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Quantization of Solitons and the Restricted Sine-Gordon Model

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Abstract: We show how to compute form factors, matrix elements of local fields, in the restricted sine-Gordon model, at the reflectionless points, by quantizing solitons. We introduce (quantum) separated variables in which the Hamiltonians are expressed in terms of (quantum) τ-functions. We explicitly describe the soliton wave functions, and we explain how the restriction is related to an unusual hermitian structure. We also present a semi-classical analysis which enlightens the fact that the restricted sine-Gordon model corresponds to an analytical continuation of the sine-Gordon model, intermediate between sine-Gordon and KdV.

1. Introduction

About 20 years ago the work on quantization of integrable models of Quantum Field Theory started with the idea of quantizing the classical soliton solutions [1, 2]. Important results were achieved in this way, in particular for the sine-Gordon (SG) theory, the semi-classical spectrum of excitations (which happens to be exact quantum-mechanically) and the semi-classical approximation for the soliton S-matrix were found. The semi-classical S-matrix allowed to guess the exact S-matrix in the reflectionless case [2], and this was used further as a fundamental input in the boot-strap construction of the S-matrix for arbitrary coupling [3]. Later, however the idea of direct quantization of solitons was abandoned in favor of other approaches such as Bethe Ansatz and its algebraic formulation in the Quantum Inverse Scattering Method (QISM).

Whatever the original motivations and methods were, it is fair to say that the most significant results in the theory were obtained by bootstrap methods. Exact S-matrices [3, 4] and exact form factors [5, 6] were found by this method. Since

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the bootstrap calculation of the S-matrix is based on semi-classical results from quantization of solitons, and since the S-matrix defines the form factors through the set of form factor axioms [6], one may imagine that there is a direct path from quantization of solitons to form factors. Our attraction by this way of thinking does not mean, of course, that we reject the achievements of QISM which revealed the mathematical structure hidden behind the integrable models and led to the discovery of quantum groups.

In recent years two important pieces of information were added to the theory of sine-Gordon. The first of them is the relation with the perturbations of the minimal models of conformal field theory (CFT) [7, 8]. The second is the discovery of the algebra of non-local charges, which is isomorphic to the q-deformation of the universal enveloping algebra of the loop algebra $\widehat{sl_2}$ [9]. Actually these two features are closely related: the restricted sine-Gordon (RSG) model coincides with the $\Phi_{1,3}$-perturbation of minimal CFT [10, 11], but the restriction is intimately connected with the existence of the non-local symmetry.

In the present paper we shall show that the results obtained by the bootstrap methods for sine-Gordon can be understood directly by quantizing solitons. Namely, we shall interpret the form factors of the restricted sine-Gordon model as matrix elements in a quantum mechanical n-soliton system[1]. This will allow us to underline the connection between profound structures in the classical and quantum theories: τ-functions and separation of variables on the one hand and the space of the local fields on the other hand. Although we present the general structure for generic values of the coupling constant, we will reconstruct the sine-Gordon form factors only at the reflectionless points. We hope to return to the general case in another publication.

For each n-soliton solution we shall introduce pairs of conjugated variables A_i and P_i $(i = 1,\ldots,n)$, which in the quantum case satisfy Weyl commutation relations. Every local operator $\mathcal{O}$ can be considered as acting in this A-representation, and therefore can be identified with a certain operator $\mathcal{O}(A,P)$. The typical formula for the matrix element of $\mathcal{O}$ between two n-soliton states can be presented as

$$\langle B'|\mathcal{O}|B\rangle = \int \Psi(A,B')^{\dagger}\mathcal{O}(A,P)\Psi(A,B)\,d\mu(A)\,, \tag{1}$$

where $\Psi(A,B)$ is the wave-function of the state of n solitons with momenta $B_1,\ldots,B_n$. The measure $d\mu(A)$ will include a specific weight admitting a natural interpretation in the n-soliton symplectic geometry. We shall give explicit expressions for $\mathcal{O}(A,P)$ corresponding to the Virasoro primary fields.

In formula (1), the variables A will be complex. Choosing the integration domain is a non-trivial issue. It corresponds to choosing a real subvariety, which specifies the configuration space of the theory. This configuration space has a natural interpretation in the *restricted* sine-Gordon model where it can be understood as an analytical continuation of the sine-Gordon or KdV configuration space. The hermitian conjugation $\dagger$ is not the naive complex conjugation inherited from the sine-Gordon dynamics, but a more subtle one adapted to this choice of real subvariety.

The local integrals of motion will be rewritten in the A-representation. They are difference operators which can be expressed in terms of "quantum τ-functions." We will describe how to separate the variables in the associated Schrödinger equations. The fact that these Schrödinger equations are difference equations provides

[1] A side motivation for this study was to learn how to directly quantize solitons with potential applications to theories where soliton-like solutions (monopoles) are known.

just enough room for the existence of the non-local charges commuting with the Hamiltonians. Recall that these charges do not allow direct classical limit. The importance of the coexistence of two commuting Weyl subalgebras was pointed out in [12].

2. The Classical sine-Gordon Theory

2.1. Sine-Gordon solitons. In this section we introduce a few useful notations for the sine-Gordon (SG) equation and its solutions. Let $x_\pm = x \pm t$ be the light cone coordinates and $\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t)$. The sine-Gordon equation is

$$\partial_+\partial_-\varphi = 2\sin(2\varphi)\,. \tag{2}$$

It is convenient to introduce two τ-functions $\tau_\pm$, in terms of which the SG equations can be rewritten in Hirota form. The sine-Gordon field φ is related to the τ-functions by

$$\exp(i\varphi) = \frac{\tau_-}{\tau_+}\,.$$

Let us describe the τ-functions of the n-soliton solutions of the SG equation. Consider the function

$$\tau(X_1,\dots,X_n|B_1,\dots,B_n) = \det(1+V)\,, \tag{3}$$

where V is a $n\times n$ matrix with elements:

$$V_{ij} = 2\frac{B_iX_i}{B_i+B_j}\,.$$

The n-soliton τ-functions $\tau_\pm(x_-,x_+)$ are written in terms of τ as follows:

$$\tau_\pm(x_-,x_+) = \tau_\pm(X(x_-,x_+)|B)\,,$$

where

$$\tau_\pm(X|B) = \tau(\pm X|B)\,.$$

The $x_\pm$-dependence of X is quite simple:

$$X_i(x_+,x_-) = X_i\exp(2(B_ix_- + B_i^{-1}x_+))\,. \tag{4}$$

The quantities X_i and B_i are the parameters of the solitons: $\beta_i = \log(B_i)$ are the rapidities and X_i are related to the positions. For the sine-Gordon equation, they satisfy specific reality conditions. For solitons or antisolitons, the rapidity B is real and X is purely imaginary, i.e. $X = i\varepsilon e^\gamma$ with $\varepsilon = +1$ for a soliton and $\varepsilon = -1$ for an antisoliton. We shall not consider "breathers" in this paper but for completeness it should be mentioned that they correspond to pairs of complex conjugated rapidities $(B,\overline{B})$ and positions $(X,-\overline{X})$. Notice that these conditions are preserved by the dynamics.

The sine-Gordon equation is a Hamiltonian system. The symplectic form is the canonical one:

$$\Omega_{SG} = \int_{-\infty}^{+\infty} dx\,\delta\pi(x)\wedge\delta\varphi(x)$$

with $\pi(x)$ the momentum conjugated to the field $\varphi(x)$. Above, δ denotes the variation on the phase space. The space of n-solitons can be viewed as a $2n$-dimensional manifold embedded into the infinite-dimensional phase space. The restriction of the symplectic form on this finite-dimensional submanifold gives the n-soliton symplectic form. In the coordinates X_i and B_i it reads, cf. eg. [13, 14]:

$$\omega = \sum_{i=1}^{n} \frac{dX_i}{X_i} \wedge \frac{dB_i}{B_i} + \sum_{i<j} \frac{4B_iB_j}{B_i^2 - B_j^2} \frac{dB_i}{B_i} \wedge \frac{dB_j}{B_j} . \tag{5}$$

The commuting conserved quantities are precisely the B's. In the following we shall always assume that the B have been ordered: $0 < B_1 < \cdots < B_n$. A complete set of commuting Hamiltonians H_k can be chosen as the set of elementary symmetric functions

$$H_k = \sigma_k(B) . \tag{6}$$

We recall that the symmetric functions $\sigma_k(B)$ are defined by: $\prod_j (z + B_j) = \sum_k z^{n-k}\sigma_k(B)$. The local integrals of motion which are given on n-soliton solutions by

$$I_k^{\pm} = s_{\mp(2k+1)}(B) \equiv \sum_{j=1}^{n} B_j^{\mp(2k+1)}$$

can be expressed in terms of H_i. In particular for the light cone components of the energy-momentum we have $I_- \equiv I_1^- = H_1$, and $I_+ \equiv I_1^+ = H_n^{-1}H_{n-1}$.

The variable Y_j canonically conjugated to B_j is defined by $Y_j = X_j \prod_{k \neq j} (\frac{B_j - B_k}{B_j + B_k})$. The symplectic form is then written as $\omega = \sum_j \frac{dY_j}{Y_j} \wedge \frac{dB_j}{B_j}$. The equations of motion are very simple in the variables $\{Y, B\}$, which furthermore have the nice property of being separated. Other sets of variables have also been introduced, in particular those which lead to Ruijsenaars models [15, 14]. However, for the purpose of comparison with the existing exact form factor formulae we need to introduce in the next section still another set of variables.

A simple explicit expression for the τ-functions can be obtained by expanding the determinant:

$$\begin{aligned} \tau(X|B) &= 1 + \sum_{p=1}^{n} \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = p}} \prod_{i<j \in I} \beta_{ij}^2(B) \cdot \prod_{i \in I} X_i \\ &= 1 + \sum_j X_j + \sum_{i<j} \beta_{ij}^2(B) X_i X_j + \cdots \end{aligned} \tag{7}$$

with $\beta_{ij}(B) = \frac{B_i - B_j}{B_i + B_j}$. In Appendix B, we gather a few useful formulae concerning these τ-functions. In particular a very useful formula is the recursion relation satisfied by the n-solitons τ-functions:

$$\tau^{(n)}(X|B) = \tau^{(n-1)}(X|B) + \tau^{(n-1)}(\beta_{kn}^2(B)X_k|B)X_n . \tag{8}$$

2.2. The analytical variables. We now give a parametrization of the n-soliton phase space in terms of new variables $\{A, B\}$. The variables B_j can be considered as poles of the Jost function for the auxiliary linear problem, and the variables A_j are zeroes of the Jost function. The importance of these variables is better understood in the more general situation of quasi-periodic finite-zone solutions from which the soliton solutions are obtained by a limiting procedure. In the finite-zone case the analogues of B_j describe the moduli of the hyper-elliptic spectral curve, while A_j give the divisor of zeroes of the Baker–Akhiezer function. The general rule that the zeroes of the Baker–Akhiezer function give the correct set of variables for quantization was called the "magic prescription" in [16]. We discuss the relation to the finite-zone solutions in Appendix A. Because of the nice algebro-geometrical meaning of these new variables we call them the analytical variables.

The set of analytical variables $\{A,B\}$ is related to the variables $\{X,B\}$ introduced above by

$$X_j \cdot \prod_{k \neq j} \left(\frac{B_j - B_k}{B_j + B_k} \right) = \prod_{k=1}^{n} \left(\frac{B_j - A_k}{B_j + A_k} \right), \quad \text{for } j = 1, \dots, n. \tag{9}$$

This relation can be considered as a system of equations for the symmetric functions $\sigma_k(A)$ as functions of the $\{X,B\}$ variables. The solution to this system is given in Eq. (80) in Appendix B.

In these analytical variables the symplectic form becomes

$$\omega = 2 \sum_{k,j} \frac{dA_j \wedge dB_k}{A_j^2 - B_k^2}. \tag{10}$$

We shall need the Liouville measure $\omega^n = \det(\omega) dA_1 \wedge \cdots \wedge dA_n \wedge dB_1 \wedge \cdots \wedge dB_n$ with

$$\det(\omega) \equiv \det \left(\frac{1}{A_j^2 - B_k^2} \right) = \frac{\prod_{j<k} (A_j^2 - A_k^2) \prod_{j<k} (B_j^2 - B_k^2)}{\prod_{k,j} (A_j^2 - B_k^2)}. \tag{11}$$

It is useful to write the non-vanishing Poisson brackets,

$$\{A_i, B_j\} = \frac{\prod_{k \neq i} (B_j^2 - A_k^2) \prod_{k \neq j} (A_i^2 - B_k^2)}{\prod_{k \neq i} (A_i^2 - A_k^2) \prod_{k \neq j} (B_j^2 - B_k^2)} (A_i^2 - B_j^2).$$

Remark that the products in the right-hand side can be written in terms of cross-ratios. This is the first manifestation of the conformal properties of these variables that will reappear in the following.

One can express the τ-functions in terms of the variables A_j and B_j. The result is the following surprisingly compact formula

$$\tau_+ = 2^n \left(\prod_{j=1}^{n} B_j \right) \frac{\prod_{i<j} (A_i + A_j) \prod_{i<j} (B_i + B_j)}{\prod_{i,j} (B_i + A_j)},$$

$$\tau_- = 2^n \left(\prod_{j=1}^{n} A_j \right) \frac{\prod_{i<j} (A_i + A_j) \prod_{i<j} (B_i + B_j)}{\prod_{i,j} (B_i + A_j)}. \tag{12}$$

The symplectic form as well as the τ-functions enjoy an intriguing $A \leftrightarrow B$ duality. The proof of these formulae is given in Appendix B. They lead to a formula expressing the sine-Gordon field in the $\{A,B\}$ variables:

$$e^{i\varphi} = \frac{\tau_-}{\tau_+} = \prod_{j=1}^{n} \left(\frac{A_j}{B_j} \right). \tag{13}$$

Let us now introduce the variables P_j conjugated to A_j,

$$P_j = \prod_{k=1}^{n} \left(\frac{B_k - A_j}{B_k + A_j} \right), \quad \text{for} \quad j = 1, \ldots, n\,. \tag{14}$$

In terms of A_j and P_j the symplectic form takes the canonical form:

$$\omega = 2 \sum_{j=1}^{n} \frac{dP_j}{P_j} \wedge \frac{dA_j}{A_j}\,.$$

We can express the hamiltonians $H_k = \sigma_k(B)$ in terms of the variables $\{A, P\}$ using Eqs. (14) as a linear system for the $\sigma_k(B)$. Surprisingly, the solution of these equations can be written in terms of the τ-function as follows:

$$H_k = \sigma_k(B) = \frac{\tau_k(Z|A)}{\tau(Z|A)}, \quad \text{where } Z_j = (-1)^n P_j \prod_{k \neq j} \left(\frac{A_j + A_k}{A_j - A_k} \right). \tag{15}$$

The functions τ_k are defined by

$$\begin{aligned} &\tau_k(Z_1, \ldots, Z_n | A_1, \ldots, A_n) \\ &= \sum_{i_1 < i_2 < \cdots < i_k} A_{i_1} A_{i_2} \cdots A_{i_k} \tau(Z_1, \ldots, -Z_{i_1}, \ldots, -Z_{i_k}, \ldots, Z_n | A_1, \ldots, A_n)\,. \end{aligned} \tag{16}$$

In particular $\tau(Z|A) = \tau_0(Z|A)$ and $\tau_-(Z|A) = (\prod A_j)^{-1} \tau_n(Z|A)$. We delay the proof of Eq. (15) as it turns out to be a limiting case of a more general quantum formula which we shall prove in Appendix E. It is convenient to introduce the generating function of the τ_k. Let $T(u) = \sum_{k=0}^{n} u^k \tau_k$. It can again be expressed in terms of τ-functions as:

$$T(u) = \tau(Z(u)|A) \prod_{j=1}^{n} (1 + uA_j) \quad \text{with } Z_j(u) = \frac{1 - uA_j}{1 + uA_j} Z_j\,. \tag{17}$$

This follows from the quantum relation Eq. (53) proved in Appendix E.

Notice the unusual feature of our approach: the local integrals of motion (15) are given in terms of τ-functions.

The equations of motion written in the variables A_j are as follows:

$$\partial_- A_i = \{I_-, A_i\} = \prod_j (A_i^2 - B_j^2) \prod_{j \neq i} \frac{1}{A_i^2 - A_j^2}\,,$$

$$\partial_+ A_i = \{I_+, A_i\} = \prod_j \frac{A_i^2 - B_j^2}{B_j^2} \prod_{j \neq i} \frac{A_j^2}{A_i^2 - A_j^2}\,. \tag{18}$$

These equations provide a particular case of the general equations of motion for the divisor of the zeroes of the Baker–Akhiezer function (see Appendix A). This kind of equation is commonly used in the theory of quasi-periodic solutions of integrable equations; they go back to Neumann and Kowalevskaya. One can show directly that (18) together with (13) imply the sine-Gordon equation [17, 18].

To finish the discussion of the variables A_i we have to explain their trajectories. For one soliton the condition $|e^{i\varphi}| = 1$ shows that the A-trajectory lies on the circle of radius B. Under the classical SG dynamics of n-soliton solutions every variable A_k runs around a curve going in the lower half-plane from $-B_k$ to B_k. When $B_1 \ll B_2 \ll \cdots \ll B_n$ the trajectories are semi-circles, for finite B_k they are getting deformed, but not too much. For antisolitons these trajectories are replaced by their complex conjugate.

2.3. Reduced action and the relation to the KdV equation. Our goal is to quantize of the (R)SG theory in the soliton variables. The first step could be a semi-classical quantization. To perform it for n-soliton solutions, we need to compute the reduced action $\int^q p\, dq$ since we are restricting the system to the level of Hamiltonians, as in the Maupertuis principle. The symplectic form (11) can be rewritten as

$$\omega = d\alpha = \sum_{k=1}^{n} d\left(\log \prod_j \frac{B_j - A_k}{B_j + A_k}\right) \wedge \frac{dA_k}{A_k}\,.$$

We discuss the relation of this symplectic form to the general theory of analytical Poisson structures [19, 20] in Appendix A. The 1-form α is defined up to an exact form $dF(A,B)$. In this paper, we shall only consider the possibility of a function $F(A,B)$ independent of B (in order that α expends only on dA_j, the coordinates) and of the special form $F(A_1,\ldots,A_n) = \sum_k F_k(A_k)$ (to preserve the separability property of the A_j). Hence the most general form of α is

$$\alpha = \sum_{k=1}^{n} \log\left(\prod_j \frac{B_j - A_k}{B_j + A_k}\right) \cdot \frac{dA_k}{A_k} + \sum_k dF_k(A_k)\,. \tag{19}$$

The functions F_k have to be fixed by additional considerations.

Now we face a serious challenge: with these choices, the form α cannot arise in the full SG theory for the simple reason that the reduced action $S(A_1,\ldots,A_n)$ constructed from α

$$S(A_1,\ldots,A_n) = \sum_{k=1}^{n} \int^{A_k} \log\left(\prod_j \frac{B_j - A}{B_j + A}\right) \cdot \frac{dA}{A} + F_k(A_k)$$

cannot be made real along the sine-Gordon soliton A_j-trajectories by any choice of F_k. This circumstance looks very discouraging and the variables A_j seem to be useless. But, as we shall later see, these are exactly the variables in which the comparison with the quantum form factor formulae is straightforward. We shall argue that the choices made for α correspond in fact to the restricted sine-Gordon (RSG) theory. For this, we first need to describe the connection with the KdV equation.

Let us discuss briefly the relation between SG and KdV solitons. The KdV equation allows soliton solutions in the form

$$u(x_-) = \partial_-^2 \log \tau(x_-,0)\,,$$

where τ is exactly the same as before. In other words, the KdV soliton τ-functions are identical to the SG soliton τ-functions but with all the chiral coordinates x_+ set to zero. The difference between the two cases lies in the reality conditions: the variables X_j which were imaginary for sine-Gordon become real in the KdV case.

Before going into the details of these reality conditions, let us discuss the relation between the two cases in the fully complexified situation.

In the variables A_j the equations for KdV solitons are exactly the same as for one chirality of SG (see Appendix A):

$$\partial_- A_i = \prod_j (A_i^2 - B_j^2) \prod_{j \neq i} \frac{1}{A_i^2 - A_j^2} \,. \tag{20}$$

The field u reads in these variables as

$$u = 2\left(\sum_j A_j^2 - \sum_j B_j^2\right). \tag{21}$$

It should be stressed that all the KdV fields are expressed in terms of even powers of A_j. The relation between the SG field φ and the KdV field u is given by the Miura transformation:

$$u(x_-) = -(\partial_- \varphi(x_-,0))^2 - i\partial_-^2 \varphi(x_-,0)\,. \tag{22}$$

It is a nice exercise to check directly that (21) follows from $e^{i\varphi} = \prod_j \frac{A_j}{B_j}$ using Eq. (20).

As it has been said we are rather interested in the RSG than in the SG theory. The RSG model describes the $\Phi_{1,3}$-perturbation of the minimal model of conformal field theory (CFT). The reason why the KdV equation appears in the context of two-dimensional CFT is well known: the second Poisson structure of KdV is a classical limit of the operator product expansion for the light-cone component of the energy-momentum tensor in CFT. (For the minimal models of CFT the classical limit is understood as the limit $c \to -\infty$.) The second Poisson structure for KdV is:

$$\{u(x_-), u(x'_-)\} = \delta'(x_- - x'_-)(u(x_-) + u(x'_-)) + \delta'''(x_- - x'_-)\,. \tag{23}$$

The KdV field u is identified with the classical limit of T_{--}. As explained in Appendix A, it turns out that the second KdV Poisson structure restricted to the soliton manifold in the $\{A, B\}$ variables is *identical* to the symplectic structure (10) that we have derived from the sine-Gordon theory. It is remarkable that the conformal KdV Poisson structure appears in the (massive) sine-Gordon model when restricted to the soliton solutions. Again, this is only true on the soliton sub-manifold.

Thus we are considering the light-cone hamiltonian picture, and the lines $x_+ =$ const. are the space directions. The coordinate x_+ has to be considered as time. Globally we cannot introduce the x_+ dynamics in KdV theory, but this can be done perfectly on the n-soliton solutions: the corresponding Hamiltonian is $I^+ = \sum B_j^{-1}$ and the equations of motion have the familiar form

$$\partial_+ A_i = \prod_j \frac{A_i^2 - B_j^2}{B_j^2} \prod_{j \neq i} \frac{A_j^2}{A_i^2 - A_j^2} \,. \tag{24}$$

For real solutions the x_+-dynamics of A_i is organized as follows (see Appendix A). Each A_i ($i \leqq n-1$) moves inside either of two segments (B_i, B_{i+1}) and $(-B_{i+1}, -B_i)$, the points B_j and $-B_j$ being identified (recall that the observables depend on A_i^2 only). The point A_n moves in two segments (B_n, ∞) and $(-\infty, -B_n)$ the points $\pm\infty$ being identified, notice that ∞ is a regular point for Eq. (24).

Let us now normalize the 1-form (19) in order that it is real over x_+-trajectories. To do that we have first to fix the branch of logarithms. For further convenience we put n cuts over semi-circles in the upper-half plane connecting the points B_j and $-B_j$ and require that the logarithm

$$\log\left(\prod_j \frac{B_j - A}{B_j + A}\right)$$

is real when $-B_1 < A < B_1$. With these conventions one easily figures out that the following choice of F_k corresponds to a 1-form α, real over x_+-trajectories:

$$\alpha = \sum_{k=1}^{n}\left(\log\left(\prod_j \frac{B_j - A_k}{B_j + A_k}\right) - k\pi i\right)\cdot\frac{dA_k}{A_k}\,. \tag{25}$$

The reduced action is real as well because the trajectories are real.

Returning to the relation with the sine-Gordon model, it is obvious from (22) that real SG solutions do not correspond to real KdV solutions. One important consequence of the comparison with the form factor formula is that we are actually considering an analytical continuation of the sine-Gordon and KdV dynamics. More precisely, the semi-classical quantization exactly brings out the 1-form α as defined above in Eq. (25). But the trajectories for A_k lie on the semi-circle of radius $B_k < |A_k| < B_{k+1}$. In other words, the 1-form needed for the quantization of the RSG model is obtained by analytical continuation of the 1-form (25). In this analytical continuation we identify the variable A_i which runs from B_i to $-B_i$ in the SG case with the variable running inside $(B_i, B_{i+1}) \cup (-B_i, -B_{i+1})$ in the KdV case (we put $B_{n+1} \equiv \infty$). Selecting the trajectories in the complex A-plane is just choosing the real phase space of the theory.

Following our logic the form (25) must be understood as a restriction to the n-soliton submanifold of a globally defined 1-form related to the Poisson structure (23). We are not able to define such a global 1-form, but we conjecture that it exists. We can only refer to certain self-consistency checks (such as reality of (25)) to support this conjecture.

3. The SG Form Factors in the Absence of Reflection of Solitons

3.1. Sine-Gordon versus Restricted Sine-Gordon. The sine-Gordon equation follows from the action:

$$S = \frac{\pi}{\gamma}\int \mathscr{L}\, d^2x, \qquad \mathscr{L} = (\partial_\mu\varphi)^2 + m^2(\cos(2\varphi) - 1)\,,$$

where γ is the coupling constant, $0 < \gamma < \pi$. The free fermion point is at $\gamma = \frac{\pi}{2}$. In the quantum theory, the relevant coupling constant is

$$\xi = \frac{\pi\gamma}{\pi - \gamma}\,.$$

We shall always use the constant ξ, which plays the role of the Planck constant. Only the mass is renormalized but not the coupling constant γ [22].

The SG theory is invariant under the quantum affine loop algebra $U_{\widehat{q}}(\widehat{sl_2})$ with $\widehat{q} = \exp(i\frac{2\pi^2}{\xi})$. This symmetry algebra is generated by the topological charge and four non-local charges of spin $s = \pm\frac{\pi}{\xi}$. (By convention, $\widehat{q}$ here is the inverse square of that of ref. [9].)

The canonical stress tensor of the SG theory is $T^{SG}_{\mu\nu} = (\partial_\mu\varphi)(\partial_\nu\varphi) - \frac{g_{\mu\nu}}{2}\mathscr{L}$. The SG theory contains two subalgebras of local operators which, as operator algebras are generated by $\exp(i\varphi)$ and $\exp(-i\varphi)$ respectively. Let us concentrate on one of them, say the one generated by $\exp(i\varphi)$. It is known that this subalgebra can be considered independently of the rest of the operators as the operator algebra of the theory with the modified energy-momentum tensor:

$$T^{\mathrm{mod}}_{\mu\nu} = T^{SG}_{\mu\nu} + i\pi\sqrt{\frac{6}{\xi(\pi+\xi)}}\varepsilon_{\mu,\mu'}\varepsilon_{\nu,\nu'}\partial_{\mu'}\partial_{\nu'}\varphi \,. \tag{26}$$

This modification changes the trace of the stress tensor which is now

$$T^{\mathrm{mod}}_{\mu\mu} = m^2\exp(2i\varphi)\,.$$

This modification corresponds to the restricted Sine-Gordon theory (RSG). For rational $\frac{\xi}{\pi}$ it describes the $\Phi_{1,3}$-perturbations of the minimal models of CFT, but it can be considered for generic values of ξ as well. The central charge and the dimension of the operator $\Phi_{1,3}$ are given by

$$c = 1 - \frac{6\pi^2}{\xi(\pi+\xi)}, \qquad \Delta_{1,3} = \frac{\xi-\pi}{\xi+\pi}\,.$$

Modifying the stress tensor as in Eq. (26) modifies the Lorentz boost and hence the spin of the non-local charges. Under this modification two of these charges become spinless. Together with the topological charge they then form a representation of the finite quantum algebra $U_{\hat{q}}(sl_2)$. The restriction of the Sine-Gordon model consists in gauging away this symmetry subalgebra. The physical states of the RSG model are annihilated by these spin-less non-local charges. The physical operators of the RSG model are those which commute with these charges. In particular, $e^{i\varphi}$ which commute with them is a physical operator, but $e^{-i\varphi}$ is not since it does not commute with these charges.

The asymptotic states of the RSG theory are constructed as follows. Consider the states in the SG theory containing n solitons and n anti-solitons:

$$|\beta_1,\beta_2,\ldots,\beta_{2n}\rangle_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{2n}}\,,$$

where β_j are the rapidities of particles, and ε_j is $+$ or $-$ for soliton or anti-soliton respectively. For the RSG theory one introduces the states

$$|\beta_1,\beta_2,\ldots,\beta_{2n}\rangle\rangle_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{2n}} = \exp\left(\frac{\pi}{2\xi}\sum_j \varepsilon_j\beta_j\right)|\beta_1,\beta_2,\ldots,\beta_{2n}\rangle_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{2n}}\,. \tag{27}$$

The extra factor in $|\beta_1,\beta_2,\ldots,\beta_{2n}\rangle\rangle$ is an echo of the modification (26) of the stress tensor. In the new basis the S-matrix becomes manifestly invariant under $U_{\hat{q}}(sl_2)$. The asymptotic states of the RSG model are then the $U_{\hat{q}}(sl_2)$-scalar in the Hilbert space spanned by the states $|\beta_1,\beta_2,\ldots,\beta_{2n}\rangle\rangle$.

In the restricted theories one generally cannot introduce a positively defined Hermitian structure. Obviously the SG hermitian conjugation $*$ maps the RSG model into the symmetric restricted model constructed from $\exp(-i\varphi)$. On the other hand the SG charge conjugation ($\varphi \to -\varphi$) is also broken in the RSG model. However, one can introduce an anti-linear involution for the RSG model as the combined CT reflection. For any local operator the combined transformation (which we denote by $\dagger$) corresponds to hermitian conjugation and reflection $\varphi \to -\varphi$. This operation $\dagger$ does not give a positively defined scalar product for the SG theory, but it does not lead to contradiction if one stays within the RSG theory. In particular, since the modified stress tensor $T_{\mu\nu}^{\mathrm{mod}}$ is hermitic in the RSG theory, one has

$$(e^{i\varphi})^\dagger = e^{i\varphi} . \tag{28}$$

This is a simple but fundamental remark.

3.2. Form factor formulae. In what follows we shall consider the case $\xi = \frac{\pi}{\nu}$ for $\nu = 1, 2, \dots$, when the reflection of solitons and anti-solitons is absent. In these cases, the two parameters q and $\widehat{q}$ are

$$q = e^{i\frac{\pi}{\nu}}, \qquad \widehat{q} = 1 .$$

The S-matrix is diagonal and given by

$$S(\beta) = \prod_{j=1}^{\nu-1} \frac{\sinh \frac{1}{2}(\beta + \frac{\pi i}{\nu} j)}{\sinh \frac{1}{2}(\beta - \frac{\pi i}{\nu} j)} = \prod_{j=1}^{\nu-1} \left(\frac{Bq^j - 1}{B - q^j} \right) .$$

We shall use the following notations:

$$B = \exp(\beta), \qquad b = \exp\left(\frac{2\pi}{\xi}\beta\right) = \exp(2\nu\beta) .$$

Consider any local operator $\mathcal{O}(x)$ for RSG. Its form factors are defined by

$$f_{\mathcal{O}}(\beta_1, \beta_2, \dots, \beta_{2n})_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2n}} = \langle\langle 0|\mathcal{O}(0)|\beta_1, \beta_2, \dots, \beta_{2n}\rangle\rangle_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2n}} . \tag{29}$$

The form factors are given by the formulae

$$\begin{aligned} & f_{\mathcal{O}}(\beta_1, \beta_2, \dots, \beta_{2n})_{-\cdots-+\cdots+} \\ &= e^{(-\frac{1}{2}(\nu(n-1)-n)\sum_j \beta_j)} \prod_{i<j} \zeta(\beta_i - \beta_j) \prod_{i=1}^{n} \prod_{j=n+1}^{2n} \frac{1}{\sinh \frac{\pi}{\xi}(\beta_j - \beta_i - \pi i)} \\ &\quad \times \widehat{f}_{\mathcal{O}}(\beta_1, \beta_2, \dots, \beta_{2n})_{-\cdots-+\cdots+} . \end{aligned} \tag{30}$$

The function $\zeta(\beta)$ is regular in the physical strip $0 < \operatorname{Im} \beta < \pi$, and satisfies $\zeta(-\beta) = \zeta(\beta - 2i\pi) = S(\beta)\zeta(\beta)$. It can be found in [6]. We shall not consider this prefactor in Eq. (30) since it is the same for all operators: it is related to the normalization of the wave function which we hope to discuss sometime. The most interesting part of the form factor is given by

$$\begin{aligned} \widehat{f}_{\mathcal{O}}(\beta_1, \beta_2, \dots, \beta_{2n})_{-\cdots-+\cdots+} &= \frac{1}{(2\pi i)^n} \int_C dA_1 \cdots \int_C dA_n \prod_{i=1}^{n} \prod_{j=1}^{2n} \psi(A_i, B_j) \\ &\quad \times \prod_{i<j} (A_i^2 - A_j^2) L_{\mathcal{O}}(A_1, \dots, A_n | B_1, \dots, B_{2n}) \prod_{i=1}^{n} a_i^{-i} . \end{aligned} \tag{31}$$

Some comments are needed for this formula. The function $\psi(A,B)$ is given by

$$\psi(A,B) = \prod_{j=1}^{\nu-1} (B - Aq^{-j}) \,. \tag{32}$$

This function satisfies the difference equation:

$$\psi(Aq,B) = \left(\frac{B-A}{B+qA}\right) \psi(A,B) \,. \tag{33}$$

As usual we define $a = A^{2\nu}$. The contour C is drawn around the point $A = 0$. Notice that the right-hand side of the formula (31) does not depend on the partition of the particles into solitons and anti-solitons; this is a peculiarity of reflectionless case.

Different local operators are defined by different functions $L_{\mathcal{O}}(A_1,\dots,A_n|B_1,\dots,B_{2n})$. These functions are symmetric polynomials of $A_1,\dots,A_n$. For the primary operators $\Phi_{2k} = \exp(2ki\varphi)$ and their Virasoro descendants, $L_{\mathcal{O}}$ are symmetric Laurent polynomials of $B_1,\dots,B_{2n}$. For the primary operators $\Phi_{2k+1} = \exp((2k+1)i\varphi)$, they are symmetric Laurent polynomials of $B_1,\dots,B_{2n}$ multiplied by $\prod B_j^{\frac{1}{2}}$. Our definition of the fields Φ_m is related to the notations coming from CFT as follows: Φ_m corresponds to $\Phi_{1,m+1}$. The explicit form of the polynomials $L_{\mathcal{O}}$ for the primary operators is as follows:

$$L_{\Phi_m}(A_1,\dots,A_n|B_1,\dots,B_{2n}) = \prod_{i=1}^{n} A_i^m \prod_{j=1}^{2n} B_j^{-\frac{m}{2}} \,.$$

We explain in Appendix C that this definition agrees with the formulae for the form factors of the operators Φ_1 and Φ_2 given in [6]. We shall return to this definition in the following sections.

Usually the formulae for the form factors are written in a slightly different way [6, 24]. First, the integrals are $(n-1)$-fold, second, instead of the polynomials $\prod (A_i^2 - A_j^2) L_{\mathcal{O}}(A_1,\dots,A_n|B_1,\dots,B_{2n})$ under the integral, one usually has polynomials of the type $\prod (A_i - A_j)\widehat{L}_{\mathcal{O}}(A_1,\dots,A_{n-1}|B_1,\dots,B_{2n})$ with the limitation $\deg_{A_i}(\widehat{L}_{\mathcal{O}}) \leqq n$. The conventional form of the integrands is important for generalization to the case of the arbitrary coupling constant because, in this case, the integrals become more complicated and the above limitation is needed for convergence. In Appendix C we explain briefly how to rewrite formula (31) in the conventional way.

There are two important facts that we learn from the calculations of Appendix C.

1. The substitution into (31) of the polynomial $L_I(A_1,\dots,A_n) = 1$ which corresponds to the unit operator, gives zero because there is no simple pole in the contour integral, in agreement with the fact that the matrix element of the unit operator between the vacuum and an excited state vanishes. Also if we try to substitute into (31) the functions

$$L_{\Phi_{-m}}(A_1,\dots,A_n|B_1,\dots,B_{2n}) = \prod_j A_j^{-m} \prod_j B_j^{\frac{m}{2}} \,,$$

which must correspond to the operators $\Phi_{-m} = \exp(-mi\varphi)$, the integral vanishes. This means exactly that our formulae suit rather the RSG than the SG model. This does not mean that the formulae for the form factors of Φ_{-m} do not exist: they are obtained by SG charge conjugation, we want to say only that these formulae cannot be obtained by putting $L_{\Phi_{-m}}$ into the integral formula.

2. Consider any polynomial $M(A_1;A_2,\dots,A_n)$, anti-symmetric with respect to $A_2,\dots,A_n$. The value of the integral (31) does not change if we add to

$\prod(A_i^2-A_j^2)L_{\mathcal{O}}(A_1,\dots,A_n)$ an "exact form"

$$\sum_k(-1)^k\left(M(A_k;A_1,\dots,\widehat{A}_k,\dots,A_n)\prod_j(B_j+A_k)\right.$$
$$\left.-qM(qA_k;A_1,\dots,\widehat{A}_k,\dots,A_n)\prod_j(B_j-A_k)\right), \tag{34}$$

where $\widehat{A}_k$ means that A_k is omitted.

4. The Semi-Classical Analysis

4.1. The semi-classical analytical quantization. In this section we show that the exact formulae for the form factors in the semi-classical limit can be obtained from the semi-classical quantization of solitons in the analytical variables. We shall proceed to the exact quantization of solitons in the next section.

We shall consider the matrix elements of the operator $\mathcal{O}$ calculated between two n-soliton states, instead of the ones calculated between the vacuum and the state with n solitons and n anti-solitons. This is because they have a more direct semi-classical interpretation. Crossing symmetry relates the connected parts of these two kinds of matrix elements. The essential piece of the n solitons to n solitons form factor is given by

$$\widehat{f}_{\mathcal{O}}(\beta_1',\dots,\beta_n'|\beta_1,\dots,\beta_n)$$
$$=\frac{1}{(2\pi i)^n}\int_C dA_1\cdots\int_C dA_n\prod_{i=1}^n\prod_{j=1}^n\psi(A_i,-B_j')\psi(A_i,B_j)\prod_{i<j}(A_i^2-A_j^2)\prod_{i=1}^n a_i^{-i}$$
$$\times L_{\mathcal{O}}(A_1,\dots,A_n|-B_1',\dots,-B_n',B_1,\dots,B_n)\,, \tag{35}$$

where $\psi(A,B)$ is defined in Eq. (32). We would like to present this formula as

$$\int_{\mathcal{D}}\Psi(A,B')^{\dagger}\mathcal{O}(A,P)\Psi(A,B)W(A)\,dA\,, \tag{36}$$

where the various terms should be interpreted as follows. The function $\Psi(A,B)$ is the n-soliton wave function in the Schrödinger A-representation. The scalar product in the Hilbert space is written in terms of a non-trivial weight $W(A)$ and a possibly non-standard conjugation $\dagger$. The integration domain $\mathcal{D}$ has to be compared to the configuration space of the classical theory. The quantity $\mathcal{O}(A,P)$ is the operator $\mathcal{O}$ realized in terms of A and the conjugated variables P. The detailed discussion of the operators A and P will be given in the next section. Let us first make a semi-classical step in this direction.

Consider the semi-classical approximation of the function $\psi(A,B)$. Asymptotically we have (see Appendix D)

$$\psi(A,B)\sim_{\nu\to\infty}\frac{B^{\nu}}{\sqrt{B^2-A^2}}\exp\left(\frac{\nu}{i\pi}\int_0^A\log\left(\frac{B-A}{B+A}\right)A^{-1}dA\right), \tag{37}$$

where the integral is taken over a contour which does not cross the cut going along the semi-circle in the upper-half plane from B to $-B$. The logarithm is real when $-B < A < B$. An important property of the reflectionless situation is that the integrals over different contours of this kind give the same result when substituted into the exponent. Indeed we have,

$$\int_{\gamma} \log\left(\frac{B-A}{B+A}\right) A^{-1} dA = -2\pi^2 ,$$

where the contour γ is drawn around the cut. Therefore, when ν is integer there is no ambiguity in the exponential in Eq. (37). This is similar to the topological ambiguity of the WZNW action.

Let us now construct the semi-classical wave-function in the A-representation for given values of B. The general rule is [21]:

$$\tilde{\Psi}(A) \simeq (d\mu(A))^{\frac{1}{2}} \exp \frac{1}{i\hbar} S(A) , \tag{38}$$

where $d\mu(A) \equiv \omega^n$ is the Liouville measure at the point A. Let us give some explanation. Integrable models provide examples of a situation which is usually described in textbooks only for the case of one degree of freedom (i.e. when there is only one integral of motion: the energy). In our case in order to construct the semi-classical wave-function in the A-representation, we have to consider the polarization corresponding to the A variables and the integrals of motion B. In these variables the Liouville measure can be written as

$$d\mu(A,B) = [\det \omega(A,B)]\, dA_1 \wedge \cdots \wedge dA_n \wedge dB_1 \wedge \cdots \wedge dB_n .$$

So $(d\mu(A))^{\frac{1}{2}}$ has to be understood as $(\det \omega(A,B))^{\frac{1}{2}}$. In this measure the terms which depend only on B will be omitted since they correspond to the normalization of the wave-functions which we do not consider.

In Eq. (38), $S(A)$ is the semi-classical action. We choose as the semi-classical action the analytical continuation of the KdV reduced action discussed in the previous section, which in the analytical variables A is $S(A) = \sum_k s_k(A_k)$ with

$$s_k(A_k) = \int_0^{A_k} \log\left(\prod_j \frac{B_j - A}{B_j + A}\right) \cdot \frac{dA}{A} - \pi i k \log(A_k) , \tag{39}$$

where the integrals are again taken in the plane with cuts going along the semi-circle from B_j to $-B_j$ for all j. The ambiguity in the definition of the contours is irrelevant for semi-classical quantization at the reflectionless points as it has been explained above. For the case of the generic coupling constant we cannot manage with the variables A_i, instead the variables $\alpha_i = \log(A_i)$ will have to be considered. We shall return to this point in a future publication.

We now have all the necessary ingredients: the classical action and the Liouville measure. So, up to normalization depending only on the B_j, the semi-classical wave function is

$$\tilde{\Psi}(A_1,\ldots,A_n|B_1,\ldots,B_n) \simeq \left(\prod_{i<j}(A_i^2 - A_j^2)\right)^{\frac{1}{2}} \left(\prod_{i=1}^{n}\prod_{j=1}^{n}(A_i^2 - B_j^2)\right)^{-\frac{1}{2}}$$

$$\times \exp\left(\frac{\nu}{i\pi}\left(\sum_{k=1}^{n}\int_0^{A_k} \log\left(\prod_j \frac{B_j - A}{B_j + A}\right) \cdot \frac{dA}{A} - \pi i k \log(A_k)\right)\right) . \tag{40}$$

The Planck constant is identified with $\xi = \frac{\pi}{\nu}$. We shall divide $\tilde{\Psi}$ into the wave function Ψ and a B_j-independent piece which will be put into the integration measure:

$$\tilde{\Psi}(A_1,\dots,A_n|B_1,\dots,B_n) = (W(A_1,\dots,A_n))^{\frac{1}{2}}\,\Psi(A_1,\dots,A_n|B_1,\dots,B_n)\,,$$

where

$$W(A_1,\dots,A_n) = \prod_{i<j}(A_i^2 - A_j^2)\prod_{k=1}^{n} a_k^{-k}\,,$$

$$\Psi(A_1,\dots,A_n|B_1,\dots,B_n) = \prod_{i=1}^{n}\prod_{j=1}^{n}\psi(A_i,B_j)\,,$$

where we recall that $a_k = A_k^{2\nu}$.

Notice that choosing the reduced action as we did, i.e. as the analytical continuation of the KdV action, was crucial for producing the factors $\prod_k a_k^{-k}$ in the weight $W(A)$. Identifying $\Psi(A,B)^\dagger = \Psi(A,-B)$ and the integration domain $\mathscr{D}$ with the contour integrals over C we see that the form factors in the semi-classical limit admit a representation as in Eq. (36).

It remains to find a more physical and geometrical interpretation of the integration domain and of the conjugation $\dagger$ in the semi-classical regime. Under the classical dynamics of n-soliton solutions, every variable A_k runs around a curve going in the lower half-plane anti-clockwise from $-B_k$ to B_k. As we already know there are two difficulties:

1. The action $s(A_k)$ is not real along the classical trajectory.
2. The trajectory itself is complex, and its exact geometry depends on B_k which must become the spectral data after the quantization.

On the other hand the exact quantum formula show that the integration domains should be small circle around the origin. This suggests the following prescription for fixing these problems.

1. Because of quantum-mechanically possible penetration through the barrier one has to consider the closed trajectories, i.e. to add to the classical trajectory of A_k the piece running in the upper half-plane anti-clockwise from B_k to $-B_k$. Classically this extra piece corresponds to an anti-soliton trajectory.

2. In order to eliminate the geometrical dependence of the trajectory of A_k on B_j (leaving the topological one) we replace it by an arbitrary closed curve along which $B_k < |A_k| < B_{k+1}$. There is one more reason for considering the configuration space as the one composed of closed cycles. If we regularize the theory putting it onto a large but finite interval with periodic boundary conditions the trajectories of A_j will become the closed cycles of the same type as described above (see Appendix A).

3. Last but not least, the reality condition is taken as follows:

$$s_k(A_k) = \overline{s_k(\overline{A_k})} \equiv s(A_k)^\dagger\,. \tag{41}$$

Let us explain that this condition is satisfied by (39). If we take

$$\int_0^{A_k}\log\left(\prod_j\frac{B_j - A}{B_j + A}\right)\cdot\frac{dA}{A} \quad\text{and}\quad \int_0^{\overline{A_k}}\log\left(\prod_j\frac{B_j - A}{B_j + A}\right)\cdot\frac{dA}{A}$$

over conjugated contours, then (41) would be true if the integrands were continuous over these contours. However, we defined the logarithms in such a way that they have cuts in the upper half plane, and to reach A_k we have to cross k cuts. The addition of $-\pi ik\log(A_k)$ in (39) is needed to compensate the corresponding discrepancy.

The function $\Psi^\dagger$ is constructed from $s^\dagger$. The choice of the conjugation $\dagger$ ensures that $\Psi(A|B')^\dagger = \Psi(A|-B')$ as it should be.

Let us discuss the condition (41). This condition provides the usual reality condition

$$s_k(A_k) = \overline{s_k(A_k)}$$

if A_k is real and such that $B_k < A_k < B_{k+1}$. The latter segment is the trajectory of A_k which corresponds to real n-soliton solutions of the KdV equation. So, we are dealing with an analytical continuation of KdV mechanics, as it has been explained in Sect. 2.3. This fits perfectly with our assumption that we are actually doing not SG but RSG model, which has to be considered as a continuation of KdV. Also the fact that the involution (41) connects the soliton and the anti-soliton part of the trajectory has a physical explanation: as we already said RSG is neither C nor T invariant, so one has to consider the combined CT invariance. The involution (41) corresponds exactly to the combined CT reflection.

It now is clear that the formula (35) allows a semi-classical interpretation by means of this semi-classical wave function. We stress once again that this semi-classical analysis forces us to introduce the analytical continuation (41). The insertion of a local operator corresponds to the insertion of a polynomial $L_{\mathcal{O}}$ which can also be understood from the expression of the classical local observables in n-soliton variables. We shall return to this point in the section on exact quantization.

Let us emphasize one circumstance. In order to put together two semi-classical wave-functions $\Psi(A,B)$ and $\Psi(A,B')^\dagger$ into the matrix element, one has to be sure that the contours of A_k can be identified. Recall that for these two wave functions the contours are inside $B_k < |A_k| < B_{k+1}$ and $B'_k < |A_k| < B'_{k+1}$ respectively. This shows that the semi-classical quantization works nicely if the soliton states are not very far from each other in the following sense. The sets B_j and B'_j are ordered $B_k < B_{k+1}$ and $B'_k < B'_{k+1}$. The semi-classical formulae are applicable if in addition $B'_k < B_{k+1}$ and $B_k < B'_{k+1}$. In the next section we shall discuss this point further.

4.2. Stationary point calculation of the semi-classical integrals. In this section we shall show that the semi-classical integrals for the form factors are actually given by stationary point contributions. This result, which is interesting by itself, will also provide some justification of the compactification of the configuration space made before.

It is convenient to return to the form factor taken between the vacuum and the state with n solitons and n anti-solitons. One can show that the semi-classical limit allows the necessary analytical continuation. Let us consider the exact quantum formula:

$$\widehat{f}_{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_{2n})_{-\cdots-+\cdots+} = \frac{1}{(2\pi i)^n}\int_{C_1} dA_1\cdots\int_{C_n} dA_n \prod_{i=1}^{n}\prod_{j=1}^{2n}\psi(A_i,B_j)\prod_{i<j}(A_i^2-A_j^2)$$
$$\times L_{\mathcal{O}}(A_1,\ldots,A_n|B_1,\ldots,B_{2n})\prod_{i=1}^{n}a_i^{-i}. \tag{42}$$

In the quantum formula the contours of integration are arbitrary. However, to perform the classical limit we need to take them as follows: if $A_k \in C_k$ then $B_{2k} < |A_k| < B_{2k+1}$. The point is that the asymptotic of $\psi(A,B)$ is not a single-valued function of A, so one cannot move the contours of integration after the asymptotic of the integrand is calculated. The above prescription for the contours agrees with the semi-classical considerations of the previous section. The justification of this choice of contours follows from further stationary phase calculations. The momentum B_i corresponds to either soliton or anti-soliton. The matrix element is related to that considered in the previous section by analytical continuation. The concluding remarks of the previous section show that it is exactly the choice of cycles described above which corresponds to the semi-classical matrix element, and the momenta B_j are partitioned into pairs B_{2i-1}, B_{2i} such that in every pair we find one soliton and one anti-soliton.

The asymptotic of the integrand is written down using the formulae from the previous section. Let us consider the integral with respect to A_k. It contains the divergent exponent:

$$\exp\left(\frac{v}{i\pi}\left(\int\limits_0^{A_k} \log\prod_j\left(\frac{B_j - A_k}{B_j + A_k}\right) dA_k - 2\pi ik\log(A_k)\right)\right),$$

where the last term comes from $\prod_k a_k^{-k}$. One has to consider the stationary phase point at $A = -iF_k$ which solves the equation

$$\log\prod_j\left(\frac{B_j - iF_k}{B_j + iF_k}\right) = 2\pi ik\ .$$

It is easy to see that all F_k are real and positive, moreover $B_{2k} < F_k < B_{2k+1}$ for $k = 1,\dots,n-1$, $F_n = \infty$. The calculation of the second derivatives shows that the integrals over A_k for $k = 1,\dots,n-1$ are given by stationary point contributions coming from $-iF_k$ if the contour of integration goes through the point F_k being topologically equivalent to a circle lying inside the domain $B_{2k} < |A_k| < B_{2k+1}$. Obviously the contours C_k from (42) can be drawn like this. The point ∞ for the n^{th} integral is not a true stationary point, but for the n^{th} integral the divergent exponent vanishes when $A_n \to \infty$, and the integral is sitting on the residue just like in the exact quantum calculation (Appendix C).

The equations for the stationary points can be summarized into the Bethe Ansatz type equation

$$\prod_j\left(\frac{B_j - iF}{B_j + iF}\right) = 1\ .$$

Consider any polynomial $M(A_1;A_2,\dots,A_n)$ anti-symmetric with respect to $A_2,\dots,A_n$. The main contribution into the asymptotic of the integral (42) does not change if we add to $\prod(A_i^2 - A_j^2)L_{\mathscr{C}}(A_1,\dots,A_n)$ the expression

$$\sum_k (-1)^k M(A_k;A_1,\dots,\widehat{A}_k,\dots,A_n)\left(\prod(B_j + A_k) - \prod(B_j - A_k)\right) \tag{43}$$

because it vanishes on the stationary point if $k \leqq n-1$, and cancels residue if $k = n$. Comparing this formula with the exact quantum formula (34) we observe the following nice circumstance. The semi-classics is of course not exact like it happens in geometrical quantization, however one can think of the exact quantum relation (34) as of deformation of (43) which in turn is nothing but the consequence of the equation for the stationary points. The factorization over (43) and (34) leave the same number of independent polynomials of A_k.

The stationary phase calculation fits nicely with the classical picture. The point $-iF_k$ lies in the region acceptable by analytical continuation from the classical soliton trajectory. This fact gives an important justification of our semi-classical methods.

5. Exact Analytical Quantization of Solitons

5.1. Hilbert space and Hermitian structure. In this section we do not specify ξ to the values $\frac{\pi}{\nu}$ until the last subsection in which we shall reconstruct the RSG form factors. Consider the variables $\alpha_j = \log(A_j)$ and $\pi_j = \log(P_j)$. The symplectic form is canonical in terms of these variables:

$$\omega = 2\sum_{j=1}^{n} d\pi_j \wedge d\alpha_j \, .$$

We quantize them in a canonical way: α_j act by multiplication and π_j by differentiation, i.e. $\alpha_j \to \alpha_j$ and $\pi_j \to i\xi\frac{\partial}{\partial\alpha_j}$. Here ξ plays the role of Planck constant. The operators A and P are defined as:

$$A_j = \exp(\alpha_j), \qquad P_j = \exp\left(i\xi\frac{\partial}{\partial\alpha_j}\right) .$$

They satisfy Weyl commutation relations with $q = \exp(i\xi)$:

$$\begin{aligned} &P_j \cdot A_j = qA_j \cdot P_j, \quad \text{with } q = e^{i\xi} \, , \\ &P_k \cdot A_j = A_j \cdot P_k, \quad j \neq k \, . \end{aligned} \tag{44}$$

This definition is rather formal since we have not yet defined the Hilbert space of functions of α_j. Staying on the same formal level one realizes the following important circumstance [12]. There is another pair of operators

$$a_j = \exp\left(\frac{2\pi}{\xi}\alpha_j\right), \qquad p_j = \exp\left(2\pi i\frac{\partial}{\partial\alpha_j}\right)$$

which represent the Weyl algebra but with the dual quantum parameter $\widehat{q} = \exp(\frac{2\pi^2 i}{\xi})$, associated to the non-local symmetry algebra $U_{\widehat{q}}(\widehat{sl_2})$:

$$\begin{aligned} &p_j \cdot a_j = \widehat{q}^2 \, a_j \cdot p_j, \quad \text{with } \widehat{q} = \exp\left(\frac{2\pi^2 i}{\xi}\right), \\ &p_k \cdot a_j = a_j \cdot p_k, \quad j \neq k \, . \end{aligned} \tag{45}$$

The operators a_j and p_j commute with A_j and P_j. The existence of two dual algebras will be crucial in the following.

All these operators act on wave functions $\Psi(\alpha)$. In the reflectionless case $\Psi(\alpha)$ is a single-valued function of A's so we shall write $\Psi(A)$ in that case. The operators A_j act by multiplication whereas P_j act by shifting the argument of Ψ by $i\xi$:

$$P_j\Psi(\alpha_1,\ldots,\alpha_j,\ldots,\alpha_n) = \Psi(\alpha_1,\ldots,\alpha_j + i\xi,\ldots,\alpha_n) \, .$$

To complete the representation of the canonical commutation relations in the Hilbert space, we also need to introduce the scalar product. In order to take into account the reality condition $(e^{i\varphi})^\dagger = e^{i\varphi}$, specific to the RSG model, we define the scalar product in a rather unusual way. Let $\Psi_1(A)$ and $\Psi_2(A)$ be two wave functions, then:

$$\langle \Psi_1|\Psi_2\rangle = \int_C d\alpha_1 \cdots \int_C d\alpha_n \prod_i A_i \prod_{i<j} (A_i^2 - A_j^2)\Psi_1^\dagger(\alpha)Q(a_1,\dots a_n, p_1,\dots, p_n)\Psi_2(\alpha)\,, \tag{46}$$

where the hermitian conjugate wave function $\Psi_1^\dagger(\alpha)$ is defined by:

$$\Psi_1^\dagger(\alpha) = \overline{\Psi_1(\overline{\alpha})}\,. \tag{47}$$

Notice that although this definition involves two complex conjugations, it is anti-linear as it should be. The operator $Q(a_1,\dots,a_n, p_1,\dots, p_n)$ inserted into the integral is not specified here, except for a simple constraint arising from the condition $\overline{\langle \Psi_1|\Psi_2\rangle} = \langle \Psi_2|\Psi_1\rangle$. The exact form of this operator has to be fixed from additional requirements. The role of this operator is similar to that of screening operators in CFT. This analogy is not just a formal coincidence as it will be explained later. Due to commutativity of A, P with a, p, the particular form of the polynomial Q is irrelevant for formal properties of the operators A, P. The contour C is complicated for generic ξ, but reduces to small circle around the origin in the reflectionless case.

As usual, given an operator $\mathcal{O}$, its hermitic conjugated operator $\mathcal{O}^\dagger$ is defined by $\langle(\mathcal{O}^\dagger\Psi_1)|\Psi_2\rangle = \langle\Psi_1|(\mathcal{O}\Psi_2)\rangle$. For the canonical operators A_j and P_j this gives:

$$A_j^\dagger = A_j\,, \tag{48}$$

$$P_j^\dagger = q\prod_{k\neq j}\left(\frac{q^2A_j^2 - A_k^2}{A_j^2 - A_k^2}\right)\cdot P_j = qP_j\cdot\prod_{k\neq j}\left(\frac{A_j^2 - A_k^2}{q^{-2}A_j^2 - A_k^2}\right). \tag{49}$$

It is an interesting check to verify that these relations are compatible with the Weyl commutation relations (44). The relation for $A_j^\dagger$ is obvious from the definition. The formula for $P_j^\dagger$ can be deduced as follows:

$$\begin{aligned}\langle\Psi_1|(P_j\Psi_2)\rangle &= \int_C d\alpha_1\cdots\int_C d\alpha_n \prod_i A_i\prod_{i<j}(A_i^2 - A_j^2)\\ &\quad\times\overline{\Psi_1(\overline{\alpha})}Q(a_1,\dots,a_n,p_1,\dots,p_n)\Psi_2(\dots,\alpha_j + i\xi,\dots)\\ &= \overline{q}\int_C d\alpha_1\cdots\int_C d\alpha_n\prod_i A_i\prod_{i<j}(A_i^2 - A_j^2)\prod_{k\neq j}\left(\frac{\overline{q}^2A_j^2 - A_k^2}{A_j^2 - A_k^2}\right)\overline{\Psi_1(\dots,\overline{\alpha}_j + i\xi,\dots)}\\ &\quad\times Q(a_1,\dots a_n,p_1,\dots,p_n)\Psi_2(\alpha)\,.\end{aligned}$$

The second equality follows from the first by changing variables $\alpha_j \to \alpha_j + i\xi$ which is possible if Ψ is regular as a function of α_j between C and $C + i\xi$. To obtain these formulae we crucially use the fact that q is of modulus one, i.e. $q^{-1} = \overline{q}$. (They will not be correct for real values of q.)

The fact that A_j is hermitian simply expresses the fact that the field $e^{i\varphi}$ is a real field in the RSG theory since it has to be identified with the field Φ_{13} of the minimal conformal model. In other words, the quantum variables A_j are real variables leaving on the unit circle!

5.2. Hamiltonians and quantum τ-functions. We now introduce the quantum version of the hamiltonians H_k defined in Eq. (15). Since the classical hamiltonians are complicated functions of A and P, we have to specify the order of the operators in the quantum formula. We choose the following minimal deformation:

$$\mathcal{T}(P'|A)H_k = \mathcal{T}_k(P'|A) \tag{50}$$

with $P' = (-1)^n P$ and

$$\begin{aligned}\mathcal{T}_k(P|A) &= \sum_{i_1 < i_2 \cdots < i_k} \mathcal{T}(P_1, \ldots, -P_{i_1}, \ldots, -P_{i_k}, \ldots, P_n|A) A_{i_1} A_{i_2} \cdots A_{i_k} \\ &= \sum_{i_1 < i_2 \cdots < i_k} A_{i_1} A_{i_2} \cdots A_{i_k} \mathcal{T}(P_1, \ldots, -qP_{i_1}, \ldots, -qP_{i_k}, \ldots, P_n|A) .\end{aligned} \tag{51}$$

We shall refer to the operator $\mathcal{T}(P|A)$ as the "quantum τ-function." Its explicit expression is:

$$\mathcal{T}(P|A) = 1 + \sum_{p=1}^{n} (-q)^{\frac{p(p-1)}{2}} \sum_{\substack{I \subset \{1, \ldots, n\} \\ |I| = p}} \left(\prod_{\substack{i \in I \\ j \notin I}} \gamma_{ij} \right) \cdot \prod_{i \in I} P_i \tag{52}$$

with

$$\gamma_{ij} = \frac{qA_i + A_j}{A_i - A_j} .$$

Alternatively, $\mathcal{T}(P|A)$ can be recursively defined by Eq. (84) in Appendix E. In the limit $q \to 1$ this formula coincides with the classical τ-functions $\tau(Z|A)$ as defined in Eq. (7). For two particles:

$$\mathcal{T}^{(2)}(P|A) = 1 + \frac{qA_1 + A_2}{A_1 - A_2} P_1 + \frac{qA_2 + A_1}{A_2 - A_1} P_2 - qP_1P_2 .$$

The operators $\mathcal{T}(P|A)$, but not the hamiltonians H_n, are closely related to the Mac Donald difference operators [23]. In particular the terms in $\mathcal{T}(P|A)$ of fixed homogeneity degree in P form a family of commuting difference operators. More precisely, the two generating functions $\mathcal{T}(uP_1, \ldots, uP_n|A)$ and $\mathcal{T}(vP_1, \ldots, vP_n|A)$ commute for any u and v. Note also that $\mathcal{T}_n(P|A) = \mathcal{T}(-P|A) \cdot (\prod_i A_i) = (\prod_i A_i) \mathcal{T}(-qP|A)$. It is convenient to introduce the generating function $\widehat{T}(u)$ for the quantum τ-functions $\mathcal{T}_k : \widehat{T}(u) = \sum_{k=0}^{n} u^k \mathcal{T}_k$. As shown in Appendix E, we have the following expression for $\widehat{T}(u)$:

$$\widehat{T}(u) = \mathcal{T}(\widehat{P}(u)|A) \cdot \prod_{j=1}^{n} (1 + uA_j) \quad \text{with } \widehat{P}_j(u) = P_j \cdot \frac{1 - uA_j}{1 + uA_j} . \tag{53}$$

This can be used to relate the generating functions of the symmetric functions of the B and A operators as:

$$\prod_{j=1}^{n} (1 + uB_j) \cdot \mathcal{T}(P|A) = \mathcal{T}(\widehat{P}(u)|A) \cdot \prod_{j=1}^{n} (1 + uA_j) .$$

It is important to realize that all the hamiltonians H_k commute with the operators a_j and p_j, since they are functions of the A and P only. As we already pointed out, the operators a_j and p_j are associated to the quantum affine symmetry $U_{\widehat{q}}(\widehat{sl_2})$. Thus, the fact that the Schrödinger equation is a difference equation just leave enough room for this non-local symmetry. This symmetry does not have any straightforward classical meaning, it corresponds to the choice of topologically different components of the classical configuration space (different cycles). The way of encoding this topological information into quantum formulae through the algebra which commute with all local observables is, in our opinion, a very interesting feature of quantization of solitons.

The conditions for the H_k to be hermitian are bilinear identities on the quantum τ-functions:

$$\mathscr{T}(P|A)\mathscr{T}_k(P|A)^\dagger = \mathscr{T}_k(P|A)\mathscr{T}(P|A)^\dagger \,. \tag{54}$$

Furthermore, the quantum τ-functions behave nicely under hermitian conjugation. More precisely, the definition of $P_j^\dagger$ implies that the quantum τ-functions are not hermitian for our scalar product but that their hermitian conjugates are again quantum τ-functions:

$$\mathscr{T}(\lambda_1 P_1, \dots, \lambda_n P_n|A)^\dagger = \mathscr{T}(q\lambda_1 P_1, \dots, q\lambda_n P_n|A)$$

for any real parameters $\lambda_1, \dots, \lambda_n$. Here, once again we use the fact that q is of modulus one. In particular, $\mathscr{T}(P|A)^\dagger = \mathscr{T}(qP|A)$. This allows us to rewrite Eq. (54) directly in terms of the quantum τ-functions. For example, for $k = n$ we have $\mathscr{T}_n(P|A)^\dagger = (\prod_i A_i)^\dagger \mathscr{T}(-P|A)^\dagger = (\prod_i A_i)\mathscr{T}(-qP|A)$. The hermiticity condition of H_n is then equivalent to

$$\mathscr{T}(P|A)\mathscr{T}(-P|A) = \mathscr{T}(-P|A)\mathscr{T}(P|A) \,,$$

which is a consequence of the identification of the quantum tau function as the generating function of the Mac Donald difference operators. Besides the case $k = n$, we also checked the hermiticity conditions (54) for all two-particle hamiltonians.

The conditions that the hamiltonians H_k commute can also be rewritten as bilinear identities for the quantum τ-functions:

$$\mathscr{T}_k(P|A)\mathscr{T}_l(P|A)^\dagger = \mathscr{T}_l(P|A)\mathscr{T}_k(P|A)^\dagger \,. \tag{55}$$

To rewrite these conditions in a simple form, we used the hermiticity of the hamiltonians. The hermiticity condition (54) corresponds to Eq. (55) with $l = 0$. Equation (55) can be rewritten as a bilinear identity on the quantum generating function $\widehat{T}(u)$, cf. Appendix E, Eq. (86). We are missing a complete algebraic proof of these conditions for arbitrary n. Although we do not have any doubt that it is true because the hamiltonians admit simultaneous eigenfunctions.

5.3. Quantum separation of variables and soliton wave functions. One of the magic aspects of these quantum hamiltonians is that the corresponding Schrödinger equations admit a separation of variables. As a consequence, they admit simultaneous eigenfunctions. Consider the set of n Schrödinger equations for a state $\Psi(\alpha|\beta)$:

$$\sigma_k(B) \cdot \mathscr{T}(P'|A)\Psi(\alpha|\beta) = \mathscr{T}_k(P'|A)\Psi(\alpha|\beta), \quad \text{for } k = 1, \dots, n \tag{56}$$

with eigenvalues $\sigma_k(B)$. (Recall that $P' = (-1)^n P$.) As shown in Appendix E, we can look for eigenfunctions $\Psi(\alpha|\beta)$ in a factorized form:

$$\Psi(\alpha|\beta) = \widehat{\psi}(\alpha_1|\beta)\widehat{\psi}(\alpha_2|\beta)\cdots\widehat{\psi}(\alpha_n|\beta) \,, \tag{57}$$

provided the function $\widehat{\psi}(\alpha|\beta)$, which depends only on one of the α variables, is a solution of the following separated difference equation:

$$P_j \cdot \widehat{\psi}(\alpha_j|\beta) = \widehat{\psi}(\alpha_j + i\xi|\beta) = \prod_{k=1}^{n} \left(\frac{B_k - A_j}{B_k + qA_j} \right) \widehat{\psi}(\alpha_j|\beta) . \tag{58}$$

This is the quantum analogue of the classical separation of variables. A remarkable fact of Eq. (58) is that its solution can again be factorized into the product of functions depending separately on only one β_k:

$$\widehat{\psi}(\alpha_j|\beta) = \prod_{k=1}^{n} \psi(\alpha_j|\beta_k) ,$$

where $\psi(\alpha|\beta)$ satisfies Eq. (33). This probably reflects the duality symmetry between the α and β variables. It is clear that the function $\psi(\alpha|\beta)$ is defined by the difference equation up to multiplication by any $i\xi$-periodic function. The way to fix this ambiguity is the following: one requires that the function $\psi(\alpha|\beta)$ is regular for $0 < \mathrm{Im}(\alpha) < 2\pi$, $\psi(\alpha|\beta) = O(\exp((\pi/\xi - 1)\alpha)$ when $\alpha \to +\infty$.

5.4. Exact form factors in the reflectionless case. Let us return to the case $\xi = \pi/\nu$ for integer ν. In this case the wave-functions $\Psi(\alpha)$ are $2\pi i$-periodic, that is why we denote them by $\Psi(A)$. Moreover $\Psi(A)$ are polynomials of A, so we take the Hilbert space as the space of polynomials. The operators p_j are identically equal to 1 because they correspond to shift of α's by $2\pi i$. Thus the formula for the scalar product must be of the form

$$\langle \Psi_1 | \Psi_2 \rangle = \int_C dA_1 \cdots \int_C dA_n \prod_{i<j} (A_i^2 - A_j^2) \Psi_1^\dagger(A) Q(a_1, \ldots, a_n) \Psi_2(A) .$$

In the reflectionless case the contour C are small contour around the origin. We recall that for $\xi = \frac{\pi}{\nu}$ with ν integer the function $\psi(\alpha|\beta)$ is

$$\psi(\alpha|\beta) = \prod_{j=1}^{\nu-1} (B - Aq^{-j}) .$$

It is time to discuss the local operators in this A-representation. As we have seen from the exact quantum formulae, in order to insert the primary field Φ_m into the matrix element one has to put under the integral the expression

$$\prod_{j=1}^{n} (B'_j)^{-\frac{m}{2}} \prod_{j=1}^{n} A_j^m \prod_{j=1}^{n} B_j^{-\frac{m}{2}} .$$

On the other hand, in the classical theory the fields $\Phi_m = e^{im\varphi}$ are represented on the n-soliton solutions by $\Phi_m = \prod_j A_j^m \prod_j B_j^{-m}$. After the quantization we have a self-adjoint operator $H_n = H_n(A,P)$ such that $H_n \Psi(A,B) = (\prod_j B_j) \Psi(A,B)$. So, the comparison of the quantum and classical formulae shows that the classical expression $\prod A_j^m H_n^{-m}$ must be ordered for quantization as follows:

$$\Phi_m(A,P) = H_n^{-m/2}(A,P) \left(\prod_{j=1}^{n} A_j^m \right) H_n^{-m/2}(A,P) . \tag{59}$$

This ordering prescription ensures that Φ_m is a real field, $\Phi_m^\dagger = \Phi_m$, since H_n and A_j are hermitic. We hope that the same notation for the operator Φ_m acting in all the space of states of RSG or restricted to n-soliton subspace is not misleading.

Now we can fix the function $Q(a)$. The contours of integration are drawn around $A = 0$, so $Q(a)$ has to be taken as $\prod a_k^{-m_k}$: positive powers of a_k would give zero scalar product. It is also clear that the m_k's must all be different for the anti-symmetry coming from $\prod_{i<j}(A_i^2 - A_j^2)$. Considering the form of the function ψ one realizes that if one of the m_k is greater than n then the matrix element corresponding to the operator Φ_1 vanishes because the contour of integration with respect to A_k can be moved to infinity. Thus we are left with the only possible choice

$$m_k = k\ .$$

This is exactly what we have in the formulae known from bootstrap.

6. Concluding Remarks

Let us describe possible directions of future developments.

We have constructed the local integrals of motion in terms of the operators A and P. One can consider the descendants of the primary fields with respect to these operators. However, to consider the full space of local fields we need to construct the Virasoro algebras in terms of A and P and to consider the descendants of the primary fields. We are sure that it can be done. In this way we must be able to identify the SG local operators described in [24] with those coming from the CFT description. For these computations we do not really need to study deeper the situation of generic coupling constant since the formulae from Subsect. 5.2 are absolutely general.

There is another point for which the consideration of the generic coupling constant is important. We have seen that there is a dual Weyl algebra composed of a and p which commute with the operators A and P. It is easy to argue that the non-local integrals can be expressed in terms of a and p. The commutativity of local and non-local integrals follows from the commutativity of A, P with a, p. The non-local charges represent the quantum loop algebra $U_{\hat{q}}(\widehat{sl_2})$. It would be very interesting to find their expressions in terms of a and p. On the other hand the non-local charges in the conformal limit correspond to screening operators.

We want also to remind the reader that we were actually not working with complete form factors. We omitted certain multipliers which are the same for all operators, the normalization of the wave functions in the logic of this paper. We hope to explain how this piece appears in our approach in a further publication. Here we just would like to stress the analogy with the method of orbits in coadjoint representations. Every orbit is quantized independently giving an irreducible representation of the group, but combining them in the regular representation requires to take into account the Plancherel measure: this is exactly the analog of the omitted normalization factors.

There are amusing coincidences between many tools used in this paper and those existing in the works on lattice quantization of SG [25]: the Weyl algebras, the functions of the type of $\psi(\alpha, \beta)$. The latter are now called quantum dilogarithms because they provide a deformation of dilogarithm functions. It might be possible that this coincidence is not occasional. The classical soliton is similar to the step-function, so one can imagine that exact quantum mechanics of n-solitons is related to the theory on the lattice with n sites.

These is also a no-empty intersection with recent works on the Calogero models, Mac Donald polynomials and affine Hecke algebras. We realize that techniques very similar to those we used to deal with the quantum τ-function, i.e. the generating function of Mac Donald difference operators, may be used to separate the variables in these operators. One potential application of this remark would be a more detailed description of the algebra of the Mac Donald polynomials.

7. Appendix A: The Analytical Variables and Finite Zone Solutions

Let us explain the origin of the A variables as a remnant of the parametrization of the finite zone solutions, when they degenerate to the soliton case. We shall do it in the simplest case of the KdV equation, cf. e.g. [27]. We start with an hyperelliptic curve Γ of genus n, and a divisor D of degree n on it.

$$\Gamma : s^2 = R(\lambda), \quad R(\lambda) = \prod_{j=0}^{2n} (\lambda - \lambda_j)\,,$$
$$D := (\nu_1, \nu_2, \ldots, \nu_n)\,.$$

We describe Γ as a two sheeted cover of the λ plane. We put cuts on the real axis on the intervals $(-\infty, \lambda_0)$ and $(\lambda_{2i-1}, \lambda_{2i})$, $i = 1, \ldots, n$. The quantities ν_i in the divisor D denote the λ coordinates of the points of the divisor. One should keep in mind that to specify the points themselves, one has to choose the sheet above $\lambda = \nu_i$. With these data we construct the Baker–Akhiezer function which is the unique function with the following analytical properties:

- It has an essential singularity at the point P_+ above infinity: $\psi(x, \lambda) = e^{kx}(1 + O(1/k))$ with $k = \sqrt{\lambda}$.
- It has n simple poles outside P_+. The divisor of these poles is D.

Considering the quantity $-\partial_x^2 \psi + \lambda\psi$, we see that it has the same analytical properties as ψ itself, apart for the first normalization condition. Hence, because ψ is unique, there exists a function $u(x)$ such that

$$-\partial_x^2 \psi + u(x)\psi + \lambda\psi = 0\,. \tag{60}$$

We recognize the usual linear system associated to the KdV equation. One can give various explicit constructions of the Baker–Akhiezer function. The most popular one is in terms of theta functions. However, for our purpose, another representation is more suitable. Let us introduce the divisor $Z(x)$ of the zeroes of the Baker–Akhiezer function. It is of degree n:

$$Z(x) := (\mu_1(x), \mu_2(x), \ldots, \mu_n(x))\,.$$

One can find the equations of motion for the divisor $Z(x)$. Consider the function $\partial_x\psi/\psi$. It is a meromorphic function on Γ, it has poles at the points $\mu_i(x)$ and behaves like $k + O(1/k)$ in the vicinity of the point P_+. Hence we can write

$$\frac{\partial_x \psi}{\psi} = \frac{\sqrt{R(\lambda)} + Q(x, \lambda)}{\prod_{i=1}^{g} (\lambda - \mu_i(x))}\,, \tag{61}$$

where Q is a polynomial of degree $n-1$ in λ. We determine Q by requiring that $\frac{\partial_x \psi}{\psi}$ has a pole above $\lambda = \mu_i(x)$ only on one of the two sheets (say $\sqrt{R(\lambda)}$). Then

$$Q(x, \mu_i(x)) = \sqrt{R(\mu_i(x))} \,. \tag{62}$$

Hence

$$Q(x, \lambda) = \sum_i \sqrt{R(\mu_i(x))} \frac{\prod_{j \neq i} (\lambda - \mu_j(x))}{\prod_{j \neq i} (\mu_i(x) - \mu_j(x))} \,. \tag{63}$$

On the other side, in the vicinity of $\mu_i(x)$, we have:

$$\frac{\partial_x \psi}{\psi} = -\frac{\partial_x \mu_i(x)}{\lambda - \mu_i(x)} + O(1) \,. \tag{64}$$

Comparing Eq. (61) and Eq. (64), we get the equation of motions:

$$\partial_x \mu_i(x) = -2 \frac{\sqrt{R(\mu_i(x))}}{\prod_{j \neq i} (\mu_i(x) - \mu_j(x))} \,. \tag{65}$$

One can now reconstruct the Baker–Akhiezer function itself. Indeed, inserting Eq. (65) into (61) we get:

$$\frac{\partial_x \psi}{\psi} = \frac{\sqrt{R(\lambda)}}{P(\lambda, x)} - \frac{1}{2} \sum_i \frac{1}{\lambda - \mu_i(x)} \partial_x \mu_i(x) \,,$$

where the polynomial $P(\lambda, x)$ is defined as $P(\lambda, x) = \prod_i (\lambda - \mu_i(x))$. Therefore [27]

$$\frac{\psi(\lambda, x)}{\psi(\lambda, x_0)} = \sqrt{\frac{P(\lambda, x)}{P(\lambda, x_0)}} \exp \left(\int_{x_0}^{x} \frac{\sqrt{R(\lambda)}}{P(\lambda, x)} dx \right) \,. \tag{66}$$

One can also reconstruct the potential $u(x)$ directly in terms of the data $\mu_i(x)$ and λ_j. Inserting back Eq. (66) into Eq. (60), we get the polynomial identity

$$R = -\frac{1}{2} P P'' + \frac{1}{4} P'^2 + (u + \lambda) P^2 \,.$$

Comparing the terms λ^{2n} we obtain

$$u = 2 \sum_{i=1}^{n} \mu_i(x) - \sum_{i=0}^{2n} \lambda_j \,. \tag{67}$$

We are now ready to analyze the soliton limit. It corresponds to the following limiting configuration:

$$\lambda_0 = 0, \qquad \lambda_{2j-1} = \lambda_{2j} = B_j^2; \qquad \lambda = A^2; \qquad \mu_j = A_j^2 \,. \tag{68}$$

In this limiting configuration, Eq. (67) reduces to Eq. (21). Under the condition (68), the full curve Γ becomes the A plane but with the points B_j and $-B_j$, $j = 1, \ldots, n$ identified. In this limit we have

$$\sqrt{R(\lambda)}|_{\lambda = A^2} = -A \prod_{j=1}^{n} (A^2 - B_j^2), \qquad P(\lambda, x)|_{\lambda = A^2} = \prod_{j=1}^{n} (A^2 - A_j^2) \,.$$

The equations of motion for the A_i become (compare with Eq. (20))

$$\partial_x A_i = \prod_j (A_i^2 - B_j^2) \prod_{j \neq i} \frac{1}{A_i^2 - A_j^2} \,. \tag{69}$$

We can also obtain the degeneration of the Baker–Akhiezer function. Noticing that

$$\left. \frac{\sqrt{R(\lambda)}}{P(\lambda,x)} \right|_{\lambda=A^2} = -A - \frac{1}{2} \sum_i \left(\frac{1}{A - A_i} + \frac{1}{A + A_i} \right) \partial_x A_i \,,$$

we can integrate Eq. (66) to get

$$\frac{\psi(A,x)}{\psi(A,x_0)} = \prod_{j=1}^{n} \left(\frac{A - A_j(x)}{A - A_j(x_0)} \right) \exp(-A(x - x_0)) \,. \tag{70}$$

As we see, in this limit the Baker–Akhiezer function contains a single exponential factor instead of two as one would expect from the second order linear equation Eq. (60). This corresponds exactly to the fact that for soliton solutions the potential u in Eq. (60) is reflectionless. It is manifest in Eq. (70) that the variables A_j are the zeroes of the Baker–Akhiezer function.

As we said, the points B_j and $-B_j$ are identified. This means that the Baker–Akhiezer function should satisfy the conditions $\psi(B_i,x) = \psi(-B_i,x)$, $i = 1,\ldots,n$. Writing these conditions, we get

$$\prod_{j=1}^{n} \frac{B_i - A_j(x)}{B_i + A_j(x)} = Y_i e^{2B_i x} \,, \tag{71}$$

where Y_i depends only on the initial conditions at $x = x_0$. Comparing with Eqs. (4, 9), we recognize the solution of the equations of motion.

In the sine-Gordon soliton case, we identify $x \equiv x_-$. The x_+ dependence is simply reintroduced by replacing in Eq. (71)

$$e^{2B_i x} \longrightarrow e^{2B_i x_- + 2B_i^{-1} x_+} \,.$$

Let us discuss now the motion of the divisor $Z(x_-,x_+)$. In the finite zone case, the point $\mu_i(x_-)$ has a quasi-periodic motion on the real axis in the interval $\lambda_{2i-2} \leqq \mu_i(x_-) \leqq \lambda_{2i-1}$.

Consider the x_+ motion in the soliton case when we have only two points. When $x_+ = -\infty$, the points A_1 and A_2 start from B_1 and B_2 respectively. When x_+ increases, the points A_1 and A_2 start to move to the right. Notice that the point $A = \infty$ is regular for Eq. (69), so the point A_2 passes smoothly from $+\infty$ to $-\infty$, and then continues to move towards $-B_2$ which is reached at some time x_*. At the *same* time x_*, A_1 reaches B_2 so that in the right-hand side of Eq. (71) the pole at $A_2 = -B_2$ is cancelled by the zero at $A_1 = B_2$. Hence everything remains finite for a finite time x_*. At this time, the point A_1 jumps to $-B_2$ and ends its motion at the point $-B_1$. Similarly, A_2 jumps to B_2 and continues its motion up to $-B_2$ again. The case of generic n is similar. Altogether, A_i starts at B_i and ends at $-B_i$.

Equation (70) is easily related to the Jost solution of Eq. (60). In our context, the Jost solution is defined by the normalization condition $\lim_{x_- \to -\infty} \psi_{\text{Jost}}(x_-) = \exp(-Ax_-)$. Hence, we find

$$\psi_{\text{Jost}}(x_-) = \lim_{x_0 \to -\infty} e^{-Ax_0} \frac{\psi_{\text{Baker}}(x_-)}{\psi_{\text{Baker}}(x_0)} = \prod_{j=1}^{n} \left(\frac{A - A_j(x_-)}{A - B_j} \right) \exp(-Ax_-) \,,$$

where we used the fact that $\lim_{x_0 \to -\infty} A_j(x_0) = B_j$. Therefore B_j are the poles of the Jost solution and $A_j(x)$ are its zeroes.

Finally, we would like to discuss the important question of the Poisson structures of the KdV equation. As we know, we have a whole hierarchy of these structures. One can describe the restrictions of the symplectic forms to the manifold of finite zone solutions [27]. Let $\Omega^{(k)}$ be the restricted k^{th} symplectic form. Then we have in terms of analytical variables [19, 20]:

$$\Omega^{(k)} = \sum_{i=1}^{n} d\mathscr{P}(\mu_i) \wedge \frac{d\mu_i}{\mu_i^{k-1}} , \tag{72}$$

where $\mathscr{P}(\lambda)$ is the pseudo momentum defined by:

$$\mathscr{P}(\lambda) = \log \frac{\psi(\lambda, x = L)}{\psi(\lambda, x = -L)} .$$

To compute the quasi-momentum in the soliton limit, we choose the normalization point to be $x_0 = -L$, and we send $L \to \infty$. According to the previous discussion, we have $A_i(L) \to -B_i$, $A_i(-L) \to B_i$. Using Eq. (70), we get

$$\mathscr{P}(A) = -2LA - \log \prod_j \left(\frac{B_j - A}{B_j + A} \right) \bmod i\pi .$$

Hence,

$$\Omega^{(k)} = 2 \sum_{i=1}^{n} d \log \prod_j \left(\frac{B_j - A_i}{B_j + A_i} \right) \wedge \frac{dA_i}{A_i^{2k-3}} .$$

The form used in Eq. (25) corresponds to $k = 2$ i.e., to the second Hamiltonian structure of KdV. We recall once more that the second Hamiltonian structure of the KdV equation is precisely the Virasoro algebra.

8. Appendix B: From the $\{X, B\}$ to the $\{A, B\}$ Variables

Before explaining the proof of the formula for the τ-functions in the $\{A, B\}$ variables we need to gather a few facts concerning the τ-functions. The τ-functions are defined by the determinant (3): $\tau_\pm = \det(1 \pm V)$. Most of the proof will be recursive using the recursion relation (8) satisfied by them, cf. e.g. ref. [26]:

$$\tau^{(n)}(X|B) = \tau^{(n-1)}(X|B) + \tau^{(n-1)}(\beta_{kn}^2(B) X_k|B) X_n . \tag{73}$$

For comparison with the quantum formula, it is also useful to know the explicit expression of the τ-function not in the X_j variables but in the variables $Y_j = X_j \prod_{k \neq j} \left(\frac{B_j - B_k}{B_j + B_k} \right)$:

$$\tau^{(n)}(Y|B) = 1 + \sum_{p=1}^{n} \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I| = p}} \prod_{\substack{i \in I \\ j \notin I}} \beta_{ij}^{-1}(B) \cdot \prod_{i \in I} Y_i \tag{74}$$

with $\beta_{ij}(B) = \frac{B_i - B_j}{B_i + B_j}$.

Let us now prove the formula (12) for the τ-functions. We recall them to ease the reading:

$$\tau_+^{(n)} = 2^n \left(\prod_{j=1}^{n} B_j \right) \frac{\prod_{i<j}(A_i + A_j) \prod_{i<j}(B_i + B_j)}{\prod_{i,j}(B_i + A_j)} ,$$

$$\tau_-^{(n)} = 2^n \left(\prod_{j=1}^{n} A_j \right) \frac{\prod_{i<j}(A_i + A_j) \prod_{i<j}(B_i + B_j)}{\prod_{i,j}(B_i + A_j)} . \tag{75}$$

The upper index n refers to the n-soliton solutions. Eqs. (75) are two identities between rational functions of A and B once the expression of X as a function of A and B,

$$Y_j^{(n)} = X_j^{(n)} \cdot \prod_{k \neq j} \left(\frac{B_j - B_k}{B_j + B_k} \right) = \prod_{k=1}^{n} \left(\frac{B_j - A_k}{B_j + A_k} \right) \tag{76}$$

has been inserted into the τ-functions.

Let us denote by $\hat{\tau}_\pm^{(n)}(A|B)$ the τ-functions with X_j expressed in terms of A and B. Since $\tau_\pm^{(n)}(X|B)$ are symmetric in X_j, so is $\hat{\tau}_\pm^{(n)}(A|B)$ as a function of A_j. Since permuting the B permutes the X, the functions $\hat{\tau}_\pm^{(n)}(A|B)$ are also symmetric in the B. Thus, the identities (75) are equalities between rational functions symmetric in A and B.

Let us first show that $\hat{\tau}_\pm^{(n)}(A|B)$ has poles only at $A_j + B_k = 0$. In view of the explicit expressions of the τ-functions and of the $X_j^{(n)}$, $\hat{\tau}_\pm^{(n)}(A|B)$ has potentially simple poles at $A_j + B_k = 0$ and $B_j \pm B_k = 0$. The expression (74) of the τ-function in terms of Y shows that there are no poles at $B_j + B_k = 0$. Similarly, the expression (7) of the τ-functions in terms of X shows that the potential poles at $B_j - B_k = 0$ are associated to X_j and X_k. Using twice the recursion relation (73) shows that these poles cancel against $\beta_{ij}^2(B)$. Thus, $\hat{\tau}_\pm^{(n)}(A|B)$ can be written as

$$\hat{\tau}_\pm^{(n)}(A|B) = \frac{Q_\pm^{(n)}(A|B)}{\prod_{j,k}(A_j + B_k)} ,$$

where $Q_\pm^{(n)}(A|B)$ are polynomials, symmetric in A and B, and of degree at most n in each variable.

To prove the identities (75) it is then enough by symmetry to show that the functions have identical residues and that they coincide at particular points. Again by symmetry, it is enough to check it for the pole at $A_n + B_n = 0$ and at the point $A_n = B_n$. To keep the size of this appendix reasonable we shall describe the proof for τ_+ only. The proof for τ_- is similar. We shall prove it by induction assuming that the identities are true up to $n-1$ solitons (they are obviously true in the one soliton case). The recursive proof is based on the three following relations:

i) The τ-function satisfies the recursion relation (73).

ii) The r.h.s of Eq. (75) satisfies the following recursion relation:

$$\mathrm{rhs}^{(n)} = \left(\frac{2B_n}{A_n + B_n} \right) \prod_{j \neq n} \frac{(A_n + A_j)(B_n + B_j)}{(A_n + B_j)(B_n + A_j)} \cdot \mathrm{rhs}^{(n-1)} . \tag{77}$$

iii) As functions of A and B the variables X_j satisfy the following recursion relations:

$$X_j^{(n)} = \left(\frac{B_n + B_j}{B_j - B_n}\right)\left(\frac{B_j - A_n}{B_j + A_n}\right) X_j^{(n-1)} \quad \text{for } j = 1, \ldots, n-1\,, \tag{78}$$

$$X_n^{(n)} = \left(\frac{B_n - A_n}{B_n + A_n}\right) \prod_{j \neq n} \left(\frac{B_n + B_j}{B_n - B_j}\right)\left(\frac{B_n - A_j}{B_n + A_j}\right)\,, \tag{79}$$

where $X_j^{(n-1)}$ is independent of A_n, B_n.

Let us first compare the values of both sides of Eq. (75) at the point $A_n = B_n$. Using Eq. (77), we have for the r.h.s.:

$$\mathrm{rhs}^{(n)}|_{A_n=B_n} = \mathrm{rhs}^{(n-1)}\,.$$

For the other side we remark that $X_n^{(n)}$ vanishes for $A_n = B_n$. Therefore the recursion relation (73) for the τ-functions implies that

$$\mathrm{lhs}^{(n)}|_{A_n=B_n} = \mathrm{lhs}^{(n-1)}\,.$$

Thus, both sides of Eq. (75) coincide at $A_n = B_n$.

Let us now compute the residue at the simple pole $A_n = -B_n$. The pole in $\hat{\tau}_+^{(n)}(A|B)$ comes from the pole of $X_n^{(n)}$ at $A_n = -B_n$. Its residue is

$$\mathrm{Res}\,(X_n^{(n)})|_{A_n=-B_n} = 2B_n \prod_{j \neq n} \left(\frac{B_n + B_j}{B_n - B_j}\right)\left(\frac{B_n - A_j}{B_n + A_j}\right)\,.$$

Furthermore, Eq. (78) gives for $1 \leqq j \leqq n-1$:

$$X_j^{(n)}|_{A_n=-B_n} = \beta_{jn}^{-2}(B) X_j^{(n-1)}\,.$$

Therefore the factor $\beta_{jn}^2(B)$ cancels in the recursion relation (73) and we get

$$\mathrm{Res}\,(\hat{\tau}_+^{(n)}(A|B))|_{A_n=-B_n} = \mathrm{Res}\,(X_n^{(n)})|_{A_n=-B_n} \cdot \hat{\tau}_+^{(n-1)}(A|B))\,.$$

On the other hand, Eq. (77) implies

$$\begin{aligned}\mathrm{Res}(\mathrm{rhs}^{(n)})|_{A_n=-B_n} &= 2B_n \prod_{j \neq n} \frac{(A_j - B_n)(B_n + B_j)}{(B_j - B_n)(B_n + A_j)} \cdot \mathrm{rhs}^{(n-1)} \\ &= \mathrm{Res}\,(X_n^{(n)})|_{A_n=-B_n} \cdot \mathrm{rhs}^{(n-1)}\,.\end{aligned}$$

Thus, the residues of both sides of Eq. (75) at $A_n = -B_n$ coincide by the induction hypothesis. This concludes the proof of the identities (75).

Let us now consider the formula expressing the symmetric functions $\sigma_k(A)$ as functions of the $\{X, B\}$ variables. As pointed out in the main text, the defining relations (9) or (76) can be considered as a system of equations for the symmetric functions of A. We claim that the solution to this system can be written as:

$$\sigma_k(A) = \frac{\tau_k(X'|B)}{\tau(X'|B)}\,, \tag{80}$$

where $X' = (-1)^n X$ and

$$\tau_k(X_1,\dots,X_n|B_1,\dots,B_n)$$
$$= \sum_{i_1<i_2<\cdots<i_k} B_{i_1}B_{i_2}\cdots B_{i_k}\tau(X_1,\dots,-X_{i_1},\dots,-X_{i_k},\dots,X_n|B_1,\dots,B_n)\,.$$

Let us compare Eqs. (9) and (14). A quick look reveals that they are identical provided we exchange A with B and Y with P. Hence, the proof of Eq. (80) is identical to the proof of Eq. (15), which is a limiting case of the quantum formula whose proof is given in Appendix E.

This somewhat mysterious formula can be related to well known results connecting the Baker function and the tau function. One first reintroduces all the times by the substitution $X_i \to X_i(t) = X_i(0)\exp(2\xi(B_i,t))$, where $\xi(A,t) = \sum_i A^{2i-1}t_{2i-1}$. Next, using the generating function Eq. (17) we have

$$\prod_i (1+uA_i) = \sum_k u^k \frac{\tau_k}{\tau} = \prod_j (1+uB_j)\frac{\tau(\frac{1-uB_i}{1+uB_i}X_i|B)}{\tau(X|B)}\,.$$

On the other hand, the multi-time Baker function reads

$$\frac{\psi(A,t)}{\psi(A,t^{(0)})} = \prod_i \left(\frac{1-A_i(t)/A}{1-A_i(t^{(0)})/A}\right) e^{-\xi(A,t)+\xi(A,t^{(0)})}\,.$$

Combining these two formulae, we can identify the Baker function as

$$\psi(A,t) = \frac{\tau(\frac{1+B_i/A}{1-B_i/A}X_i|B)}{\tau(X|B)}e^{-\xi(A,t)}\,.$$

Since $\frac{1+B_i/A}{1-B_i/A} = \exp(2\sum_i \frac{1}{2n-1}\left(\frac{B_i}{A}\right)^{2n-1})$ we find

$$\psi(A,t) = \frac{\tau(t_{2n-1} + \frac{1}{2n-1}\frac{1}{A^{2n-1}})}{\tau(t)}e^{-\xi(A,t)}\,,$$

and we recognize the well known Sato formula.

9. Appendix C: Information about the Integral Formulae

In this appendix we explain how the formulae for the form factors given in Sect. 3 agrees with the conventional ones [6]. We have to show how to reduce the number of integrals and the degree of polynomials.

In order to reduce the number of integrals by one let us consider the integral with respect to A_n in which we would like to move the integration contour to infinity. When $A_n \to \infty$, one has

$$\prod_{j=1}^{2n} \psi(A_n,B_j)\prod_{i<n}(A_i^2 - A_n^2)a_n^{-n} = A_n^{-2}\left(1 - \frac{q+1}{q-1}A_n^{-1}\sum_j B_j + O(A_n^{-2})\right)\,. \quad (81)$$

So, the integral over A_n is

$$\frac{1}{2\pi i}\int_C A_n^{-2}\left(1-\frac{q+1}{q-1}A_n^{-1}\sum_j B_j+O(A_n^{-2})\right)$$
$$\times L_{\mathcal{O}}(A_1,\dots,A_{n-1},A_n|B_1,\dots,B_{2n})\,dA_n\,. \tag{82}$$

It is important that there are no contributions with $a_n = A_n^{2\nu}$ which means that the integral is essentially independent of the coupling constant (it depends on ν only through the constants like $\frac{q+1}{q-1}$ in (82)). The first two terms written in (82) are sufficient to calculate this integral for

$$L_{\Phi_1}(A_1,\dots,A_n|B_1,\dots,B_{2n})=\prod_j A_j\prod_j B_j^{-\frac{1}{2}}\,,$$

and

$$L_{\Phi_2}(A_1,\dots,A_n|B_1,\dots,B_{2n})=\prod_j A_j^2\prod_j B_j^{-1}\,,$$

the results being respectively

$$\prod_{i=1}^{n-1}A_i\prod_j B_j^{-\frac{1}{2}}\quad\text{and}\quad\frac{q+1}{q-1}\left(\sum_j B_j\right)\prod_{i=1}^{n-1}A_i^2\prod_j B_j^{-1}\,.$$

This calculation gives agreement with the formulae from [6]. To calculate the integral over A_n for a higher operator Φ_m one has to take into account higher contributions to (81), it would be nice to find these primary operators among those described in [24].

Let us now explain why the reduction of the degree of polynomials is possible. The formula for $\widehat{f_{\mathcal{O}}}$ is composed of one-fold integrals of the type

$$\int_C\prod_{j=1}^{2n}\psi(A,B_j)a^{-k}L(A)\,dA\,.$$

One can reduce the degree of the polynomial $L(A)$ using the following circumstance. Due to the fact that the function $\psi(A,B)$ satisfies Eq. (33),

$$\psi(Aq,B)=\frac{B-A}{B+qA}\psi(A,B)\,,$$

the polynomials of the form

$$L(A)=M(A)\prod_j(B_j+A)-qM(Aq)\prod_j(B_j-A)\simeq 0 \tag{83}$$

for any polynomial $M(A)$. Here $L(A)\simeq 0$ means that $L(A)$ produces zero when substituted into the integral. This allows for any given polynomial $L(A)$ to find such a polynomial $L'(A)$ such that $L(A)\simeq L'(A)$ and $\deg(L')\leqq 2n-1$. To do that one has to find a polynomial $M(A)$ such that

$$L(A)=M(A)\prod_j(B_j+A)-qM(Aq)\prod_j(B_j-A)+L'(A)$$

with $\deg(L')\leqq 2n-1$. This is always possible by induction.

10. Appendix D: A Semi-Classical Limit

We start with

$$\psi(A,B) = \prod_{j=1}^{\nu-1} (B - Aq^{-j}); \quad q = e^{i\frac{\pi}{\nu}} .$$

We recall the obvious formula

$$\sum_{n=1}^{N-1} f(n\Delta) = \frac{1}{\Delta}\sum_{n=0}^{N-1} \frac{f(n\Delta)+f((n+1)\Delta)}{2}\Delta - \frac{1}{2}(f(0)+f(N))$$

$$= \frac{1}{\Delta}\int_0^{N\Delta} f(x)\,dx - \frac{1}{2}(f(0)+f(N)) + O(\Delta) .$$

Using this formula we find

$$\log\psi = \frac{\nu}{\pi}\int_0^{\pi} \log(B - Ae^{-ix})\,dx - \frac{1}{2}(\log(B-A)+\log(B+A)) + O(\Delta) .$$

The integral defines an analytical function of A in the plane with a cut which is a semi-circle from B to $-B$ in the upper half plane. Remark that for $A = 0$, this integral is real and its value is $\pi \log B$. To proceed, we perform some formal manipulations on the integral. We have

$$\begin{aligned}\int_0^{\pi} \log(B - Ae^{-ix})\,dx &= i\int_{-A}^{A} \log(B+A)\frac{dA}{A} \\ &= i\int_0^{A} \log(B+A)\frac{dA}{A} - i\int_0^{-A} \log(B+A)\frac{dA}{A} \\ &= i\int_0^{A} \log\left(\frac{B+A}{B-A}\right)\frac{dA}{A} + \pi\log B ,\end{aligned}$$

where the last term has been added to normalize the function by its value at $A = 0$, thereby fixing the ambiguities of the formal manipulations. Putting everything together, we get

$$\psi(A,B) = \frac{B^{\nu}}{\sqrt{B^2 - A^2}} \exp\left[-i\frac{\nu}{\pi}\int_0^{A} \log\left(\frac{B-A}{B+A}\right)\frac{dA}{A}\right] ,$$

which is the result quoted in the text.

11. Appendix E: A Proof of the Quantum Separation of Variables

Before proving the separation of variables in the quantum theory, we present a proof for the formula of the generating function $\widehat{T}^{(n)}(u) = \sum_k u^k \mathscr{T}_k^{(n)}$. As it is defined in (52), the quantum τ-function satisfies the following recursion relation:

$$\mathscr{T}^{(n)}(P|A) =: \mathscr{T}^{(n-1)}(\gamma_{kn}P_k|A_k) + \left(\prod_{k\neq n} \gamma_{nk}\right) \mathscr{T}^{(n-1)}(\overline{\gamma}_{kn}^{-1}P_k|A_k)\,P_n : , \quad (84)$$

where

$$\gamma_{ij} = \frac{qA_i + A_j}{A_i - A_j}, \quad \text{and} \quad \bar{\gamma}_{ij} = \frac{q^{-1}A_i + A_j}{A_i - A_j} = -\bar{q}\gamma_{ji} .$$

The double dots : : mean writing the P's on the right. In the classical limit $q \to 1$ this is equivalent to the relation (8).

Let $\mathscr{T}_k^{(n)}$ be the operators defined in Eq. (51) in the n solitons case. By convention we set: $\mathscr{T}_0^{(n)} = \mathscr{T}^{(n)}(P|A)$ and $\mathscr{T}_k^{(n)} = 0$ for $k < 0$ or $k > n$. From Eq. (84), we deduce a recursion relation for the $\mathscr{T}_k$:

$$\mathscr{T}_k^{(n)} = \tilde{\mathscr{T}}_k^{(n-1)} + \tilde{\tilde{\mathscr{T}}}_k^{(n-1)} P_n + A_n(\tilde{\mathscr{T}}_{k-1}^{(n-1)} - q\tilde{\tilde{\mathscr{T}}}_{k-1}^{(n-1)} P_n) , \tag{85}$$

where

$$\tilde{\mathscr{T}}_k^{(n-1)} =: \mathscr{T}_k^{(n-1)}(\gamma_{jn}P_j|A_j) : ,$$

$$\tilde{\tilde{\mathscr{T}}}_k^{(n-1)} =: \left(\prod_{j \neq n} \gamma_{nj} \right) \mathscr{T}_k^{(n-1)}(\bar{\gamma}_{jn}^{-1}P_j|A_j) : .$$

Summing up Eqs. (85) and defining $\tilde{T}^{(n-1)}(u) = \sum_k u^k \tilde{\mathscr{T}}_k^{(n-1)}$ and $\tilde{\tilde{T}}^{(n-1)}(u) = \sum_k u^k \tilde{\tilde{\mathscr{T}}}_k^{(n-1)}$, we get

$$\begin{aligned} \widehat{T}^{(n)}(u) &= (1 + uA_n)\tilde{T}^{(n-1)}(u) + (1 - quA_n)\tilde{\tilde{T}}^{(n-1)}(u)P_n \\ &= \left(\tilde{T}^{(n-1)}(u) + \tilde{\tilde{T}}^{(n-1)}(u)P_n \left(\frac{1 - uA_n}{1 + uA_n} \right) \right) (1 + uA_n) . \end{aligned}$$

Comparing with the recursion relation (84) satisfied by the quantum τ-function $\mathscr{T}(P|A)$ proves the result quoted in Eq. (53).

Furthermore, the hermiticity properties of the quantum τ-function $\tau(P|A)$ implies:

$$\widehat{T}(u)^\dagger = \prod_{j=1}^{n} (1 + uA_j)\, \mathscr{T}(q\widehat{\widehat{P}}(u)|A), \quad \text{with } \widehat{\widehat{P}}_j(u) = \frac{1 - uA_j}{1 + uA_j} P_j .$$

We can use the generating function $\widehat{T}(u)$ to write the commutativity relation (55) in an alternative form:

$$\begin{aligned} &\mathscr{T}(\widehat{P}(u)|A) \cdot \prod_{j=1}^{n} (1 + uA_j)(1 + vA_j) \cdot \mathscr{T}(q\widehat{\widehat{P}}(v)|A) \\ &\quad = \mathscr{T}(\widehat{P}(v)|A) \cdot \prod_{j=1}^{n} (1 + vA_j)(1 + uA_j) \cdot \mathscr{T}(q\widehat{\widehat{P}}(u)|A) . \end{aligned} \tag{86}$$

Let us now give a proof of the quantum separation of variables. Assume that we are acting on the quantum hamiltonian H_k with a wave function $\Psi(\alpha|\beta)$, Eq. (57), satisfying the difference equation (58). Since the momenta operators P_j have been ordered to the right in the definition of the quantum τ-functions, this wave function will be an eigenfunction with eigenvalue $\sigma_k(B)$ if the following relation is true:

$$\sigma_k(B)\mathscr{T}^{(n)}(P^{(n)}|A) = \mathscr{T}_k^{(n)}(P^{(n)}|A) \tag{87}$$

with

$$P_j^{(n)} = \prod_{k=1}^{n} \left(\frac{A_j - B_k}{qA_j + B_k} \right) . \tag{88}$$

Recall that we define the Hamiltonians using $P' = (-1)^n P$. In Eq. (87) we used the same notation for the quantum operator $\mathcal{T}^{(n)}(P'|A)$ and the c-number function obtained by inserting the values (88) of the momenta operators P_j'. We recall that the functions $\mathcal{T}_k^{(n)}(P|A)$ are defined by:

$$\mathcal{T}_k^{(n)}(P|A) = \sum_{i_1 < i_2 \cdots < i_k} A_{i_1} A_{i_2} \cdots A_{i_k} \tau(P_1, \ldots, -qP_{i_1}, \ldots, -qP_{i_k}, \ldots, P_n|A) . \tag{89}$$

Once the specific values (88) of the momenta have been plugged into Eq. (87), this is an identity between two rational functions in the variables A and B. Both the l.h.s. and the r.h.s. of (87) are symmetric functions in A and B. They only have simple poles at $qA_j + B_k = 0$ and $A_j = A_k$, the former comes from $P_j^{(n)}$ and the latter from γ_{jk}. Due to their behavior at infinity, to prove Eq. (87) it is enough to check that these rational functions have the same residues at their poles and that they are equal at specific points. By symmetry it is enough to check the residues at $qA_n + B_n = 0$ and $A_n = A_{n-1}$, and the values at the point $A_n = B_n$. We shall do it by induction assuming that the identity (87) is true up to $n-1$ (it is of course true for $n = 1$).

i) Consider first the residue at $qA_n + B_n = 0$. This pole is associated to $P_n^{(n)}$. To compute the residues of both sides of Eq. (87) we may use the recursion relation (84). Using the fact that:

$$\left. \frac{1}{\gamma_{jn}} P_j^{(n)} \right|_{qA_n + B_n = 0} = P_j^{(n-1)}, \quad \text{for } j = 1, \ldots, n-1$$

the recursion relation (87) implies:

$$\begin{aligned} \mathrm{Res}(\mathrm{lhs})|_{qA_n+B_n=0} &= (\sigma_k^{(n-1)}(B) + B_n \sigma_{k-1}^{(n-1)}(B)) \\ &\times \left((-q)^{n-1} \prod_{j \neq n} \left(\frac{A_j - B_n}{qA_j + B_n} \right) \cdot \mathcal{T}^{(n-1)}(P^{(n-1)}|A) \right) \mathrm{Res}(P_n^{(n)}) . \end{aligned}$$

Similarly applying the recursion relation to the rhs, but distinguishing whether A_n is a marked point in the sum (89) or not, we get:

$$\begin{aligned} \mathrm{Res}(\mathrm{rhs})|_{qA_n+B_n=0} = \Bigg[& (-q)^{n-1} \prod_{j \neq n} \left(\frac{A_j - B_n}{qA_j + B_n} \right) \cdot \mathcal{T}_k^{(n-1)}(P^{(n-1)}|A) \\ & + (-q)^{n-1} B_n \prod_{j \neq n} \left(\frac{A_j - B_n}{qA_j + B_n} \right) \mathcal{T}_{k-1}^{(n-1)}(P^{(n-1)}|A) \Bigg] \mathrm{Res}(P_n^{(n)}) . \end{aligned}$$

Comparing these formula gives the equality of the residues by the induction hypothesis.

ii) Consider the residues at $A_n = A_{n-1}$. They are both vanishing, and therefore equal. The proof of the vanishing of the rhs residue is similar to the proof of vanishing of the lhs residue, so we shall only present the latter. The potential pole at $A_n = A_{n-1}$ is associated to $\gamma_{n,n-1}$. Therefore, its residue may be computed by using twice the recursion relation (84). We get

$$\mathrm{Res}(\mathrm{lhs})|_{A_n=A_{n-1}}$$

$$= \mathrm{const.} \left(\mathscr{T}^{(n-2)} \left(\frac{\gamma_{jn}}{\overline{\gamma}_{j,n-1}} P_j | A \right) P_{n-1} - \mathscr{T}^{(n-2)} \left(\frac{\gamma_{j,n-1}}{\overline{\gamma}_{jn}} P_j | A \right) P_n \right) \Bigg|_{A_n=A_{n-1}} .$$

This vanishes since we have

$$P_n^{(n)}|_{A_n=A_{n-1}} = P_{n-1}^{(n)}|_{A_n=A_{n-1}} .$$

iii) Consider now the value of the functions at the point $A_n = B_n$. At this point we have $P_n^{(n)} = 0$ and:

$$\gamma_{jn} P_j^{(n)}|_{A_n=B_n} = P_j^{(n-1)}, \quad \text{for } j = 1, \dots, n-1 .$$

Therefore, the recursion relation (84) yields to

$$\mathrm{lhs}|_{A_n=B_n} = (\sigma_k^{(n-1)}(B) + B_n \sigma_{k-1}^{(n-1)}(B)) \mathscr{T}^{(n-1)}(P^{(n-1)}|A)$$

for the left-hand side and,

$$\mathrm{rhs}|_{A_n=B_n} = \mathscr{T}_k^{(n-1)}(P^{(n-1)}|A) + B_n \mathscr{T}_{k-1}^{(n-1)}(P^{(n-1)}|A)$$

for the right-hand side. It clearly appears that they are equal by the induction hypothesis.

Collecting the points i) to iii) proves the quantum separation of variables.

The classical formula corresponds to the case $q = 1$.

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