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The Trivial Connection Contribution to Witten's Invariant and Finite Type Invariants of Rational Homology Spheres

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Abstract: We derive an analog of the Melvin–Morton bound on the power series expansion of the colored Jones polynomial of algebraically split links and boundary links. This allows us to produce a simple formula for the trivial connection contribution to Witten's invariant of rational homology spheres. We show that the n^{th} term in the $1/K$ expansion of the logarithm of this contribution is a finite type invariant of Ohtsuki order $3n$ and of at most Garoufalidis order n.

1. Introduction

Let M be a 3-dimensional manifold with an N-component link $\mathcal{L}$ inside it. We assign α_j-dimensional irreducible representations of $SU(2)$ to every component $\mathcal{L}_j$ of $\mathcal{L}$. Witten's invariant of M and $\mathcal{L}$ is given [1] by a path integral over all $SU(2)$ connections A_μ on M:

$$Z_{\alpha_1,\ldots,\alpha_N}(M,\mathcal{L};k) = \int [\mathcal{D}A_\mu] \exp\left(\frac{ik}{2\pi}S_{\mathrm{CS}}\right) \prod_{j=1}^{N} \mathrm{Tr}_{\alpha_j} \mathrm{Pexp}\left(\oint_{\mathcal{L}_j} A_\mu dx^\mu\right), \quad (1.1)$$

here S_{CS} is the Chern–Simons action

$$S_{\mathrm{CS}} = \frac{1}{2}\,\mathrm{Tr}\,\varepsilon^{\mu\nu\rho} \int_M d^3x \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right), \quad (1.2)$$

$\mathrm{Tr}_{\alpha_j} \mathrm{Pexp}\left(\oint_{\mathcal{L}_j} A_\mu dx^\mu\right)$ are traces of holonomies of A_μ along $\mathcal{L}_j$ taken in α_j-dimensional representations of $SU(2)$ and Tr of Eq. (1.2) is the trace taken in the fundamental 2-dimensional representation. In most cases instead of the integer number k we will be using

$$K = k + 2\,. \quad (1.3)$$

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According to quantum field theory, the path integral (1.1) can be calculated in the limit of $k \to \infty$ by the stationary phase approximation. The stationary points of the phase (1.2) are flat connections. The whole path integral (1.1) is presented as a sum of contributions of connected components c of the flat connection moduli space:

$$Z_{\alpha_1,\dots,\alpha_N}(M,\mathcal{L};k) = \sum_c Z^{(c)}_{\alpha_1,\dots,\alpha_N}(M,\mathcal{L};k)\,. \tag{1.4}$$

Each contribution $Z^{(c)}_{\alpha_1,\dots,\alpha_N}(M,\mathcal{L};k)$ is proportional to the classical exponent exp $(2\pi i k S^{(c)}_{\mathrm{CS}})$, $S^{(c)}_{\mathrm{CS}}$ being the Chern–Simons action of the flat connections of component c. The preexponential factor is generally an asymptotic series in k^{-1}, or equivalently, in K^{-1}.

Suppose that the manifold M is a rational homology sphere (RHS). Then the trivial connection forms a separate component of the moduli space of flat connections. Therefore it produces a distinct contribution to Witten's invariant (1.4). This contribution is known [2–4] to be of the following form:

$$Z^{(\mathrm{tr})}(M;k) = \frac{\sqrt{2}\pi}{K^{\frac{3}{2}}[\operatorname{ord} H_1(M,\mathbb{Z})]^{\frac{3}{2}}} \exp\left(\sum_{n=1}^{\infty} S_n(M)\left(\frac{i\pi}{K}\right)^n\right), \tag{1.5}$$

here $\operatorname{ord} H_1(M,\mathbb{Z})$ is the order of integer homology group and we assumed that M contained no links. We call $S_n(M)$ "perturbative invariants," because, according to quantum field theory assumptions they should be equal to the sums of $(n+1)$-loop connected Feynman diagrams, studied, e.g. in papers [5–7]. However caution is advised, because no direct mathematically rigorous evidence supporting this relation has been established yet (in fact, the results of [7] may even contradict it).

The only mathematically rigorous definition of the quantum invariant $Z_{\alpha_1,\dots,\alpha_N}(M,\mathcal{L};k)$ is the Reshetikhin–Turaev formula [13]. Its asymptotic properties at $k \to \infty$ have been studied only for a small subclass of 3d manifolds M, so Eqs. (1.4) and (1.5) are conjectures. Therefore we will use Eqs. (1.1) and (1.4) only as motivation.

We want to keep our discussion mathematically rigorous. Therefore we define the invariant $Z_{\alpha_1,\dots,\alpha_N}(S^3,\mathcal{L};k)$ of an N-component link $\mathcal{L} \subset S^3$ not by Eq. (1.1), but as

$$Z_{\alpha_1,\dots,\alpha_N}(S^3,\mathcal{L};k) = Z(S^3;k)\exp\left(\frac{i\pi}{2K}\sum_{j=1}^{N} l_{jj}(\alpha_j^2-1)\right) J_{\alpha_1,\dots,\alpha_N}(\mathcal{L};k)\,, \tag{1.6}$$

here $J_{\alpha_1,\dots,\alpha_N}(\mathcal{L};k)$ is the colored, framing independent Jones polynomial of the link $\mathcal{L}$ normalized by the condition that it is multiplicative for disconnected links, and that

$$Z(S^3;k) = \sqrt{\frac{2}{K}}\sin\left(\frac{\pi}{K}\right)\,. \tag{1.7}$$

Then we proceed to define $Z^{(\mathrm{tr})}(M;k)$ as the formal power series (1.5). First we assume by definition that

$$Z^{(\mathrm{tr})}_{\alpha_1,\dots,\alpha_N}(S^3,\mathcal{L};k) = Z_{\alpha_1,\dots,\alpha_N}(S^3,\mathcal{L};k)\,. \tag{1.8}$$

In particular, this means that

$$Z^{(\mathrm{tr})}(S^3;k) = Z(S^3;k)\,, \tag{1.9}$$

so that according to Eq. (1.5),

$$S_{2n+1}(S^3) = 0,\ \sum_{n=1}^{\infty} S_{2n}(S^3)\left(\frac{i\pi}{K}\right)^{2n} = \log\left(\frac{K}{\pi}\sin\left(\frac{\pi}{K}\right)\right)\,. \tag{1.10}$$

We define $Z^{(\mathrm{tr})}_{\alpha_1,\dots,\alpha_N}(M,\mathscr{L};k)$ for general RHS M by the surgery formulas of [8] and [10] (Definitions 2.1 and 2.2). We proved in [25] that such a definition is consistent (i.e. invariant under Kirby moves) and thus presents an infinite sequence of genuine invariants $S_n(M)$ of M. We will still call $Z^{(\mathrm{tr})}(M;k)$ and its coefficients $S_n(M)$ "perturbative invariants" in the hope that their relation to the Feynman diagrams of [5] and to the trivial connection contribution into the path integral (1.1) will be established in the future.

In this paper we will study how $Z^{(\mathrm{tr})}(M;k)$ changes under a rational surgery on an algebraically split link (i.e. a link with zero linking numbers between its components) and on a boundary link. We will derive simple surgery formulas for the invariants $S_n(M)$ and show that they are finite type invariants of Ohtsuki [14] order $3n$ and of at most Garoufalidis [15] order n.

In Sect. 2 we review the previous surgery formulas of [8] and [10] as well as Reshetikhin's formula [9] for the Jones polynomial of a link. In Sect. 3 we derive the analog of Melvin–Morton bound [11] for the power series expansion of the colored Jones polynomial of algebraically split links and boundary links. By using this bound we derive the surgery formulas for the perturbative invariants $S_n(M)$ for the case of a surgery on these classes of links. The invariants are expressed in terms of derivatives of the colored Jones polynomial and surgery data. We work out an explicit expression for $S_1(M)$ and demonstrate that it is consistent with J. Hoste's surgery formula [12] for Casson's invariant λ_{CW} if we put [8]

$$S_1(M) = 6\lambda_{\mathrm{CW}}\,. \tag{1.11}$$

We also show how to convert $S_n(M)$ into integer valued invariants $S_n^{(\mathrm{int})}(M)$.

In Sect. 4 we extend Ohtsuki and Garoufalidis definitions of finite type invariants to rational homology spheres. We also define an extra finite type invariant that we call Ohtsuki$'$. We use the results of Sect. 3 to demonstrate that the perturbative invariants $S_n(M)$ are finite type invariants of Ohtsuki order $3n$, Ohtsuki$'$order $2n$ and of at most Garoufalidis order n. Finally, in Sect. 5 we speculate about the relation of our results to Feynman diagram calculations of perturbative invariants of [5–7] and to Ohtsuki's polynomial invariant [20, 21].

2. Surgery Formulas for Knots and General Links

2.1. General Considerations. Let $\mathscr{L}$ be an N-component link in S^3. We assign rational surgeries (p_j, q_j) to its components. A rational surgery (p,q) is presented by an $SL(2,\mathbb{Z})$ matrix

$$U^{(p,q)} = \begin{pmatrix} p & r \\ q & s \end{pmatrix}, \; ps - qr = 1 , \tag{2.1}$$

whose coefficients show how to glue meridian and parallel of the solid torus to meridian and parallel of the knot complement: the meridian of the boundary of the solid torus is glued to p(meridian) + q(parallel) of the knot complement (for more details see, e.g. [8]). We denote by $M = \chi_{\mathscr{L}}(S^3)$ a manifold constructed by performing all surgeries on the components of $\mathscr{L}$. The Reshetikhin–Turaev formula [13] relates Witten's invariants of S^3 and M:

$$Z(M;k) = e^{i\phi_{\mathrm{fr}}} \sum_{1 \leqq \alpha_1,\ldots,\alpha_N \leqq K-1} Z_{\alpha_1,\ldots,\alpha_N}(S^3, \mathscr{L}; k) \prod_{j=1}^{N} \tilde{U}_{\alpha_j 1}^{(p_j,q_j)} . \tag{2.2}$$

In this formula the matrices $\tilde{U}_{\alpha\beta}^{(p,q)}$ represent the group $SL(2,\mathbb{Z})$ in the $(K-1)$-dimensional space (of level k affine $SU(2)$ characters) [3]:

$$\tilde{U}_{\alpha\beta}^{(p,q)} = i \frac{\mathrm{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4}\Phi(U^{(p,q)})}$$
$$\times \sum_{n=0}^{q-1} \sum_{\mu=\pm 1} \mu \exp\left[\frac{i\pi}{2Kq}(p\alpha^2 - 2\alpha(2Kn + \mu\beta) + s(2Kn+\mu\beta)^2)\right] , \tag{2.3}$$

here $\Phi(U^{(p,q)})$ is the Rademacher function

$$\Phi\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \frac{p+s}{q} - 12s(p,q) \tag{2.4}$$

and $s(p,q)$ is the Dedekind sum

$$s(p,q) = \frac{1}{4q} \sum_{j=1}^{q-1} \cot\left(\frac{\pi j}{q}\right) \cot\left(\frac{\pi p j}{q}\right) . \tag{2.5}$$

The phase ϕ_{fr} in Eq. (2.2) is the framing correction

$$\phi_{\mathrm{fr}} = \frac{\pi}{4} \frac{K-2}{K} \left[\sum_{j=1}^{N} \Phi(U^{(p_j,q_j)}) - 3\,\mathrm{sign}(L^{(\mathrm{tot})})\right] , \tag{2.6}$$

here $L^{(\mathrm{tot})}$ is an $N \times N$ matrix

$$L_{ij}^{(\mathrm{tot})} = l_{ij} + \frac{p_j}{q_j}\delta_{ij} , \tag{2.7}$$

l_{ij} is a linking matrix of $\mathscr{L}$ and $\mathrm{sign}(L^{(\mathrm{tot})})$ is the difference between the number of positive and negative eigenvalues of $L^{(\mathrm{tot})}$.

The formula (2.2) reflects the change in the whole invariant (1.4). In [8] we explained why its simple modification should reflect the change in the trivial connection contribution. Here we use this modification as a definition of the invariant $Z^{(\mathrm{tr})}(M;k)$, while keeping in mind the conjecture that $Z^{(\mathrm{tr})}(M;k)$ may be related to the asymptotic properties of $Z(M;k)$.

Definition 2.1. *Suppose that a RHS M is constructed by rational (p_j, q_j) surgeries on the components of an N-component link $\mathcal{L} \subset S^3$. Then the perturbative invariant of M is given by the following formula* (cf. *Eq.* (2.2)):

$$Z^{(\mathrm{tr})}(M;k) = e^{i\phi_{\mathrm{fr}}} \frac{1}{2^N} \int_{\substack{-\infty \\ [\alpha_j=0]}}^{+\infty} d\alpha_1 \cdots d\alpha_N \, Z_{\alpha_1,\ldots,\alpha_N}(S^3,\mathcal{L};k) \prod_{j=1}^{N} \hat{U}^{(p_j,q_j)}_{\alpha_j 1} . \tag{2.8}$$

Here the symbol $\int_{-\infty,[\alpha=0]}^{+\infty}$ means that we take only the stationary phase contribution of the point $\alpha = 0$ to the whole integral. The matrix $\hat{U}^{(p,q)}_{\alpha\beta}$ is obtained from $\tilde{U}^{(p,q)}_{\alpha\beta}$ by substituting $n = 0$ instead of $\sum_{n=0}^{q-1}$ in Eq. (2.3):

$$\hat{U}^{(p,q)}_{\alpha\beta} = \sqrt{\frac{2}{K|q|}} \operatorname{sign}(q) e^{-\frac{i\pi}{4}\Phi(U^{(p,q)})} \sin\left(\frac{\pi}{K}\frac{\alpha\beta}{q}\right) \times \exp\left(\frac{i\pi}{2Kq}(p\alpha^2 + s\beta^2)\right) . \tag{2.9}$$

Since the framing independent colored Jones polynomial of $\mathcal{L}$ is related to Witten's invariants by the formula (1.6) then the surgery formula (2.8) can be rewritten as

$$Z^{(\mathrm{tr})}(M;k) = Z(S^3;k) e^{i\phi_{\mathrm{fr}}} \frac{1}{2^N} \int_{\substack{-\infty \\ [\alpha_j=0]}}^{+\infty} d\alpha_1 \cdots d\alpha_N J_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k) \times \prod_{j=1}^{N} \left(e^{\frac{i\pi}{2K} l_{jj}(\alpha_j^2-1)} \hat{U}^{(p_j,q_j)}_{\alpha_j 1}\right) . \tag{2.10}$$

Another modification of this formula is especially useful for integer surgeries, i.e. when $p_j = 1$, $l_{jj} = 0$. It is obtained by shifting the integration variables

$$\alpha_j \to \alpha_j \pm \frac{1}{p_j + q_j l_{jj}} \tag{2.11}$$

and working with an even part of the shifted Jones polynomial

$$\tilde{J}_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k) = \frac{\prod_{j=1}^{N}(p_j + q_j l_{jj})}{2^N} \times \sum_{\mu_j=\pm 1} \left(\prod_{j=1}^{N} \mu_j\right) J_{\alpha_1+\frac{\mu_1}{p_1+q_1 l_{11}},\ldots,\alpha_N+\frac{\mu_N}{p_N+q_N l_{NN}}} . \tag{2.12}$$

As a result, Eq. (2.10) transforms into

$$Z^{(\mathrm{tr})}(M;k) = Z(S^3;k) \frac{i^N}{(2K)^{\frac{N}{2}}} \left(\prod_{j=1}^{N} \frac{\operatorname{sign}(q_j)}{\sqrt{|q_j|}}\right) e^{-i\pi\frac{3}{4}\operatorname{sign}(L^{(\mathrm{tot})})} e^{\frac{i\pi}{K}\Delta_{\mathrm{fr}}} \times \int_{\substack{-\infty \\ [\alpha_j=0]}}^{+\infty} d\alpha_1 \cdots d\alpha_N \exp\left[\frac{i\pi}{2K}\sum_{j=1}^{N}\left(\frac{p_j}{q_j} + l_{jj}\right)\alpha_j^2\right] \times \tilde{J}_{\alpha_1,\ldots,\alpha_N}(\mathcal{L};k) , \tag{2.13}$$

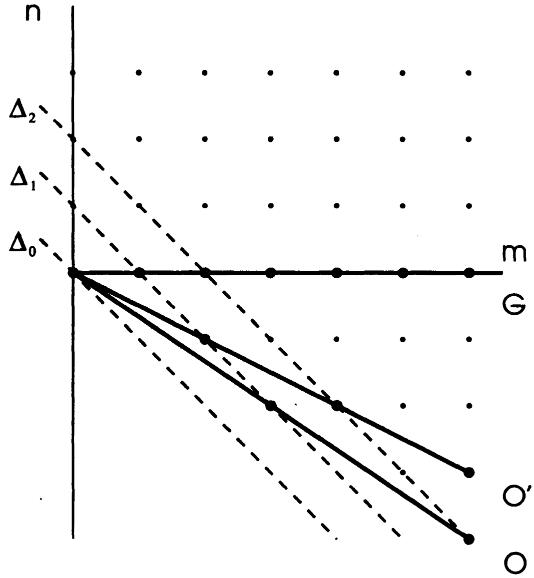

Fig. 1. The structure of power series expansions

here

$$\Delta_{\mathrm{fr}} = \frac{3}{2}\operatorname{sign}(L^{(\mathrm{tot})}) + \frac{1}{2}\sum_{j=1}^{N}\left(12s(p_j,q_j) - \left(\frac{p_j}{q_j} + l_{jj}\right) - \frac{1}{q_j(p_j + q_j l_{jj})}\right) . \tag{2.14}$$

In order to understand how Eq. (2.10) works we consider a simple example of the stationary phase calculation:

$$I(K) = \int_{\substack{-\infty \\ [a=0]}}^{+\infty} e^{iKf(a)} g(a,K)\, da\,, \tag{2.15}$$

here the functions $f(a)$ and $g(a,K)$ have a smooth analytic behavior at $a = K^{-1} = 0$ and $f'(0) = 0$. We separate the quadratic part of the exponent and then remove the odd part of the preexponential factor, because it does not contribute to the integral:

$$I(K) = e^{iKf(0)} \int_{\substack{-\infty \\ [a=0]}}^{+\infty} e^{\frac{iK}{2}f''(0)a^2} G_{\mathrm{ev}} da\,, \tag{2.16}$$

$$G(a,K) = e^{iK(f(a)-f(0)-\frac{1}{2}f''a^2)} g(a,K)\,, \tag{2.17}$$

$$G_{\mathrm{ev}} = \frac{1}{2}(G(a,K) + G(-a,K)) = \sum_{m\geqq 0} \sum_{-\frac{2}{3}m \leqq n < \infty} d_{m,n} a^{2m} K^{-n}\,. \tag{2.18}$$

The non-zero coefficients $d_{m,n}$ are depicted in Fig. 1. The inequality

$$n \geqq -\frac{2}{3}m \tag{2.19}$$

(line O in Fig. 1) comes from the fact that the expansion of the exponent $f(a) -f(0) - \frac{1}{2}f''(0)a^2$ in powers of a starts with the qubic term, while the expansion of $g(a,K)$ has only negative powers of K.

Combining Eqs. (2.16)–(2.18) we find that

$$I(K) = e^{iKf(0)} e^{\frac{i\pi}{4}\operatorname{sign}(f''(0))} \sqrt{\frac{2\pi}{K|f''(0)|}} \sum_{n=0}^{\infty} \Delta_n K^{-n} \,. \tag{2.20}$$

After being integrated with the gaussian factor $e^{i\frac{K}{2}f''(0)a^2}$, the term $d_{m,n}a^{2m}K^{-n}$ contributes to the coefficient Δ_{m+n}, so that

$$\Delta_n = \sum_{m=0}^{3n} \frac{(2m)!}{m!} \left(\frac{i}{2f''(0)}\right)^m d_{m,n-m} \tag{2.21}$$

(the terms contributing to a given Δ_n are connected by dashed lines on Fig. 1). The bound (2.19) on powers of K in the power series expansion of G_{ev} guarantees that only a finite number of terms in that expansion is required to achieve a given precision in expansion (2.20). This property makes the stationary phase calculation of the integral (2.15) quite effective. As we will see in Sect. 4, it also determines the finite type nature of the invariants $S_n(M)$.

2.2. Knot Surgery Formula. Now we come back to the surgery formula (2.10). The substitution

$$\alpha_j = Ka_j \tag{2.22}$$

puts the factors $e^{\frac{i\pi}{2K}l_{jj}(\alpha_j^2-1)}\hat{U}^{(p_j,q_j)}_{\alpha_j 1}$ of the integrand (2.10) in the form (2.15) with $f(a_j) = \frac{\pi}{2}\left(\frac{p_j}{q_j} + l_{jj}\right)a_j^2$. It remains only to put the Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathscr{L};k)$ into a similar form. If $\mathscr{L}$ has only one component, i.e. if it is a knot $\mathscr{K}$, then this is achieved by (the first part of) the Melvin–Morton conjecture, which was proven by D. Bar-Natan and S. Garoufalidis [16] (for a simple path integral proof see [8]).

Proposition 2.1. *Let $\mathscr{K}$ be a knot in S^3. Then its framing independent colored Jones polynomial has the following expansion in powers of α and K^{-1}:*

$$J_\alpha(\mathscr{K};k) = \alpha \sum_{n=0}^{\infty} \sum_{0\leqq m \leqq \frac{n}{2}} D_{m,n}\alpha^{2m}\left(\frac{i\pi}{K}\right)^n , \tag{2.23}$$

or equivalently,

$$J_{Ka}(\mathscr{K};k) = aK\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{m,n}(i\pi a)^{2m}\left(\frac{i\pi}{K}\right)^n, a = \frac{\alpha}{K},\ d_{m,n} = D_{m,n+2m}\,, \tag{2.24}$$

$$\tilde{J}_{Ka}(\mathscr{K};k) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \tilde{d}_{m,n}(i\pi a)^{2m}\left(\frac{i\pi}{K}\right)^n, \tilde{d}_{m,0} = (2m+1)d_{m,0}\,. \tag{2.25}$$

The dominant part of the expansions (2.23) *and* (2.24) *is related to the Alexander polynomial of* $\mathcal{K}$:

$$\pi a \sum_{n=0}^{\infty} D_{n,2n}(i\pi a)^{2n} \equiv \pi a \sum_{n=0}^{\infty} d_{n,0}(i\pi a)^{2n} = \frac{\sin(\pi a)}{\Delta_A(\mathcal{K}; e^{2\pi i a})}\,. \tag{2.26}$$

The Alexander polynomial $\Delta_A(\mathcal{K}; e^{2\pi i a})$ is normalized in such a way that

$$\Delta_A\left(\text{unknot}; e^{2\pi i a}\right) = 1\,,$$

$$\Delta_A(\mathcal{K}; e^{2\pi i a}) = \frac{2i\sin(\pi a)}{\tau_R(S^3\setminus \operatorname{Tub}\mathcal{K}; e^{2\pi i a})}\,, \tag{2.27}$$

here $\tau_R\left(S^3\setminus \operatorname{Tub}\mathcal{K}; e^{2\pi i a}\right)$ is the $U(1)$ Reidemeister–Ray–Singer torsion of the knot complement $S^3\setminus \operatorname{Tub}\mathcal{K}$.

The coefficients $\tilde{d}_{m,n}$ lie above the line G in Fig. 1. The formula (2.25) demonstrates that the function $\tilde{J}_{Ka}(\mathcal{K}; k)$ is of the form $g(a, K)$, that is, it has only zero or negative powers of K in its expansion. Therefore if a RHS M is constructed by a rational (p, q) surgery on a knot $\mathcal{K} \subset S^3$ then

$$Z^{(\mathrm{tr})}(M; k) = Z(S^3; k)\, i\sqrt{\frac{K}{2}}\frac{\operatorname{sign}(q)}{\sqrt{|q|}} e^{-i\pi\frac{3}{4}\operatorname{sign}(\frac{p}{q}+l_{00})} e^{\frac{i\pi}{K}\Delta_{\mathrm{fr}}} \frac{1}{p+ql_{00}}$$

$$\times \int_{-\infty\ [a=0]}^{+\infty} da \exp\left[\frac{i\pi K}{2}\left(\frac{p}{q}+l_{00}\right)a^2\right] \tilde{J}_{Ka}(\mathcal{K}; k)\,, \tag{2.28}$$

$$\Delta_{\mathrm{fr}} = \frac{1}{2}\left[12s(p,q) - \left(\frac{p}{q}+l_{00}\right) - \frac{1}{q(p+ql_{00})} + 3\operatorname{sign}\left(\frac{p}{q}+l_{00}\right)\right]. \tag{2.29}$$

The integral is calculated similarly to the one in Eq. (2.15) by integrating the terms of expansion (2.25) one by one with the gaussian factor. The result can be expressed in terms of invariants $S_n(S^3)$ of Eq. (1.10) if we recall that $\operatorname{ord} H_1(M, \mathbb{Z}) = |p + ql_{00}|$ and that $d_{00} = \tilde{d}_{00} = 1$:

$$\sum_{n=1}^{\infty} S_n(M)\left(\frac{i\pi}{K}\right)^n = \sum_{n=1}^{\infty} S_n(S^3)\left(\frac{i\pi}{K}\right)^n + \frac{i\pi}{K}\Delta_{\mathrm{fr}} + \log\left(1 + \sum_{n=1}^{\infty}\Delta_n\left(\frac{i\pi}{K}\right)^n\right), \tag{2.30}$$

$$\Delta_n = \sum_{m=0}^{n}(-1)^m\frac{(2m)!}{2^m m!}\left(\frac{q}{p+ql_{00}}\right)^m \tilde{d}_{m,n-m}\,. \tag{2.31}$$

Equation (2.30) implies that for individual perturbative corrections

$$S_n(M) = S_n(S^3) + \delta_{n1}\Delta_{\mathrm{fr}} - \sum_{\substack{m_1,\dots,m_n \geq 0 \\ m_1+2m_2+\dots+nm_n=n}} (-1)^{m_1+\dots+m_n}\frac{(m_1+\dots+m_n-1)!}{m_1!\cdots m_n!}\Delta_1^{m_1}\cdots\Delta_n^{m_n}\,. \tag{2.32}$$

We checked in [8] that the surgery formula for S_1 which follows from these equations, is proportional to Walker's formula [17] for the Casson–Walker invariant λ_{CW} of rational homology spheres. This led us to the relation (1.11).

The coefficients $D_{m,n}$ of the Taylor expansion of the Jones polynomial of a knot $\mathcal{K} \subset S^3$ are known to be Vassiliev (i.e., finite type) invariants of order n. Therefore the coefficients $d_{m,n} = D_{m,n+2m}$ are Vassiliev invariants of order $n + 2m$. The coefficients $\tilde{d}_{m,n}$ can be expressed as linear combinations of the coefficients $d_{m+l,n-2l}$, $l \geqq 0$. Therefore $\tilde{d}_{m,n}$ is also Vassiliev invariant of order $m + 2n$.

The invariants $\Delta_n(M)$ and $S_n(M)$ of the RHS $M = \chi_{\mathcal{K}}(S^3)$ can be considered as knot invariants of $\mathcal{K}$. Equations (2.31) and (2.32) present $S_n(M)$ as a linear combination of products of the coefficients $\tilde{d}_{m_i,n_i}$ such that in each product $\sum_i (n_i + m_i) = n$. As a result, for each product of $\tilde{d}_{m_i,n_i}$ appearing in the expression of $S_n(M)$

$$\sum_i (n_i + 2m_i) \leqq 2\sum_i (n_i + m_i) = 2n\,. \tag{2.33}$$

Thus we make the following conclusion:

Proposition 2.2. (*cf.* part 1 *of Question* 3 *of* [15]). *For a rational* (p,q) *surgery on a knot* $\mathcal{K} \subset S^3$, *the coefficient* $S_n(\chi_{\mathcal{K}}(S^3))$, *considered as an invariant of* $\mathcal{K}$, *is a Vassiliev invariant of order* (*at most*) $2n$.

Note that for $l_{00} = 0$ (which can be always achieved for a knot $\mathcal{K} \subset S^3$ by a suitable choice of its framing) the dependence of Δ_n of q in Eq. (2.31) is polynomial. We will use this in order to extract the coefficients of the Alexander polynomial of $\mathcal{K}$ from the invariants $S_n(\chi_{\mathcal{K}}(S^3))$ obtained by applying surgeries with different values of q. Denote by $\chi_{\mathcal{K},(p,q)}(S^3)$ the manifold constructed by applying the (p,q) surgery to $\mathcal{K}$. Consider the surgeries $(p, q + \sum_{j=1}^n \mu_j)$, $\mu_j = \pm 1$. Since

$$\sum_{\substack{\mu_j=\pm 1\\(1\leqq j\leqq n)}} \left(\prod_{j=1}^n \mu_j\right)\left(q + \sum_{j=1}^n \mu_j\right)^m = \begin{cases} 0 & \text{for } m < n \\ 2^m\, m! & \text{for } m = n\,, \end{cases} \tag{2.34}$$

we find that

$$\sum_{\substack{\mu_j=\pm 1\\(1\leqq j\leqq n)}} \left(\prod_{j=1}^n \mu_j\right) \Delta_{n'}\left(\chi_{\mathcal{K},(p,q+\sum_{j=1}^n \mu_j)}\right)$$

$$= \begin{cases} 0 & \text{for } n' < n \\ (-1)^n (2n+1)!\, p^{-n} D_{n,2n} & \text{for } n' = n\,, \end{cases} \tag{2.35}$$

here we used the fact that $\tilde{d}_{n,0} = (2n+1)d_{n,0} = (2n+1)D_{n,2n}$. Similarly, if we substitute Eq. (2.31) into Eq. (2.32) and apply the alternating sum to both of its sides, then Eq. (2.34) leads to the following proposition.

Proposition 2.3. *For a knot $\mathcal{K} \subset S^3$ the alternating sum of invariants $S_{n'}$, $2 \leqq n' \leqq n$ over the surgeries $(p, q + \sum_{j=1}^{n} \mu_j)$, $\mu_j = \pm 1$ on $\mathcal{K}$ is given by the formula*

$$\sum_{\substack{\mu_j=\pm 1\\ (1\leqq j\leqq n)}} \left(\prod_{j=1}^{n} \mu_j\right) S_{n'}\left(\chi_{\mathcal{K},(p,q+\sum_{j=1}^{n}\mu_j)}\right) = 0 \quad \textit{for } n' < n,$$

$$\sum_{\substack{\mu_j=\pm 1\\ (1\leqq j\leqq n)}} \left(\prod_{j=1}^{n} \mu_j\right) S_{n}\left(\chi_{\mathcal{K},(p,q+\sum_{j=1}^{n}\mu_j)}\right)$$

$$= - \sum_{\substack{m_1,\dots,m_n \geqq 0\\ m_1+2m_2+\dots+nm_n=n}} (-1)^{\sum_{j=1}^{n} m_j} \left(\sum_{j=1}^{n} m_j - 1\right)! \prod_{j=1}^{n} \left(\frac{(-1)^j(2j+1)!}{m_j!\, p^j} D_{j,2j}\right)^{m_j}. \tag{2.36}$$

Here the coefficients $D_{j,2j}$ of expansion (2.23) *are Vassiliev invariants of $\mathcal{K}$ of order $2j$.*

Equation (2.26) demonstrates that $D_{j,2j}$ is a "special" Vassiliev invariant: it is expressed in terms of derivatives of the Alexander polynomial $\Delta_A(\mathcal{K}; e^{2\pi i a})$. Therefore the alternating sum $\sum_{\mu_j=\pm 1,(1\leqq j\leqq n)} (\prod_{j=1}^{n} \mu_j) S_n(\chi_{\mathcal{K},(p,q+\sum_{j=1}^{n}\mu_j)})$ is also a special Vassiliev invariant of order $2n$ (cf. part 3 of Question 3 of [15]).

One might use the relation

$$\Delta_{\mathrm{fr}} = \frac{1}{2} q \quad \text{for } l_{00} = 0,\ p = 1 \tag{2.37}$$

in order to obtain the alternating sum properties of S_1:

$$\sum_{\substack{\mu_j=\pm 1\\ (1\leqq j\leqq n)}} \left(\prod_{j=1}^{n} \mu_j\right) S_1\left(\chi_{\mathcal{K},(1,q+\sum_{j=1}^{n}\mu_j)}\right) = \begin{cases} 0 & \text{for } n > 1\,, \\ q - 6D_{1,2} & \text{for } n = 1\,. \end{cases} \tag{2.38}$$

Since $1 - 6D_{1,2}$ is proportional to the second derivative of the Alexander polynomial, Eq. (2.38) is consistent with Casson's original formula for λ_{CW} of a manifold produced by a $(1, q)$ surgery on a knot in S^3.

2.3. General Link Surgery Formula. It is hard to present the colored Jones polynomial $J_{\alpha_1,\dots,\alpha_N}(\mathcal{L}; k)$ of a general link in the form of the integrand of Eq. (2.15). The closest thing to the Melvin–Morton conjecture for a general link is Reshetikhin's formula. We proved it in [9] with the help of quantum field theory arguments and Feynman diagrams. The proof of [9] can be made rigorous if one uses Kontsevich's integral (see [26] and the Appendix of [25] for details).

Proposition 2.4. *Let $\mathcal{L}$ be an N-component link in S^3. Its framing independent colored Jones polynomial can be expressed as an integral over 3-dimensional vectors $\vec{a}_j$ of the fixed length (i.e. over the co-adjoint orbits associated with*

α_j-dimensional representations of $SU(2)$ which are assigned to link components):

$$J^{(\mathrm{tr})}_{\alpha_1,\ldots,\alpha_N}(M,\mathscr{L};k) = \int_{|\vec{a}_j|=\frac{\alpha_j}{K}} \prod_{j=1}^{N} \left(\frac{K}{4\pi}\frac{d^2\vec{a}_j}{|\vec{a}_j|}\right) \exp\left(\frac{i\pi K}{2}\sum_{m=2}^{\infty} L_m(\vec{a}_1,\ldots,\vec{a}_N)\right)$$

$$\times \left[1+\sum_{\substack{l,m=0\\ l+m\neq 0}}^{\infty} K^{-l}P_{m,l}(\vec{a}_1,\ldots,\vec{a}_N)\right], \tag{2.39}$$

here $L_m(\vec{a}_1,\ldots,\vec{a}_N)$ and $P_{m,l}(\vec{a}_1,\ldots,\vec{a}_N)$ are homogeneous $SO(3)$-invariant polynomials of degree m. In particular,

$$L_2(\vec{a}_1,\ldots,\vec{a}_N) = 2\sum_{0\leqq i<j\leqq N} l_{ij}\vec{a}_i\cdot\vec{a}_j\,, \tag{2.40}$$

l_{ij} are the linking numbers of $\mathscr{L}$.

The analog of the second part of the Melvin–Morton conjecture is the relation [9] between the polynomials L_m, $P_{m,l}$ and the multicolored Alexander polynomial of the link $\mathscr{L}$. Here we *define* the multicolored Alexander polynomial of a link as the inverse of the Reidemeister–Ray–Singer torsion of its complement:

$$\tilde{\Delta}_A(\mathscr{L};e^{2\pi i a_1},\ldots,e^{2\pi i a_N}) = \frac{1}{\tau_R(S^3\backslash \mathrm{Tub}\mathscr{L};e^{2\pi i a_1},\ldots,e^{2\pi i a_N})}\,. \tag{2.41}$$

It is simply related to the usual Alexander polynomial of the link (cf. Eq. (2.27)):

$$\Delta_A(\mathscr{L};e^{2\pi i a}) = 2i\sin(\pi a)\tilde{\Delta}_A(\mathscr{L};e^{2\pi i a},\ldots,e^{2\pi i a})\,. \tag{2.42}$$

The following conjecture was proved in [9] with the help of path integral arguments:

Conjecture 2.1. *The multicolored Alexander polynomial* (2.41) *can be expressed in terms of polynomials L_m and $P_{m,l}$:*

$$\tilde{\Delta}_A(\mathscr{L};e^{2\pi i a_1},\ldots,e^{2\pi i a_N}) = -ie^{-\frac{i\pi}{4}\,\mathrm{sign}(M_{ij,\mu\nu})}(2\pi)^{N-2}\frac{|\det M''|^{\frac{1}{2}}}{\prod_{j=1}^{N} a_j}$$

$$\times\left[1+\sum_{m=2}^{\infty} P_{m,0}(a_1\vec{n},\ldots,a_N\vec{n})\right]^{-1}, \tag{2.43}$$

here $\vec{n}$ is a unit vector. A symmetric $2N\times 2N$ matrix $M_{ij,\mu\nu}$, $1\leqq i,j\leqq N$, $\mu,\nu = 1,2$ comes from the quadratic form

$$\sum_{i,j=1}^{N}\sum_{\mu,\nu=1,2} M_{ij,\mu\nu}(a_1,\ldots,a_N)x_\mu^{(i)}x_\nu^{(j)}\,, \tag{2.44}$$

which is extracted from the exponent of Eq. (2.39),

$$\sum_{m=2}^{\infty} L_m(\vec{a}_1,\ldots,\vec{a}_N)\,, \tag{2.45}$$

after performing a substitution

$$\vec{a}_j = a_j \left(\vec{n} + \vec{x}_j - \frac{1}{2} \vec{n}(\vec{x}_j^2) \right), \qquad \vec{x}_j \cdot \vec{n} = 0 \,. \tag{2.46}$$

Here $x_\mu^{(j)}$ are coordinates of the vector $\vec{x}_j$ which is orthogonal to $\vec{n}$. M'' is a $(N-1)\times(N-1)$ matrix obtained from $M_{ij,\mu\nu}$ by crossing out two columns and two rows to which diagonal elements $M_{ii,11}$ and $M_{jj,22}$ belong ($\det M''$ *does not depend on the choice of i and j*).

The polynomials $L_m(\vec{a}_1,\ldots,\vec{a}_N)$ appear to be related to Milnor's linking numbers $l^{(\mu)}_{j_1,\ldots,j_m}$ of the link $\mathcal{L}$. If the order m Milnor's linking numbers are well defined, then, based on path integral arguments, we conjectured in [9] that

$$L_m(\vec{a}_1,\ldots,\vec{a}_N) = \frac{(i\pi)^{m-2}}{m} \sum_{1\leq j_1,\ldots,j_m\leq N} l^{(\mu)}_{j_1,\ldots,j_m} \operatorname{Tr}(\vec{\sigma}\cdot\vec{a}_{j_1})\cdots(\vec{\sigma}\cdot\vec{a}_{j_m}) \,, \tag{2.47}$$

here $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ is a 3d vector of Pauli matrices. In particular,

$$L_3(\vec{a}_1,\ldots,\vec{a}_N) = -\frac{2\pi}{3} \sum_{i,j,k=1}^{N} l^{(\mu)}_{ijk} \vec{a}_i\cdot(\vec{a}_j\times\vec{a}_k) \,, \tag{2.48}$$

$$L_4(\vec{a}_1,\ldots,\vec{a}_N) = \frac{\pi^2}{3} \sum_{i,j,k,l=1}^{N} \left(l^{(\mu)}_{ijkl} - l^{(\mu)}_{kijl} \right)(\vec{a}_i\times\vec{a}_j)\cdot(\vec{a}_k\times\vec{a}_l) \,. \tag{2.49}$$

The proof of the latter relations (2.48) and (2.49) in [9] can be made rigorous by the use of Kontsevich's integral (see [26] and the Appendix of [25] for details).

In our future calculations we will also need the polynomial

$$P_{2,0}(\vec{a}_1,\ldots,\vec{a}_N) = \sum_{i,j=1}^{N} p_{ij} \vec{a}_i\cdot\vec{a}_j \,. \tag{2.50}$$

A combination of integrals over $\alpha_j = K a_j$ in Eq. (2.10) with the integrals over directions of vectors $\vec{a}_j$ in Eq. (2.39) produces the link surgery formula which serves as the basic definition of perturbative invariants (see [25] and references therein for details):

Definition 2.2. *If a RHS M is constructed by rational (p_j, q_j) surgeries on an N-component link $\mathcal{L}\subset S^3$, then*

$$Z^{(\mathrm{tr})}(M;k) = Z(S^3;k)(2K^3)^{\frac{N}{2}} \left(\prod_{j=1}^{N} \frac{\operatorname{sign}(q_j)}{\sqrt{|q_j|}} \right) e^{-i\pi\frac{3}{4}\operatorname{sign}(L^{(\mathrm{tot})})}$$

$$\times \exp\left[\frac{i\pi}{2K} \left(3\operatorname{sign}\left(L^{(\mathrm{tot})}\right) + \sum_{j=1}^{N} \left(12 s(p_j,q_j) - \frac{p_j}{q_j} - l_{jj} \right) \right) \right]$$

$$\times \int_{\substack{-\infty \\ [\vec{a}_j=0]}}^{+\infty} \left(\prod_{j=1}^{N} \frac{d^3\vec{a}_j}{4\pi} \right) \exp\left[\frac{i\pi K}{2} \left(\sum_{i,j=1}^{N} L^{(\mathrm{tot})}_{ij} \vec{a}_i\cdot\vec{a}_j \right. \right.$$

$$+\sum_{m=3}^{\infty} L_m(\vec{a}_1,\ldots,\vec{a}_N)\Bigg)\Bigg]$$

$$\times\left(\prod_{j=1}^{N}\frac{\sin\left(\pi\frac{|\vec{a}_j|}{q_j}\right)}{|\vec{a}_j|}\right)\left[1+\sum_{\substack{l,m=0\\ l+m\neq 0}}^{\infty} K^{-l}P_{m,l}(\vec{a}_1,\ldots,\vec{a}_N)\right]. \tag{2.51}$$

By switching from integration over a_j (Cartan subalgebra) to $\vec{a}_j$ (the whole Lie algebra) we managed to put the surgery formula in the recognizable stationary phase form (2.15). However we paid a heavy price: the invariants $S_n(M)$ are no longer expressed in terms of derivatives of the original colored Jones polynomial $J_{\alpha_1,\ldots,\alpha_N}(\mathscr{L};k)$, as was the case for Eqs. (2.30), (2.31). Instead we first have to present the polynomial in the form (2.39) in order to use the coefficients of polynomials L_m, $P_{m,l}$ in actual computation of the integral (2.28). This is a big disadvantage of Eq. (2.28) since we do not know of any effective way to find the polynomials L_m, $P_{m,l}$ of a link (most of them are not even unambiguously defined by Eq. (2.39), see [9] for details). This problem can be circumvented to a certain degree by the "step-by-step" procedure of [25]. Moreover, as we will see in Sect. 3, for some special classes of links it is possible to go back from Eq. (2.51) to the formula similar to Eqs. (2.30), (2.31).

3. Special Links

Now we will concentrate on studying the Jones polynomials and surgery formulas of some special classes of links.

Definition 3.1. *An N-component link $\mathscr{L}\subset S^3$ RHS is an algebraically split link (ASL) if linking numbers between its components are zero:*

$$l_{ij}=0,\quad 1\leqq i<j\leqq N. \tag{3.1}$$

A link is a special algebraically split link (SASL) if in addition to (3.1) *all of its triple Milnor's linking numbers are zero:*

$$l_{ijk}^{(\mu)}=0,\quad 1\leqq i,j,k\leqq N. \tag{3.2}$$

A link is a boundary link (BL) if one can choose Seifert surfaces for its components in such a way that they do not intersect.

Note that all Milnor's linking numbers of a boundary link are zero.

Algebraically split links in relation to Witten's invariant were studied by H. Murakami and T. Ohtsuki [18–21]. In particular, they showed that any integer homology sphere can be constructed by integer surgeries $(1,q_j)$ on an ASL in S^3. They also proved that any rational homology sphere can be constructed by rational surgeries $(p_j,1)$ on an ASL up to a connected sum of lens spaces. This means that instead of a desired RHS M we may end up with a connected sum $M\#L_{p_1',1}\#\cdots\#L_{p_n',1}$. This suits our purposes since $Z^{(\mathrm{tr})}(M;k)$ has a simple behavior under connected sum:

$$Z^{(\mathrm{tr})}(M_1\#M_2;k)=\frac{Z^{(\mathrm{tr})}(M_1;k)Z^{(\mathrm{tr})}(M_2;k)}{Z(S^3;k)}\,, \tag{3.3}$$

while the trivial connection contribution to Witten's invariant of a lens space was calculated by L. Jeffrey [3]:

$$Z^{(\mathrm{tr})}(L_{p,q};K)=\sqrt{\frac{2}{K|p|}}\sin\left(\frac{\pi}{K|p|}\right)e^{-\frac{6\pi i}{K}s(q,p)}\,. \tag{3.4}$$

3.1. The Jones Polynomial of Special Links. We are going to calculate the expansion of $J_{\alpha_1,\ldots,\alpha_N}(\mathscr{L};k)$ in powers of α_j (or equivalently, in powers of $a_j=\alpha_j/K$) and K^{-1} with the help of Reshetikhin's formula (2.39). We expand the exponential of that formula in powers of polynomials L_m and then integrate over the directions of vectors $\vec{a}_j$ according to the following formulas:

$$\int_{|\vec{a}|=a}\frac{d^2\vec{a}}{4\pi|\vec{a}|}a_{\mu_1}\cdots a_{\mu_{2n+1}}=0\,, \tag{3.5}$$

$$\int_{|\vec{a}|=a}\frac{d^2\vec{a}}{4\pi|\vec{a}|}a_{\mu_1}\cdots a_{\mu_{2n}}=\frac{a^{2n+1}}{(2n+1)!}\sum_{s\in S_{2n}}\delta_{\mu_{s(1)}\mu_{s(2)}}\cdots\delta_{\mu_{s(2n-1)}\mu_{s(2n)}}\,, \tag{3.6}$$

here S_{2n} is a symmetry group of $2n$ elements.

For an ASL, $L_2=0$. As a result, each positive power of K coming from the expansion of the exponent of Eq. (2.39) carries with it at least three powers of phases a_j. For a SASL, $L_2=L_3=0$. Therefore each power of K carries at least four powers of phases a_j. Finally, for a BL all $L_m=0$, so the exponential is trivial and the power series expansion of its Jones polynomial contains only negative powers of K (apart from the overall pre-factor K^N). A "slope index" si($\mathscr{L}$) defined for special links as

$$\mathrm{si}(\mathscr{L})=\begin{cases}\frac{2}{3} & \text{for ASL}\\ \frac{1}{2} & \text{for SASL}\\ 0 & \text{for BL}\end{cases} \tag{3.7}$$

allows us to formulate these results as a universal formula. It is the analog of Melvin–Morton bound for knots.

Proposition 3.1. *The trivial connection contribution to the Jones polynomial of a special link $\mathscr{L}\subset S^3$ has the following expansion in powers of K^{-1} and $a_j=\alpha_j/K$:*

$$\begin{aligned}J_{\alpha_1,\ldots,\alpha_N}(\mathscr{L};k)&=\left(\prod_{j=1}^N\alpha_j\right)\sum_{n=0}^{\infty}\sum_{m\leq\frac{n}{2-\mathrm{si}(\mathscr{L})}}D_{m,n}(\alpha_1,\ldots,\alpha_N)\left(\frac{i\pi}{K}\right)^n\\&=K^N\left(\prod_{j=1}^N a_j\right)\sum_{m\geq0}\sum_{n\geq-\mathrm{si}(\mathscr{L})m}d_{m,n}(a_1,\ldots,a_N)(i\pi)^{2m+n}K^{-n}\,,\end{aligned} \tag{3.8}$$

while for the "shifted" Jones polynomial defined by Eq. (2.12),

$$\tilde{J}_{\alpha_1,\ldots,\alpha_N}(\mathscr{L};k)=\sum_{m\geqq 0}\ \sum_{n\geqq -\mathrm{si}(\mathscr{L})m}\tilde{d}_{m,n}(a_1,\ldots,a_N)(i\pi)^{2m+n}K^{-n}\,. \tag{3.9}$$

In these formulas $D_{m,n}$, $d_{m,n}$ and $\tilde{d}_{m,n}$ are even homogeneous polynomials of degree $2m$:

$$D_{m,n}(\alpha_1,\ldots,\alpha_N)=\sum_{\substack{m_1,\ldots,m_N\geqq 0\\ m_1+\cdots+m_N=m}} D^{(m,n)}_{m_1,\ldots,m_n}\alpha_1^{2m_1}\cdots\alpha_N^{2m_N}\,, \tag{3.10}$$

$$d_{m,n}(a_1,\ldots,a_N)=\sum_{\substack{m_1,\ldots,m_N\geqq 0\\ m_1+\cdots+m_N=m}} d^{(m,n)}_{m_1,\ldots,m_N}a_1^{2m_1}\cdots a_N^{2m_N}\,, \tag{3.11}$$

$$\tilde{d}_{m,n}(a_1,\ldots,a_N)=\sum_{\substack{m_1,\ldots,m_N\geqq 0\\ m_1+\cdots+m_N=m}} \tilde{d}^{(m,n)}_{m_1,\ldots,m_N}a_1^{2m_1}\cdots a_N^{2m_N}\,, \tag{3.12}$$

and $d^{(m,n)}_{m_1,\ldots,m_N}=D^{(m,2m+n)}_{m_1,\ldots,m_n}$.

The bounding lines for the polynomials $d_{m,n}$ and $\tilde{d}_{m,n}$ in Fig. 1 are O for ASL, O' for SASL and G for BL. Note that for ASL the polynomials of critical degrees $d_{3m,-2m}$, $\tilde{d}_{3m,-2m}$ come exclusively from the polynomial L_3 of Eq. (2.48), i.e. from the triple Milnor's linking numbers $l^{(\mu)}_{ijk}$, while the critical degree polynomials for SASL $d_{2m,-m}$, $\tilde{d}_{2m,-m}$ come exclusively from the polynomial L_4 of Eq. (2.49), i.e. from quartic Milnor's linking numbers $l^{(\mu)}_{ijkl}$.

We will need the polynomials $\tilde{d}_{1,0}$, $\tilde{d}_{2,-1}$ and $\tilde{d}_{3,-2}$ for the surgery formula for $S_1(M)$, therefore we are going to express them explicitly in terms of the polynomials (2.48)–(2.50). The polynomial $\tilde{d}_{0,1}$ comes from the polynomial $P_{2,0}$ in the preexponential factor of Eq. (2.39):

$$\tilde{d}_{1,0}=3d_{1,0}=-\frac{3}{\pi^2}\sum_{j=1}^{N}p_{jj}a_j^2\,. \tag{3.13}$$

The polynomial $\tilde{d}_{2,-1}$ comes from averaging the linear term in the expansion of the exponential of the 2-color part of L_4,

$$-2i\pi^3K\sum_{1\leqq i<j\leqq N} l^{(\mu)}_{iijj}(\vec{a}_i\times\vec{a}_j)\cdot(\vec{a}_i\times\vec{a}_j) \tag{3.14}$$

over the directions of $\vec{a}_i$ and $\vec{a}_j$:

$$\tilde{d}_{2,-1}=9d_{2,-1}=12\sum_{1\leqq i<j\leqq N} l^{(\mu)}_{iijj}a_i^2a_j^2\,. \tag{3.15}$$

The polynomial $\tilde{d}_{3,-2}$ comes from averaging the quadratic term of the expansion of the exponent of L_3:

$$\tilde{d}_{3,-2}=27d_{3,-2}=-12\sum_{1\leqq i<j<k\leqq N}\left(l^{(\mu)}_{ijk}\right)^2a_i^2a_j^2a_k^2\,. \tag{3.16}$$

The coefficients $p_{jj}, l^{(\mu)}_{iijj}$ and $\left(l^{(\mu)}_{ijk}\right)^2$ also appear as derivatives of the Alexander polynomials of 1-, 2- and 3-component sublinks of $\mathscr{L}$. To see this, we recall [9] the procedure of removing a link component from Eq. (2.39). To remove a particular component $\mathscr{L}_i$ of $\mathscr{L}$ we have to substitute

$$\alpha_i \equiv K a_i = 1 \tag{3.17}$$

and integrate over the directions of $\vec{a}_i$. We do this by expanding the exponential of Eq. (2.39) in all the monomials of polynomials L_m which depend on $\vec{a}_i$. As a result of the substitution (3.18), these monomials become of order K^{-1} or less. Therefore this procedure preserves the overall structure of Eq. (2.39) and leads directly to the representation (2.39) of the Jones polynomial of $\mathscr{L}\backslash\mathscr{L}_i$. We see that the coefficients of monomials of L_m, which are independent of $\vec{a}_i$, do not change. Since $L_2 = 0$, the monomials of $P_{2,0}$ which do not contain $\vec{a}_i$, are not modified either.

Suppose that we remove all components of $\mathscr{L}$ except $\mathscr{L}_j$. Then according to the Melvin–Morton conjecture (2.26), the Alexander polynomial of $\mathscr{L}_j$ in the standard normalization (2.27) has the following power series expansion:

$$\Delta_A(M, \mathscr{L}_j; e^{2\pi i a_j}) = 1 - a_j^2\left(p_{jj} + \frac{\pi^2}{6}\right) + O(a_j^4) = 1 + \frac{z^2}{4\pi^2}\left(p_{jj} + \frac{\pi^2}{6}\right) + O(z^4)\,, \tag{3.18}$$

here we used the standard variable z for the Alexander polynomial:

$$z = -2i\sin(\pi a)\,. \tag{3.19}$$

If following J. Hoste [12] we denote by $\phi_1(\mathscr{L})$ a coefficient in front of $z^{\#\mathscr{L}+1}$ in the power series expansion of the single-colored Alexander polynomial $\Delta_A(M, \mathscr{K}; e^{2\pi i a})$ ($\#\mathscr{L}$ is the number of components of $\mathscr{L}$), then according to Eq. (3.18),

$$\phi(\mathscr{L}_j) = \frac{1}{4\pi^2}\left(p_{jj} + \frac{\pi^2}{6}\right)\,. \tag{3.20}$$

The relation between the coefficients $l^{(\mu)}_{iijj}$, $\left(l^{(\mu)}_{ijk}\right)^2$ and the derivatives of the Alexander polynomial of 2- and 3-component sublinks of $\mathscr{L}$ were established by T. Cochran [22]. Here we show how to derive the same relations from Conjecture 2.1.

Suppose that we remove all components of $\mathscr{L}$, except for $\mathscr{L}_i$ and $\mathscr{L}_j$. Since $l_{ij} = 0$, then according to Eqs. (2.43) and (2.42) the power series expansion of the single-colored Alexander polynomial starts with the term

$$\Delta_A(M, \mathscr{L}_i, \mathscr{L}_j; e^{2\pi i a}) = 8i\pi^3 l^{(\mu)}_{iijj} a^3 + O(a^5) = l^{(\mu)}_{iijj} z^3 + O(z^5)\,, \tag{3.21}$$

so that

$$\phi_1(\mathscr{L}_i, \mathscr{L}_j) = l^{(\mu)}_{iijj}\,. \tag{3.22}$$

Finally for a 3-component sublink $\mathcal{L}_i \cup \mathcal{L}_j \cup \mathcal{L}_k$ of $\mathcal{L}$

$$\Delta_A(M, \mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k; e^{2\pi i a}) = 16\pi^4 \big(l_{ijk}^{(\mu)}\big)^2 a^4 + O(a^6) = \big(l_{ijk}^{(\mu)}\big)^2 z^4 + O(z^6)\,, \tag{3.23}$$

and

$$\phi_1(\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k) = \big(l_{ijk}^{(\mu)}\big)^2\,. \tag{3.24}$$

A combination of Eqs. (3.13), (3.15), (3.16) with Eqs. (3.20), (3.22), (3.24) leads to the following relations between the derivatives of the shifted Jones polynomial and the derivatives of the Alexander polynomials of sublinks:

$$\tilde{d}_{1,0} = -12 \sum_{j=1}^{N} \left(\phi_1(\mathcal{L}_j) - \frac{1}{24}\right) a_j^2\,, \tag{3.25}$$

$$\tilde{d}_{2,-1} = 12 \sum_{1 \leq i < j \leq N} \phi_1(\mathcal{L}_i, \mathcal{L}_j) a_i^2 a_j^2\,, \tag{3.26}$$

$$\tilde{d}_{3,-2} = -12 \sum_{1 \leq i < j < k \leq N} \phi_1(\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k) a_i^2 a_j^2 a_k^2\,. \tag{3.27}$$

3.2. A Surgery Formula for Special Links. Proposition 3.1 guarantees that for all three classes of special links – ASL, SASL and BL – the expansion of the (shifted) Jones polynomial is similar to that of function (2.17). As a result, we can calculate the stationary phase contribution of the point $a_j = 0$ to the integral in the surgery formula

$$Z^{(\mathrm{tr})}(M;k) = Z(S^3;k) i^N \left(\frac{K}{2}\right)^{\frac{N}{2}} \left(\prod_{j=1}^{N} \frac{\mathrm{sign}(q_j)}{\sqrt{|q_j|}}\right) e^{-i\pi\frac{3}{4}\sum_{j=1}^{N} \mathrm{sign}\left(\frac{p_j}{q_j}+l_{jj}\right)} e^{\frac{i\pi}{K}\Delta_{\mathrm{fr}}}$$
$$\times \frac{1}{\prod_{j=1}^{N}(p_j + q_j l_{jj})} \int_{\substack{-\infty \\ [a_j=0]}}^{+\infty} da_1 \cdots da_N$$
$$\times \exp\left[\frac{i\pi K}{2} \sum_{j=1}^{N} \left(\frac{p_j}{q_j} + l_{jj}\right) a_j^2\right] \tilde{J}_{Ka_1,\dots,Ka_N}(\mathcal{L};k)\,, \tag{3.28}$$

$$\Delta_{\mathrm{fr}} = \frac{1}{2} \sum_{j=1}^{N} \left[12 s(p_j, q_j) - \left(\frac{p_j}{q_j} + l_{jj}\right) - \frac{1}{q_j(p_j + q_j l_{jj})} + 3\,\mathrm{sign}\left(\frac{p_j}{q_j} + l_{jj}\right)\right] \tag{3.29}$$

by substituting the expansion (3.9) and integrating term by term.

Proposition 3.2. *Let M be a RHS obtained by (p_j, q_j) rational surgeries on components of an N-component special (ASL, SASL or BL) link $\mathcal{L} \subset S^3$. Then the* *invariants* $S_n(M)$ *and* $S_n(S^3)$ *are related by the following equation*:

$$\sum_{n=1}^{\infty} S_n(M)\left(\frac{i\pi}{K}\right)^n = \sum_{n=1}^{\infty} S_n(S^3)\left(\frac{i\pi}{K}\right)^n + \frac{i\pi}{K}\Delta_{\mathrm{fr}} + \log\left(1+\sum_{n=1}^{\infty}\Delta_n\left(\frac{i\pi}{K}\right)^n\right), \tag{3.30}$$

$$\Delta_n = \sum_{m=0}^{\overline{\frac{n}{1-\mathrm{si}(\mathscr{L})}}} \frac{(-1)^m}{2^m} \sum_{\substack{m_1,\dots,m_N \geqq 0\\ m_1+\dots+m_N=m}} \tilde{d}^{(m,n-m)}_{m_1,\dots,m_N} \prod_{j=1}^{N} \frac{(2m_j)!}{m_j!}\left(\frac{q_j}{p_j+q_j l_{jj}}\right)^{m_j}. \tag{3.31}$$

The individual invariants are related by the formula

$$S_n(M) = S_n(S^3) + \delta_{n1}\Delta_{\mathrm{fr}} - \sum_{\substack{m_1,\dots,m_n \geqq 0\\ m_1+2m_2+\dots+nm_n=n}} (-1)^{m_1+\dots+m_n}\frac{(m_1+\dots+m_n-1)!}{m_1!\cdots m_n!}\Delta_1^{m_1}\cdots\Delta_n^{m_n}. \tag{3.32}$$

The case of S_1 is especially interesting since we demonstrated in [8] that its knot surgery formula coincides with K. Walker's formula for Casson's invariant λ_{CW} of RHS if we set Eq. (1.11). It is obvious from Eq. (3.32) that for S_1 one needs to know only Δ_1, which for a general ASL is expressed by Eq. (3.31) in terms of $\tilde{d}_{1,0}, \tilde{d}_{2,-1}$ and $\tilde{d}_{3,-2}$ (note that $\tilde{d}_{0,1} = d_{0,1} = 0$, this follows from the condition $J_{1,\dots,1}(\mathscr{L};k) = 1$). Thus combining Eqs. (3.25)–(3.27) with Eq. (3.31) we find that

$$S_1(M) = \Delta_{\mathrm{fr}} + 12\left(\sum_{j=1}^{N}\frac{\phi_1(\mathscr{L}_j)-\frac{1}{24}}{\frac{p_j}{q_j}+l_{jj}} + \sum_{1\leqq i<j\leqq N}\frac{\phi_1(\mathscr{L}_i,\mathscr{L}_j)}{(\frac{p_i}{q_i}+l_{ii})(\frac{p_j}{q_j}+l_{jj})} + \sum_{1\leqq i<j<k\leqq N}\frac{\phi_1(\mathscr{L}_i,\mathscr{L}_j,\mathscr{L}_k)}{(\frac{p_i}{q_i}+l_{ii})(\frac{p_j}{q_j}+l_{jj})(\frac{p_k}{q_k}+l_{kk})}\right). \tag{3.33}$$

If we recall the Dedekind sum identity

$$s(p,q)+s(q,p) = \frac{p^2+q^2+1}{12pq} - \frac{1}{4}\mathrm{sign}(pq), \tag{3.34}$$

then it is not hard to check that for an integer surgery (i.e. when $p_j = 1$, $l_{jj} = 0$) the substitution (1.11) transforms Eq. (3.33) into J. Hoste's formula [12] for Casson's invariant

$$\lambda_{\mathrm{CW}}(M) = 2\left(\sum_{j=1}^{N} q_j\phi_1(\mathscr{L}_j) + \sum_{1\leqq i<j\leqq N} q_iq_j\phi_1(\mathscr{L}_i,\mathscr{L}_j) + \sum_{1\leqq i<j<k\leqq N} q_iq_jq_k\phi_1(\mathscr{L}_i,\mathscr{L}_j,\mathscr{L}_k)\right). \tag{3.35}$$

This is another confirmation of the general relation (1.11).

3.3. Integer Valued Invariants. The surgery formulas (3.30)–(3.32) suggest that the invariants $S_n(M)$ are rational numbers. In fact, we can convert them into integers by multiplying them by factors that depend only on n and $\operatorname{ord} H_1(M, \mathbb{Z})$. We are going to present a rather rough estimate of the necessary factors.

Let M be a RHS constructed by rational (p_j, q_j) surgeries on an ASL $\mathcal{L}$ in S^3. The framing independent colored Jones polynomial $J_{\alpha_1,\dots,\alpha_N}(S^3, \mathcal{L}; k)$ has integer coefficients in front of the positive and negative powers of $e^{\frac{i\pi}{K}}$. We will expand the polynomial $J_{\alpha_1,\dots,\alpha_N}(S^3, \mathcal{L}; k)$ in powers of K in two steps. First, we introduce a variable

$$x = e^{\frac{i\pi}{K}} - 1 = \left(\frac{i\pi}{K}\right) \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{i\pi}{K}\right)^n . \tag{3.36}$$

Since

$$e^{-\frac{i\pi}{K}} = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n , \tag{3.37}$$

we see that both variables $e^{\frac{i\pi}{K}}$ and $e^{-\frac{i\pi}{K}}$ have integer coefficient expansions in powers of x. Then a simple relation (3.36) between x and $(\frac{i\pi}{K})$ together with Eq. (3.8) imply the following expansion of the Jones polynomial:

$$J_{\alpha_1,\dots,\alpha_N}(\mathcal{L}; k) = \left(\prod_{j=1}^{N} \alpha_j\right) \sum_{n=0}^{\infty} \sum_{m \leq \frac{3}{4}n} \tilde{D}_{m,n}(\alpha_1, \dots, \alpha_N) x^n , \tag{3.38}$$

here $\tilde{D}_{m,n}(\alpha_1, \dots, \alpha_N)$ are homogeneous polynomials of degree $2m$:

$$\tilde{D}_{m,n} = \sum_{\substack{m_1,\dots,m_N \geq 0 \\ m_1 + \cdots + m_N = m}} \tilde{D}^{(m,n)}_{m_1,\dots,m_n} \alpha_1^{2m_1} \cdots \alpha_N^{2m_N} . \tag{3.39}$$

The polynomials

$$\left(\prod_{j=1}^{N} \alpha_j\right) \sum_{m \leq \frac{3}{4}n} \tilde{D}_{m,n}(\alpha_1, \dots, \alpha_N) \tag{3.40}$$

are odd in all their variables. They also have integer values when all the variables $\alpha_1, \dots, \alpha_N$ are integer. This means that the polynomials (3.40) can be presented as sums of products of elementary binomial polynomials of odd degree:

$$P_{2m+1}(\alpha) = \frac{\alpha(\alpha^2 - 1)(\alpha^2 - 4) \cdots (\alpha^2 - m^2)}{(2m+1)!} . \tag{3.41}$$

In other words,

$$\begin{aligned} &\left(\prod_{j=1}^{N} \alpha_j\right) \sum_{m \leq \frac{3}{4}n} \tilde{D}_{m,n}(\alpha_1, \dots, \alpha_N) \\ &= \sum_{\substack{0 \leq m_j \leq M_j \\ (1 \leq j \leq N)}} C^{(n)}_{m_1,\dots,m_N} P_{2m_1+1}(\alpha_1) \cdots P_{2m_N+1}(\alpha_N), \quad C_{m_1,\dots,m_N} \in \mathbb{Z} . \end{aligned} \tag{3.42}$$

Here the numbers M_j are the maximum values of the powers m_j appearing in Eq. (3.39) for all $m \leqq \frac{3}{4}n$. Since $\sum_{j=1}^N m_j = m$, then the number of $m_j \neq 0$ in each term of Eq. (3.42) is not greater than m. Therefore $\sum_{1 \leqq j \leqq N, m_j \neq 0}(2m_j + 1) \leqq 3m$ and $\frac{(3m)!}{\prod_{j=1}^N (2m_j+1)!} \in \mathbb{Z}$. Taking into account that the coefficients of the polynomial $\prod_{j=1}^N ((2m_j + 1)!\, P_{2m_j+1}(\alpha))$ are integer, we conclude that

$$(3m)!\, \tilde{D}^{(m,n)}_{m_1,\dots,m_n} \in \mathbb{Z}\,. \tag{3.43}$$

The estimate (3.43) does not mean that prime divisors of the denominator of $\tilde{D}^{(m,n)}_{m_1,\dots,m_n}$ can go as high as $3m$. Indeed, the source of denominators is $(2m+1)!$ in Eq. (3.41). Since $M_j \leqq m$, we see that prime divisors of the denominator of $\tilde{D}^{(m,n)}_{m_1,\dots,m_n}$ are less than $2m+1$. Another estimate can be obtained with the help of the following.

Proposition 3.3. *For the coefficients* $d^{(m,n)}_{m_1,\dots,m_N}$ *participating in the power series expansion* (3.11), *the indices* m_j *can not be bigger than* $m+n$:

$$m_j \leqq m + n, \quad 0 \leqq j \leqq N\,. \tag{3.44}$$

Recall that n can be negative. In that case the bound (3.44) is stronger than an obvious relation $m_j \leqq m$.

Proposition 3.3 follows easily from Proposition 2.3 of [9], which states that the power of any vector $\vec{a}_j$ in a polynomial $L_m(\vec{a}_1, \dots, \vec{a}_N)$ of Reshetikhin's formula (2.39) is not greater than $m-2$. A calculation of the integrals in Eq. (2.39) leads to the inequality (3.44).

Adapting Proposition 3.36 to the coefficients $\tilde{D}^{(m,n)}_{m_1,\dots,m_n}$ we come to the following simple corollary:

Corollary 3.1. *For the coefficients* $\tilde{D}^{(m,n)}_{m_1,\dots,m_n}$ *of the polynomials* $\tilde{D}_{m,n}$, *which participate in the expansion* (3.38), *each index* m_j *can not be bigger than* $n-m$:

$$m_j \leqq n - m, \quad 1 \leqq j \leqq N\,. \tag{3.45}$$

Note that $m_j \leqq m$ and therefore $m_j \leqq \frac{n}{2}$.

A combination of this corollary with Eq. (3.42) leads to the following:

Proposition 3.4. *For the coefficients* $C^{(n)}_{m_1,\dots,m_N}$ *of Eq.* (3.42) *there is an upper bound on the maximum value of individual indices*

$$m_j \leqq n - \sum_{i=1}^N m_i\,. \tag{3.46}$$

Suppose that there exists a coefficient $C^{(n)}_{m_1,\dots,m_N}$ for which the inequality (3.46) is not true, say, for m_1. Then the highest degree monomial of the corresponding polynomial $C^{(n)}_{m_1,\dots,m_N} P_{2m_1+1}(\alpha_1) \cdots P_{2m_N+1}(\alpha_N)$ violates the inequality (3.45). Therefore it has to be canceled by monomials of other polynomials $C^{(n)}_{m_1',\dots,m_N'} P_{2m_1'+1}(\alpha_1) \cdots P_{2m_N'+1}(\alpha_N)$ for which apparently $m_j' \geqq m_j$, $1 \leqq j \leqq N$ and $\sum_{j=1}^N m_j' > \sum_{j=1}^N m_j$. But the index m_1' of these monomials again violates the inequality (3.46), so we need to go to higher values of $\sum_{j=1}^N m_j$ for new cancellations. Since $\sum_{j=1}^N m_j \leqq n$, this process can not be completed. The contradiction proves the proposition.

The inequality (3.46) indicates that the prime divisors of denominators of the coefficients of the polynomial in the r.h.s. of Eq. (3.42) can not be bigger than $2(n - \sum_{j=1}^N m_j) + 1$.

In order to find the polynomials $D_{m,n}(\alpha_1, \ldots, \alpha_N)$ of the expansion (3.8) we substitute the relation (3.36) between x and K into Eq. (3.38). The contributions to the polynomial $D_{m,n}$ come from the polynomials $\tilde{D}_{m,n-l}$, $l \geqq 0$:

$$D_{m,n} = \sum_{0 \leqq l \leqq n - \frac{4}{3}m} C_l \tilde{D}_{m,n-l} \,. \tag{3.47}$$

The numbers C_l are rational, their denominators come from the denominator $(n+1)!$ of Eq. (3.36). It is easy to see that C_l has a common denominator $(2l)!$. As a result,

$$\left[2n - \frac{8}{3}m\right]! \, (3m)! \, D^{(m,n)}_{m_1,\ldots,m_n} \in \mathbb{Z} \,. \tag{3.48}$$

Here $[x]$ is the integer part of x. The polynomials $d_{m,n}$ come from $D_{m,n+2m}$: $d_{m,n} = D^{(m,n+2m)}_{m_1,\ldots,m_N}$. Therefore

$$\left[2n + \frac{4}{3}m\right]! \, (3m)! \, d^{(m,n)}_{m_1,\ldots,m_N} \in \mathbb{Z} \,. \tag{3.49}$$

The polynomial $\tilde{d}_{m,n}$ comes from the polynomials $d_{m+l,n-2l}$, $l \geqq 0$ through the shift of Eq. (2.12). The coefficient $\tilde{d}^{(m,n)}_{m_1,\ldots,m_N}$ comes from the coefficients $d^{(m+l,n-2l)}_{m_1+l_1,\ldots,m_N+l_N}$, $l_j \geqq 0$, $\sum_{j=1}^N l_j = l$. The bound on powers of the series (3.8) implies that $\frac{2}{3}(m+l) \geqq 2l - n$, that is, $l \leqq [\frac{3}{4}n + \frac{1}{2}m]$. Since

$$\prod_{j=1}^N (p_j + q_j l_{jj}) = \operatorname{ord} H_1(M, \mathbb{Z}) \,, \tag{3.50}$$

we conclude that

$$\left[2n + \frac{4}{3}m\right]! \, \left[\frac{9}{4}n + \frac{9}{2}m\right]! \, (\operatorname{ord} H_1(M,\mathbb{Z}))^{2\max\{l_j\}} \, \tilde{d}^{(m,n)}_{m_1,\ldots,m_N} \in \mathbb{Z} \,. \tag{3.51}$$

Since $\frac{2}{3}m \geqq -n$ in $\tilde{d}_{m,n}$ because of expansion (3.9), Eq. (3.31) shows that

$$2^{3n}(2n)! \, (9n)! \, (\operatorname{ord} H_1(M,\mathbb{Z}))^{2\max\{l_j\}+\max\{m_j\}} \Delta_n \in \mathbb{Z} \,. \tag{3.52}$$

Applying the inequality (3.44) to the coefficients $d^{(m+l,n-m-2l)}_{m_1+l_1,\ldots,m_N+l_N}$, which produce the coefficients $\tilde{d}^{(m,n-m)}_{m_1,\ldots,m_N}$ of Eq. (3.31), we find that

$$m + l - m_j - l_j \geqq m + 2l - n \,, \tag{3.53}$$

so that

$$m_j + 2l_j \leqq n - l + l_j \leqq n \,. \tag{3.54}$$

The equality may be achieved if $l = l_j$. Therefore

$$2\max\{l_j\} + \max\{m_j\} = n \tag{3.55}$$

and

$$2^{3n}(2n)!\,(9n)!\,(\operatorname{ord} H_1(M,\mathbb{Z}))^n \Delta_n \in \mathbb{Z}\,. \tag{3.56}$$

The smallest denominator of each coefficient of the sum of Eq. (3.32) divides $m_1 + \cdots + m_N$. Therefore $n!$ may be selected as their common denominator and

$$2^{3n} n!\,(2n)!\,(9n)!\,(\operatorname{ord} H_1(M,\mathbb{Z}))^n (S_n(M) - S_n(S^3)) \in \mathbb{Z}\,. \tag{3.57}$$

After using the expression (1.10) for $S_n(S^3)$ we come to the following conclusion:

Proposition 3.5. *The modified invariants $S_n^{(\mathrm{int})}(M)$ of an RHS M are integer*:

$$S_n^{(\mathrm{int})}(M) \equiv 2^{3n} n!\,(2n)!\,(9n)!\,(\operatorname{ord} H_1(M,\mathbb{Z}))^n S_n(M) \in \mathbb{Z}\,. \tag{3.58}$$

We might have been too generous in our choice of the numerical factor $2^{3n}n!\,(2n)!\,(9n)!$ in this definition of $S_n^{(\mathrm{int})}(M)$. However the calculations for Seifert manifolds (see, e.g. [8]) suggest that our choice of the power of $\operatorname{ord} H_1(M,\mathbb{Z})$ is minimal.

Recall that the factor $(9n)!$ in Eq. (3.58) originates from $(3m)!$ in Eq. (3.43). We noted there that it was needed to remove the denominators of the polynomials $P_{2m_j+1}(\alpha_j)$ in the r.h.s. of Eq. (3.42), whose prime divisors did not exceed $2(n - \sum_{j=1}^N m_j) + 1$. The coefficients $\Delta_{n'}$ receive contributions from the polynomials $P_{2m_1+1}(\alpha_1)\cdots P_{2m_N+1}(\alpha_N)$ for which $n - \sum_{j=1}^N m_j \leqq n'$. Therefore the factor $(9n)!$ in Eq. (3.58) accounts for prime divisors which are not greater than $2n+1$. Thus we see that

$$(\operatorname{ord} H_1(M,\mathbb{Z}))^n S_n(M) \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2n+1}\right] \tag{3.59}$$

(cf. similar results for Ohtsuki's invariants [20, 21]).

The estimate (3.59) can be improved slightly if we note that the factor $\frac{1}{2n+1}$ comes from the highest degree polynomials $P_{2n+1}(\alpha_j)$ which may appear in the r.h.s. of Eq. (3.42) for the polynomials $\tilde{D}_{m,n+m}$ contributing to $S_n(M)$ through $\Delta_n(M)$. In other words, the term in the r.h.s. of Eq. (3.42) containing P_{2n+1} will carry a factor $C_{m_1,\ldots,m_N}^{(n+\sum_{j=1}^N m_j)}$. A simple power counting indicates that only the highest degree monomial of the corresponding polynomial

$$C_{m_1,\ldots,m_N}^{(n+\sum_{j=1}^N m_j)} P_{2m_1+1}(\alpha_1)\cdots P_{2m_N+1}(\alpha_N) \tag{3.60}$$

does contribute to $S_n(M)$. Therefore we have to follow the transformations of only the highest degree monomial $\frac{\alpha_j^{2n+1}}{(2n+1)!}$ of $P_{2n+1}(\alpha_j)$. It moves unchanged from $(\prod_{j=1}^N \alpha_j)\tilde{D}_{m,n+m}$ to $(\prod_{j=1}^N \alpha_j)D_{m,n+m}$ and transforms into $\frac{a_j^{2n+1}}{(2n+1)!}$ inside $(\prod_{j=1}^N a_j)$ $d_{m,n-m}$. A transfer to $\tilde{d}_{m,n-m}$ requires a substitution of $a_j + \frac{1}{Kp_j}$ instead of a_j. The highest even power term in $\frac{(a_j+\frac{1}{Kp_j})^{2n+1}}{(2n+1)!}$ is $\frac{1}{Kp_j}\frac{a_j^{2n}}{(2n)!}$. Thus we see that the highest divisor of the denominator reduced to $2n$ and we can make an improved estimate.

Proposition 3.6. *The highest divisor of denominator of* $[\operatorname{ord} H_1(M, \mathbb{Z})]^n S_n(M)$ *is* $2n$:

$$[\operatorname{ord} H_1(M, \mathbb{Z})]^n S_n(M) \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2n}\right] . \tag{3.61}$$

4. Finite Type Invariants

4.1. Definitions Let $\mathcal{L}$ be an N-component link in a 3-manifold M. We assign rational surgeries (p_j, q_j) to all of its components. The new manifold constructed by performing all these surgeries is denoted as $\chi_{\mathcal{L}}(M)$.

T. Ohtsuki [14] and S. Garoufalidis [15] gave the following definitions of finite type invariants of integer homology spheres ($\mathbb{Z}$HS) (we add here an extra type which we call Ohtsuki$'$):

Definition 4.1. *A topological invariant* λ *of integer homology spheres is a finite type invariant of at most Ohtsuki (Ohtsuki$'$, Garoufalidis) order* N *if for any* $N' > N$ *and any* N'*-component ASL (SASL, BL)* $\mathcal{L} \subset S^3$ *with surgeries* $(\pm 1, 1)$ *assigned to its components, the following alternating sum over the surgeries performed on sublinks* $\mathcal{L}' \subset \mathcal{L}$ *(including* $\mathcal{L}$ *itself) is equal to zero:*

$$\sum_{\mathcal{L}' \subset \mathcal{L}} (-1)^{\#\mathcal{L}'} \lambda(\chi_{\mathcal{L}'}(S^3)) = 0 , \tag{4.1}$$

here $\#\mathcal{L}'$ *is the number of components of* $\mathcal{L}'$.

The invariant λ *is of Ohtsuki (Ohtsuki$'$, Garoufalidis) order* N $(\lambda \in O_N\ (O'_N, G_N))$ *if* λ *is of at most order* N *and not of at most order* $N - 1$.

T. Ohtsuki proved that his classes O_1, O_2 were empty, while Casson's invariant of $\mathbb{Z}$HS was the only representative of his class O_3. S. Garoufalidis proved that Casson's invariant was the only representative of his class G_1. He also conjectured that

$$O_{3n+1} = O_{3n+2} = \emptyset, \qquad O_{3n} = G_n . \tag{4.2}$$

We extend Definition 4.1 of finite type invariants to rational homology spheres by substituting "arbitrary rational surgeries (p_j, q_j)" instead of "surgeries $(\pm 1, 1)$" in that definition. We also conjecture that

$$O'_{2n+1} = \emptyset, \qquad O'_{2n} = G_n . \tag{4.3}$$

4.2. An Upper Estimate of Finite Type Order. Our first goal is to show that perturbative invariants $S_n(M)$ are of finite type.

Proposition 4.1. *The invariants* $S_n(M)$ *of a RHS* M *are finite type of at most Ohtsuki order* $3n$*, at most Ohtsuki$'$ order* $2n$ *and at most Garoufalidis order* n.

Our proof is based on an observation that the difference $S_n(M) - S_n(S^3)$, as it comes from Eqs. (3.32) and (3.31), is sensitive to at most $\frac{n}{1-\operatorname{si}(\mathcal{L})}$ surgeries simultaneously. The word "simultaneously" means that this difference can be presented as a sum of terms, each of which can be sensitive to at most $\frac{n}{1-\operatorname{si}(\mathcal{L})}$ surgeries on link components of $\mathcal{L}$.

Let L be an N-component link in S^3 with rational (p_j, q_j) surgeries assigned to its components. Let $\mathcal{L}' \subset \mathcal{L}$ be a sublink of $\mathcal{L}$ which does not contain a particular component, say, $\mathcal{L}_1$. Consider a contribution of the term

$$d^{(m,n)}_{0,m_2,\ldots,m_N} a_2^{2m_2} \cdots a_N^{2m_N}, \quad m = m_2 + \cdots + m_N \tag{4.4}$$

from the Jones polynomial of $\mathcal{L}$, which does not depend on the phase a_1, to the coefficients Δ of Eq. (3.30), as it comes from the surgery on the link $\mathcal{L}'$ or on another link $\mathcal{L}' \cup \mathcal{L}_1$.

Lemma 4.1. *The contribution of the term* (4.4) *to the coefficients* Δ *of Eq.* (3.9) *is the same for a surgery on* $\mathcal{L}'$ *or* $\mathcal{L}' \cup \mathcal{L}_1$.

The proof is a direct calculation with Eqs. (2.12) and (3.31). Consider a surgery on $\mathcal{L}'$. First we have to "remove" the link components of $\mathcal{L} \backslash \mathcal{L}'$ from the Jones polynomial of $\mathcal{L}$. This is achieved by fixing the corresponding colors: $a_j = \alpha_j/K = 1/K$. In particular, we set

$$a_1 = \frac{1}{K}\,. \tag{4.5}$$

This substitution transforms the term

$$K^N \left(\prod_{j=1}^{N} a_j \right) d^{(m,n)}_{0,m_2,\ldots,m_N} a_2^{2m_2} \cdots a_N^{2m_N} \tag{4.6}$$

of Eq. (3.2) into

$$K^{N-1} \left(\prod_{j=2}^{N} a_j \right) d^{(m,n)}_{0,m_2,\ldots,m_N} a_2^{2m_2} \cdots a_N^{2m_N}\,. \tag{4.7}$$

If a surgery is performed on $\mathcal{L}' \cup \mathcal{L}_1$ then the variable a_1 is treated differently. First we make a part of the shift (2.11) and rescaling of Eq. (2.12) that are related to a_1. They convert the term (4.6) again into the expression (4.7). Then we perform an integral over a_1 in Eq. (3.28) which, according to Eq. (3.31) has no effect at all because m_1=0. This proves the lemma.

Now we count the powers. An even homogeneous polynomial $d_{m,n}(a_1, \ldots, a_N)$ in power series expansion (3.8) is of order $2m$, so each of its monomials (3.11) depends on at most m different colors. Therefore their contribution is sensitive to at most m surgeries simultaneously. As a result of the shifts (2.11) of Eq. (2.12), each polynomial $\tilde{d}_{m,n}$ of the power series expansion (3.9) receives the contributions of the polynomials $d_{m+l,n-2l}$, $l \geqq 0$. According to Eq. (3.31), a polynomial $\tilde{d}_{m,n}$ contributes to Δ_{m+n}. Therefore a term Δ_n in the surgery formula (3.30) receives the contributions of the polynomials $d_{m+l,n-m-2l}$, $l, m \geqq 0$. The most surgery sensitive contribution comes from the highest possible value of $m + l$ for a given n. Because of the power bound on expansion (3.8), it comes from the polynomial $d_{\frac{n}{1-\mathrm{si}(\mathcal{L})}, -\frac{\mathrm{si}(\mathcal{L})}{1-\mathrm{si}(\mathcal{L})} n}$. Such contribution is sensitive to at most $\frac{n}{1-\mathrm{si}(\mathcal{L})}$ surgeries simultaneously.

Now it is easy to see that the products $\Delta_1^{m_1} \cdots \Delta_n^{m_n}$ with $m_1 + 2m_2 + \cdots + nm_n = n$ can be presented as sums of terms, each of which is sensitive to at most

$\frac{n}{1-\mathrm{si}(\mathscr{L})}$ surgeries. Therefore if we calculate a sum

$$\tilde{S}_n(\mathscr{L}) = \sum_{\mathscr{L}' \subset \mathscr{L}} (-1)^{\#\mathscr{L}'} S_n(\chi_{\mathscr{L}'}(S^3)) \tag{4.8}$$

for an $(\frac{n}{1-\mathrm{si}(\mathscr{L})} + 1)$-component link, then each term in Eq. (3.32) will be insensitive to at least one surgery, so that it will be canceled in the alternating sum. Now it only remains to check that

$$\frac{n}{1 - \mathrm{si}(\mathscr{L})} = \begin{cases} 3n & \text{for ASL} \\ 2n & \text{for SASL} \\ n & \text{for BL} \end{cases} . \tag{4.9}$$

This proves Proposition (4.1).

4.3. An Exact Estimate of Ohtsuki Order. It follows from our proof of Proposition 4.1 that the most surgery sensitive contribution to an invariant $S_n(\chi_{\mathscr{L}}(M))$ comes from the most color-diverse monomials

$$c_{j_1,\ldots,j_m}^{\left(\frac{n}{1-\mathrm{si}(\mathscr{L})}, -\frac{\mathrm{si}(\mathscr{L})}{1-\mathrm{si}(\mathscr{L})} n\right)} a_{j_1}^2 \cdots a_{j_m}^2 \tag{4.10}$$

of the polynomial $d_{\frac{n}{1-\mathrm{si}(\mathscr{L})}, -\frac{\mathrm{si}(\mathscr{L})}{1-\mathrm{si}(\mathscr{L})} n}$. In case of ASL (SASL) these monomials have a clear geometrical origin: according to Eq. (2.39) they come exclusively from triple (quartic) Milnor's linking numbers. This allows us to make a precise estimate of Ohtsuki (Ohtsuki$'$) order of $S_n(M)$.

Proposition 4.2. *The invariants S_n are of Ohtsuki (Ohtsuki$'$) order $3n$ ($2n$):*

$$S_n \in O_{3n}, \qquad S_n \in O'_{2n} . \tag{4.11}$$

We will present the proof for Ohtsuki invariants. The proof for Ohtsuki$'$ invariants is similar. From now on $\mathscr{L}$ in a ASL and $\mathrm{si}(\mathscr{L}) = 2/3$.

To prove the proposition we need an effective algorithm of computing the link invariant $\tilde{S}_n(\mathscr{L})$ defined by Eq. (4.8), for the case of $\#\mathscr{L} = 3n$. Since as we have observed, the only non-zero contributions to $\tilde{S}_n(\mathscr{L}, M)$ come from triple Milnor's linking numbers of $\mathscr{L}$, we may use a simplified version of the general link surgery formula (2.51) combined with Eq. (1.5),

$$\begin{aligned}
&\exp\left(\sum_{n=1}^{\infty} S_n(M)\left(\frac{i\pi}{K}\right)^n\right) \\
&\approx \exp\left(\sum_{n=1}^{\infty} S_n(S^3)\left(\frac{i\pi}{K}\right)^n\right)\left(\frac{K}{2}\right)^{\frac{3}{2}N}\left|\prod_{j=1}^{N}\frac{p_j + q_j l_{jj}}{q_j}\right|^{\frac{3}{2}} e^{-i\pi\frac{3}{4}\sum_{j=1}^{N}\mathrm{sign}\left(\frac{p_j}{q_j}+l_{jj}\right)} \\
&\times \int_{\substack{-\infty \\ [\vec{a}_j=0]}}^{+\infty} d^3\vec{a}_1 \cdots d^3\vec{a}_N \exp \\
&\times \left[\frac{i\pi K}{2}\left(\sum_{j=1}^{N}\left(\frac{p_j}{q_j} + l_{jj}\right)\vec{a}_j^2 - \frac{2}{3}\pi\sum_{i,j,k=1}^{N} l_{ijk}^{(\mu)}\,\vec{a}_i\cdot(\vec{a}_j \times \vec{a}_k)\right)\right] .
\end{aligned} \tag{4.12}$$

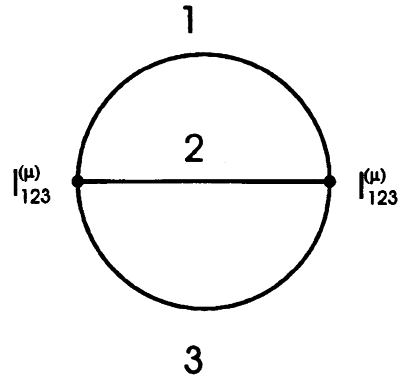

Fig. 2. The group weight diagram for S_1

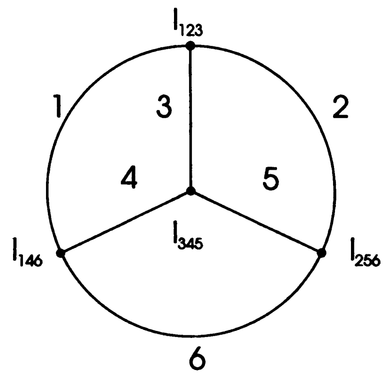

Fig. 3. A group weight diagram for S_2

Among other things we made a substitution

$$\frac{\sin\left(\pi\frac{|\vec{a}_j|}{q_j}\right)}{|\vec{a}_j|} \approx \frac{\pi}{q_j} \tag{4.13}$$

in Eq. (2.51). This approximation is justified for our purposes. It amounts to retaining only the contribution of $d_{m,n}$ among all polynomials $d_{m+l,n-2l}$ contributing to $\tilde{d}_{m,n}$.

Taking a logarithm of the integral in Eq. (4.12) is a standard exercise in combinatorics of Feynman diagrams, only in this case the combinatorics is applied to a finite dimensional integral (4.12) rather than to a path integral of quantum field theory. The difference between the old and new invariants is presented as a sum over diagrams

$$S_n(M) - S_n(S^3) = \sum_{\mathrm{dgr}\in \mathrm{Dgr}_{n+1}(\mathscr{L})} \Delta_{\mathrm{dgr}} \,. \tag{4.14}$$

Here $\mathrm{Dgr}_n(\mathscr{L})$ is a set of connected n-loop diagrams with trivalent vertices (see Figs. 2 and 3): each vertex represents a non-zero triple Milnor's linking number $l^{(\mu)}_{ijk}$ while each edge represents a link component. Since we are interested only in the contribution of the most color-diverse monomials (4.10) coming from the polynomials $d_{3n,-2n}$, we should also require that each component of the link should be represented at most only once as an edge in any particular diagram. As a result, each triple Milnor's linking number will also appear at most once, except for the diagram of Fig. 2, where the same number appears twice. Note that a set of participating triple linking numbers completely determines the diagram.

A contribution Δ_{dgr} of a diagram dgr $\in \text{Dgr}_n(\mathscr{L})$ is calculated by expanding the exponential of the cubic part of the exponent of Eq. (4.12) in participating vertices (we take linear terms for all diagrams except Fig. 2) and calculating the gaussian integrals over $d^3\vec{a}_j$. Since

$$\left(\frac{K}{2}\right)^{\frac{3}{2}} \left|\frac{q}{p+ql}\right|^{-\frac{3}{2}} e^{-i\pi\frac{3}{4}\operatorname{sign}\left(\frac{p}{q+l}\right)} \int d^3\vec{a}\, a_\mu a_\nu \exp\left[\frac{i\pi K}{2}\left(\frac{p}{q}+l\right)\vec{a}^2\right]$$
$$= \frac{i}{\pi K}\frac{q}{p+ql}\delta_{\mu\nu}, \tag{4.15}$$

the contribution Δ_{dgr} of an n-loop diagram is given by the formula

$$\Delta_{\text{dgr}} = 4^{n-1}\frac{1}{1+\delta_{n2}} W_{\text{dgr}} \left(\prod_{j\in E_{\text{dgr}}} \frac{q_j}{p_j+q_j l_{jj}}\right)\left(\prod_{(i,j,k)\in V_{\text{dgr}}} l_{ijk}^{(\mu)}\right). \tag{4.16}$$

Here E_{dgr} is a set of link components appearing as edges, V_{dgr} is a set of triple linking numbers appearing as vertices (with their multiplicities) and W_{dgr} is a group theoretical weight factor. It is calculated by assigning antisymmetric tensors $\varepsilon_{\mu_1\mu_2\mu_3}$ (i.e. Lie algebra structure constants) to every vertex and contracting indices along the edges. This prescription eliminates 1-particle reducible diagrams, i.e. the diagrams that can be split in disconnected parts by removing one edge.

Consider now the calculation of $\tilde{S}_n(\mathscr{L})$ for a $3n$-component ASL $\mathscr{L}$. The $(n+1)$-loop diagrams which contribute to the difference

$$S_n(\chi_{\mathscr{L}'}(S^3)) - S_n(S^3), \tag{4.17}$$

contain $3n$ edges and therefore require $3n$ link components to saturate them. Therefore of all sublinks $\mathscr{L}' \subset \mathscr{L}$ the difference (4.17) is non-zero only for $\mathscr{L}' = \mathscr{L}$. Then Eq. (4.8) implies that

$$\tilde{S}_n(\mathscr{L}) = (-1)^{3n} \sum_{\text{dgr}\in\text{Dgr}_{n+1}(\mathscr{L})} \Delta_{\text{dgr}}$$
$$= (-4)^n \frac{1}{1+\delta_{n1}}\left(\prod_{j=1}^{N}\frac{q_j}{p_j+q_j l_{jj}}\right) \sum_{\text{dgr}\in\text{Dgr}_{n+1}(\mathscr{L})} W_{\text{dgr}} \prod_{(i,j,k)\in V_{\text{dgr}}} l_{ijk}^{(\mu)}. \tag{4.18}$$

Now we can prove that $S_n \in O_{3n}$. Indeed, consider an $(n+1)$-loop diagram dgr such that $W_{\text{dgr}} \neq 0$ (it is easy to find an example). Then similarly to [14] draw a $3n$-component ASL in S^3 with Borromean-type junction for every vertex of dgr. Since we kept Borromean junctions to the minimum, then the set $\text{Dgr}_{n+1}(\mathscr{L})$ of this link contains only the original diagram dgr. Therefore the sum (4.18) contains only one term which is non-zero. This proves that the invariant S_n is not of at most Ohtsuki order $3n-1$.

A similar analysis can be carried out to show that $S_n \in O'_{2n}$. The Feynman diagrams will have 4-valent vertices coming from the quartic Milnor's linking numbers $l^{(\mu)}_{ijkl}$.

5. Discussion

So far the only known examples of Vassiliev invariants of links have been the derivatives of colored Jones polynomials corresponding to various Lie groups. Therefore one might conjecture that for rational homology spheres the only finite type invariants will be perturbative invariants S_n. Thus it is possible that the properties of S_n are universal properties of finite type invariants defined by Definition 4.1. In particular, one might hope that the relations (4.2), (4.3) which follow so naturally from Fig. 1, are indeed true. Each dashed line in Fig. 1 represents a finite type invariant (or, rather, a set of invariants of the same order) of RHS. Its order is equal to the m-coordinate of the intersection of its dashed line with Ohtsuki, Ohtsuki$'$and Garoufalidis boundary lines. The main reason for us to introduce the type Ohtsuki$'$was that similarly to O and G, the line O' intersects all dashed lines at integer points.

Not all Ohtsuki diagrams [14] appear in our sets Dgr_n. We require the diagrams to be closed (no 1-valent vertices) and 1-particle irreducible. If someone could prove that these conditions do follow from Definition 4.1, then it might be easier to show that $G_n \subset O_{3n}$ along the lines suggested in [15].

It is easy to see that our diagrams and their group weight factors W_{dgr} coincide with those appearing in the Feynman diagram calculations of [5, 6 and 7]. This is what one might expect since according to quantum field theory, the invariants S_n should come from $(n+1)$-loop Feynman diagrams. We present here intuitive arguments on how the Feynman diagrams may transform into the diagrams of Eq. (4.14).

Consider a Feynman diagram, say, the one in Fig. 2, with the edges representing the $(1,1)$ bilocal form gauge particle propagators $\Omega_{1,1}$. The whole expression is equal to

$$\int d^3x_1 d^3x_2 \Omega_{1,1}(x_1,x_2)\Omega_{1,1}(x_1,x_2)\Omega_{1,1}(x_1,x_2)\,. \tag{5.1}$$

Suppose that we make a rational (p_j, q_j) surgery on a knot $\mathcal{K}_1$. How does the propagator $\Omega_{1,1}(x_1,x_2)$ change outside the tubular neighborhood, on which the surgery is performed? Since $\Omega_{1,1}$ measures the linking numbers

$$\mathrm{lk}(\mathcal{K},\mathcal{K}') = \oint_{\mathcal{K}} dx_1 \oint_{\mathcal{K}'} dx_2 \Omega_{1,1}(x_1,x_2) \tag{5.2}$$

and we know how the linking numbers change under a rational surgery on $\mathcal{K}_1$,

$$\mathrm{lk}(\mathcal{K},\mathcal{K}') \to \mathrm{lk}(\mathcal{K},\mathcal{K}') + \frac{p}{q}\,\mathrm{lk}(\mathcal{K},\mathcal{K}_1)\,\mathrm{lk}(\mathcal{K}',\mathcal{K}_1)\,, \tag{5.3}$$

we may suggest that the propagator $\Omega_{1,1}(x_1,x_2)$ acquires an extra piece

$$\Omega_{1,1}(x_1,x_2) \to \Omega_{1,1}(x_1,x_2) + \frac{p}{q}\oint_{\mathcal{K}_1} dy_1 \oint_{\mathcal{K}_1} dy'_1 \Omega_{1,1}(x_1,y_1)\Omega_{1,1}(x_2,y'_1)\,. \tag{5.4}$$

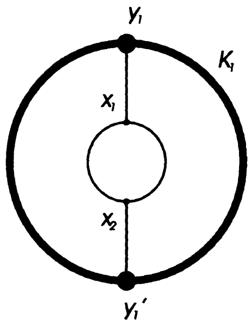

Fig. 4. A Feynman diagram contributing to ϕ_1

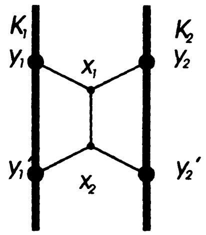

Fig. 5. A Feynman diagram contributing to $l^{(\mu)}_{1122}$

As a result, the change in the integral (5.1) is proportional to

$$\int d^3x_1 d^3x_2 \Omega_{1,1}(x_1,x_2)\Omega_{1,1}(x_1,x_2) \oint_{\mathscr{K}_1} dy_1 \oint_{\mathscr{K}_1} dy_1' \Omega_{1,1}(x_1,y_1)\Omega_{1,1}(x_2,y_1') . \quad (5.5)$$

The corresponding diagram is drawn in Fig. 3. It is known [23] to contribute to the coefficient $d_{1,0}$ of the power series expansion (2.24) of $J_{Ka}(\mathscr{K}_1)$ and to the second derivative $\phi_1(\mathscr{K}_1)$ of the Alexander polynomial of $\mathscr{K}_1$. These contributions are exactly cancelled by the ghost loop. Still this relation between the diagram of Fig. 4 and the surgery change in the diagram of Fig. 2 is in line with our expectations that the latter represents the Casson–Walker invariant, whose surgery formula includes, among other terms, the derivative $\phi_1(\mathscr{K}_1)$.

Let us perform a second rational surgery on another knot $\mathscr{K}_2$, such that $\mathrm{lk}(\mathscr{K}_1,\mathscr{K}_2) = 0$ since we want to work only with ASL. Then the change in the integral (5.5) comes from "breaking" the second propagator $\Omega_{1,1}(x_1,x_2)$:

$$\int d^3x_1 d^3x_2 \Omega_{1,1}(x_1,x_2) \left(\oint_{\mathscr{K}_1} dy_1 \oint_{\mathscr{K}_1} dy_1' \Omega_{1,1}(x_1,y_1)\Omega_{1,1}(x_2,y_1') \right)$$
$$\times \left(\oint_{\mathscr{K}_2} dy_2 \oint_{\mathscr{K}_2} dy_2' \Omega_{1,1}(x_1,y_2)\Omega_{1,1}(x_2,y_2') \right) . \quad (5.6)$$

The corresponding diagram is drawn in Fig. 5. Note that we could not break the propagators $\Omega_{1,1}(x_1,y_1)\Omega_{1,1}(x_2,y_1')$ in the expression (5.5) because the result would

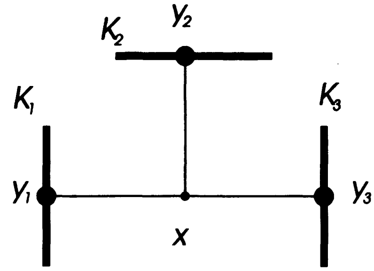

Fig. 6. A Feynman diagram contributing to $l^{(\mu)}_{123}$

be proportional to

$$\oint_{\mathcal{K}_1} dy_1 \oint_{\mathcal{K}_2} dy_2 \Omega_{1,1}(y_1, y_2) = \mathrm{lk}(\mathcal{K}_1, \mathcal{K}_2)\,. \tag{5.7}$$

The diagram of Fig. 5 contributes [9] to the quartic Milnor's linking number $l^{(\mu)}_{1122}$. This is what one may expect from Eq. (3.33) in view of relation (3.22).

As we make a third surgery on $\mathcal{K}_3$, the change in the integral (5.6) comes from breaking the last remaining propagator $\Omega_{1,1}(x_1, x_2)$:

$$\left(\int d^3x \oint_{\mathcal{K}_1} dy_1 \Omega_{1,1}(x, y_1) \oint_{\mathcal{K}_2} dy_2 \Omega_{1,1}(x, y_2) \oint_{\mathcal{K}_3} dy_3 \Omega_{1,1}(x, y_3)\right)^2 . \tag{5.8}$$

The corresponding diagram of Fig. 6 represents [9] a contribution to the triple Milnor's linking number $l^{(\mu)}_{123}$ (cf. Eqs. (3.24) and (3.33)). Its square corresponds to the original diagram of Fig. 2, but now each vertex represents a triple Milnor's linking number rather than a cubic term in the Chern–Simons action (1.2) and each edge represents a link component instead of a propagator $\Omega_{1,1}$. Note also that after three surgeries we ran out of propagators to break. This indicates that the original Feynman diagram may represent the finite type invariant of Ohtsuki order 3.

This procedure can be applied with similar results to any Feynman diagram containing no ghost propagators $\Omega_{0,2}$. However a complete analysis of all diagrams and all contributions (including the interiors of tubular neighborhoods) seems to be much more complicated. Still the careful calculations along these lines may shed some light on the contradiction between the relation (1.11) and the results of [7].

Consider for a moment integer surgeries on a special link, that is, the surgeries for which $p_j = 1$, $l_{jj} = 0$. Then it is easy to see that the changes in perturbative invariants described by Eqs. (3.31) and (3.32) have a polynomial dependence on the integer numbers q_j. The degree of the polynomial for S_n is $\frac{n}{1-\mathrm{si}(\mathcal{L})}$. The highest degree terms come from the coefficients lying on the dashed lines corresponding to Δ_n. These facts may lead to another definition of finite type invariants that would use alternating sums over surgeries performed with varying values of q_j.

Finally we would like to comment on the relation between perturbative invariants S_n and Ohtsuki's invariants λ_n introduced in [20 and 21]. We show in [24] that Ohtsuki's polynomial

$$\tau(M) = \sum_{n=0}^{\infty} \lambda_n (q-1)^n \tag{5.9}$$

is proportional to the trivial connection contribution $Z^{(\mathrm{tr})}(M;k)$ if we make a substitution $q = \exp(\frac{2\pi i}{K})$:

$$\tau(M) = \frac{(\frac{\pi}{K})}{\sin(\frac{\pi}{K})} \exp\left[\sum_{n=1}^{\infty} S_n(M)\left(\frac{i\pi}{K}\right)^n\right]. \tag{5.10}$$

The proof follows from the analysis of the surgery formulas which is similar to the one performed in this paper. The same expansion formulas (3.8) and (3.9) have to be used in conjunction with finite gaussian sums rather than with gaussian integrals of Eq. (3.28).

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