, Definition II.1.12].

In this paper we shall show how this same C*-algebra arises as the "non-commutative" space of a set of one dimensional almost periodic tilings of $\mathbb{R}$.

For each irrational number α, $0<\alpha<1$, let T_α be the space of tilings obtained from the projection method using a line of slope α. We shall classify the tilings in T_α as follows. Given $\mathbf{T}\in T_\alpha$ we choose a tile t in $\mathbf{T}$ and construct in an explicit way a sequence (x_i) in $X_\alpha=\{(x_i)\,|\,x_i\in\{0,1,2,3,\ldots,a_i\}$ and $x_{i+1}=0$ whenever $x_i=a_i\}$, where $\alpha=[0;a_1,a_2,a_3,\ldots]$ is the continued fraction expansion of α. The sequence of X_α constructed from $(t,\mathbf{T})$ depends on the choice of the tile t. So we put on X_α the smallest equivalence relation so that the sequence obtained from $(t,\mathbf{T})$ is equivalent to the sequence obtained from $(t',\mathbf{T})$ for any other tile $t'\in\mathbf{T}$. By putting a topology and a Haar system on this relation we construct a C*-algebra and show that it is the irrational rotation C*-algebra A_α.

A number of authors have considered C*-algebras associated with almost periodic tilings. This paper was motivated by the observation of Connes [3, II.3] that the space of Penrose tilings are classified by the space $\{(x_i)\,|\,x_i\in\{0,1\}$ and $x_{i+1}=0$ whenever $x_i=1\}$ $(=X_{\frac{\sqrt{5}-1}{2}}$ in our notation) modulo the equivalence relation of tail equivalence. Connes then shows that the C*-algebra of this equivalence relation is the simple AF C*-algebra $AF_{\frac{\sqrt{5}-1}{2}}$ with $K_0=\mathbb{Z}+\frac{\sqrt{5}-1}{2}\mathbb{Z}$ (as an additive subgroup of $\mathbb{R}$) and positive cone $(\mathbb{Z}+\frac{\sqrt{5}-1}{2}\mathbb{Z})_+$. In [5] J. Kellendonk considers C*-algebras associated with almost periodic tilings, however the algebras constructed are the C*-crossed products associated with an action of $\mathbb{Z}$ on a Cantor set and thus have $K_1=\mathbb{Z}$. In [1] Anderson and Putnam consider C*-algebras associated with substitution tilings. While our tilings are also substitution tilings, the substitution rule will (in general) change at each iteration; thus the tilings considered here are different from those analysed by Anderson and Putnam.

An interesting feature of our construction is that there is a sub-relation $\mathscr{R}_\alpha\subseteq R_\alpha$. $\mathscr{R}_\alpha=\{(x,y)\in X_\alpha\times X_\alpha\,|\,x$ is tail equivalent to $y\}$. The topology of R_α restricted to $\mathscr{R}_\alpha$ is a Hausdorff topology and $\mathscr{R}_\alpha$ is a principal r-discrete groupoid. We shall show that $C^*(\mathscr{R}_\alpha)$ is a simple AF-algebra with the same ordered K_0 group as A_α.

Let us now describe in detail the plan of the paper. In Sect. 1 we give a brief overview of the tilings under consideration; full details will be published separately [6].

In Sect. 2 we put a topology on the relation $\mathscr{R}_\alpha$, of tail equivalence on X_α, and show that it yields a principal r-discrete groupoid whose C*-algebra is AF and we show that its ordered K_0 is $(\mathbb{Z}+\alpha\mathbb{Z},(\mathbb{Z}+\alpha\mathbb{Z})_+)$ with the class of the identity equal to 1.

In Sect. 3 we describe an isomorphism φ between $S^1_{\mathbf{N}\alpha}$ and X_α, where $S^1_{\mathbf{N}\alpha}$ is the Cantor set obtained by disconnecting the circle S^1 along the forward orbit of 0: $\{0,\alpha,2\alpha,3\alpha,\ldots\}$. On the space $S^1_{\mathbf{N}\alpha}$ there is the partial homeomorphism of adding α modulo 1 with domain $S^1_{\mathbf{N}\alpha}\backslash\{-\alpha\}$. We construct a partial homeomorphism Θ on X_α, such that φ intertwines Θ and the partial homeomorphism on $S^1_{\mathbf{N}\alpha}$. The relation $x\sim\Theta(x)$ on X_α is exactly tail equivalence.

In Sect. 4 we put a topology on the relation R_α and construct a continuous onto map $\Phi:R_\alpha\to S^1\rtimes_\alpha\mathbb{Z}$ such that $\Phi^*:C_{oo}(S^1\rtimes_\alpha\mathbb{Z})\to C_{oo}(R_\alpha)$ is an isomorphism of vector spaces, where $C_{oo}(R_\alpha)$ is the space of functions whose support is the closure of a compact set.

In Sect. 5 we construct a Haar system $\{\mu^x\}$ on R_α and use this to put the structure of a $*$-algebra on $C_{oo}(R_\alpha)$. Then we show that Φ^* is a $*$-homomorphism. This then implies that $C^*(R_\alpha, \mu)$ is isomorphic to A_α.

1. The Tilings

The tilings we consider are doubly infinite sequences $\{t_i\}_{i=-\infty}^{\infty}$, where $t_i \in \{\mathbf{a}, \mathbf{b}\}$ and which satisfy three axioms.

(A_1): the letter **a** is isolated: if $t_i = \mathbf{a}$ then $t_{i-1} = t_{i+1} = \mathbf{b}$.

(A_2): there is an integer n such that between **a**'s there are either n or $n+1$ **b**'s.

A sequence which satisfies (A_1) and (A_2) is *composable*. Given a composable sequence **T** we can produce a new sequence **T**$'$ by *composition*: each segment beginning with an **a** and followed by n **b**'s gets replaced by a **b**, and each segment beginning with an **a** and followed by $n+1$ **b**'s gets replaced by **ba**.

$$\mathbf{a}\underbrace{\mathbf{bbb}\ldots\mathbf{b}}_{n} \mapsto \mathbf{b} \quad \text{and} \quad \mathbf{a}\underbrace{\mathbf{bbb}\ldots\mathbf{b}}_{n+1} \mapsto \mathbf{ba}$$

Axioms (A_1) and (A_2) are exactly what are needed in order to compose a sequence. The third axiom is then:

(A_3): each composition of the sequence produces a composable sequence.

We shall call a sequence satisfying axioms (A_1), (A_2), and (A_3) a *cutting sequence*, following C. Series [6].

A cutting sequence may be constructed by choosing a slope α and a y-intercept β for a line $\mathbf{L} : y = \alpha x + \beta$. We mark by an **a** each intersection of the line **L** with the horizontal lines $y = i$ for $i \in \mathbb{Z}$ and by a **b** the intersection of **L** with the vertical line $x = j$ for $j \in \mathbb{Z}$. This produces along **L** a sequence of **a**'s and **b**'s.

If a line **L** passes through a point (m, n) in $\mathbb{Z}^2$ we call it *singular* for at (m, n) an **a** and a **b** coincide. Such a line produces a *singular pair*: two cutting sequences $\mathbf{T}^+$ and $\mathbf{T}^-$. In the upper sequence $\mathbf{T}^+$ all coinciding **a**'s and **b**'s are written with the **a** preceding the **b**; in $\mathbf{T}^-$ all coinciding **a**'s and **b**'s are written with the **a** following the **b**.

Via composition we may associate with a cutting sequence a real number $0 < \alpha < 1$ which we call the *slope* of the tiling. Let **T** be a cutting sequence. Let $\mathbf{T}_2$

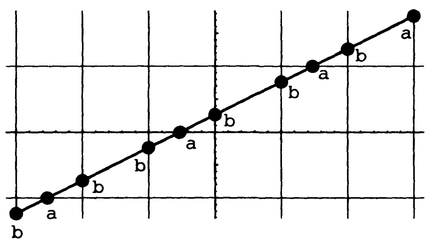

Fig. 1.

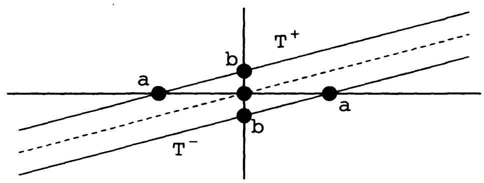

Fig. 2.

be the cutting sequence obtained from $\mathbf{T}_1 = \mathbf{T}$ by composition. In general, let $\mathbf{T}_{k+1}$ be the cutting sequence obtained from $\mathbf{T}_k$ by composition. For each i there is, by axiom (A_2), an integer n_i such that in $\mathbf{T}_i$ there are between adjacent $\mathbf{a}$'s either n_i or n_{i+1} $\mathbf{b}$'s. This produces a sequence of non-negative integers $\{n_1, n_2, n_3, \ldots\}$. Let α be the real number with continued fraction expansion $[0; n_1, n_2, n_3, \ldots]$, adopting the convention that a trailing sequence of 0's is dropped. Let T_α be the set of cutting sequences of slope α.

A line of slope α will produce a cutting sequence of slope α, moreover for each cutting sequence of slope α there is a β (not unique) such that the line $y = \alpha x + \beta$ will produce the given cutting sequence.

Motivated by the classification (see [3]) of Penrose tilings by sequences of 0's and 1's where a 1 must be followed by a 0, modulo tail equivalence, we can classify the cutting sequences of slope α by sequences of integers. If α is rational then there is up to translation only one cutting sequence and it is periodic.

Suppose that $0 < \alpha < 1$ and α is irrational. Let $[0; a_1, a_2, a_3, \ldots]$ be the continued fraction expansion of α. Let $X_\alpha = \{(x_i)_{i=1}^\infty \mid x_i \in \{0, 1, 2, \ldots, a_i\}$ and $x_i = a_i$ implies that $x_{i+1} = 0\}$. We give X_α the topology it inherits as a subspace of $\prod_{i=1}^\infty \{0, 1, 2, \ldots, a_i\}$ with the product topology. X_α becomes a separable totally disconnected metrizable space, i.e. a Cantor set. When $\alpha = \frac{\sqrt{5}-1}{2}$, X_α is the space which classifies the Penrose tilings.

Suppose $\mathbf{T} \in T_\alpha$ is a cutting sequence of slope α and t is a letter in $\mathbf{T}$. Let $\mathbf{T}_1 = \mathbf{T}$, and $\mathbf{T}_i$ be the sequence of cutting sequences obtained by composition. The letter $t \in \mathbf{T}$ will be absorbed into a letter t_2 of $\mathbf{T}_2$, this letter t_2 will be absorbed into a letter t_3 of $\mathbf{T}_3$.

$$\begin{array}{cccc}
\underbrace{\mathbf{ab}\boxed{\mathbf{b}}\mathbf{b}\ldots\mathbf{b}} & t_i & \in & \mathbf{T}_i \\
 & \downarrow & & \\
\mathbf{ba}\boxed{\mathbf{b}}\mathbf{ab} & t_{i+1} & \in & \mathbf{T}_{i+1}
\end{array}$$

Letting $t_1 = t$ we obtain a sequence $\{t_i\}_{i=1}^\infty$ with $t_i \in \mathbf{T}_i$. The sequence $(x_i) \in X_\alpha$ associated with the pair $(\mathbf{T}, t)$ is constructed as follows. If $t_i = \mathbf{a}$ then $x_i = 0$, if $t_i = \mathbf{b}$ then x_i is the number of $\mathbf{b}$'s between t_i and the first $\mathbf{a}$ to the left of t_i. In the example above $x_i = 1$ and $x_{i+1} = 0$. This describes a map from $\{(\mathbf{T}, t) \mid t \in \mathbf{T} \in T_\alpha\}$ to X_α. If t and t' are in $\mathbf{T}$ then we will obtain two sequences (x_i) and (x_i') in X_α which will be in general different. If $\mathbf{T}$ is not singular then (x_i) and (x_i') will be *tail equivalent*, i.e. there is an integer k such that $x_i = x_i'$ for $i > k$. If α is irrational and $\mathbf{T}$ is singular, this may not happen.

Let us denote by $0^+ = (0, a_2, 0, a_4, \ldots)$, $0^- = (a_1, 0, a_3, 0, \ldots)$, and $-\alpha = (a_1 - 1, a_2 - 1, a_3 - 1, a_4 - 1, \ldots)$ three sequences in X_α. If $\mathbf{T}$ is a $\mathbf{T}^+$ then each (x_i) will be tail equivalent to either 0^+ or $-\alpha$. If $\mathbf{T}$ is a $\mathbf{T}^-$ then each (x_i) will be tail equivalent to either 0^- or $-\alpha$.

Suppose now that on the set of cutting sequences with slope α, T_α, we say that $\mathbf{T}_1$ is equivalent to $\mathbf{T}_2$, if by shifting $\mathbf{T}_1$ a finite number of letters to the left or right it agrees with $\mathbf{T}_2$ and that we decree that the upper and lower sequences for a singular line are equivalent (as in fact they only differ by a single transposition of an **a** and a **b** at the one singular point). Transferring this relation to X_α it becomes the relation R_α generated by tail equivalence and $0^+ \sim 0^- \sim -\alpha$.

In [6] we prove that the map from T_α to X_α is onto and tail equivalence plus $0^+ \sim 0^- \sim -\alpha$ classifies the tilings of slope α.

2. $K_0(C^*(\mathscr{R}))$

In this section we calculate K_0 of the AF C*-algebra $C^*(\mathscr{R})$ and show that it is equal to $(\mathbb{Z}+\alpha\mathbb{Z}, (\mathbb{Z}+\alpha\mathbb{Z})_+,\ [1])$. The equivalence relation $\mathscr{R}$ defines an AF groupoid, and thus this C*-algebra is AF (see Renault [7, Proposition III.1.5]). We shall follow the construction given by Connes [3, II.3].

Let $0<\alpha<1$ be irrational and $[0; a_1, a_2, a_3, \ldots]$ be its continued fraction expansion. Let $X_\alpha = \{(x_i)_{i=1}^\infty \mid x_i \in \{0,1,2,3,\ldots,a_i\}$ and $x_i = a_i$ implies $x_{i+1}=0\}$ and $\mathscr{R}_\alpha = \{(x,y) \in X_\alpha \times X_\alpha \mid \text{there is } k \text{ such that } x_i = y_i \text{ for } i>k\}$. To simplify the notation we shall write X for X_α and $\mathscr{R}$ for $\mathscr{R}_\alpha$, as α will be fixed throughout this section.

We construct a topology on $\mathscr{R}$ as follows. Suppose $(x,y) \in \mathscr{R}$ for each k such that $x_i = y_i$ for $i>k$ we construct a basic neighbourhood $\mathscr{U}(x,y,k) = \{(a,b) \in \mathscr{R} \mid a_i = x_i \text{ and } b_i = y_i \text{ for } 1 \leqq i \leqq k \text{ and } a_i = b_i \text{ for } i>k\}$.

Suppose $(x,y) \in \mathscr{R}$, and $x_i = y_i$ for $i>k$, also $(x',y') \in R$ and $x'_i = y'_i$ for $i>k'$, and $k'>k$. Then either $\mathscr{U}(x,y,k)$ and $\mathscr{U}(x',y',k')$ are disjoint or $\mathscr{U}(x',y',k') \subseteq \mathscr{U}(x,y,k)$. For suppose $(a,b) \in \mathscr{U}(x,y,k) \cap \mathscr{U}(x',y',k')$. Then $a_i = x_i$ for $1 \leqq i \leqq k$ and $a_i = x'_i$ for $1 \leqq i \leqq k'$. Hence $x_i = x'_i$ for $1 \leqq i \leqq k$. Similarly $y_i = y'_i$ for $1 \leqq i \leqq k$. Since $a_i = b_i$ for $i>k$, we have $x'_i = y'_i$ for $i>k$. Thus $(x',y') \in \mathscr{U}(x,y,k)$, so $\mathscr{U}(x',y',k') \subseteq \mathscr{U}(x,y,k)$. Thus the set $\{\mathscr{U}(x,y,k)\}$ forms a base for a topology of $\mathscr{R}$.

By defining $r(x,y) = (x,x)$ and $d(x,y) = (y,y)$, R becomes an r-discrete principal groupoid in the sense of Renault [7, I.Sect. 1 and I.Sect. 2]. The sets $\mathscr{U}(x,y,k)$ are compact open $\mathscr{R}$-sets in that both r and d are one-to-one when restricted to $\mathscr{U}(x,y,k)$.

$C^*(\mathscr{R})$ will be the completion of the space of continuous functions on R with compact support with respect to a norm that we will presently construct.

Let $\mathscr{R}^{(k)} = \{(x,y) \in X \times X \mid x_i = y_i \text{ for } i>k\}$. Then $\mathscr{R} = \bigcup_k \mathscr{R}^{(k)}$. If $(x,y) \in \mathscr{R}^{(k)}$ then $\mathscr{U}(x,y,k') \subseteq \mathscr{R}^{(k)}$ for some $k' \leqq k$. So $\mathscr{R}^{(k)}$ is an open subset of $\mathscr{R}$. If $x_i \neq y_i$ for some $i>k$ then $\mathscr{U}(x,y,k')$ is disjoint from $\mathscr{R}^{(k)}$, where k' ($>k$) is such that $x_i = y_i$ for $i>k'$. Hence $\mathscr{R}^{(k)}$ is also closed in $\mathscr{R}$. Since $\mathscr{U}(x,y,k) \subseteq \mathscr{R}^{(k)}$ we see that $\mathscr{R}$ has the inductive limit topology associated with the sequence

$$\mathscr{R}^{(0)} \subseteq \mathscr{R}^{(1)} \subseteq \mathscr{R}^{(2)} \subseteq \cdots \subseteq \mathscr{R}\,.$$

Let us show that each $\mathscr{R}^{(k)}$ is compact. In doing so we shall see that $\mathscr{R}^{(k)}$ is an elementary groupoid in the terminology of Renault [7, p. 123]. First we develop some notation. Let $X^{(k)} = \{(x_i)_{i=k+1}^\infty \mid x_i \in \{0,1,2,3,\ldots,a_i\}$ and $x_i = a_i$ implies $x_{i+1}=0\}$, and $\widetilde{X}^{(k)}$ be the subset of $X^{(k)}$ consisting of those sequences which begin with 0: $\widetilde{X}^{(k)} = \{x \in X^{(k)} \mid x_{k+1} = 0\}$, $X^{(0)} = X$. Give $X^{(k)}$ and $\widetilde{X}^{(k)}$ the product topology.

Each $X^{(k)}$ and each $\widetilde{X}^{(k)}$ is compact. Let $X_{(k)} = \{(x_1, \ldots, x_k) \mid x_i \in \{0, 1, \ldots, a_k\}$ and $x_{i+1} = 0$ whenever $x_i = a_i\}$. Let $\mathscr{R}_{(k)} = \{(x, y) \in X_{(k)} \times X_{(k)} \mid x_k = y_k\}$. Give $X_{(k)}$ and $\mathscr{R}_{(k)}$ the discrete topology. Write $\mathscr{R}_{(k)}$ as the disjoint union of two groupoids $\mathscr{R}^{\sim} \cup \mathscr{R}^{\approx}$: $\mathscr{R}^{\sim} = \{(x, y) \in \mathscr{R}_{(k)} \mid x_k \neq a_k\}$ and $\mathscr{R}^{\approx} = \{(x, y) \mid x_k = a_k\}$. We shall next show that $\mathscr{R}^{(k)}$ is homeomorphic to the Cartesian product of a finite set and $X^{(k)}$. In the following lemma we put the product topology on each of $\mathscr{R}^{\sim} \times X^{(k)}$ and $\mathscr{R}^{\approx} \times \widetilde{X}^{(k)}$, and denote by $\mathscr{R}^{\sim} \times X^{(k)} \cup \mathscr{R}^{\approx} \times \widetilde{X}^{(k)}$ their topological disjoint sum.

Lemma 2.1. *The map*

$$(*) \qquad (x, y) \mapsto (((x_1, \ldots, x_k), (y_1, \ldots, y_k)), (x_{k+1}, x_{k+2}, \ldots))$$

is a homeomorphism from $\mathscr{R}^{(k)}$ to $\mathscr{R}^{\sim} \times X^{(k)} \cup \mathscr{R}^{\approx} \times \widetilde{X}^{(k)}$. So $\mathscr{R}^{(k)}$ is compact.

Proof. The map is one-to-one as, for $(x, y) \in \mathscr{R}^{(k)}$ we have $x_{k+1} = y_{k+1}$, $x_{k+2} = y_{k+2}, \ldots$. If $(x, y) \in \mathscr{R}^{\sim}$ and $z \in X_{(k)}$ then $(x_1, \ldots, x_k, z_{k+1}, \ldots)$ and $(y_1, \ldots, y_k, z_{k+1}, \ldots)$ are in X as neither x_k nor y_k is equal to a_k. Also given $(x, y) \in \mathscr{R}^{\approx}$ and $z \in \widetilde{X}^{(k)}$ the sequences $(x_1, \ldots, x_k, z_{k+1}, z_{k+2}, \ldots) = (x_1, \ldots x_{k-1}, a_k, 0, z_{k+2}, \ldots)$ and $(y_1, \ldots, y_k, z_{k+1}, \ldots) = (x_1, \ldots, y_{k-1}, a_k, 0, z_{k+2}, \ldots)$ are in X. Thus the map is onto. The map also takes the basic open sets $\mathscr{U}(x, y, k')$ (for $k' > k$) for the topology of $\mathscr{R}^{(k)}$ to basic open sets in $\mathscr{R}^{\sim} \times X^{(k)} \cup \mathscr{R}^{\approx} \times \widetilde{X}^{(k)}$. Hence $(*)$ is a homeomorphism.

$\mathscr{R}_{(k)}$ is a finite equivalence relation. Let A_k be the C*-algebra of $\mathscr{R}_{(k)}$; i.e. A_k is the complex vector space with basis $\{e_{(x,y)} \mid (x, y) \in \mathscr{R}_{(k)}\}$, with involution $e^*_{(x,y)} = e_{(y,x)}$ and product $e_{(x,y)} e_{(x',y')} = e_{(x,y')}$ if $y = x'$ and 0 otherwise. The product and involution are extended to all of A_k by linearity. We shall also find it convenient to think of $e_{(x,y)}$ as the characteristic function of the set $\{(x, y)\}$. For each k and $0 \leqq i \leqq a_k$ let m_i^k be the number of sequences of $X^{(k)}$ ending in i.

Lemma 2.2.

$$A_k \simeq M_{m_0^k}(\mathbb{C}) \oplus \cdots \oplus M_{m_{a_k}^k}(\mathbb{C}) .$$

Proof. For each $x \in X^{(k)}$ we have a projection $e_{(x,x)} \in A_k$. Moreover $e_{(x,x)} \sim e_{(y,y)}$ if and only if $x_k = y_k$. Hence $e_{(x,x)}$ and $e_{(y,y)}$ are centrally disjoint if $x_k \neq y_k$. Hence A_k has $1 + a_k$ central summands. Also for each $j \in \{0, 1, 2, \ldots, a_k\}$, $\{e_{(x,x)} \mid x_k = j\}$ is a set of pairwise orthogonal pairwise equivalent projections which sum to the central support for the j^{th} summand ($0 \leqq j \leqq a_k$). Hence the size of the j^{th} summand is m_j^k.

Define $\psi_k : A_k \otimes C(X^{(k)}) \to C(\mathscr{R}^{(k)})$ by

$$\psi_k(a \otimes f)(x, y) = a((x_1, \ldots, x_k), (y_1, \ldots, y_k)) f(x_{k+1}, x_{k+2}, \ldots) .$$

By Lemma 2 ψ_k is an isomorphism when restricted to the ideal

$$M_{m_0^k}(C(X^{(k)})) \oplus \cdots \oplus M_{m_{a_k-1}^k}(C(X^{(k)})) \oplus M_{m_{a_k}^k}(C(\widetilde{X}^{(k)})) \subseteq A_k \otimes C(X^{(k)}) .$$

Let $C_{oo}(\mathscr{R})$ be the continuous functions on $\mathscr{R}$ with compact support. If $f \in C_{oo}(\mathscr{R})$, then there is k such that the support of f is contained in $\mathscr{R}^{(k)}$, since $\{\mathscr{R}^{(k)}\}_k$ is an open cover of $\mathscr{R}$. Thus f is in the subspace $C(\mathscr{R}^{(k)})$. Hence $C_{oo}(\mathscr{R}) = \bigcup_k C(\mathscr{R}^{(k)})$. Thus $\mathscr{R}$ is what Renault [7, p. 123] calls an AF groupoid.

Next we shall recall the $*$-algebra structure on $C_{oo}(\mathscr{R})$. Suppose f and g are in $C(\mathscr{R}^{(k)})$. We define $f^* \in C(\mathscr{R}^{(k)})$ by $f^*(x,y) = \overline{f(y,x)}$ and $f * g$ in $C(\mathscr{R}^{(k)})$ by $f * g(x,y) = \sum_{(x,z)\in\mathscr{R}^{(k)}} f(x,z)g(z,y)$. The sum is finite because, for given x and k, $\{z \in X \mid (x,z) \in \mathscr{R}^{(k)}\}$ is finite. Each subspace $C(\mathscr{R}^{(k)})$ is a $*$-subalgebra.

$A_k \otimes C(X^{(k)})$ has a unique C*-norm, and thus so does

$$M_{m_0^k}(C(X^{(k)})) \oplus \cdots \oplus M_{m_{a_k-1}^k}(C(X^{(k)})) \oplus M_{m_{a_k}^k}(C(\widetilde{X}^{(k)})) .$$

Hence $C(\mathscr{R}^{(k)})$ has a unique C*-norm. Thus $C_{oo}(\mathscr{R})$ has a unique C*-norm.

Definition 2.3. *$C^*(\mathscr{R})$, the C*-algebra of the equivalence relation $\mathscr{R}$, is the completion of $C_{oo}(\mathscr{R})$ with respect to its unique norm.*

To calculate the K_0 group of $C^*(\mathscr{R})$ we have to carefully analyse the inclusion maps $i : C(\mathscr{R}^{(k)}) \to C(\mathscr{R}^{(k+1)})$ in terms of the maps ψ_k. For $(x,y) \in \mathscr{R}_{(k)}$ let $S(x,y) = \{(\tilde{x},\tilde{y}) \mid (\tilde{x},\tilde{y}) \in \mathscr{R}_{(k+1)}$ and $x_i = \tilde{x}_i$, $y_i = \tilde{y}_i$ for $1 \leqq i \leqq k\}$. Define $\varphi_k : A_k \otimes C(\mathscr{R}^{(k)}) \to A_{k+1} \otimes C(\mathscr{R}^{(k+1)})$ by

$$\varphi_k(e_{(x,y)} \otimes f)(a_{k+2}, a_{k+3}, \ldots) = \sum_{(\tilde{x},\tilde{y})\in S(x,y)} e_{(\tilde{x},\tilde{y})} f(\tilde{x}_{k+1}, a_{k+2}, \ldots) .$$

Lemma 2.4. *The diagram*

$$\begin{tikzcd} C(\mathscr{R}^{(k)}) \arrow[r, "i"] & C(\mathscr{R}^{(k+1)}) \\ A_k \otimes C(X^{(k)}) \arrow[u, "\psi_k"] \arrow[r, "\varphi_k"'] & A_{k+1} \otimes C(X^{(k+1)}) \arrow[u, "\psi_{k+1}"'] \end{tikzcd}$$

is commutative.

Proof. It is enough to check commutativity on the elementary tensors: $e_{(x,y)} \otimes f \in A_k \otimes C(X^{(k)})$. For $(a,b) \in \mathscr{R}$ we have

$$\begin{aligned} &\psi_{k+1}(\varphi_k(e_{(x,y)} \otimes f))(a,b) \\ &= \begin{cases} \sum_{(\tilde{x},\tilde{y})\in S(x,y)} e_{(\tilde{x},\tilde{y})}((a_1,\ldots,a_{k+1}),(b_1,\ldots,b_{k+1})f(\tilde{x}_{k+1},a_{k+2},\ldots) \\ \quad \text{when } a_i = b_i \text{ for } i > k+1 \text{ and} \\ 0 \quad \text{otherwise} \end{cases} \\ &= \begin{cases} f(\tilde{x}_{k+1},a_{k+2},\ldots) & a_i = x_i,\ b_i = y_i \text{ for } 1 \leqq i \leqq k+1 \\ & \quad \text{and } a_i = b_i \text{ for } i > k+1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} f(a_{k+1},a_{k+2},\ldots) & a_i = x_i,\ b_i = y_i \text{ for } 1 \leqq i \leqq k \\ & \quad \text{and } a_i = b_i \text{ for } i > k \\ 0 & \text{otherwise} \end{cases} \\ &= \psi_k(e_{(x,y)} \otimes f)(a,b) . \end{aligned}$$

Note that φ_k carries $A_k \otimes 1$ into $A_{k+1} \otimes 1$. It is these maps that will enable us to calculate $K_0(C^*(\mathscr{R}))$. For we shall denote by A the limit of the inductive sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

and show that $A \simeq C^*(\mathscr{R})$ and then use the maps $\{\varphi_k\}$ to calculate $K_0(A)$. So we shall identify, where convenient, A_k with $A_k \otimes 1$. With this identification we have a sequence of commutative diagrams:

$$\begin{tikzcd}
C(\mathscr{R}^{(1)}) \arrow[r, "i"] & C(\mathscr{R}^{(2)}) \arrow[r, "i"] & \cdots \arrow[r] & C^*(\mathscr{R}) \\
A_1 \arrow[u, "\psi_1"] \arrow[r, "\varphi_1"'] & A_2 \arrow[u, "\psi_2"] \arrow[r, "\varphi_2"'] & \cdots \arrow[r] & A \arrow[u, "\psi"]
\end{tikzcd}$$

Lemma 2.5. *ψ is an isomorphism.*

Proof. We shall show that the range of $\psi : \bigcup_k A_k \to C_{oo}(\mathscr{R})$ is dense. Let $f \in C(X^{(k)})$ and $\varepsilon > 0$ be given. For each $x \in X^{(k)}$ choose j_x such that on $O(x, j_x) = \{a \in X^{(k)} \mid a_i = x_i \text{ for } k \leqq i \leqq k + j_x - 1\}$, f varies by less than ε, i.e. $|f(y) - f(x)| < \varepsilon$ for $y \in O(x, j_x)$. Then by the compactness of $X^{(k)}$, we may cover $X^{(k)}$ by a finite number of these sets $\{O(x_1, j_{x_1}), \ldots, O(x_N, j_{x_N})\}$; since these sets are open and closed we may re-arrange them into a cover $\{O_1, \ldots, O_K\}$ of pairwise disjoint open and closed sets, with $O_j \subseteq O(x_{i(j)}, j_{x_{i(j)}})$. Thus

$$\left\| f - \sum_{1 \leqq j \leqq K} f(x_{i(j)}) \chi_{O_j} \right\| < \varepsilon .$$

Let $j_{\max} = \max\{j_{x_1}, \ldots, j_{x_N}\}$. Now $\varphi_{j_{\max}-1} \circ \cdots \circ \varphi_k(1_{A_k} \otimes \chi_{O_j}) \in A_{j_{\max}} \otimes 1 \subseteq A_{j_{\max}} \otimes C(X^{(j_{\max})})$. Thus $\varphi_{j_{\max}-1} \circ \cdots \circ \varphi_k(e_{(x,y)} \otimes \chi_{O(x_i, j_{x_i})})$ is within ε of an element of $A_{j_{\max}} \otimes 1$. Hence for each element $f \in C(\mathscr{R}^{(k)})$ and $\varepsilon > 0$ there is j and $\tilde{f} \in A_j \otimes 1$ such that $\|f - \psi_j(\tilde{f})\| < \varepsilon$. Hence the range of ψ is dense.

Each central projection in A_k produces one copy of $\mathbb{Z}$ in $K_0(A_k)$. Thus $K_0(A_k) \simeq \mathbb{Z}^{1+a_k}$ with positive cone $\mathbb{Z}_+^{1+a_k} = \{(z_0, \ldots, z_{a_k}) \mid z_i \geqq 0\}$.

Lemma 2.6. *Under the identification of $K_0(A_k)$ with $\mathbb{Z}^{1+a_k}$,*

$$[\varphi_k] : K_0(A_k) \to K_0(A_{k+1})$$

is represented by the $1 + a_{k+1} \times 1 + a_k$ matrix

$$T_k = \begin{pmatrix} 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 1 & 0 \end{pmatrix},$$

i.e. $T_{ij} = \begin{cases} 1 & j < 1 + a_k \text{ or } i = 1 \\ 0 & j = 1 + a_k \text{ and } i > 1. \end{cases}$

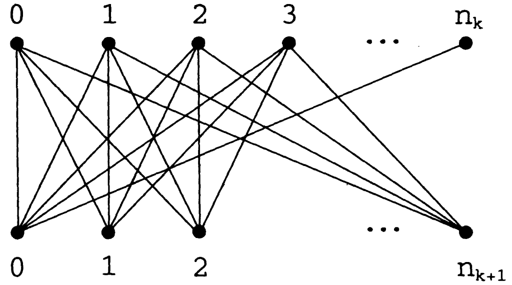

Fig. 3. The Bratteli diagram for the inclusion of A_k into A_{k+1}.

Proof. We only have to show that there is a map of multiplicity one from each central summand of A_k to each central summand of A_{k+1} with the exception of the last summand $M_{m^k_{a_k}}(\mathbb{C})$ of A_k. In the latter case we must show that $M_{m^k_{a_k}}(\mathbb{C})$ gets mapped only to the first summand $M_{m_0^{k+1}}(\mathbb{C})$ of A_{k+1} and that this map has multiplicity one.

Suppose $x \in X_{(k)}$ and $x_k \neq a_k$. Then $S(x,x) = \{((x,0),(x,0)),\ldots,((x,a_k),(x,a_k))\}$; i.e. the sequence x in $X_{(k)}$ can be extended to a sequence (x,i) in $X_{(k+1)}$ by adding any $i \in \{0,1,\ldots,a_{k+1}\}$ to the end of x. Hence in the sum $\varphi_k(e_{(x,x)}) = \sum_{(\tilde{x},\tilde{x})\in S(x,x)} e_{(\tilde{x},\tilde{x})}$ there is one term in each of the $1+a_{k+1}$ summands of A_{k+1}.

Suppose $x \in X_{(k)}$ and $x_k = a_k$. Then x can be extended only by adding a 0, so $S(x,x) = \{((x,0),(x,0))\}$. Thus the last summand of A_k only gets mapped into the first of A_{k+1} and with multiplicity one.

The Bratteli diagram for the inclusion of A_k into A_{k+1} can be described as follows. There are $1+a_k$ vertices on level k and an edge between the i^{th} vertex of the k^{th} level to the j^{th} vertex of the $k+1^{\text{st}}$ level if a sequence in $X_{(k)}$ ending in i can be extended to one in $X_{(k+1)}$ by appending a j.

For each k let $\xi_1^{(k)} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $\xi_2^{(k)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ be vectors in $\mathbb{Z}^{1+a_k}$. Then

$$T_k\xi_1^{(k)} = \begin{pmatrix} a_k+1 \\ a_k \\ \vdots \\ a_k \end{pmatrix} = a_k\xi_1^{(k+1)} + \xi_2^{(k+1)} \quad \text{and} \quad T_k\xi_2^{(k)} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \xi_1^{(k+1)}.$$

So let $\Xi^k \subseteq \mathbb{Z}^{1+a_k}$ be the span of $\{\xi_1^{(k)}, \xi_2^{(k)}\}$. Since the rank of T_k is two, we see that Ξ^{k+1} is the range of T_k and $\mathbb{Z}^{1+a_k} = \ker(T_k) \oplus \Xi^k$. Let $P = \{(m,n) \mid m\xi_1^{(k)} + n\xi_2^{(k)} \in \Xi^k_+\} = \{(m,n) \mid m \geqq 0 \text{ and } m+n \geqq 0\}$. Define a map $\Xi^k \to \mathbb{Z}^2$ by $m\xi_1^{(k)} + n\xi_2^{(k)} \mapsto (m,n)$. The positive part of Ξ^k gets mapped to P. Relative to the standard basis $\{\binom{1}{0}, \binom{0}{1}\}$ of $\mathbb{Z}^2$ we have $T_k = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$. Hence we have a sequence

$$\mathbb{Z}^2 \xrightarrow{T_1} \mathbb{Z}^2 \xrightarrow{T_2} \mathbb{Z}^2 \xrightarrow{T_2} \cdots$$

with positive cone P at each term. Recall that $A_1 = \overbrace{\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{1+a_k}$ and so the class of 1 in $K_0(A_1)$ is $\xi_1^{(1)} \in \Xi^1 \subseteq \mathbb{Z}^{1+n_1}$. Under the map from Ξ^1 to $\mathbb{Z}^2$ $\xi_1^{(1)}$ is sent to $\binom{1}{0}$. We shall compute $K_0(C^*(\mathscr{R})$ using the following diagram – where

$$S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_1 = S_0^{-1}T_1^{-1}, \dots, S_k = S_0^{-1}T_1^{-1}\cdots T_k^{-1}\,.$$

$$(**) \qquad \begin{tikzcd} \binom{1}{0} \in \mathbb{Z}^2 \arrow[r, "T_1"] \arrow[d, "S_0"'] & \mathbb{Z}^2 \arrow[r, "T_2"] \arrow[d, "S_1"'] & \mathbb{Z}^2 \arrow[r, "T_3"] \arrow[d, "S_2"'] & \cdots \\ \mathbb{Z}^2 \arrow[r, equal] & \mathbb{Z}^2 \arrow[r, equal] & \mathbb{Z}^2 \arrow[r, equal] & \cdots \end{tikzcd}$$

Since $T_k \cdots T_1 S_0 = \begin{pmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{pmatrix}$, where $p_0 = 0$, $p_1 = 1, \dots, p_{k+1} = a_{k+1}p_k + p_{k-1}$ and $q_0 = 1$, $q_1 = a_1, \dots, q_{k+1} = a_{k+1}q_k + q_{k-1}$, $S_k = (-1)^{k+1}\begin{pmatrix} q_{k-1} & -q_k \\ -p_{k-1} & p_k \end{pmatrix}$. Let $\eta_k = S_{k+1}\binom{1}{0} = (-1)^k\binom{q_k}{-p_k}$ and $\mu_k = S_{k+1}\binom{1}{-1} = (-1)^k\binom{q_k+q_{k+1}}{-(p_k+p_{k+1})}$. Let $P_\alpha = \{(m,n) \in \mathbb{Z}^2 \mid \alpha m + n > 0\}$.

Lemma 2.7. *For all k, η_k and μ_k are in P_α, and P_α is generated by $\{\eta_k\}_k$.*

Proof. Since $\frac{p_{2k}}{q_{2k}} < \alpha$, we have $\alpha q_{2k} + (-p_{2k}) > 0$; thus $\eta_{2k} \in P_\alpha$. Also since $\frac{p_{2k+1}}{q_{2k+1}} > \alpha$, we have $\alpha(-q_{2k+1}) + p_{2k+1} > 0$; thus $\eta_{2k+1} \in P_\alpha$. We apply the same argument to the inequalities $\frac{p_{2k}}{q_{2k}} < \frac{p_{2k+1}+p_{2k}}{q_{2k+1}+q_{2k}} < \frac{p_{2k+2}}{q_{2k+2}} < \alpha$ to conclude that $\mu_{2k} = \binom{q_{2k+1}+q_{2k}}{-(p_{2k+1}+p_{2k})} \in P_\alpha$. The inclusion of $\mu_{2k-1} = \binom{-(q_{2k}+q_{2k-1})}{p_{2k}+p_{2k-1}}$ is proved using the inequalities $\alpha < \frac{p_{2k+1}}{q_{2k+1}} < \frac{p_{2k}+p_{2k-1}}{q_{2k}+q_{2k-1}} < \frac{p_{2k-1}}{q_{2k-1}}$.

Finally let us show that P_α is generated by $\{\eta_k\}_k$. Since $\binom{m}{n} = (mp_{2k+1} - nq_{2k+1})\eta_{2k} + (mp_{2k} + nq_{2k})\eta_{2k+1}$ it suffices to show that whenever (m,n) is in P_α there is large enough k so that $mp_{2k+1} + nq_{2k+1}$ and $mp_{2k} + nq_{2k}$ are positive. This can always be done; for if $m \geqq 0$ choose k so that $\frac{-n}{m} < \frac{p_{2k}}{q_{2k}} < \alpha$, and if $m < 0$ choose k so that $\alpha < \frac{p_{2k+1}}{q_{2k+1}} < \frac{n}{-m}$.

Theorem 2.8.

$$(K_0(C^*(\mathscr{R}))K_0(C^*(\mathscr{R}))_+, [1]) \simeq (\mathbb{Z} + \alpha Z, (\mathbb{Z} + \alpha Z)_+, 1)\,.$$

Proof. By the diagram $(**)$ $K_0(A) \simeq \mathbb{Z}^2$. Under this mapping the positive cone gets sent to $\bigcup_k S_k(P)$. In Lemma 2.7 we have shown that this union is exactly P_α. The class of $1, \binom{1}{0}$ in the upper left-hand corner of $(**)$ gets sent to $\binom{0}{1}$ in $\mathbb{Z}^2$. Thus $(K_0(A), K_0(A)_+, [1]) \simeq (Z^2, P_\alpha, \binom{0}{1})$. Now map $\mathbb{Z}^2$ to $\mathbb{R}$ by $(m,n) \mapsto \alpha m + n$. This order isomorphism sends $(\mathbb{Z}^2, P_\alpha, \binom{0}{1})$ onto $((\mathbb{Z} + \alpha Z, (\mathbb{Z} + \alpha Z)_+, 1)$.

Remark 2.9. Let us conclude by showing how the Bratteli diagram for A may be given an order making it an ordered Bratteli diagram in the sense of Herman, Putnam, and Skau [4, Sect. 2] so that X is homeomorphic to the path space X. This ordered Bratteli diagram is not simple in that there are two minimal paths and one maximal path. In this case the Veršik transformation is a partial homeomorphism. Two paths are tail equivalent if and only if a power of the Veršik

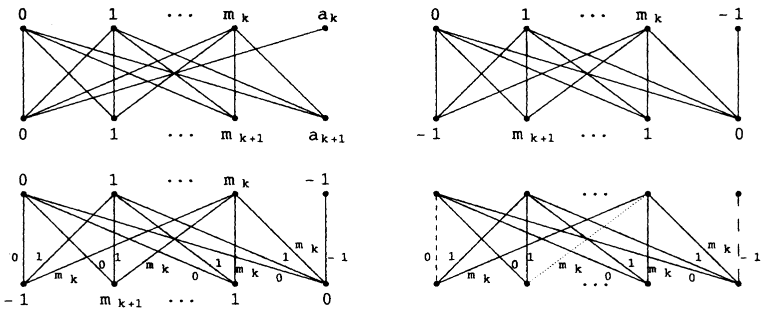

Fig. 4. Construction of the ordered Bratteli diagram for $\mathcal{R}$. In the figure $m_k = a_k - 1$. In the upper left we have the original diagram. In the upper right we have reversed the order of the lower row and changed the a_k's to -1's. In the lower left we have added the ordering to the edges. In the lower right we have marked $-\alpha$ with a dotted line and 0^- and 0^+ with dashed lines. Assuming that k is odd, 0^- is to the right.

transformation takes one of them tothe other and thus $\mathcal{R}$ is the equivalence relation arising from this partial homeomorphism. In the next section we shall show that there is a homeomorphism from X to $S^1_{\mathbf{N}\alpha}$, the Cantor set obtained by cutting the circle along the forward orbit of 0 under rotation by $2\pi\alpha$, such that the Veršik transformation is exactly rotation by $2\pi\alpha$. To simplify the notation let $m_k = a_k - 1$ and $Y = \{(y_i)_{i=1}^{\infty} \mid y_i \in \{-1, 0, 1, \ldots, m_k\}$ and $y_{i+1} = 0$ whenever $y_i = -1\}$. X and Y are homeomorphic by rewriting all a_k's as -1's. Under our new notation the vertices of the k^{th} row of our Bratteli diagram are $V_k = \{-1, 0, 1, \ldots, m_k\}$ and the edges between V_k and V_{k+1} are $E_k = \{(i,j) \in V_k \times V_{k+1} \mid j = 0$ whenever $i = -1\}$. We put an order on E_k by saying $(i_1, j) \leqq (i_2, j)$ whenever $i_1 \leqq i_2$. We set $V_0 = \{0\}$ and $E_0 = \{(0,i) \mid i \in V_1\}$. A path on this diagram is thus a sequence $\{(0,i_1),(i_1,i_2),(i_2,i_3),\ldots\}$, i.e. a point of Y.

Denote by 0^- the path $(-1,0,-1,0,\ldots)$, by 0^+ the path $(0,-1,0,-1,\ldots)$, and by $-\alpha$ the path $(m_1,m_2,m_3,\ldots)$. Under the homeomorphism in Sect. 3, these points get sent to the points 0^-, 0^+, and $-\alpha$ in $S^1_{\mathbf{N}\alpha}$ respectively, hence our notation. In path notation $-\alpha = \{(0,m_1),(m_1,m_2),(m_2,m_3),\ldots\}$. (m_i, m_{i+1}) is the maximal edge ending at m_{i+1}. So $-\alpha$ is maximal and must be the only maximal path. In path notation $0^- = \{(0,-1),(-1,0),(0,-1),\ldots\}$. $(-1,0)$ is the minimal edge ending at 0 because -1 is the minimal index, and $(0,-1)$ is the minimal edge ending at -1 because there is no edge $(-1,-1)$. Thus 0^- is a minimal path and by the same argument 0^+ is another minimal path.

If $p = \{(0,i_1),(i_1,i_2),\ldots\}$ is a minimal path then i_k is 0 or -1. To be minimal then we must have -1 whenever possible, i.e. every other entry. Hence 0^- and 0^+ are the only minimal paths.

Let us recall the Veršik transformation. Suppose $y \in Y$ and $y \neq -\alpha$. Let k be the first k such that $(y_1, y_2, \ldots, y_k) = (m_1, m_2, m_3, \ldots, m_k)$ and $y_{k+1} < m_{k+1}$. Then $(y_i) \mapsto (y_i')$, where $y_i' = y_i$ for $i > k+1$, $y_{k+1}' = 1 + y_{k+1}$, and

$$(y_1', y_2', \ldots, y_k') = \begin{cases} (-1,0,-1,\ldots,-1,0,-1) & \text{if } k \text{ is odd and } y_{k+1} = -1 \\ (0,-1,0,\ldots,0,-1,0) & \text{if } k \text{ is odd and } y_{k+1} \neq -1 \\ (-1,0,\ldots,-1,0) & \text{if } k \text{ is even and } y_{k+1} \neq -1 \\ (0,-1,\ldots,0,-1) & \text{if } k \text{ is even and } y_{k+1} = -1\,, \end{cases}$$

i.e. $y'_k = -1$ if $y'_{k+1} = 0$ and $y'_k = 0$ otherwise, and we then extend backwards to y'_1 by an alternating sequence of 0's and -1's.

3. The Space X_α

Suppose α is an irrational number between 0 and 1. Let $\alpha = [0; a_1, a_2, a_3, \ldots]$ be the continued fraction expansion of α and let $m_i = a_i - 1$. For a real number x, $[x]$ denotes the unique integer such that $[x] \leqq x < [x] + 1$. Note that $[-x] = -(1 + [x])$. Let $\{x\} = x - [x]$. Let

$$\begin{aligned}
\alpha_0 &= 1 \\
\alpha_1 &= \alpha \\
\alpha_2 &= 1 - a_1\alpha = \alpha\{\alpha^{-1}\} \\
\alpha_3 &= \alpha_1 - a_2\alpha_2 = \alpha_2\{\alpha_1\alpha_2^{-1}\} \\
&\vdots \\
\alpha_{n+1} &= \alpha_{n-1} - a_n\alpha_n = \alpha_n\{\alpha_{n-1}\alpha_n^{-1}\}\,. \\
&\vdots
\end{aligned}$$

Let

$$\begin{aligned}
q_{-1} &= 0 & q_2 &= q_0 + a_2 q_1 \\
q_0 &= 1 & &\vdots \\
q_1 &= q_{-1} + a_1 q_0 & q_{n+1} &= q_{n-1} + a_{n+1} q_n\,,
\end{aligned}$$

be the usual denominators of the convergents in the continued fraction expansion of α. Note that *modulo* 1 $\alpha_{i+1} = (-1)^i q_i \alpha$.

Let us construct the space $S^1_{\mathbb{N}\alpha}$. $S^1_{\mathbb{N}\alpha}$ is obtained by disconnecting the circle at the points of $\mathbb{N}\alpha$. $S^1_{\mathbb{N}\alpha}$ is an inverse limit $S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S^1_{\mathbb{N}\alpha}$. $S_0 = S^1$ $S_1 = S^1$ cut at the point 0α, i.e. as a topological space $S_1 = [0,1]$ except we relabel the end points as 0^+ and 0^- respectively. S_2 is obtained by cutting S_1 at the point α, i.e. $S_2 = [0^+, \alpha^-] \cup [\alpha^+, 0^-]$. In general S_{n+1} is obtained from S_n by cutting S_n at the point $n\alpha$. As an alternative description S_n is the maximal ideal space of the C^*-algebra obtained by adjoining the projections $\chi_{[0,n\alpha]}$ to $C(S^1)$.

Let $\pi : S^1_{\mathbb{N}\alpha} \to S^1$ be the canonical map, i.e. the map which sends $m\alpha^\pm$ to $m\alpha$ and leaves the other points alone. We shall also need the larger space $S^1_{\mathbb{Z}\alpha}$, which is constructed in the same way as $S^1_{\mathbb{N}\alpha}$ except that we cut along all the points of the orbit of α.

Given $x \in \mathbb{R}$ and $y \in S^1_{\mathbb{Z}\alpha}$ we define

$$x + y = \begin{cases} \pi(x+y)^+ & \text{if } y = \pi(y)^+ \\ \pi(x+y)^- & \text{if } y = \pi(y)^- \end{cases}\,,$$

$$xy = \begin{cases} \pi(xy)^+ & \text{if } y = \pi(y)^+ \text{ and } x > 0 \text{ or } y = \pi(y)^- \text{ and } x < 0 \\ \pi(xy)^- & \text{if } y = \pi(y)^- \text{ and } x > 0 \text{ or } y = \pi(y)^+ \text{ and } x > 0 \end{cases}\,.$$

We shall also write $-m\alpha^+$ to mean $(-m\alpha)^+$; on the other hand $-(m\alpha^+) = -m\alpha^-$.

Recall that $X_\alpha = \{(x_i)_{i=1}^\infty \mid x_i \in \{0,1,\ldots,a_i\}$ and if $x_i = a_i$ then $x_{i+1} = 0\}$. We shall define a map $\varphi : S^1_{\mathbb{N}\alpha} \to X_\alpha$ as follows. To do this we first extend the floor map to $\mathbb{R}$ cut along $\mathbb{Z}$: $[n^+] = n$, $[n^-] = n - 1$.

Given $\beta \in S^1_{\mathbb{N}\alpha}$ let, $\beta_1 = \beta$ and $x_1 = [\beta_1/\alpha_1]$, then

$$\beta_2 = \begin{cases} (1+x_1)\alpha_1 - \beta_1 & x_1 < a_1 \\ \alpha_0 - \beta_1 & x_1 = a_1 \end{cases},$$

and $x_2 = [\beta_2/\alpha_2]$. Supposing $\beta_1,\ldots,\beta_n$ and $x_1,\ldots,x_{n-1}$ to be already defined we let $x_n = [\beta_n/\alpha_n]$ and

$$\beta_{n+1} = \begin{cases} (1+x_n)\alpha_n - \beta_n & x_n < a_n \\ \alpha_{n-1} - \beta_n & x_n = a_n \end{cases}.$$

Note that if $x_n = a_n$ then $a_n\alpha_n \leqq \beta_n < \alpha_{n-1} = a_n\alpha_n + \alpha_{n+1}$ so $\beta_{n+1} = \alpha_{n-1} - \beta_n < \alpha_{n+1}$. Hence $x_{n+1} = 0$. Thus $(x_i) = \varphi(\beta) \in X_\alpha$.

Examples 3.1.

(i) Let $\beta = 1 - \alpha = \alpha_0 - \alpha_1$. Then $x_1 = a_1 - 1$ and so $\beta_2 = a_1\alpha_1 - \beta_1 = \alpha_1 - \alpha_2$. Suppose $\beta_k = \alpha_{k-1} - \alpha_k$. Then $x_k = [\beta_k/\alpha_k] = [\alpha_{k-1}/\alpha_k] - 1 = a_k - 1$, and $\beta_{k+1} = (1+x_k)\alpha_k - \beta_k = a_k\alpha_k - (\alpha_{k-1} - \alpha_k) = \alpha_k - (\alpha_{k-1} - a_k\alpha_k) = \alpha_k - \alpha_{k+1}$. Hence by induction $x_k = a_k - 1$ for all k.

(ii) Let $\beta = 0^+$. Then $x_1 = 0$, $\beta_2 = \alpha_1^-$, $x_2 = [\alpha_1/\alpha_2] = a_2$, and $\beta_3 = 0^+$. If $\beta_{2k-1} = 0^+$ then $x_{2k-1} = 0$, $\beta_{2k} = \alpha_{2k-1} - \beta_{2k-1} = \alpha_{2k-1}^-$, $x_{2k} = [\alpha_{2k-1}/\alpha_{2k}] = a_{2k}$, and $\beta_{2k+1} = \alpha_{2k-1} - \beta_{2k} = 0^+$. Thus $(x_1,x_2,x_3,x_4,\ldots) = (0,a_2,0,a_4,\ldots)$.

(iii) Let $\beta = 0^- = \alpha_0^-$. Then $x_1 = a_1$, $\beta_2 = \alpha_0 - \beta_1 = 0^+$, $x_2 = 0$, and $\beta_3 = \alpha_2^-$. Suppose $\beta_{2k-1} = \alpha_{2k-2}^-$. Then $x_{2k-1} = a_{2k-1}$, $\beta_{2k} = \alpha_{2k-2} - \beta_{2k-1} = 0^+$, $x_{2k} = 0$, and $\beta_{2k+1} = \alpha_{2k} - \beta_{2k} = \alpha_{2k}^-$.

$S^1_{\mathbb{N}\alpha}$ has the inductive limit topology and X_α has the product topology; thus both are Cantor sets. We shall show that φ is a homeomorphism such that

(i) $\varphi(m\alpha^+)$ is tail equivalent to $(0,a_2,0,a_4,\ldots)$,
(ii) $\varphi(m\alpha^-)$ is tail equivalent to $(a_1,0,a_3,0,\ldots)$,
(iii) $\varphi(-n\alpha)$ is tail equivalent to $(a_1-1,a_2-1,a_3-1,\ldots)$.

To prove this we shall adopt (with a small modification) the notation of Sinai [9, Lecture 9]. If $x,y \in S^1_{\mathbb{N}\alpha}$, $[x,y]$ means the oriented interval which begins at x and ends at y where $S^1_{\mathbb{N}\alpha}$ has the usual counter-clockwise orientation.
Let

$$\Delta_1^{-1} = [0^+,0^-],$$

$$\Delta_1^n = \begin{cases} [q_n\alpha^+,0^-] & n \text{ odd} \\ [0^+,q_n\alpha^-] & n \text{ even} \end{cases},$$

$$\Delta_i^n = \begin{cases} [(i-1+q_n)\alpha^+,(i-1)\alpha^-] & n \text{ odd} \\ [(i-1)\alpha^+,(i-1+q_n)\alpha^-] & n \text{ even} \end{cases}.$$

These are intervals in $S^1_{\mathbb{N}\alpha}$. If we apply π to these intervals we obtain the closure of the intervals in S^1 used by Sinai. The same arguments apply to $S^1_{\mathbb{N}\alpha}$ and thus:

Theorem 3.2 (Sinai [9, Lecture 9, Theorem 1]).

(i) *For each* n,

$$\mathscr{P}_n = \{\Delta_1^{n-1}, \ldots, \Delta_{q_n}^{n-1}, \Delta_1^n, \ldots, \Delta_{q_{n-1}}^n\}$$

is a partition of $S^1_{\mathbf{N}\alpha}$ *into disjoint open and closed sets.*

(ii) *For each* n *and* $1 \leqq i \leqq q_n$,

$$\Delta_i^{n-1} = \Delta_{i+q_{n-1}}^n \cup \Delta_{i+q_{n-1}+q_n}^n \cup \cdots \cup \Delta_{i+q_{n-1}+m_{n+1}q_n}^n \cup \Delta_i^{n+1}$$

and the sets in this partition are disjoint.

Let us show that the sequence (x_i) constructed above can be obtained from the partitions $\{\mathscr{P}_n\}_{n=1}^{\infty}$.

Theorem 3.3. *For* $\beta \in S^1_{\mathbf{N}\alpha}$, x_n *and* β_{n+1} *can be computed using the partition*

$$\mathscr{P}_{n-1} = \{\Delta_1^{n-2}, \ldots, \Delta_{q_{n-1}}^{n-2}, \Delta_1^{n-1}, \ldots, \Delta_{q_{n-2}}^{n-1}\}$$

to decompose $S^1_{\mathbf{N}\alpha}$ *as follows.*

(i) *If* $\beta \in \Delta_i^{n-1} = [s^+, t^-]$ *for* $1 \leqq i \leqq q_{n-2}$ *then* $x_n = 0$ *and*

$$\beta_{n+1} = \begin{cases} \beta - s & n \text{ even} \\ t - \beta & n \text{ odd} \end{cases} .$$

(ii) *If* $\beta \in \Delta_i^{n-2}$ *for* $1 \leqq i \leqq q_{n-1}$ *then write (by Theorem* 3.2) $\Delta_i^{n-2} = \Delta_i^n \cup (\bigcup_{j=0}^{m_n} \Delta_{i+q_{n-2}+jq_{n-1}}^{n-1})$.

(a) *If* $\beta \in \Delta_{i+q_{n-2}+jq_{n-1}}^{n-1} = [s^+, t^-]$, *then* $x_n = j$ *and*

$$\beta_{n+1} = \begin{cases} \beta - s & n \text{ even.} \\ t - \beta & n \text{ odd} \end{cases} .$$

(b) *If* $\beta \in \Delta_i^n = [s^+, t^-]$ *then* $x_n = a_n$ *and*

$$\beta_{n+1} = \begin{cases} \beta - s & n \text{ even} \\ t - \beta & n \text{ odd} \end{cases} .$$

Proof. For $n = 1$ we use $\mathscr{P}_0 = \{\Delta_1^{-1}\}$. This excludes case (i). So we write (since $q_{-1} = 0$)

$$\Delta_1^{-1} = \Delta_1^0 \cup \Delta_{1+q_0}^0 \cup \cdots \cup \Delta_{1+m_1q_0}^0 \cup \Delta_1^1 .$$

If $\beta \in \Delta_{1+jq_0}^0 = [j\alpha^+, (j+1)\alpha^-]$ then $j\alpha^+ \leqq \beta \leqq (j+1)\alpha^-$ so $j \leqq [\beta/\alpha] < j+1$. Hence $x_1 = j$, and $\beta_2 = (1+j)\alpha - \beta = t - \beta$. If $\beta \in \Delta_1^1 = [a_1\alpha^+, 0^-] = [a_1\alpha^+, 1^-]$, $a_1\alpha^+ \leqq \beta \leqq 1^-$ and so $a_1 \leqq [\beta/\alpha]$, thus $x_1 = a_1$ and $\beta_2 = 1 - \beta = t - \beta$.

Now suppose the theorem holds for $n = k$. Let us prove it for $n = k+1$. To compute x_{k+1} we use the partition $\mathscr{P}_k = \{\Delta_1^{k-1}, \ldots, \Delta_{q_k}^{k-1}, \Delta_1^k, \ldots, \Delta_{q_{k-1}}^k\}$.

(i) Suppose $\beta \in \Delta_i^k$ for some $1 \leqq i \leqq q_{k-1}$. Then

$$\beta \in \Delta_i^k \subseteq \Delta_i^{k-2} = \Delta_i^k \cup \Delta_{i+q_{k-2}}^{k-1} \cup \cdots \cup \Delta_{i+q_{k-2}+m_kq_{k-1}}^{k-1} .$$

So by the induction hypothesis $x_k = a_k$, hence $x_{k+1} = 0$ as required. To prove the claim about β_{k+2} there are two cases to consider: k even or k odd.

Let us suppose k is even. Then $\Delta_i^k = [(i-1)\alpha^+, (i-1+q_k)\alpha^-]$ and $\beta_{k+1} = \beta - (i-1)\alpha$. Then

$$\begin{aligned}\beta_{k+2} &= \alpha_{k+1} - \beta_{k+1} = (-1)^k q_k\alpha - \beta + (i-1)\alpha\\ &= (i-1+q_k)\alpha^- - \beta = t - \beta\,,\end{aligned}$$

as required.

If k is odd then $\Delta_i^k = [(i-1+q_k)\alpha^+, (i-1)\alpha^-]$ and $\beta_{k+1} = (i-1)\alpha - \beta$. Then

$$\begin{aligned}\beta_{k+2} &= \alpha_{k+1} - \beta_{k+1} = (-1)^k q_k\alpha + \beta - (i-1)\alpha\\ &= \beta - (i-1+q_k)\alpha = \beta - s\,,\end{aligned}$$

as required.

(ii) Suppose $\beta \in \Delta_i^{k-1}$ for some $1 \leqq i \leqq q_k$.

(a) Suppose $\beta \in \Delta^k_{i+q_{k-1}+jq_k}$ for $0 \leqq j \leqq m_{k+1}$, again there are two cases depending on the parity of k. First suppose k is even. Then $\Delta_i^{k-1} = [(i-1+q_{k-1})\alpha^+, (i-1)\alpha^-]$ and $\beta_{k+1} = \beta - (i-1+q_{k-1})\alpha$. Since

$$x_{k+1} = [\beta_{k+1}/\alpha_{k+1}] = [\beta_{k+1}/q_k\alpha]$$

we have

$$(x_{k+1}q_k)\alpha^+ \leqq \beta_{k+1} \leqq (1+x_{k+1})q_k\alpha^-\,,$$

i.e.

$$(i-1+q_{k-1}+x_{k+1}q_k)\alpha^+ \leqq \beta \leqq (i-1+q_{k-1}+(1+x_{k+1})q_k)\alpha^-\,,$$

hence $\beta \in \Delta^k_{i+q_{k-1}+x_{k+1}q_k}$ as required and

$$\begin{aligned}\beta_{k+2} &= (1+x_{k+1})\alpha_{k+1} - \beta_{k+1}\\ &= (-1)^k(1+x_{k+1})q_k\alpha + (i-1+q_{k-1}) - \beta\\ &= (i-1+q_{k-1}+(1+x_{k+1})q_k)\alpha - \beta = t - \beta\end{aligned}$$

as required. Suppose k is odd. Then $\Delta_i^{k-1} = [(i-1)\alpha^+, (i-1+q_{k-1})\alpha^-]$ and $\beta_{k+1} = (i-1+q_{k-1})\alpha - \beta$. Since

$$x_{k+1} = [\beta_{k+1}/\alpha_{k+1}] = [\beta_{k+1}/-q_k\alpha]\,,$$

we have

$$-x_{k+1}q_k\alpha^+ \leqq \beta_{k+1} \leqq -(1+x_{k+1})q_k\alpha^-\,,$$

thus

$$-(i-1+q_{k-1}+x_{k+1}q_k)\alpha^+ \leqq -\beta \leqq -(i-1+q_{k-1}+(1+x_{k+1})q_k)\alpha^-\,,$$

hence

$$-(i-1+q_{k-1}+(1+x_{k+1})q_k)\alpha^+ \leqq \beta \leqq (i-1+q_{k-1}+x_{k+1}q_k)\alpha^-\,,$$

i.e.

$$\beta \in \Delta^k_{i+q_{k-1}+x_{k+1}q_k}$$

as required, and

$$\begin{aligned}\beta_{k+2} &= (1+x_{k+1})\alpha_{k+1} - \beta_{k+1} \\ &= \beta - (i-1+q_{k-1}+(1+x_{k+1})q_k)\alpha = \beta - s\,,\end{aligned}$$

as required.

(b) Suppose $\beta \in \Delta_i^{k+1}$ $1 \leqq i \leqq q_k$. Again we consider the two cases; k even and k odd. Suppose k is even. Then $\Delta_i^{k-1} = [(i-1+q_{k-1})\alpha^+, (i-1)\alpha^-]$ and $\beta_{k+1} = \beta - (i-1+q_{k-1})\alpha$. As $\Delta_i^{k+1} = [(i-1+q_{k+1})\alpha^+, (i-1)\alpha^-]$ we have $(i-1+q_{k+1})\alpha^+ \leqq \beta \leqq (i-1)\alpha^-$, i.e. $(i-1+q_{k-1}+a_{k+1}q_k)\alpha^+ \leqq \beta$, hence

$$a_{k+1}\alpha_{k+1}^+ = a_{k+1}q_k\alpha^+ \leqq \beta - (i-1+q_{k-1})\alpha = \beta_{k+1}\,.$$

Thus $a_{k+1} \leqq [\beta_{k+1}/\alpha_{k+1}] = x_{k+1}$. So $x_{k+1} = a_{k+1}$ as required. Also

$$\begin{aligned}\beta_{k+2} &= \alpha_k - \beta_{k+1} = -q_{k-1}\alpha - \beta + (i-1+q_{k-1})\alpha \\ &= (i-1)\alpha - \beta = t - \beta\,,\end{aligned}$$

as required.

Finally, let us consider the case of k odd. As before $\Delta_i^{k-1} = [(i-1)\alpha^t, (i-1+q_{k-1})\alpha^+]$, $\beta_{k+1} = (i-1+q_{k-1})\alpha - \beta$. As $\Delta_i^{k+1} = [(i-1)\alpha^+, (i-1+q_{k+1})\alpha^-]$ we have $(i-1)\alpha^+ \leqq \beta \leqq (i-1+q_{k+1})\alpha^- = (i-1+q_{k-1}+a_{k+1}q_k)\alpha^-$. So $(i-1+q_{k-1})\alpha - \beta_{k+1} \leqq (i-1+q_{k-1}+a_{k+1}q_k)\alpha^-$, i.e. $a_{k+1}\alpha_{k+1}^+ = -a_{k+1}q_k\alpha^+ \leq \beta_{k+1}$, so $x_{k+1} = a_{k+1}$ as required. Also

$$\begin{aligned}\beta_{k+2} &= \alpha_k - \beta_{k+1} = q_{k-1}\alpha - (i-1+q_{k-1}) + \beta \\ &= \beta - (i-1)\alpha = \beta - s\end{aligned}$$

as required.

Definition 3.4. *In the formula for β_{n+1} given above, β_{n+1} is the distance of β from the left (n even) or right (n odd) of an interval in the partition P_n. Call this element of $\mathcal{P}_n$ the n^{th} interval of β.*

Lemma 3.5.

(i) *$\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$.*

(ii) *If $a \leqq m < q_{n-1} + q_n$ then $m\alpha^-$ and $m\alpha^+$ are in different partition elements of $\mathcal{P}_n$.*

(iii) *For each n let $P_n \in \mathcal{P}_n$ be the n^{th} interval of β. Then $\{\beta\} = \bigcap_{n=1}^{\infty} P_n$.*

Proof. (i) This follows from Theorem 3.2(ii).

(ii) If $0 \leqq m < q_{n-1} + q_n$ and n is odd then

$$\begin{aligned}&\text{either} \quad \Delta_{m+1}^n = [s^+, m\alpha^-] && \text{if } m < q_{n-1}\,,\\ &\text{or} \quad \Delta_{m+1-q_{n-1}}^{n-1} = [s^+, m\alpha^-] && \text{if } q_{n-1} \leqq m < q_{n-1} + q_n\,,\end{aligned}$$

and

$$\begin{aligned}&\text{either} \quad \Delta_{m+1}^{n-1} = [m\alpha^+, t^-] && \text{if } m < q_n\,,\\ &\text{or} \quad \Delta_{m+1-q_n}^n = [m\alpha^+, t^-] && \text{if } m \geqq q_n\,.\end{aligned}$$

So we have three cases:

(a) If $m < q_{n-1}$, then $m\alpha^- \in \Delta^n_{m+1}$ and $m\alpha^+ \in \Delta^{n-1}_{m+1}$.
(b) If $q_{n-1} \leqq m < q_n$, then $m\alpha^- \in \Delta^{n-1}_{m+1-q_n}$ and $m\alpha^+ \in \Delta^{n-1}_{m+1}$.
(c) If $q_n \leqq m < q_{n-1} + q_n$, then $m\alpha^- \in \Delta^{n-1}_{m+1-q_{n-1}}$ and $m\alpha^+ \in \Delta^n_{m+1-q_n}$.

By Theorem 3.2(i), in all three cases, these intervals are disjoint.

If n is even then

$$\begin{aligned} &\text{either} \quad \Delta^{n-1}_{m+1} = [s^+, m\alpha^-] \qquad \text{if } m < q_n\,, \\ &\text{or} \quad \Delta^n_{m+1-q_n} = [s^+, m\alpha^-] \qquad \text{if } q_n \leqq m < qq_{n-1} + q_n\,; \end{aligned}$$

and

$$\begin{aligned} &\text{either} \quad \Delta^n_{m+1} = [m\alpha^+, t^-] \qquad \text{if } m < q_{n-1}\,, \\ &\text{or} \quad \Delta^{n-1}_{m+1-q_{n-1}} = [m\alpha^+, t^-] \text{ if } q_{n-1} \leqq m < q_{n-1} + q_n\,. \end{aligned}$$

We can apply the same analysis to conclude that $m\alpha^+$ and $m\alpha^-$ are separated in $\mathscr{P}_n$.

(iii) By construction $\beta \in \bigcap_{n=1}^\infty P_n$. Also the diameter of $\pi(P_n) \to 0$. Thus $\pi(\bigcap_{n=1}^\infty P_n) = \{\pi(\beta)\}$. If $\beta \notin \mathbb{N}\alpha$ then $\beta = \pi(\beta)$ and we are done. If $\beta \in \mathbb{N}\alpha$ then by part (ii) of this lemma $m\alpha^+$ and $m\alpha^-$ eventually lie in different intervals so we cannot have $m\alpha^+$ and $m\alpha^-$ in $\bigcap_{n=1}^\infty P_n$. Thus $\bigcap_{n=1}^\infty P_n = \{\beta\}$.

Theorem 3.6. *The map $\varphi : S^1_{\mathbb{N}\alpha} \to X_\alpha$ given by $\varphi(\beta) = (x_1, x_2, \ldots)$ is a homeomorphism.*

Proof. Since $S^1_{\mathbb{N}\alpha}$ and X_α are both compact metric spaces we only have to show that φ is continuous, one-to-one, and onto.

Suppose $P \in \mathscr{P}_n$, and $\beta \in P$. Then $\varphi(P) = \{(y_i) \mid y_i = \varphi(\beta)_i \;\; 1 \leqq i \leqq n\}$. So φ takes basic open sets to basic open sets. So φ is continuous. If $\varphi(\beta_1) = \varphi(\beta_2)$ then for each n, β_1 and β_2 have the same n^{th} interval P_n. So $\beta_1, \beta_2 \in \bigcap_{n=1}^\infty P_n$. By Lemma 3.5 $\beta_1 = \beta_2$; hence φ is one-to-one. Specifying a sequence $\{x_i\} \in X_\alpha$, specifies a path $P_n \in \mathscr{P}_n$ on the partition tree which must have non-empty intersection by the compactness of $S^1_{\mathbb{N}\alpha}$. Thus φ is onto.

We want to consider next the connection between $\varphi(\beta)$ and $\varphi(\beta + \alpha)$. As before let $\beta \in S^1_{\mathbb{N}\alpha}$ and $\varphi(\beta) = (x_1, x_2, \ldots)$. Recall that $m_i = a_i - 1$.

Lemma 3.7.

(i) $\beta \in \Delta^{k-1}_{q_k}$ *if and only if* $x_1 = m_1, \ldots, x_k = m_k$,
(ii) $\beta \in \Delta^{2k-1}_1$ *if and only if* $x_1 = a_1,\ x_2 = 0, \ldots, x_{2k-1} = a_{2k-1}, x_{2k} = 0$,
(iii) $\beta \in \Delta^{2k}_1$ *if and only if* $x_1 = 0,\ x_2 = a_2,\ x_3 = 0, \ldots, x_{2k} = a_{2k},\ x_{2k+1} = 0$.

Proof. (i) We shall prove this by induction on k. It is clear for $k = 1$. Suppose it is true for $1 \leqq k \leqq n$ and prove it for $k = n + 1$. This means we must show that $\beta \in \Delta^n_{q_{n+1}}$ if and only if $x_1 = m_1, \ldots, x_{n+1} = m_{n+1}$. By the induction hypothesis we have $x_1 = m_1, \ldots, x_n = m_n$ if and only if $\beta \in \Delta^{n-1}_{q_n}$. So we only have to show that if $\beta \in \Delta^{n-1}_{q_n}$ then, $x_{n+1} = m_{n+1}$ if and only if $\beta \in \Delta^n_{q_n}$. Now to compute x_{n+1} we use the partition $\mathscr{P}_n = \{\Delta^{n-1}_1, \ldots, \Delta^{n-1}_{q_n}, \Delta^n_1, \ldots, \Delta^n_{q_{n-1}}\}$. We are already assuming

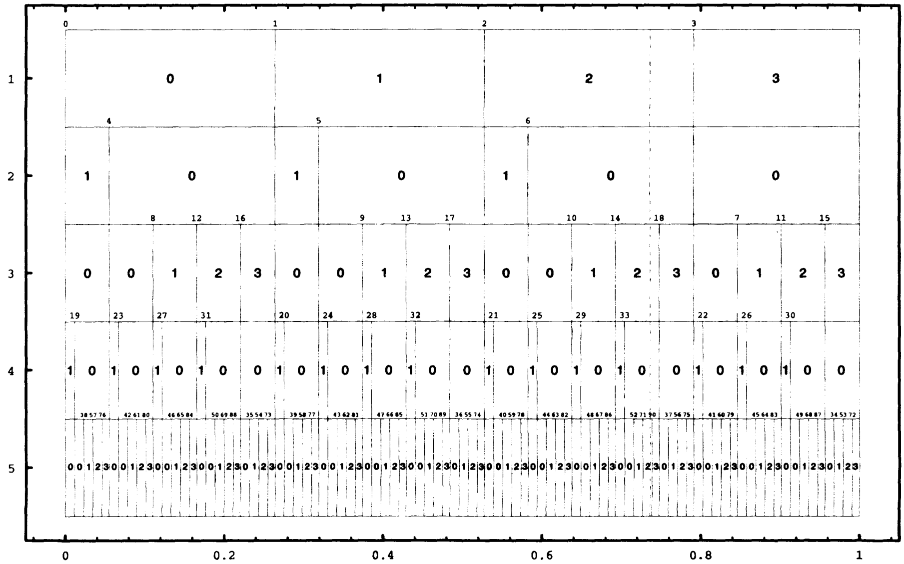

Fig. 5. The decomposition of $S_N^1\alpha$ is shown along the horizontal axis and the first five terms of X_α are shown on the vertical axis. In this example $\alpha = \frac{\sqrt{21}-3}{6} = [0;3,1,3,1,\ldots]$. The expansion of $-\alpha$ is shown by the dashed line.

that $\beta \in \Delta_{q_n}^{n-1}$, so by Theorem 3.3 we decompose $\Delta_{q_n}^{n-1}$ as

$$\Delta_{q_n+q_{n-1}}^{n} \cup \Delta_{q_n+q_{n-1}+q_n}^{n} \cup \cdots \cup \Delta_{q_n+q_{n-1}+m_{n+1}q_n}^{n} \cup \Delta_{q_n}^{n+1} .$$

Now $x_{n+1} = m_{n+1}$ if and only if $\beta \in \Delta_{q_n+q_{n-1}+m_{n+1}q_n}^{n} = \Delta_{q_{n+1}}^{n}$, as required.

(ii) We prove this by induction on k. For $k=1$ we must show that $\beta \in \Delta_1^1$ if and only if $x_1 = a_1$ and $x_2 = 0$. This is straightforward. Suppose that we have proved the claim for $1 \leqq k \leqq n$, and we shall prove it for $k = n+1$. So by the induction hypothesis $\beta \in \Delta_1^{2n-1}$ if and only if $x_1 = a_1$, $x_2 = 0, \ldots, x_{2n-1} = a_{2n-1}$, and $x_{2n} = 0$. So we only have to show that for $\beta \in \Delta_1^{2n-1}$, $\beta \in \Delta_1^{2n+1}$ if and only if $x_{2n+1} = a_{2n+1}$ (and hence $x_{2n+2} = 0$). To compute x_{2n+1} we use the partition $\mathscr{P}_{2n} = \{\Delta_1^{2n-1}, \ldots, \Delta_{q_{2n}}^{2n-1}, \Delta_1^{2n}, \ldots, \Delta_{q_{2n-1}}^{2n}\}$. Since $\beta \in \Delta_1^{2n-1}$ we decompose Δ_1^{2n-1} as

$$\Delta_{1+q_{2n-1}}^{2n} \cup \Delta_{1+q_{2n-1}+q_{2n}}^{2n} \cup \cdots \cup \Delta_{1+q_{2n-1}+m_{2n+1}q_{2n}}^{2n} \cup \Delta_1^{2n+1} .$$

By Theorem 3.3, $\beta \in \Delta_1^{2n+1}$ if and only if $x_{2n+1} = a_{2n+1}$ as required.

(iii) We shall again prove this by induction. For $k=0$ we must show that $\beta \in \Delta_1^0 = [0^+, \alpha^-]$ if and only if $x_1 = 0$; but this is clear from the definitions.

Suppose we have proved the claim for $1 \leqq k \leqq n$ and we shall prove it for $k = n+1$. Since we have that $\beta \in \Delta_1^{2n}$ if and only if $x_1 = 0$, $x_2 = a_2, \ldots, x_{2n} = a_{2n}, x_{2n+1} = 0$, we only have to show that for $\beta \in \Delta_1^{2n}$, $\beta \in \Delta_1^{2n+2}$ if and only if $x_{2n+2} = a_{2n+2}$. To compute x_{2n+2} we use the partition $\mathscr{P}_{2n+1} = \{\Delta_1^{2n}, \ldots, \Delta_{q_{2n+1}}^{2n}, \Delta_1^{2n+1}, \ldots, \Delta_{q_{2n}}^{2n+1}\}$. We are assuming that

$$\beta \in \Delta_1^{2n} = \Delta_{1+q_{2n}}^{2n+1} \cup \cdots \cup \Delta_{1+q_{2n}+m_{2n+2}q_{2n+1}}^{2n+1} \cup \Delta_1^{2n+2} .$$

By Theorem 3.3, $\beta \in \Delta_1^{2n+2}$ if and only if $x_{2m+2} = a_{2n+2}$, as required.

Theorem 3.8. *Let* $-\alpha \neq \beta \in S^1_{\mathbf{N}\alpha}$, $\varphi(\beta) = (x_1, x_2, \ldots)$ *and* $\varphi(\beta + \alpha) = (y_1, y_2, \ldots)$. *If* $(x_1, \ldots, x_k) = (m_1, \ldots, m_k)$ *but* $x_{k+1} \neq m_k$ *then*

$$(y_1, \ldots, y_{k+1}) = \begin{cases} (0, a_2, 0, a_4, \ldots, 0, a_{k-2}, 0, a_k, 0) & x_{k+1} = a_{k+1} \text{ and } k \text{ even} \\ (a_1, 0, a_3, 0, \ldots, 0, a_{k-2}, 0, a_k, 0) & x_{k+1} = a_{k+1} \text{ and } k \text{ odd} \\ (a_1, 0, a_3, 0, \ldots, 0, a_{k-1}, 0, 1 + x_{k+1}) & x_{k+1} < m_{k+1} \text{ and } k \text{ even} \\ (0, a_2, 0, a_4, \ldots, 0, a_{k-1}, 0, 1 + x_{k+1}) & x_{k+1} < m_{k+1} \text{ } k \text{ and odd} \end{cases}$$

and $y_i = x_i$ *for* $i > k + 1$.

Proof. Suppose $x_{k+1} = a_{k+1}$. By Lemma 3.7,

$$\beta \in \Delta_{q_k}^{k-1} = \Delta_{q_k+q_{k-1}}^{k} \cup \Delta_{q_k+q_{k-1}+q_k}^{k} \cup \cdots \cup \Delta_{q_k+q_{k-1}+m_{k+1}q_k}^{k} \cup \Delta_{q_k}^{k+1} .$$

By Theorem 3.3, $\beta \in \Delta_{q_k}^{k+1}$. Thus

$$\beta + \alpha \in \Delta_{1+q_k}^{k+1} \subseteq \Delta_1^k = \Delta_{1+q_k}^{k+1} \cup \cdots \cup \Delta_{1+q_k+m_{k+2}q_{k+1}}^{k+1} \cup \Delta_1^{k+2} ,$$

and so by Theorem 3.3 again $y_{k+1} = 0$. By Lemma 3.7

$$(y_1, y_2, \ldots, y_k) = \begin{cases} (0, a_2, 0, a_4, \ldots, 0, a_k) & k \text{ even} \\ (a_1, 0, a_3, 0, \ldots, 0, a_k) & k \text{ odd} \end{cases} .$$

If $\Delta_{q_k}^{k+1} = [s^+, t^-]$ then $\Delta_{1+q_k}^{k+1} = [(s+\alpha)^+, (t+\alpha)^-]$. As

$$\beta_{k+2} = \begin{cases} \beta - s & k+1 \text{ even} \\ t - \beta & k+1 \text{ odd} \end{cases}$$

and

$$(\beta + \alpha)_{k+2} = \begin{cases} \beta + \alpha - (s + \alpha) & k+1 \text{ even} \\ (t+\alpha) - (\beta + \alpha) & k+1 \text{ odd} \end{cases} ,$$

we see that β_{k+2} is unchanged and hence $y_i = x_i$ for $i > k+1$.

Now suppose $x_{k+1} < m_{k+1}$. By Lemma 3.7,

$$\beta \in \Delta_{q_k}^{k-1} = \Delta_{q_k+q_{k-1}}^{k} \cup \Delta_{q_k+q_{k-1}+q_k}^{k} \cup \cdots \cup \Delta_{q_k+q_{k-1}+m_{k+1}q_k}^{k} \cup \Delta_{q_k}^{k+1} .$$

Since $x_{k+1} < m_{k+1}$, $\beta \in \Delta_{q_{k-1}+(1+x_{k+1})q_k}^{k}$. Thus

$$\beta + \alpha \in \Delta_{1+q_{k-1}+(1+x_{k+1})q_k}^{k} \subseteq \Delta_1^{k-1} = \Delta_{1+q_{k-1}}^{k} \cup \Delta_{1+q_{k-1}+q_k}^{k} \cup \cdots \cup \Delta_{1+q_{k-1}+m_{k+1}q_k}^{k} \cup \Delta_1^{k+1} .$$

Hence $y_{k+1} = 1 + x_{k+1}$ and since $\beta + \alpha \in \Delta_1^{k-1}$, we have by Lemma 3.7 that

$$(y_1, \ldots, y_k) = \begin{cases} (a_1, 0, a_3, 0, \ldots, a_{k-1}, 0) & k \text{ even} \\ (0, a_2, 0, a_4, \ldots, a_{k-1}, 0) & k \text{ odd} \end{cases} .$$

Writing $\Delta^k_{q_{k-1}+(1+x_{k+1})q_k}$ as $[s^+, t^-]$ we have $\Delta^k_{1+q_{k-1}+(1+x_{k+1})q_k} = [(s+\alpha)^+, (t+\alpha)^-]$. As

$$\beta_{k+1} = \begin{cases} \beta - s & k+1 \text{ even} \\ t - \beta & k+1 \text{ odd} \end{cases}$$

and

$$(\beta+\alpha)_{k+2} = \begin{cases} (\beta+\alpha) - (s+\alpha) & k+1 \text{ even} \\ (t+\alpha) - (\beta+\alpha) & k+1 \text{ odd} \end{cases}$$

we see that $\beta_{k+2} = (\beta+\alpha)_{k+2}$ and hence $x_i = y_i$ for $i > k+1$.

Corollary 3.9. *Suppose $\beta, \gamma \in S^1_{\mathbf{N}\alpha}$ and $\varphi(\beta) = \{x_i\}$ and $\varphi(\gamma) = \{y_i\}$. Let P and Q in $\mathscr{P}_k$ be the k^{th} intervals of β and γ respectively.*

(i) *If there is k such that $x_i = y_i$ for $i > k$ then there is $n \in \mathbb{Z}$ such that $\beta = \gamma + n\alpha$ and $|n| < q_k + q_{k-1}$. Moreover if $|P| \leqq |Q|$ then $\mathscr{U}(x, y, k) = \{(\varphi(\mu), \varphi(\nu)) \mid \mu \in P$ and $\nu = \mu + n\alpha\}$ (see the third paragraph of Sect. 2 for the definition of $\mathscr{U}$).*

(ii) *If $\pi(\beta) \notin \mathbb{Z}\alpha$ and there is $n \in \mathbb{Z}$ such that $\beta = \gamma + n\alpha$, then there is k such that $x_i = y_i$ for $i \geqq k$.*

(iii) *If $\pi(\beta) = m\alpha$ and $\pi(\gamma) = n\alpha$ and either $m, n \geqq 0$ and β and γ have the same sign, or $m, n < 0$ then there is k such that $x_i = y_i$ for $i \geqq k$.*

Proof. (i) Since $x_i = y_i$ for $i > k$, $\beta_{k+1} = \gamma_{k+1}$; so the distance of β and γ from the corresponding endpoints (left for k even, right for k odd) of P and Q will be equal.

Suppose P and Q are of the same length. If $P = \Delta^{k-1}_p$ and $Q = \Delta^{k-1}_q$, then $1 \leqq p, q \leqq q_k$ and so $\beta = \gamma + (p-q)n$ with $|n| \leqq q_k < q_k + q_{k-1}$. If $P = \Delta^k_p$ and $Q = \Delta^k_q$ then the same argument applies except we then have $|n| \leqq q_{k-1}$.

Suppose P and Q are of different lengths. Say $P = \Delta^{k-1}_p$ and $Q = \Delta^k_q$ with $1 \leqq p \leqq q_k$ and $1 \leqq q \leqq q_{k-1}$. We know that $\beta_{k+1} = \gamma_{k+1} \leqq \alpha_{k+1}$; so $x_{k+1} = 0$. Now decomposing

$$\Delta^{k-1}_p = \Delta^k_{p+q_{k-1}} \cup \Delta^k_{p+q_{k-1}+q_k} \cup \cdots \cup \Delta^k_{p+q_{k-1}+m_{k+1}q_k} \cup \Delta^{k+1}_p ,$$

we see that $\beta \in \Delta^k_{p+q_{k-1}}$. Thus for $n = q - (p + q_{k-1})$ we have $\beta = \gamma + n\alpha$ and $|n| < q_k + q_{k-1}$ as $p + q_{k-1} \leqq q_k + q_{k-1}$ and $q \geqq 1$.

For the second assertion suppose $|P| \leqq |Q|$. Let $a = \varphi(\mu)$ and $b = \varphi(\nu)$. Then $\mu \in P$ if and only if $a_i = x_i$ for $1 \leqq i \leqq k$ and $\nu \in Q$ if and only if $b_i = y_i$ for $1 \leqq i \leqq k$. If $(a, b) \in \mathscr{U}(x, y, k)$ then $\mu \in P$ and $\varphi(\mu)_i = \varphi(\nu)_i$ for $i > k$. Hence $\nu = \mu + n\alpha$. Conversely if $\mu \in P$ and $\nu = \mu + n\alpha$ then $\nu \in Q$, so $a_i = x_i$ and $b_i = y_i$ for $1 \leqq i \leqq k$. Also if $p = \Delta^k_p$ and $Q = \Delta^{k-1}_q$, then $P + n\alpha \subseteq Q$ so $a_i = b_i$ for $i > k$.

(ii) and (iii) Theorem 3.8 showed that as long as $-\alpha \notin \{\gamma, \gamma+\alpha, \gamma+2\alpha, \ldots, \gamma + (n-1)\alpha\}$ then for $1 \leqq i \leqq n$, $\varphi(\gamma + (i-1)\alpha)$ and $\varphi(\gamma + i\alpha)$ agree from some point onwards.

4. The Relation R_α

Suppose again that $0 < \alpha < 1$ is irrational with continued fraction expansion $a = [0; a_1, a_2, a_3, \ldots]$. Let X_α be the Cantor set constructed in Sect. 2. $R_\alpha \subseteq X_\alpha \times$

X_α will be the equivalence relation on X_α generated by tail equivalence and $(a_1, 0, a_3, 0, \ldots) \sim (0, a_2, 0, a_4, \ldots) \sim (a_1 - 1, a_2 - 1, a_3 - 1, \ldots)$. In this section we shall construct a locally Hausdorff topology on R_α and a surjective continuous map $\Phi : R_\alpha \to S_1 \times \mathbb{Z}$ such that

(i) the diagram

$$\begin{array}{ccc} R_\alpha & \xrightarrow{\Phi} & S^1 \times \mathbb{Z} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ X_\alpha & \xrightarrow[\pi]{} & S^1 \end{array}$$

commutes where p_1 is the projection onto the first factor; and

(ii) $\Phi^* : C(S^1 \times \mathbb{Z}) \to C(R_\alpha)$ is a linear bijection.

Recall that $S^1_{\mathbf{N}\alpha}$ is S^1 cut along the forward orbit of α. Rotating by α is a partial homeomorphism on $S^1_{\mathbf{N}\alpha}$, defined on $S^1_{\mathbf{N}\alpha} \backslash \{-\alpha\}$. Let us denote this partial homeomorphism by Θ. In Theorem 3.8 we showed that there is a partial homeomorphism on X_α and that the bijection $\varphi : S^1_{\mathbf{N}\alpha} \to X_\alpha$ intertwines the actions. Therefore we shall denote by Θ as well, the partial homeomorphism on $X_\alpha : \varphi \circ \Theta \circ \varphi^{-1}$.

We shall find it convenient to identify, via φ, points of $S^1_{\mathbf{N}\alpha}$ with their corresponding sequences in X_α. In particular

$$0^+ = (0, a_2, 0, a_4, \ldots)$$

$$0^- = (a_1, 0, a_3, 0, \ldots)$$

$$-\alpha = (m_1, m_2, m_3, m_4, \ldots)$$

recalling that in Sect. 2, m_i was defined to be $a_i - 1$ (and the computations in Example 2.1).

Definition 4.1.

(i) *For $x, y \in X_\alpha$, x and y are tail equivalent, $x \sim_t y$, if there is k such that $x_i = y_i$ for $i > k$.*

(ii) *R_α is the smallest equivalence relation on X_α containing $\{(x, y) \,|\, x \sim_t y\} \cup \{(0^+, 0^-), (0^+, -\alpha)\}$.*

Remark 4.2. Explicitly $(x, y) \in R_\alpha$ if either, x and y are tail equivalent, or each of x and y are tail equivalent to one of $\{0^+, 0^-, -\alpha\}$.

The topology on R_α will be constructed from a basis made from three families of sets. The first family is the one constructed in Sect. 2 giving the topology on $\mathcal{R}_\alpha : \{\mathcal{U}(x, y, k) \,|\, (x, y) \in \mathcal{R}_\alpha\}$. These form a neighbourhood base for the points of $\mathcal{R}_\alpha$. For points in $R_\alpha \backslash \mathcal{R}_\alpha$ we will introduce two new families; basic neighbourhoods of points of the form $(m\alpha^\pm, n\alpha^\mp)$ with $m, n \geqq 0$ will be denoted $\mathcal{V}(n\alpha^\pm, n\alpha^\mp, k)$, and basic neighbourhoods of points of the form $(m\alpha^\pm, -n\alpha)$ or $(-n\alpha, m\alpha^\pm)$ for $m \geqq 0$ and $n > 0$ will be denoted by $\mathcal{W}(m\alpha^\pm, -n\alpha, k)$ or $\mathcal{W}(-n\alpha, m\alpha^\pm, k)$ as the case demands. To describe $\mathcal{V}$ we need to construct some open sets in $\mathcal{R}_\alpha$. $\mathcal{U}^o(x, y, k)$ is an open subset of $\mathcal{U}(x, y, k)$ and should be thought of as being constructed by removing both endpoints of the interval – if $\mathcal{U}(x, y, k)$ were equal to $[a, b]$, then $\mathcal{U}^o(x, y, k)$ would be (a, b).

Definition 4.3. *Suppose $(x,y)\in\mathcal{R}_\alpha$ and let β and γ in $S^1_{\mathbf{N}\alpha}$ be the pre-images of x and y respectively, with $\beta=\gamma+n\alpha$ for $|n|<q_k+q_{k-1}$ (as in Corollary 3.9). Let $P_1=[s_1^+,t_1^-]$ and $P_2=[s_2^+,t_2^-]$ be the intervals in $\mathcal{P}_k$ containing β and γ respectively.*

If $|P_1|\leqq|P_2|$ let $\mathcal{U}^o(x,y,k)=\{(\varphi(\mu),\varphi(\nu))\mid\mu\in(s_1^+,t_1^-)$ and $\mu=\nu+n\alpha\}$. If $|P_2|\leqq|P_1|$ let $\mathcal{U}^o(x,y,k)=\{(\varphi(\mu),\varphi(\nu))\mid\nu\in(s_2^+,t_2^-)$ and $\mu=\nu+n\alpha\}$.

Definition 4.4. *Suppose $x=m\alpha^\pm$, $y=n\alpha^\pm$ for $m,n\geqq 0$ and k is large enough that $m,n<q_{k-1}+q_k$. Let $\mathcal{V}(x,y,k)=\mathcal{U}^o(m\alpha^+,n\alpha^+,k)\cup\mathcal{U}^o(m\alpha^-,n\alpha^-,k)\cup\{(x,y)\}$.*

Before constructing $\mathcal{W}$ we shall make some preparations.

Lemma 4.5.

(i) $\Theta((\Delta^k_{q_{k+1}}\cup\Delta^{k+1}_{q_k})\backslash\{-\alpha\})=(\Delta_1^{k+1}\cup\Delta_1^k)\backslash\{0^+,0^-\}$.

(ii)

$$\Theta(\Delta^{k-1}_{q_k}\backslash\{-\alpha\})=(\Delta^k_{1+q_k+q_{k-1}}\cup\Delta^k_{1+2q_k+q_{k-1}}\cup\cdots\cup\Delta^k_{1+q_{k-1}+m_{k+1}q_k}\cup\Delta_1^{k+1}\cup\Delta_1^k)\backslash\{0^+,0^-\}\,.$$

(iii) *If $0\leqq n<q_k$ and $1\leqq m<q_k$ then*

$$\Delta^{k-1}_{q_k-(m-1)}\cap\{-\alpha,-2\alpha,\dots,-(m+n-1)\}=\{-m\alpha\}\,.$$

Proof. (i) Suppose k is even

$$(\Delta^k_{q_{k+1}}\cup\Delta^{k+1}_{q_k})\backslash\{-\alpha\}=[(q_{k+1}-1)\alpha^+,-\alpha)\cup(-\alpha,(q_{k+1}+q_k-1)\alpha^-]\cup[(q_{k+1}+q_k-1)\alpha^+,(q_k-1)\alpha^-]\,.$$

Thus

$$\Theta(\Delta^k_{q_{k+1}}\backslash\{-\alpha\}\cup\Delta^{k+1}_{q_k})=[q_{k+1}\alpha^+,0^-)\cup(0^+,q_k\alpha^-]=\Delta_1^{k+1}\backslash\{0^-\}\cup\Delta_1^k\backslash\{0^+\}\,.$$

The proof is the same for k odd.

(ii)

$$\Delta^{k-1}_{q_k}\backslash\{-\alpha\}=\Delta^k_{q_k+q_{k-1}}\cup\Delta^k_{2q_k+q_{k-1}}\cup\cdots\cup\Delta^k_{m_{k+1}q_k+q_{k-1}}\cup\Delta^k_{q_{k+1}}\backslash\{-\alpha\}\cup\Delta^{k+1}_{q_k}\,.$$

So by (i)

$$\Theta(\Delta^{k-1}_{q_k}\backslash\{-\alpha\})=\Delta^k_{1+q_k+q_{k-1}}\cup\Delta^k_{1+q_{k-1}+2q_k}\cup\cdots\cup\Delta^k_{1+q_{k-1}+m_{k+1}q_k}\cup(\Delta_1^{k+1}\cup\Delta_1^k)\backslash\{0^+,0^-\}\,.$$

(iii) As $m<q_k$, $-m\alpha\in\Delta^{k-1}_{q_k-(m-1)}$ and $\Delta^{k-1}_{q_k-(m-1)}$ is disjoint from $\{-\alpha,-2\alpha,\dots,-(m-1)\alpha\}$. Thus we are reduced to showing that $\Delta^k_{q_k-(m-1)}$ is disjoint from $\{-(m+1)\alpha,\dots,-(m+n-1)\alpha\}$. If $-j\alpha\in\Delta^{k-1}_{q_k-(m-1)}$ for some $m+1\leqq j\leqq m+n-1$ then $-(j-1)\alpha\in\Delta^{k-1}_{q_k-(m-1)}$. So we may assume that $m=1$. Thus we must show that $\Delta^{k-1}_{q_k}$ is disjoint from $\{-2\alpha,\dots,-n\alpha\}$ which is true as long as $n<q_k$.

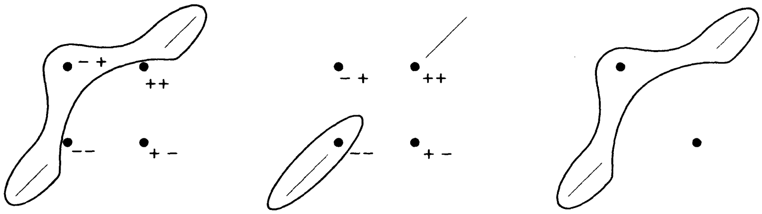

Fig. 6. From left to right: the neighbourhoods $\mathscr{V}$, $\mathscr{U}$, and $\mathscr{W}$.

Definition 4.6. *Given positive integers m and n, choose k such that $m, n < q_k + q_{k-1}$; let $\mathscr{W}(-m\alpha, n\alpha^{\pm}, k) = \{(a,b) \mid a \in \Delta^{k-1}_{q_k-(m-1)} \backslash \{-m\alpha\}$ and $b = \Theta^{m+n}(a)\} \cup \{(-m\alpha, n\alpha^{\pm})\}$. By Lemma 4.7(iii) Θ^{m+n} is defined on $\Delta^{k-1}_{q_k-(m-1)} \backslash \{-m\alpha\}$ and so the definition makes sense. We let $\mathscr{W}(n\alpha^{\pm}, -m\alpha, k) = \mathscr{W}(-m\alpha, n\alpha^{\pm}, k)^{-1}$, where $(x,y)^{-1} = (y,x)$ for any $(x,y) \in R_\alpha$. We let $\mathscr{W}^o(-m\alpha, n\alpha^{\pm}, k)$ be the subset of $\mathscr{W}(-m\alpha, n\alpha^{\pm}, k)$ obtained by deleting the endpoints of $\Delta^{k-1}_{q_k-(m-1)}$ in the construction above.*

Lemma 4.7. *If $\mathscr{U}(x,y,k)$ meets $\mathscr{W}(x',y',k')$ then*

(i) *$\mathscr{W}(x',y',k') \subseteq \mathscr{U}(x,y,k)$ if $k \leqq k'$, or*

(ii) *$\mathscr{U}(x,y,k) \subseteq \mathscr{W}(x',y',k')$ if $k' \leqq k$ and x' is not in the k^{th} set of x, or*

(iii) *$\mathscr{U}(x,y,k) \cap \mathscr{W}(x',y',k') = \mathscr{W}(x',y',k)$ if $k' \leqq k$ and x' is in the k^{th} set of x.*

Proof. By taking inverses, if necessary, we may assume that $x' = -m\alpha$. Let P and Q be respectively the k^{th} sets of x and y with $|P| \leqq |Q|$. Suppose $\mathscr{U}(x,y,k) = \{(u,v) \mid u \in P,\ v = \Theta^n(u)\}$. Then P and $\Delta^{k'}_{q_{k'}-(m-1)}$ meet, and $\mathscr{U}(x,y,k) \cap \mathscr{W}(x',y',k') = \{(u,v) \mid u \in P \cap \Delta^{k'}_{q_{k'}-(m-1)} \backslash \{-m\alpha\}$ and $v = \Theta^n(u)\}$. So if $k \leqq k'$ then $\Delta^{k'}_{q_{k'}-(m-1)} \subseteq P$ and $\mathscr{W}(x',y',k') \subseteq \mathscr{U}(x,y,k)$. If $k' \leqq k$ and $x' \notin P$ then $P \subseteq \Delta^{k'}_{q_{k'}-(m-1)} \backslash \{-m\alpha\}$ and thus $\mathscr{U}(x,y,k) \subseteq \mathscr{W}(x',y',k')$. If $k' \leqq k$ and $x' \in P$ then $\mathscr{U}(x,y,k) \cap \mathscr{W}(x',y',k') = \{(u,v) \mid u \in P \backslash \{-m\alpha\}$ and $v = \Theta^n(u)\} = \mathscr{W}(x',y',k)$. The case when $|Q| \leqq |P|$ is handled similarly.

Theorem 4.8. *The sets $\{\mathscr{U}, \mathscr{V}, \mathscr{W}\}$ form a basis for a locally Hausdorff topology on R_α.*

Proof. The sets $\{\mathscr{U}\}$ are a basis of $\mathscr{R}_\alpha$. By construction $\{\mathscr{V}, \mathscr{W}\}$ covers $R_\alpha \backslash \mathscr{R}_\alpha$ so we just have to show that the intersection of two subsets of $\{\mathscr{U}, \mathscr{V}, \mathscr{W}\}$ is an open subset of $\mathscr{R}_\alpha$ or is a neighbourhood of type $\mathscr{V}$ or $\mathscr{W}$. A $\mathscr{V}$ neighbourhood is the union of two $\mathscr{U}^o$'s and a point not in $\mathscr{R}_\alpha$. Thus any intersection of the form $\mathscr{U} \cap \mathscr{V}$ is an open subset of $\mathscr{R}_\alpha$.

By Lemma 4.7 the intersection of a $\mathscr{W}$ and a $\mathscr{U}$-type neighbourhood must be a $\mathscr{U}$ neighbourhood or a $\mathscr{W}$ neighbourhood. Also if $\mathscr{V}(x,y,k)$ meets $\mathscr{W}(x',y',k')$, then the intersection must be a union of $\mathscr{U}$-type neighbourhoods since no point $(-n\alpha, m\alpha^{\pm})$ or $(m\alpha^{\pm}, -n\alpha)$ is in any $\mathscr{V}$-neighbourhood nor is any $(m\alpha^{\pm}, n\alpha^{\mp})$ in any $\mathscr{W}$-neighbourhood. The intersection of two $\mathscr{V}$ neighbourhoods (or $\mathscr{W}$ neighbourhoods) is either a $\mathscr{V}$ neighbourhood (respectively a $\mathscr{W}$ neighbourhood) or an open set in $\mathscr{R}_\alpha$, i.e. a $\mathscr{U}^o$ neighbourhood or the union of two $\mathscr{U}^o$ neighbourhoods. Thus the family $\{\mathscr{U}, \mathscr{V}, \mathscr{W}\}$ forms a basis for a topology of R_α.

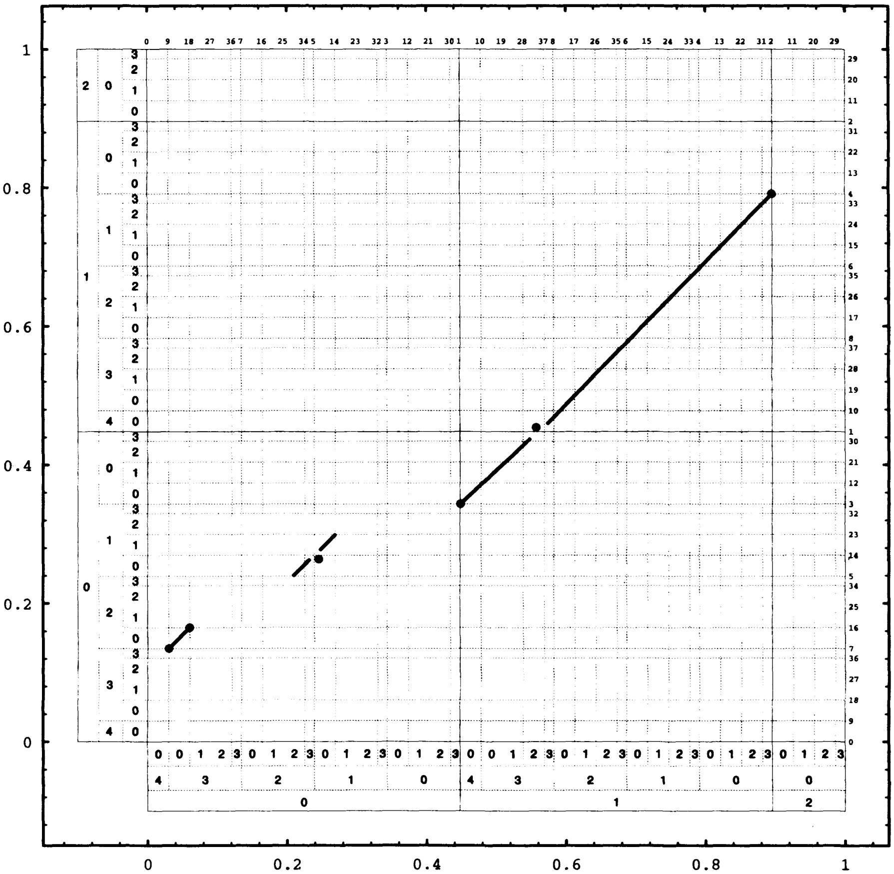

Fig. 7. The diagram shows some sub-basic neighbourhoods of $R_\alpha \subseteq X_\alpha \times X_\alpha$. In this example $\alpha = [0; 2, 4, 3, \ldots]$. From left to right are shown $\mathscr{U}(9\alpha^+, 7\alpha^+, 3)$, $\mathscr{V}(5\alpha^+, 14\alpha^-, 2)$, and $\mathscr{W}(-\alpha, \alpha^+, 1)$.

Each $\mathscr{U}$ neighbourhood is Hausdorff, thus each point of $\mathscr{R}_\alpha$ has a Hausdorff neighbourhood. Hence each $\mathscr{V}$ neighbourhood is Hausdorff, being the union of a point and two $\mathscr{U}^o$ neighbourhoods. If $(x, y) \in \mathscr{W}(-m\alpha, n\alpha^\pm, k)$ and $(x, y) \neq (-m\alpha, n\alpha^\pm)$, then we may choose $k' > k$ large enough that x and $-m\alpha$ lie in different elements of $\mathscr{P}_k$. Thus $\mathscr{U}(x, y, k')$ and $\mathscr{W}(-m\alpha, n\alpha^\pm, k')$ will be disjoint. Since any other two points of $\mathscr{W}(-m\alpha, n\alpha^\pm, k)$ lie in $\mathscr{R}_\alpha$, they can be separated. So we can conclude that $\mathscr{W}(-m\alpha, n\alpha^\pm, k)$ is also Hausdorff. Hence the topology just constructed is locally Hausdorff.

Definition 4.9. *Define $\Phi : R_\alpha \to S^1 \times \mathbb{Z}$ as follows:*

If $x \sim_t y$, in which case there is $n \in \mathbb{Z}$ such that $y = \Theta^n(x)$, then let $\Phi(x, y) = (\pi(x), n)$.

If $m, n \geqq 0$, let $\Phi(m\alpha^+, n\alpha^-) = \Phi(m\alpha^-, n\alpha^+) = (m\alpha, n - m)$.

If $m > 0$ and $n \geqq 0$, let $\Phi(-m\alpha, n\alpha^+) = \Phi(-m\alpha, n\alpha^-) = (-m\alpha, m + n)$ and $\Phi(n\alpha^+, -m\alpha) = \Phi(n\alpha^-, -m\alpha) = (n\alpha, -(m + n))$.

Proposition 4.10. *If $\Phi(x,y) \neq \Phi(x',y')$ then (x,y) and (x',y') can be separated by disjoint open sets.*

If $\Phi(x,y) = \Phi(x',y')$ but $(x,y) \neq (x',y')$ then there is an open set containing (x,y) but not (x',y'); however (x,y) and (x',y') cannot be separated by disjoint open sets.

Proof. Suppose $\Phi(x,y) = (\pi(x),n)$ and $\Phi(x',y') = (\pi(x'),n')$. If $n \neq n'$ then every basic neighbourhood of (x,y) will be disjoint from every basic neighbourhood of (x',y'). If $\pi(x) \neq \pi(x')$ we may choose k large enough so that the basic neighbourhoods of (x,y) and (x',y') are defined and the elements of $\mathscr{P}_k$ containing $\{\pi(x)^+, \pi(x)^-\}$ are disjoint from the elements of $\mathscr{P}_k$ containing $\{\pi(x')^+$ and $\pi(x')^-\}$. The basic neighbourhoods of (x,y) and (x',y') will be disjoint.

Suppose $\Phi(x,y) = \Phi(x',y')$. There are two cases. First (x,y), $(x',y') \in \{(m\alpha^+, (n-m)\alpha^+), (m\alpha^+,(n-m)\alpha^-), (m\alpha^-,(n-m)\alpha^+), (m\alpha^-,(n-m)\alpha^-)\}$. The basic $\mathscr{V}$-neighbourhoods contain exactly one of the two points, $(-,-),(+,+)$. So there are basic neighbourhoods which contain one of the four points but none of the other three. Also the basic $\mathscr{V}$-neighbourhoods of each of these points (for any k) all meet the basic $\mathscr{U}$ neighbourhoods, so these points cannot be separated by disjoint open sets.

The second case is that (x,y), $(x',y') \in \{(-m\alpha,(m+n)\alpha^+), (-m\alpha, (m+n)\alpha^-)\}$ (after taking inverses if necessary). The basic $\mathscr{W}$-neighbourhoods contain only one of these two points but any two of them meet. Thus one can find an open set containing a given point but not the other, but one cannot separate these points with disjoint open sets.

Proposition 4.11. *$\Phi : R_\alpha \to S^1 \times \mathbb{Z}$ is continuous.*

Proof. Let $T \subseteq S^1$ be open and $(x,y) \in \Phi^{-1}(T \times \{n\})$.

First suppose $\pi(x) \notin \mathbb{Z}\alpha$. Then there is k and $P \in \mathscr{P}_k$ such that $\pi(P) \subseteq T$. Suppose $y \in Q \in \mathscr{P}_k$. Then $\Phi(\mathscr{U}(x,y,k))$ is either $\pi(P) \times \{n\}$ or $\pi(\Theta^{-n}(Q)) \times \{n\}$ whichever is smaller. Hence $\mathscr{U}(x,y,k) \subseteq \Phi^{-1}(T \times \{n\})$.

Secondly suppose $\pi(x), \pi(y) \in \mathbb{N}\alpha$. Then choose k such that there are $P,P' \in \mathscr{P}_k$ with $\pi(P), \pi(P') \subseteq T$ and $\pi(x)^+ \in P$ and $\pi(x)^- \in P'$. Then $\mathscr{U}(\pi(x)^+,\pi(y)^+,k)$, $\mathscr{U}(\pi(\alpha)^-,\pi(y)^-,k) \subseteq \Phi^{-1}(T \times \{n\})$. Hence $\mathscr{V}(x,y,k) \subseteq \Phi^{-1}(T \times \{n\})$.

Thirdly suppose $x = -m\alpha$ for some $m > 0$. Then choose k such that there is $P \in \mathscr{P}_k$ with $x \in P$ and $\pi(P) \subseteq T$. Then $\mathscr{W}(x,y,k) \subseteq \Phi^{-1}(T \times \{n\})$.

Since neighbourhoods of the form $T \times \{n\}$, with T open in S^1 form a base for the topology, the proposition is proved.

Proposition 4.12. *If $\Phi(x,y) = \Phi(x',y')$ then $f(x,y) = f(x',y')$ for all continuous functions $f : R_\alpha \to \mathbb{C}$.*

Proof. Let $c = f(x,y)$ and $c' = f(x',y')$ and suppose $c \neq c'$. Then there are neighbourhoods $\mathscr{U}$ of (x,y) and $\mathscr{U}'$ of (x',y') such that

$$|f(x,y) - f(u,v)| < \frac{|c-c'|}{2} \quad \text{for } (u,v) \in \mathscr{U}$$

and

$$|f(x',y') - f(u',v)| < \frac{|c-c'|}{2} \quad \text{for } (u',v') \in \mathscr{U}' .$$

By Proposition 4.10 $\mathscr{U} \cap \mathscr{U}'$ is not empty. Suppose $(u,v) \in \mathscr{U} \cap \mathscr{U}'$. Then

$$\begin{aligned} |c-c'| &= |f(x,y) - f(x',y')| \\ &\leqq |f(x,y) - f(u,v)| + |f(u,v) - f(x',y')| \\ &\leqq \frac{|c-c'|}{2}\,. \end{aligned}$$

This contradiction shows that we must have $c = c'$.

Theorem 4.13. *$\Phi^* : C(S^1 \times \mathbb{Z}) \to C(R_\alpha)$ is a linear bijection.*

Proof. Φ^* is injective since Φ is surjective. Suppose $f \in C(R_\alpha)$. By Proposition 4.12 there is $\tilde{f} : S^1 \times \mathbb{Z} \to \mathbb{C}$, such that $f = \tilde{f} \circ \Phi$.

Let $\varepsilon > 0$ and $(\pi(x), n) \in S^1 \times \mathbb{Z}$ be given. We shall show that $\tilde{f}$ is continuous at $(\pi(x), n)$. Suppose first that $\pi(x) \notin \mathbb{Z}\alpha$ and choose y such that $(x,y) \in R_\alpha$. Choose k such that $|f(x,y) - f(x',y')| < \varepsilon$ for $(x',y') \in \mathscr{U}^o(x,y,k)$. Now $\Phi(\mathscr{U}^o(x,y,k))$ is open and for $(t,n) \in \Phi(\mathscr{U}^o(x,y,k))$, $|\tilde{f}(\pi(x),n) - \tilde{f}(t,n)| < \varepsilon$. Thus $\tilde{f}$ is continuous at $(\pi(x), n)$.

Now suppose $\pi(x) \in \mathbb{N}\alpha$, choose y such that $(x,y) \in R_\alpha \backslash \mathscr{R}_\alpha$ and k such that $|f(x,y) - f(x',y')| < \varepsilon$ for $(x',y') \in \mathscr{V}(x,y,k)$. Again $\Phi(\mathscr{V}(x,y,k))$ is open and for $(t,n) \in \Phi(\mathscr{V}(x,y,k))$, $|\tilde{f}(\pi(x),n) - \tilde{f}(t,n)| < \varepsilon$. Thus $\tilde{f}$ is continuous at $(\pi(x), n)$.

Finally suppose $\pi(x) \in -\mathbb{N}\alpha$. Choose y such that $(x,y) \in R_\alpha$ and k such that $|f(x,y) - f(x',y')| < \varepsilon$ for $(x',y') \in \mathscr{W}^o(x,y,k)$. Since $\Phi(\mathscr{W}^o(x,y,k))$ is open we have again that $\tilde{f}$ is continuous at $(\pi(x), n)$.

Remark 4.14. As shown in Proposition 4.10, the topology on R_α is not Hausdorff. By a *compact* subset of R_α we mean a set satisfying the Borel–Lebesgue axiom: every open cover has a finite subcover. These sets are called *quasi-compact* by Bourbaki [1, Chapter 1, Sect. 9]. The set $\{(x,x) \mid x \in X_\alpha\}$ is an open compact subset of R_α which is not closed and whose closure is not compact.

Lemma 4.15. *For each compact $J \subseteq S^1 \times \mathbb{Z}$, the inverse image $\Phi^{-1}(J) \subseteq R_\alpha$ is the closure of the compact set $\Phi^{-1}(J) \cap \mathscr{R}_\alpha$.*

Proof. We may suppose that J has no isolated points. Let $K = \Phi^{-1}(J) \cap \mathscr{R}_\alpha$. K is a compact subset of $\mathscr{R}_\alpha$ and thus a compact subset of R_α. Let $x \in \Phi^{-1}(J) \backslash K$.

Suppose that x (or x^{-1}) is of the form $(m\alpha^+, n\alpha^-)$ for $m,n \geqq 0$. Then $\mathscr{V}(m\alpha^+, n\alpha^-, k) \backslash \{x\} \subseteq \mathscr{R}_\alpha$ and $\Phi(\mathscr{V}(m\alpha^+, n\alpha^-, k))$ must meet $J \backslash \{\Phi(x)\}$, as $\Phi(\mathscr{V}(m\alpha^+, n\alpha^-, k))$ is open. Hence $\mathscr{V}(m\alpha^+, n\alpha^-, k)$ meets K. The same argument applies to the case when x (or x^{-1}) is of the form $(m\alpha^+, -n\alpha)$ for $m \geqq 0$ and $n > 0$, as Φ carries $\mathscr{W}^o$ neighbourhoods into open subsets of $S^1 \times \mathbb{Z}$. In either case $x \in K^-$.

Theorem 4.16. *Suppose $f \in C(R_\alpha)$ and $g \in C(S^1 \times \mathbb{Z})$ is the unique function such that $f = g \cdot \Phi$. The support of f is the closure of a compact set if and only if the support of g is compact.*

Proof. Suppose supp(g) is compact. So supp$(g) \subseteq S^1 \times \{-n, \ldots, n\}$ for some n. Since $\Phi^{-1}(S^1 \times \{-n, \ldots, n\}) \cap \mathscr{R}_\alpha$ is compact, $\Phi^{-1}(\text{supp}(g)) \cap \mathscr{R}_\alpha$ is compact.

Now $\Phi^{-1}\{y \mid g(y) \neq 0\} \subseteq \Phi^{-1}(\{y \mid g(y) \neq 0\}^-) = \Phi^{-1}(\mathrm{supp}(g))$. Thus $\mathrm{supp}(f) = (\Phi^{-1}\{y \mid g(y) \neq 0\})^- \subseteq \Phi^{-1}(\mathrm{supp}(g))$, and thus $\mathrm{supp}(f) \cap \mathscr{R}_\alpha$ is a closed subset of the compact set $\Phi^{-1}(\mathrm{supp}(g)) \cap \mathscr{R}_\alpha$.

Now suppose $x \in \mathrm{supp}(f) \backslash \mathscr{R}_\alpha$. We have two cases to consider: $x = (m\alpha^+, n\alpha^-)$ (or its inverse) for $m, n \geqq 0$, or $x = (m\alpha^\pm, -m\alpha)$ (or its inverse) for $m \geqq 0$ and $n > 0$.

In the first case let $\tilde{x} = (m\alpha^+, n\alpha^+)$. Then $f(x) = f(\tilde{x})$. So if $f(x) \neq 0$ then $f \neq 0$ on some $\mathscr{U}$ neighbourhood of $\tilde{x}$. Hence x has a $\mathscr{V}$ neighbourhood which meets $\mathrm{supp}(f) \cap \mathscr{R}_\alpha$. If $f(x) = 0$ then every $\mathscr{V}$ neighbourhood of x meets $\mathrm{supp}(f) \cap \mathscr{R}_\alpha$, since every $\mathscr{V}$ neighbourhood of x contains a point y for which $f(y) \neq 0$ and such a point must be in $\mathscr{R}_\alpha$ as a $\mathscr{V}$ neighbourhood only contains one point not in $\mathscr{R}_\alpha$ – x in our case. Thus $x \in (\mathrm{supp}(f) \cap \mathscr{R}_\alpha)^-$.

In the second case we proceed similarly. If $f(x) \neq 0$ then $f \neq 0$ on some $\mathscr{W}$ neighbourhood of x. x will be the only point of such $\mathscr{W}$ neighbourhoods not in $\mathscr{R}_\alpha$, so each $\mathscr{W}$ neighbourhood of x meets $\mathrm{supp}(f) \cap \mathscr{R}_\alpha$. If $f(x) = 0$ then every $\mathscr{W}$ neighbourhood of x meets $\mathrm{supp}(f) \cap \mathscr{R}_\alpha$, since every $\mathscr{W}$ neighbourhood of x contains a point y for which $f(y) \neq 0$ and such a point must be in $\mathscr{R}_\alpha$ as a $\mathscr{W}$ neighbourhood only contains one point not in $\mathscr{R}_\alpha$ – x in our case. Thus $x \in (\mathrm{supp}(f) \cap \mathscr{R}_\alpha)^-$. Hence in either case $x \in (\mathrm{supp}(f) \cap \mathscr{R}_\alpha)^-$, and thus $\mathrm{supp}(f)$ is the closure of the compact set $\mathrm{supp}(f) \cap \mathscr{R}_\alpha$.

Now suppose $\mathrm{supp}(f) = (\Phi\{y \mid g(y) \neq 0\})^-$ is compact. Then $\Phi((\Phi^{-1}\{y \mid g(y) \neq 0\})^-) = \Phi(\mathrm{supp}(f))$ is compact. Hence $\{y \mid g(y) \neq 0\}$ is a subset of the compact set $\Phi(\mathrm{supp}(f))$, and thus $\mathrm{supp}(g)$ is compact.

Definition 4.17. *$C_{oo}(R_\alpha)$ is the space of continuous functions on R_α whose support is the closure of a compact set. By Theorem* 4.13, *$\Phi^*(C_{oo}(S^1 \times \mathbb{Z})) = C_{oo}(R_\alpha)$, the continuous functions on $S^1 \times \mathbb{Z}$ with compact support.*

5. The C*-Algebra C*(R_α, μ)

We construct a Haar system μ on R_α and show that the C*-algebra $C^*(R_\alpha, \mu)$ is isomorphic to A_α the irrational rotation C*-algebra for the angle $2\pi\alpha$.

Definition 5.1. *For $x \in X_\alpha$ let $R_\alpha^x = \{(x,y) \mid (x,y) \in R_\alpha\}$. Define a measure μ^x on R_α^x by setting $\mu^x(x,y) = 1$ if $y \notin \{m\alpha^\pm \mid m \geqq 0\}$ and $\mu^x(x,y) = 1/2$ if $y \in \{m\alpha^\pm \mid m \geqq 0\}$. Let $\mu = \{\mu^x\}_{x \in X_\alpha}$. We make $S^1 \times \mathbb{Z}$ into a groupoid in the usual way: $(x,m)(y,n) = (x, m+n)$ provided $y = x + m\alpha$ (modulo 1), $(x,n)^{-1} = (x + n\alpha, -n)$. Let ν^x be counting measure on $(S^1 \times \mathbb{Z})^x = \{(x,n) \mid n \in \mathbb{Z}\}$. Then $\nu = \{\nu^x\}_{x \in S^1}$ is a Haar system on $S^1 \times \mathbb{Z}$.*

We shall adopt the notation of Renault [7, Chapter 1, Definition 2.2]: for $f \in C_{oo}(R_\alpha)$ and $x \in X_\alpha$ let

$$\mu(f)(x) = \sum_{(x,y) \in R_\alpha} f(x,y)\mu^x(x,y),$$

and for $g \in C_{oo}(S^1 \times \mathbb{Z})$ and $y \in S^1$ let

$$\nu(g)(y) = \sum_{n \in \mathbb{Z}} g(y,n).$$

Lemma 5.2. *Given $g \in C_{oo}(S^1 \times \mathbb{Z})$ let $f = g \circ \Phi$. For $x \in X_\alpha$,*

$$\mu(f)(x) = \nu(g)(\pi(x)),$$

and $\mu(f)$ is continuous on X_α. Also μ is left invariant:

$$\sum_{(y_1,y_2)\in R_\alpha^{x_2}} f((x_1,x_2)(y_1,y_2))\mu^{x_2}(y_1,y_2) = \sum_{(y_1,y_2)\in R_\alpha^{x_1}} f(y_1,y_2)\mu^{x_1}(y_1,y_2).$$

Proof. We shall break the proof into two cases.

(i) $\pi(x) \notin \mathbb{Z}\alpha$. Then $R_\alpha^x = \{(x, \Theta^n(x)) \mid n \in \mathbb{Z}\}$, $\Phi : R_\alpha^x \to (S^1 \times \mathbb{Z})^{\pi(x)}$ is bijection and both μ^x and $\nu^{\pi(x)}$ are counting measures.

(ii) $\pi(x) \in \mathbb{Z}\alpha$. Then

$$\begin{aligned}
\mu(f)(x) &= \sum_{(x,y)\in R_\alpha} f(x,y)\mu^x(x,y) \\
&= \sum_{m\geqq 0} (f(x,m\alpha^+)\mu^x(x,m\alpha^+) + f(x,m\alpha^-)\mu^x(x,m\alpha^-)) \\
&\quad + \sum_{m>0} f(x,-m\alpha)\mu^x(x,-m\alpha) \\
&= \sum_{m\geqq 0} g(\pi(x),m) + \sum_{m>0} g(\pi(x),m) \\
&= \nu(g)(\pi(x)).
\end{aligned}$$

Hence $\mu(f) = \nu(g) \circ \pi$ is continuous on X_α.

As for the last claim note that $\mu^x(x,y)$ depends only on y. Thus

$$\begin{aligned}
\sum_{(y_1,y_2)\in R_\alpha^{x_2}} f((x_1,x_2)(y_1,y_2))\mu^{x_2}(y_1,y_2) &= \sum_{y_2\sim x_2} f((x_1,x_2)(x_2,y_2))\mu^{x_2}(x_2,y_2) \\
= \sum_{y_2\sim x_2} f(x_1,y_2)\mu^{x_2}(x_2,y_2) &= \sum_{y_2\sim x_2} f(x_1,y_2)\mu^{x_1}(x_1,y_2) \\
= \sum_{(y_1,y_2)\in R_\alpha^{x_1}} f(y_1,y_2)\mu^{x_1}(y_1,y_2).
\end{aligned}$$

Definition 5.3. *We give $C_{oo}(R_\alpha)$ an involution and product by defining*

$$f^*(x,y) = \overline{f(y,x)}$$

and

$$f_1 * f_2(x,z) = \sum_{y\sim x} f_1(x,y)f_2(y,z)\mu^x(x,y).$$

As the algebra of continuous functions on a topological groupoid $C_{oo}(S^1 \times \mathbb{Z})$ has the involution and product:

$$g(y,n)^* = \overline{g(y+n\alpha,-n)}$$

and

$$g_1 * g_2(y,n) = \sum_{m\in\mathbb{Z}} g_1(y,m)g_2(y+m\alpha,n-m).$$

Since the functions have support the closure of a compact set the sums are always finite.

Proposition 5.4. *Suppose f_1 and f_2 are in $C_{oo}(R_\alpha)$, and g_1 and g_2 are in $C_{oo}(S^1 \times \mathbb{Z})$ with $f_1 = g_1 \circ \Phi$ and $f_2 = g_2 \circ \Phi$. Then $f_1^* = g_1^* \circ \Phi$ and $f_1 * f_2 = (g_1 * g_2) \circ \Phi$. Hence Φ^* is a $*$-homomorphism.*

Proof. Suppose $\Phi(x,z) = (\pi(x), m)$, then $\Phi(z,x) = (\pi(x) + m\alpha, -m)$. Thus

$$\begin{aligned} g_1^*(\Phi(x,z)) &= g_1^*(\pi(x), m) = \overline{g_1(\pi(x) + m\alpha, -m)} \\ &= \overline{f_1(z,x)} = f_1^*(x,z)\,. \end{aligned}$$

Hence Φ^* is a $*$-linear map.

To verify that Φ^* is a homomorphism we consider two cases.

(i) Suppose $\pi(x) \notin \mathbb{Z}\alpha$. Then $R_\alpha^x = \{(x, \Theta^n(x) \mid n \in \mathbb{Z}\}$, μ^x is counting measure, and the restriction of Φ to R_α^x is one-to-one. Also there is $m \in \mathbb{Z}$ such that $z = \Theta^m(x)$. Thus

$$\begin{aligned} f_1 * f_2(x,z) &= f_1 * f_2(x, \Theta^m(x)) \\ &= \sum_{n\in\mathbb{Z}} f_1(x, \Theta^n(x)) f_2(\Theta^n(x), \Theta^m(x)) \\ &= \sum_{n\in\mathbb{Z}} g_1(\pi(x), n) g_2(\pi(x) + n\alpha, m - n) \\ &= g_1 * g_2(\pi(x), m) = g_1 * g_2(\Phi(x,z))\,. \end{aligned}$$

(ii) Suppose $\pi(x) \in \mathbb{Z}\alpha$, and $\Phi(x,z) = (\pi(x), m)$. Then

$$\begin{aligned} f_1 * f_2(x,z) &= \sum_{n\geqq 0} (f_1(x, n\alpha^+) f_2(n\alpha^+, z) + f_1(x, n\alpha^-) f_2(n\alpha^-, z))/2 \\ &\quad + \sum_{n>0} f_1(x, -n\alpha) f_2(-n\alpha, z) \\ &= \sum_{n\geqq 0} g_1(\pi(x), n) g_2(\pi(x) + n\alpha, m - m) \\ &\quad + \sum_{n>0} g_1(\pi(x), n) g_2(\pi(x) + n\alpha, m - n) \\ &= g_1 * g_2(\pi(x), m) = g_1 * g_2(\Phi(x,z))\,. \end{aligned}$$

Thus Φ^* is a $*$-homomorphism.

Definition 5.5. *We give $C_{oo}(R_\alpha)$ the topology of uniform convergence on the closures of compact sets, and $C_{oo}(S^1 \times \mathbb{Z})$ the topology of uniform convergence on compact sets.*

Proposition 5.6. *$\Phi^* : C_{oo}(R_\alpha) \to C_{oo}(S^1 \times \mathbb{Z})$ is a homeomorphism.*

Proof. Let $f_0 \in C_{oo}(R_\alpha)$ and $g_0 = \Phi^*(f_0)$. A basic neighbourhood of f_0 is given by $\mathcal{U}(f_0, K, \varepsilon) = \{f \in C_{oo}(R_\alpha) \mid |f(x) - f_0(x)| < \varepsilon$ for $x \in K^-\}$, where $K \subseteq R_\alpha$ is compact and $\varepsilon > 0$. Thus

$$\begin{aligned} \Phi^*(\mathcal{U}(f, K, \varepsilon)) &= \{g \in C_{oo}(S^1 \times \mathbb{Z}) \mid |g(\Phi(x)) - g_0(\Phi(x))| < \varepsilon \text{ for } x \in K^-\} \\ &= \{g \in C_{oo}(S^1 \times \mathbb{Z}) \mid |g(y) - g_0(y)| < \varepsilon \text{ for } y \in \Phi(K)\}\,. \end{aligned}$$

This is a basic neighbourhood of g_0 in $C_{oo}(S^1 \times \mathbf{Z})$.

Conversely, given $J \subseteq S^1 \times \mathbf{Z}$ compact and $\varepsilon > 0$, let

$$\mathscr{U}(g_0, J, \varepsilon) = \{g \in C_{oo}(S^1 \times \mathbf{Z}) \mid |g(y) - g_0(y)| < \varepsilon \text{ for } y \in J\}.$$

By Proposition 4.15 there is a compact set $K \subseteq R_\alpha$ such that $\Phi(K) = J$. So $\mathscr{U}(g_0, J, \varepsilon) = \Phi^*(\mathscr{U}(f_0, K, \varepsilon)$.

Definition 5.7. *Following Renault* [7, Definition 1.3] *we define a norm* $\|\cdot\|_I$ *on* $C_{oo}(R_\alpha)$ *such that* $\|f^*\|_I = \|f\|_I$ *and* $\|f_1 * f_2\| \leqq \|f_1\|_I \|f_2\|_I$. *Let*

$$\|f\|_{I,r} = \sup_{x \in X_\alpha} \sum_{y \sim x} |f(x,y)| \mu^x(x,y),$$

$$\|f\|_{I,l} = \sup_{x \in X_\alpha} \sum_{y \sim x} |f(y,x)| \mu^x(x,y),$$

$$\|f\|_I = \max\{\|f\|_{I,r}, \|f\|_{I,l}\}.$$

Remark 5.8. Note that $\|f^*\|_{I,r} = \|f\|_{I,l}$, so $\|f^*\|_I = \|f\|_I$. Also

$$\begin{aligned}
\|f_1 * f_2\|_{I,r} &= \sup_{x \in X_\alpha} \sum_{y \sim x} |f_1 * f_2| \mu^x(x,y) \\
&= \sup_{x \in X_\alpha} \sum_{y \sim x} \left| \sum_{z \sim x} f_1(x,z) f_2(z,y) \mu^x(x,z) \right| \mu^x(x,y) \\
&\leqq \sup_{x \in X_\alpha} \sum_{y \sim x} \sum_{z \sim x} |f_1(x,z)|\ |f_2(z,y)| \mu^x(x,z) \mu^x(x,y) \\
&= \sup_{x \in X_\alpha} \sum_{z \sim x} |f_1(x,z)| \left(\sum_{y \sim x} |f_2(z,y)| \mu^x(x,y) \right) \mu^x(x,y) \\
&= \sup_{x \in X_\alpha} \sum_{z \sim x} |f_1(x,z)| \left(\sum_{y \sim x} |f_2(z,y)| \mu^z(z,y) \right) \mu^x(x,z) \\
&\leqq \sup_{x \in X_\alpha} \sum_{z \sim x} |f_1(x,z)| \left(\sup_{z \in X_\alpha} \sum_{y \sim x} |f_2(z,y)| \mu^z(z,y) \right) \mu^x(x,z) \\
&\leqq \|f_1\|_{I,r} \|f_2\|_{I,r}.
\end{aligned}$$

Also

$$\|f_1 * f_2\|_{I,l} = \|f_2^* * f_1^*\|_{I,r} \leqq \|f_2^*\|_{I,r} \|f_1^*\|_{I,r} = \|f_1\|_{I,l} \|f_2\|_{I,l}.$$

Hence $\|f_1 * f_2\|_I \leqq \|f_1\|_I \|f_2\|_I$.

Definition 5.9. *A $*$-representation of $C_{oo}(R_\alpha)$ on a Hilbert space $\mathscr{H}$ is a continuous $*$-homomorphism from $C_{oo}(R_\alpha)$ to $\mathscr{B}(\mathscr{H})$ when $C_{oo}(R_\alpha)$ has the topology of uniform convergence on the closure of compact sets and $\mathscr{B}(\mathscr{H})$ has the strong operator topology. A $*$-representation π is bounded if $\|\pi(f)\| \leqq \|f\|_I$ for all f in $C_{oo}(R_\alpha)$. We place a C^*-norm on $C_{oo}(R_\alpha)$ by setting $\|f\| = \sup\{\|\pi(f)\| \mid \pi$ is a bounded $*$-representation of $C_{oo}(R_\alpha)\}$. $C^*(R_\alpha, \mu)$ is the completion of $C_{oo}(R_\alpha)$ with respect to this norm.*

Theorem 5.10.

$$C^*(R_\alpha, \mu) \simeq A_\alpha \,.$$

Proof. We have already shown that as topological $*$-algebras Φ^* is a homeomorphic $*$-isomorphism from $C_{oo}(R_\alpha)$ to $C_{oo}(S^1 \times \mathbb{Z})$. Let us show that Φ also preserves the norm $\|\cdot\|_I$. Let $f \in C_{oo}(R_\alpha)$ and $g = \Phi^*(f) \in C_{oo}(S^1 \times \mathbb{Z})$. Then

$$\begin{aligned} \|f\|_{I,r} &= \sup_{x\in X_\alpha} \sum_{y\sim x} |f(x,y)|\mu^x(x,y) = \sup_{x\in X_\alpha} \mu(|f|)(x) \\ &= \sup_{x\in X_\alpha} \nu(|\Phi^*(f)|)(\pi(x)) \quad \text{(by Lemma 5.2)} \\ &= \sup_{y\in S^1} \nu(|g|)(y) = \|g\|_{I,r} \,. \end{aligned}$$

Thus $\|\Phi^*(f)\|_I = \|f\|_I$. Since the completion of $(C_{oo}(S^1 \times \mathbb{Z}), \|\cdot\|)$ is A_α, the proof is complete.

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