

Necessary Conditions for Existence of Non-Degenerate Hamiltonian Structures

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Abstract: The necessary criteria are pointed out for the existence of Hamiltonian and bi-Hamiltonian non-degenerate structures for a nonlinear system of partial differential equations of first order. The results are formulated in terms of the new invariants of the intrinsic geometry, introduced in this paper, connected with the Nijenhuis and Haantjes tensors of a (1,1) tensor field.

1. Introduction

This paper is devoted to the investigation of the intrinsic geometry of systems of nonlinear partial differential equations of first order

$$u_t^i = \sum_{j=1}^n A_j^i(u^1, \dots, u^n) u_y^j. \quad (1.1)$$

As it is known, systems (1.1) arise in numerous classical problems of gas dynamics and mathematical physics [1–4, 7–9, 12–15].

Riemann pointed out in his classical work [1] that the system (1.1) is closely connected with the (1,1) tensor field $A_j^i(u^1, \dots, u^n)$ defined on the Euclidean space R^n with the coordinates u^1, \dots, u^n .

Geometry of the vector fields of eigenvectors of the operators $A_j^i(u^1, \dots, u^n)$ has been studied in famous papers by Nijenhuis [5] and Haantjes [6].

Hamiltonian systems (1.1) and the associated structures of the Poisson brackets were investigated in [7–9, 14, 15] along with their applications to the theory of the Whitham equations.

Tensor fields $A_j^i(u^1, \dots, u^n)$ were considered in [5, 6] as vector-valued differential 1-forms and also as fields of operators defined on the tangent bundle $T(M^n)$. The Nijenhuis tensor $N_{jk}^i(u^1, \dots, u^n)$ and the Haantjes tensor $H_{jk}^i(u^1, \dots, u^n)$ were considered as the vector-valued differential 2-forms.

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We consider in this paper the Nijenhuis and Haantjes tensors as the laws of alternating bilinear multiplications in the tangent bundle. Therefore the Nijenhuis and Haantjes tensors define on the tangent bundle $T(M^n)$ the deformations of the structures of non-associative algebras, which appear to be Lie algebras for several important systems (1.1). This point of view leads us to the definition of the analogues of the Cartan-Killing form on the tangent bundle $T(M^n)$, countable sets of differential 1-forms ω_k and 2-forms α_k and β_k and (1, 3) tensors J_N and J_H which we name the Jacobi tensors. These (1, 3) tensors characterize the deviation of the algebraic structures defined by the Nijenhuis and Haantjes tensors from the Lie algebra structures. We define also invariant polynomials $P_N(v)$ and $P_H(v)$ on the tangent bundle and the associated fibrations of the algebraic submanifolds V_N and V_H embedded into the tangent bundle $T(M^n)$.

In terms of these constructions we point out several necessary criteria for the existence of non-degenerate Hamiltonian or bi-Hamiltonian structures for a system (1.1). For example, if system (1.1) has a non-degenerate Hamiltonian structure then the invariant polynomials $P_H(v)$ are even, $P_H(v) = P_H(-v)$ and the fibration of the algebraic submanifolds V_H is invariant under the involution $v \rightarrow -v$. For $n = 4$ existence of a non-degenerate Hamiltonian structure implies that the Haantjes tensor $H(u, v)$ defines a deformation of Lie algebra structures in the tangent bundle $T(M^4)$. For $n = 3$ these Lie algebras have to be either simple or commutative. The corresponding Cartan-Killing form $(u, v)_H$ defines the metric on the manifold M^3 that has to be conformally flat and therefore has to satisfy the classical Weyl-Schouten equations.

The existence of two non-degenerate Hamiltonian structures implies that the Haantjes tensor is reducible and is necessarily zero if the two structures are in general position. In the last case the Hamiltonian system possesses the Riemann invariants and is integrable by the generalized hodograph transformation.

These necessary criteria can be checked by a direct calculation of the Haantjes tensor for any system of partial differential equations (1.1).

The efficiency of the geometric methods of this paper is demonstrated for the equations of classical gas dynamics, for the perturbations of the Benney equations and for the certain matrix partial differential equations of the form (1.1).

2. Gauge Invariance of the Haantjes Tensor

1. Let \tilde{u} and \tilde{v} be two vector fields on a manifold M^n with a (1, 1) tensor field A_j^i . Let u and v are values of the vector fields \tilde{u} and \tilde{v} in a point P . The Nijenhuis tensor $N(u, v)$ [5] is defined by the formula

$$N(u, v) = A^2[\tilde{u}, \tilde{v}] + [A\tilde{u}, A\tilde{v}] - A([A\tilde{u}, \tilde{v}] + [\tilde{u}, A\tilde{v}]), \tag{2.1}$$

where $[\tilde{x}, \tilde{y}]$ is the commutator of vector fields \tilde{x} and \tilde{y} . The Nijenhuis tensor does not depend on a choice of vector fields \tilde{u} and \tilde{v} extending the tangent vectors u and v in the point P . Obviously the Nijenhuis tensor is alternating and has the following entries (in a local map x^1, \dots, x^n):

$$N_{jk}^i = \frac{\partial A_k^i}{\partial x^\alpha} A_j^\alpha - \frac{\partial A_j^i}{\partial x^\alpha} A_k^\alpha + \frac{\partial A_j^\alpha}{\partial x^k} A_\alpha^i - \frac{\partial A_k^\alpha}{\partial x^j} A_\alpha^i. \tag{2.2}$$

The Haantjes tensor $H(u, v)$ [6] is defined by the formula

$$H(u, v) = A^2N(u, v) + N(Au, Av) - A(N(Au, v) + N(u, Av)) \tag{2.3}$$

and also is alternating. Haantjes tensor has the following entries:

$$H_{jk}^i = A_\alpha^i A_\beta^\alpha N_{jk}^\beta + N_{\alpha\beta}^i A_j^\alpha A_k^\beta - A_\alpha^i N_{\beta k}^\alpha A_j^\beta - A_\alpha^i N_{j\beta}^\alpha A_k^\beta, \tag{2.4}$$

where one must substitute formulae (2.2).

The Nijenhuis and Haantjes tensors define the alternating products in the tangent space which lead to structures of non-associative algebras in the tangent space. Thus in the tangent bundle $T(M^n)$ we have a deformation of structures of non-associative and alternating algebras.

II. We define two linear operators N_u and H_u depending on vector u which are analogues of the operator ad_u for the Lie algebras:

$$N_u(v) = N(u, v), \quad H_u(v) = H(u, v). \tag{2.5}$$

These two operators in view of formula (2.3) are connected by the relation

$$H_u = [N_{Au} - AN_u, A]. \tag{2.6}$$

From (2.6) we get

$$\text{Tr}H_u = 0, \quad \text{Tr}(H_u A^k) = 0 \tag{2.7}$$

for all integers k .

We define two symmetric scalar products $(u, v)_N$ and $(u, v)_H$ which are analogues of the Cartan-Killing forms for the Lie algebras:

$$(u, v)_N = \text{Tr}(N_u N_v), \quad (u, v)_H = \text{Tr}(H_u H_v). \tag{2.8}$$

III.

Proposition 1. *If two (1, 1) tensor fields \tilde{A}_j^i and A_j^i are connected by the relation*

$$\tilde{A}_j^i(x) = f(x)A_j^i(x) + g(x)\delta_j^i, \tag{2.9}$$

where $f(x)$ and $g(x)$ are arbitrary functions on the manifold M^n then the corresponding Haantjes tensors and scalar products $(u, v)_H$ are connected by the formulae

$$\tilde{H}(u, v) = f^4(x)H(u, v), \tag{2.10}$$

$$(u, v)_{\tilde{H}} = f^8(x)(u, v)_H. \tag{2.11}$$

Proof. Let us first assume that $\tilde{A}(x) = f(x)A(x)$. From the definition of the Nijenhuis tensor (2.1) we have

$$\tilde{N}_u(v) = f^2N_u(v) + f(Au)(f)Av - fu(f)A^2v - f(Av)(f)Au + fv(f)A^2u. \tag{2.12}$$

Hence we get

$$\tilde{M}_u = \tilde{N}_{\tilde{A}u} - \tilde{A}\tilde{N}_u = f^3(N_{Au} - AN_u) + f^2(A^2u)(f)A - 2f^2(Au)(f)A^2 + f^2u(f)A^3.$$

This expression leads to the equality

$$\tilde{H}_u = [\tilde{M}_u, \tilde{A}] = f^4[N_{Au} - AN_u, A] = f^4H_u,$$

that proves (2.10) in the case $\tilde{A} = fA$.

Let us suppose now that $\tilde{A} = A + g1$. From (2.7) we get

$$\tilde{N}_u(v) = N_u(v) + (Au)(g)v - u(g)Av - (Av)(g)u + v(g)Au. \tag{2.13}$$

Hence we obtain

$$\tilde{M}_u = \tilde{N}_{\tilde{A}u} - \tilde{A}\tilde{N}_u = N_{Au} - AN_u + (A^2u)(g)1 - 2(Au)(g)A + u(g)A^2.$$

Thus the equality follows

$$\tilde{H}_u = [\tilde{M}_u, \tilde{A}] = [N_{Au} - AN_u, A] = H_u.$$

Therefore the formula (2.10) is proven for the general case (2.9).

The formula (2.11) follows from (2.10) and the definition of the scalar product $(u, v)_H = \text{Tr}(H_u H_v)$.

The derived formula (2.10) means that the Haantjes tensor $H(u, v)$ in the contrast with the Nijenhuis tensor $N(u, v)$ possesses the gauge invariance (2.10) under the transformations (2.9) of the (1, 1) tensor field $A_j^i(x)$.

IV. Let $e_k(x)$ and $e_s(x)$ are smooth vector fields of eigenvectors of the operator $A_j^i(x)$, corresponding to the eigenvalues $\lambda_k(x)$ and $\lambda_s(x)$. The known Nijenhuis formula [5] follows straightforward from the definition (2.1),

$$N(e_k, e_s) = (A - \lambda_k)(A - \lambda_s)[e_k, e_s] + (\lambda_k - \lambda_s)(e_k(\lambda_s)e_s + e_s(\lambda_k)e_k). \tag{2.14}$$

The formulae (2.3) and (2.4) imply

$$H(e_k, e_s) = (A - \lambda_k)^2(A - \lambda_s)^2[e_k, e_s]. \tag{2.15}$$

For the bilinear tensor

$$M(u, v) = N(Au, v) - AN(u, v), \tag{2.16}$$

we obtain from (2.14)

$$M(e_k, e_s) = -(A - \lambda_k)^2(A - \lambda_s)[e_k, e_s] + (\lambda_k - \lambda_s)^2 e_k(\lambda_s)e_s. \tag{2.17}$$

The gauge invariance of the Haantjes tensor easily follows from the formula (2.15) for a (1, 1) tensor field $A_j^i(x)$ having n real and distinct eigenvalues. Indeed, operator $\tilde{A}_j^i(x)$ (2.9) has the same eigenvectors $e_1(x), \dots, e_n(x)$ with the eigenvalues

$$\tilde{\lambda}_k(x) = f(x)\lambda_k(x) + g(x).$$

Therefore we obtain from the formula (2.15)

$$\tilde{H}(e_k, e_s) = (\tilde{A} - \tilde{\lambda}_k)^2(\tilde{A} - \tilde{\lambda}_s)^2[e_k, e_s] = f^4(x)H(e_k, e_s).$$

For the general case of complex eigenvalues and non-diagonal Jordan normal form of $A_j^i(x)$ one needs the direct proof given above.

V. We denote $N_B(u, v)$ and $H_B(u, v)$ the Nijenhuis and Haantjes tensors determined by an operator tensor field $B_j^i(x)$. Let

$$R = R(A, \lambda(x)) = (\lambda(x) - A)^{-1} = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}} \quad (2.18)$$

be the resolvent operator for an operator $A_j^i(x)$.

Proposition 2. *The relations hold*

$$N_{A^{-1}}(u, v) = A^{-2}N_A(A^{-1}u, A^{-1}v), \quad (2.19)$$

$$M_{A^{-1}}(u, v) = -A^{-3}M_A(A^{-2}u, A^{-1}v), \quad (2.20)$$

$$H_{A^{-1}}(u, v) = A^{-4}H_A(A^{-2}u, A^{-2}v), \quad (2.21)$$

$$H_R(u, v) = R^4H_A(R^2u, R^2v). \quad (2.22)$$

Proof. Let us first prove (2.19)–(2.21) for an operator tensor field $A(x)$ having real and distinct eigenvalues $\lambda_1, \dots, \lambda_n$. The inverse operator A^{-1} has the same eigenvectors e_1, \dots, e_n and eigenvalues $\lambda_1^{-1}, \dots, \lambda_n^{-1}$. From the Nijenhuis formula (2.14) we obtain

$$\begin{aligned} N_{A^{-1}}(e_k, e_s) &= (A^{-1} - \lambda_k^{-1})(A^{-1} - \lambda_s^{-1})[e_k, e_s] \\ &\quad + (\lambda_k^{-1} - \lambda_s^{-1})(e_k(\lambda_s^{-1})e_s + e_s(\lambda_k^{-1})e_k) \\ &= A^2\lambda_k^{-1}\lambda_s^{-1}(A - \lambda_k)(A - \lambda_s)[e_k, e_s] \\ &\quad + (\lambda_k - \lambda_s)(\lambda_k^{-1}e_k(\lambda_s)\lambda_s^{-3}e_s + \lambda_s^{-1}e_s(\lambda_k)\lambda_k^{-3}e_k) \\ &= A^{-2}N_A(A^{-1}e_k, A^{-1}e_s). \end{aligned} \quad (2.23)$$

Formula (2.19) follows from (2.23) in view of the bilinearity of the Nijenhuis tensor $N(u, v)$. Formula (2.19) was derived by an indirect method in [5].

Formula (2.17) leads to the equality

$$\begin{aligned} M_{A^{-1}}(e_k, e_s) &= -(A^{-1} - \lambda_k^{-1})^2(A^{-1} - \lambda_s^{-1})[e_k, e_s] \\ &\quad + (\lambda_k^{-1} - \lambda_s^{-1})^2e_k(\lambda_s^{-1})e_s \\ &= A^{-3}\lambda_k^{-2}\lambda_s^{-1}(A - \lambda_k)^2(A - \lambda_s)[e_k, e_s] \\ &\quad - (\lambda_k - \lambda_s)^2\lambda_k^{-2}e_k(\lambda_s)\lambda_s^{-1}e_s \\ &= -A^{-3}M_A(A^{-2}e_k, A^{-1}e_s), \end{aligned} \quad (2.24)$$

that implies the formula (2.20).

From the formula (2.15) we get

$$\begin{aligned} H_{A^{-1}}(e_k, e_s) &= (A^{-1} - \lambda_k^{-1})^2(A^{-1} - \lambda_s^{-1})^2[e_k, e_s] \\ &= A^{-4}\lambda_k^{-2}\lambda_s^{-2}(A - \lambda_k)^2(A - \lambda_s)^2[e_k, e_s] \\ &= A^{-4}H_A(A^{-2}e_k, A^{-2}e_s), \end{aligned} \quad (2.25)$$

and hence formula (2.21) follows.

Operators $A(x)$ with distinct and real eigenvalues form a domain \mathcal{O} in the space of all linear operators. The expressions (2.19)–(2.21) analytically depend on the entries $A_j^i(x)$ and their derivatives. Therefore the validity of the formulae

(2.19)–(2.21) in the entire domain \mathcal{O} of the operators $A(x)$ implies their validity for arbitrary operators $A(x)$.

Formula (2.22) follows from (2.21) and from the gauge invariance (2.10) of the Haantjes tensor

$$H_{\lambda(x)-A}(u, v) = H_A(u, v) \tag{2.26}$$

for any smooth function $\lambda(x)$.

Remark 1. From (2.22) we obtain that the Haantjes tensor for the resolvent operator $R(A, \lambda(x))$ (2.18) depends analytically on the function $\lambda(x)$ and does not depend on its derivatives. Formula (2.22) after the substitution of the power series (2.18) leads to an infinite family of identities connected with the Haantjes tensor $H_A(u, v)$.

3. Jacobi Tensors and Lie Algebra Structures

I. We define the following alternating functions of pairs of tangent vectors u and $v \in T_x(M^n)$, values of which are linear operators on the tangent space $T_x(M^n)$:

$$J_N(u, v) = N_{N(u, v)} - [N_u, N_v], \tag{3.1}$$

$$J_H(u, v) = H_{H(u, v)} - [H_u, H_v]. \tag{3.2}$$

Obviously J_N and J_H form (1,3) tensors

$$J_N(u, v)w = J_{N_{ijk}}^s u^i v^j w^k, \quad J_H(u, v)w = J_{H_{ijk}}^s u^i v^j w^k. \tag{3.3}$$

Tensor J_H is gauge invariant as well as the Haantjes tensor, see (2.10). After any transformation (2.9) one gets

$$\tilde{J}_H(u, v) = f^8(x)J_H(u, v). \tag{3.4}$$

Formulae (3.1) and (3.2) are similar to the definition of the Riemann tensor in the Riemannian geometry

$$R(u, v) = \nabla_{[u, v]} - [\nabla_u, \nabla_v]. \tag{3.5}$$

The tensors J_N and J_H have another meaning: they characterize the deviation of the algebraic structures defined by the tensors N and H from the Lie algebra structures. Indeed, from (3.1) and (3.2) we get

$$J_N(u, v)w = N(N(u, v), w) + N(N(v, w), u) + N(N(w, u), v), \tag{3.6}$$

$$J_H(u, v)w = H(H(u, v), w) + H(H(v, w), u) + H(H(w, u), v). \tag{3.7}$$

Thus if tensor $J_N \equiv 0$ then (3.6) is the Jacobi identity and hence tensor $N(u, v)$ defines the Lie algebra structure in the tangent space. Therefore we name tensors J_N and J_H the Jacobi tensors. These tensors have the following symmetries :

$$J(u, v)w = -J(v, u)w, \tag{3.8}$$

$$J(u, v)w = J(v, w)u = J(w, u)v. \tag{3.9}$$

In the index form equality (3.7) implies

$$J_{H_{ijk}}^s = H_{ij}^\alpha H_{\alpha k}^s + H_{jk}^\alpha H_{\alpha i}^s + H_{ki}^\alpha H_{\alpha j}^s. \tag{3.10}$$

Symmetries (3.8) and (3.9) lead to the general equality

$$J_{Hijk}^s = \text{sign}(\sigma) J_{H\sigma(i)\sigma(j)\sigma(k)}^s, \tag{3.11}$$

where σ is an arbitrary permutation of the indices i, j and k (the same for J_N).

From the identity $\text{Tr}H_u = 0$ and the formulae (3.2) and (3.11) we obtain that all contractions of the Jacobi tensor J_H vanish:

$$J_{Hij\alpha}^\alpha = J_{Hi\alpha j}^\alpha = J_{H\alpha ij}^\alpha = 0. \tag{3.12}$$

Contractions of the Jacobi tensor J_{Nijk}^s define the differential 2-form $\alpha_{N0}(u, v)$

$$J_{Nij\alpha}^\alpha = -J_{Ni\alpha j}^\alpha = J_{N\alpha ij}^\alpha = \alpha_{N0ij}. \tag{3.13}$$

II. For a (1,1) tensor field A_j^i on a manifold M^n of small dimension we prove the following facts.

Proposition 3. For $n = 2$ the Jacobi tensor vanishes $J_N = 0$ and the Haantjes tensor $H = 0$ (hence $J_H = 0$). For $n = 3$ the Jacobi tensor $J_H = 0$ and the Jacobi tensor $J_N = 0$ if the differential form $\alpha_{N0}(u, v) = 0$.

Proof. For $n = 2$ the equalities $J_{Hijk}^s = 0$ follow from the symmetry properties (3.11) as well as equalities $H_{ij}^k = 0$ follow from $H_{ij}^k = -H_{ji}^k$ and $H_{i\alpha}^\alpha = 0$.

For $n = 3$ the equalities $J_{Hikj}^s = 0$ follow from (3.11) and (3.12) and the equalities $J_{Nijk}^s = 0$ follow analogously from (3.13) if $\alpha_{N0}(u, v) = 0$.

Corollary 1. A manifold M^n with a (1,1) tensor field $A_j^i(x)$ for $n = 2, 3$ possesses a deformation of Lie algebra structures in the tangent bundle $T(M^n)$ defined for $n = 2$ by the Nijenhuis tensor N_{ij}^k and for $n = 3$ by the Haantjes tensor H_{ij}^k .

For $n = 3$ let $e_1(x), e_2(x), e_3(x)$ be the eigenvectors of the operator $A_j^i(x)$. From the formula (2.15) we obtain

$$H(e_1, e_2) = h_3 e_3, \quad H(e_2, e_3) = h_1 e_1, \quad H(e_3, e_1) = h_2 e_2. \tag{3.14}$$

These formulae describe, in view of the relations (2.7) $\text{Tr}H_u = 0$, the canonical form of the commutators of all 3-dimensional Lie algebras, determined by the Haantjes tensors $H(u, v)$.

III. We define two bilinear symmetric tensors

$$\begin{aligned} L_1(u, v) &= N(Au, v) - N(u, Av), \\ L_2(u, v) &= H(Au, v) - H(u, Av). \end{aligned} \tag{3.15}$$

These tensors define two deformations of structures of commutative but not associative algebras in the tangent bundle $T(M^n)$.

Tensors

$$L_3(u, v) = N(Au, v) + N(u, Av), \tag{3.16}$$

$$L_4(u, v) = H(Au, v) + H(u, Av). \tag{3.17}$$

are alternating.

We define two bilinear tensors

$$K(u, v) = N(u, Av) - AN(u, v), \tag{3.18}$$

$$M(u, v) = -K(v, u) = N(Au, v) - AN(u, v). \tag{3.19}$$

The following relations hold:

$$K(Au, v) - AK(u, v) = H(u, v), \tag{3.20}$$

$$M(u, Av) - AM(u, v) = H(u, v). \tag{3.21}$$

Thus for the corresponding linear operators $K_u(v) = K(u, v)$ and $M_u(v) = M(u, v)$ we obtain

$$K_u = [N_u, A], \quad K_{Au} - AK_u = H_u, \tag{3.22}$$

$$M_u = N_{Au} - AN_u, \quad [M_u, A] = H_u. \tag{3.23}$$

If the Haantjes tensor $H(u, v) = 0$ then we get from (3.22) for any polynomial $P(A)$,

$$K_{P(A)u} = P(A)K_u. \tag{3.24}$$

Thus if $P(A)u = 0$ then $P(A)K_u = 0$. Also when $H(u, v) = 0$ we get from (3.23) that all operators M_u commute with the operator A .

4. Differential Forms and Conservation Laws

I. We define the following differential 1-forms

$$\omega_k(u) = \text{Tr}(A^k N_u), \quad k \geq 0 \tag{4.1}$$

and differential 2-forms

$$\alpha_{Nk}(u, v) = \omega_k(N(u, v)) = \text{Tr}(A^k N_{N(u,v)}), \quad k \geq 0, \tag{4.2}$$

$$\alpha_{Hk}(u, v) = \omega_k(H(u, v)) = \text{Tr}(A^k N_{H(u,v)}), \quad k \geq 0, \tag{4.3}$$

$$\beta_{Nk}(u, v) = \text{Tr}(A^k [N_u, N_v]), \quad k \geq 1, \tag{4.4}$$

$$\beta_{Hk}(u, v) = \text{Tr}(A^k [H_u, H_v]), \quad k \geq 1, \tag{4.5}$$

It is also useful to define the generating forms depending on an arbitrary parameter λ , for the countable families of forms (4.1)–(4.5), for example 1-form $w(u, \lambda)$

$$\omega(u, \lambda) = \text{Tr}((\lambda - A)^{-1} N_u) = \sum_{k=0}^{\infty} \frac{\omega_k(u)}{\lambda^{k+1}}. \tag{4.6}$$

Remark 2. A necessary criteria for the Nijenhuis or Haantjes tensors to determine a Lie-algebraic structure in the tangent bundle $T(M^n)$ consist of the vanishing of the forms $\alpha_{Nk} - \beta_{Nk} = 0$ (for N) and $\beta_{Hk} = 0$ (for H) for all $k \geq 0$. Indeed, these vanishings follow from (3.1) and (3.2) if $J_N = 0$ or $J_H = 0$. Hence we get in view of Proposition 3 and formulae (2.7) that for $n = 3$ differential 2-forms $\beta_{Hk}(u, v) = 0$ for all k and if the differential 2-form $\alpha_{N0}(u, v) = 0$, then $\alpha_{Nk}(u, v) = \beta_{Nk}(u, v)$ for all k .

Proposition 4. *Differential 1-form ω_k satisfies the relation*

$$\omega_k(v) = Av \left(\frac{1}{k+1} \text{Tr}A^{k+1} \right) - v \left(\frac{1}{k+2} \text{Tr}A^{k+2} \right) \quad (4.7)$$

for any tangent vector v . If the differential form ω_k vanishes then all solutions $u^i(t, x)$ of the system (1.1) satisfy the conservation law

$$\frac{\partial}{\partial t} \left(\frac{1}{k+1} \text{Tr}A^{k+1}(u^1, \dots, u^n) \right) = \frac{\partial}{\partial x} \left(\frac{1}{k+2} \text{Tr}A^{k+2}(u^1, \dots, u^n) \right). \quad (4.8)$$

Proof. Expression (4.7) after the substitution of the formulae (4.1) and (2.2) turns into an analytic relation between the entries of the operator $A_j^i(u^1, \dots, u^n)$ and their derivatives. Therefore it is sufficient to prove formula (4.7) for the domain of operators $A_j^i(u^1, \dots, u^n)$ having real and distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Let e_1, \dots, e_n be the corresponding eigenvectors. From the Nijenhuis formula (2.14) we obtain

$$\begin{aligned} \omega_k(e_j) &= \text{Tr}(A^k N_{e_j}) = \sum_{s=1}^n (\lambda_j - \lambda_s) e_j(\lambda_s) \lambda_s^k \\ &= Ae_j \left(\sum_{s=1}^n \frac{1}{k+1} \lambda_s^{k+1} \right) - e_j \left(\sum_{s=1}^n \frac{1}{k+2} \lambda_s^{k+2} \right) \\ &= Ae_j \left(\frac{1}{k+1} \text{Tr}A^{k+1} \right) - e_j \left(\frac{1}{k+2} \text{Tr}A^{k+2} \right). \end{aligned} \quad (4.9)$$

Formula (4.7) is a consequence of the formula (4.9) for an arbitrary tangent vector $v = v^1 e_1 + \dots + v^n e_n$.

We represent the expression (4.8) for an arbitrary solution $u^i(t, x)$ of the system (1.1) $u_t^i = A_j^i(u^1, \dots, u^n) u_x^j$ in the form

$$\frac{\partial}{\partial u^i} \left(\frac{1}{k+1} \text{Tr}A^{k+1}(u) \right) \frac{\partial u^i}{\partial t} - \frac{\partial}{\partial u^i} \left(\frac{1}{k+2} \text{Tr}A^{k+2}(u) \right) \frac{\partial u^i}{\partial x} \quad (4.10)$$

$$\begin{aligned} &= A_j^i u_x^j \frac{\partial}{\partial u^i} \left(\frac{1}{k+1} \text{Tr}A^{k+1}(u) \right) - u_x^i \frac{\partial}{\partial u^i} \left(\frac{1}{k+2} \text{Tr}A^{k+2}(u) \right) \\ &= \omega_k(u_x). \end{aligned} \quad (4.11)$$

Therefore if the differential form ω_k vanishes then any solution $u^i(t, x)$ satisfies the conservation law (4.8). Proposition 4 is proven.

Corollary 2. *If all differential forms ω_k vanish (for example when the Nijenhuis tensor $N(u, v) = 0$) then for any solution $u^i(t, x)$ of the system (1.1) the eigenvalues $\lambda_i(u)$ of the operator $A_j^i(u^1, \dots, u^n)$ satisfy the equations*

$$\lambda_{it} = \lambda_i \lambda_{ix}. \quad (4.12)$$

Indeed, in this case we obtain from (4.8),

$$\sum_{i=1}^n \lambda_i^k \lambda_{it} = \sum_{i=1}^n \lambda_i^{k+1} \lambda_{ix}.$$

The validity of these equations for all k implies the validity of Eq. (4.12).

5. Invariant Polynomials and Fibrations of the Algebraic Submanifolds

I. We define on the tangent space $T_x(M^n)$ two invariant polynomials

$$P_N(u, \lambda) = \det(N_u - \lambda) = \sum_{k=0}^n p_{Nk}(u) \lambda^k, \tag{5.1}$$

$$P_H(u, \lambda) = \det(H_u - \lambda) = \sum_{k=0}^n p_{Hk}(u) \lambda^k, \tag{5.2}$$

which are homogeneous functions of all arguments u and λ of degree n . Coefficients $p_{Nk}(u)$ and $p_{Hk}(u)$ are homogeneous polynomials of u of degree $n - k$. From the equalities

$$N_u(u) = 0, \quad H_u(u) = 0, \quad H_u = [M_u, A]$$

we get

$$p_{Nn}(u) = (-1)^n, \quad p_{Nn-1}(u) = (-1)^{n-1} \omega_0(u), \quad p_{N0}(u) = \det N_u = 0, \tag{5.3}$$

$$p_{Hn}(u) = (-1)^n, \quad p_{Hn-1}(u) = \text{Tr} H_u = 0, \quad p_{H0}(u) = \det H_u = 0. \tag{5.4}$$

Let $\lambda_1(u), \dots, \lambda_n(u)$ be eigenvalues of an operator N_u . The equalities hold

$$p_{Nn-2}(u) = (-1)^n \sum_{i \neq j} \lambda_i \lambda_j, \tag{5.5}$$

$$\omega_0(u) = \text{Tr} N_u = \sum_{i=1}^n \lambda_i, \quad (u, u)_N = \text{Tr}(N_u N_u) = \sum_{i=1}^n \lambda_i^2. \tag{5.6}$$

Hence we obtain

$$p_{Nn-2}(u) = \frac{(-1)}{2} \omega_0^2(u) - \frac{(-1)^n}{2} (u, u)_N, \tag{5.7}$$

and analogously

$$p_{Hn-2}(u) = -\frac{(-1)^n}{2} (u, u)_H. \tag{5.8}$$

Polynomials (5.1), (5.2) for $n = 2$ have the form

$$P_N(u, \lambda) = \lambda^2 - \omega_0(u) \lambda, \quad P_H(u, \lambda) = \lambda^2. \tag{5.9}$$

For $n = 3$ we get from (5.3)–(5.8),

$$P_N(u, \lambda) = -\lambda^3 + \omega_0(u) \lambda^2 - \frac{1}{2} (\omega_0^2(u) - (u, u)_N) \lambda, \tag{5.10}$$

$$P_H(u, \lambda) = -\lambda^3 + \frac{1}{2} (u, u)_H \lambda. \tag{5.11}$$

Operators N_u and H_u in view of (5.1) and (5.2) satisfy the algebraic equations

$$P_N(u, N_u) = 0, \quad P_H(u, H_u) = 0. \tag{5.12}$$

For $n = 3$ these equations have the form

$$N_u(N_u^2 - \omega_0(u)N_u + \frac{1}{2}\omega_0^2(u) - \frac{1}{2}(u, u)_N) = 0, \quad (5.13)$$

$$H_u(H_u^2 - \frac{1}{2}(u, u)_H) = 0. \quad (5.14)$$

II. We define in the tangent space $T_x(M^n)$ two algebraic submanifolds V_N and V_H by the equations

$$V_N : P_N(u, 1) = \det(N_u - 1) = 0, \quad (5.15)$$

$$V_H : P_H(u, 1) = \det(H_u - 1) = 0. \quad (5.16)$$

The manifolds V_N and V_H are affine parts of the projective manifolds \tilde{V}_N and \tilde{V}_H defined by the homogeneous characteristic equations

$$\tilde{V}_N : \det(N_u - \lambda) = 0, \quad \tilde{V}_H : \det(H_u - \lambda) = 0. \quad (5.17)$$

Thus we have two fibrations of algebraic manifolds V_N and V_H embedded into the tangent bundle $T(M^n)$ or into its projectivization. Complex forms of these manifolds are embedded into the complexification of the tangent bundle $T(M^n)$.

For $n = 2$ manifolds V_N are straight lines $\text{Tr}N_u = 1$ (V_H are empty). For $n = 3$ manifolds V_N and V_H are quadrics:

$$V_N : (u, u)_N - \omega_0^2(u) + 2\omega_0(u) = 2, \quad (5.18)$$

$$V_H : (u, u)_H = 2. \quad (5.19)$$

V_H is invariant under the reflection $u \rightarrow -u$.

Let G_{Hx} be the group of all linear automorphisms of the Haantjes tensor H (in a point $x \in M^n$), so $g \in G_{Hx}$ if for any $u, v \in T_x(M^n)$ we have

$$H(gu, gv) = gH(u, v). \quad (5.20)$$

Polynomial $P_H(u)$ and algebraic manifold V_H obviously are invariant under the action of the group of automorphisms G_{Hx} in $T_x(M^n)$.

The Haantjes tensor H (or N) is called reducible if all operators H_u (or N_u) have an invariant subspace $L_1 \subset T_x(M^n)$. This means that $H(T_x(M^n), L_1) \subset L_1$. In this case the polynomial $P_H(u)$ can be factored $P_H(u) = P_{H1}(u)P_{H2}(u)$ and the algebraic manifold V_H has two components, determined by equations $P_{H1}(u) = 0$ and $P_{H2}(u) = 0$.

If there exists filtration of different subspaces

$$L_1 \subset L_2 \subset L_3 \subset \dots \subset L_k = T_x(M^n), \quad (5.21)$$

invariant with respect to all operators H_u , or $H(T_x(M^n), L_\alpha) \subset L_\alpha$ for all $\alpha = 1, \dots, k$, then polynomial $P_H(u)$ can be factored into a product of k factors $P_H(u) = P_{H1}(u)P_{H2}(u)\dots P_{Hk}(u)$ and algebraic manifold V_H has k components.

If tensor H (or N) has nontrivial central subspace L_c of dimension d , i.e.,

$$H(L_c, T_x(M^n)) = 0,$$

then the polynomial $P_H(u, 1)$ (5.2) has degree at most $n - d$.

We call tensor H (or N) nilpotent, if for any u we have $H_u^k = 0$ for some k . Obviously the complexification of the manifold V_H is empty and polynomial $P_H(u) = (-1)^n$ if and only if tensor H is nilpotent.

We call vector $v \in T_x(M^n)$ N -nilpotent (analogously H -nilpotent) if operator N_v is nilpotent, $N_v^n = 0$. The set Z_N of all N -nilpotent vectors is determined by the system of $n - 1$ homogeneous equations

$$p_{N1}(u) = 0, \dots, p_{Nn-1}(u) = 0. \tag{5.22}$$

Indeed, for vectors u , satisfying (5.22) we get from (5.12),

$$P_N(u, N_u) = (-1)^n N_u^n = 0.$$

If polynomial $P_N(u, \lambda)$ can be factored into a product of n linear factors

$$P_N(u, \lambda) = (-1)^n \lambda \prod_{k=1}^{n-1} (\lambda - f_k(u)), \tag{5.23}$$

then the set Z_N is a linear subspace

$$f_1(u) = 0, \dots, f_{n-1}(u) = 0. \tag{5.24}$$

This is the case for the Benney system, studied in Sect. 9.

If all operators N_u (or H_u) have a common eigenvector e then e is N -nilpotent and therefore satisfies all Eqs. (5.22). Indeed, from the condition

$$N_u e = N(u, e) = \lambda(u)e$$

we obtain

$$N_e^2 u = N_e(N(e, u)) = -\lambda(u)N(e, e) = 0.$$

Polynomial $P_N(u, \lambda)$ in this case has a linear divisor $\lambda - \lambda(u)$.

If the two operator tensor fields \tilde{A}_j^i and A_j^i are connected by the gauge transformation (2.9) then in view of (2.10) their Haantjes operators H_u are connected by the equality $\tilde{H}_u = f^4 H_u$. Therefore the corresponding polynomials (5.2) are connected by the relation

$$P_{\tilde{H}}(u, \lambda) = P_H(f^4 u, \lambda). \tag{5.25}$$

III. Let $(v, w) = g_{\alpha\beta} v^\alpha w^\beta$ be a bilinear form in the tangent space $T_x(M^n)$, which can be indefinite and non-symmetric, but it must be non-degenerate.

Lemma 1. *If all operators H_u are skew-symmetric with respect to a bilinear form (v, w) :*

$$(H_u v, w) + (v, H_u w) = 0, \tag{5.26}$$

then the polynomial $P_H(u)$ is even and the algebraic manifold V_H is invariant under the reflection $u \rightarrow -u$. Eigenvalues of the operator H_u are symmetric with respect to zero and its image has an even dimension in a general case.

Proof. Let e_1, \dots, e_n be a basis in a tangent space $T_x(M^n)$, scalar products $(e_i, e_j) = B_{ij}$ and $H_u(e_i) = H_{uik} e_k$. Then the expression (5.26) is equivalent to the matrix equation

$$H_u B = -B H_u^t. \tag{5.27}$$

For the polynomial $P_H(u, \lambda)$ we have

$$\begin{aligned}
P_H(u, \lambda) &= \det(H_u - \lambda) = \det(B^{-1}H_u B - \lambda) = \det(-H_u^t - \lambda) \\
&= \det(H_{(-u)} - \lambda) = P_H(-u, \lambda).
\end{aligned} \tag{5.28}$$

Thus the polynomials $P_H(u, \lambda)$ and $P_H(u)$ are even and hence the algebraic manifold V_H is symmetric under the reflection and eigenvalues of an operator H_u are symmetric with respect to zero. Therefore the image space $H_u(T_x(M^n))$ has even dimension.

Lemma 2. *If all operators H_u are skew-symmetric (see (5.26)) then the Jacobi tensor $J_H(u, v)$ for any u, v satisfies the relations*

$$(J_H(u, v)x, y) + (x, J_H(u, v)y) = 0, \tag{5.29}$$

$$(J_H(u, v)x, y) = (v, J_H(x, y)u). \tag{5.30}$$

Proof. Equality (5.29) follows from (5.26) after the substitution of the formula (3.2). Equality (5.30) follows from (5.29) and properties of symmetry of the Jacobi tensor (3.8) and (3.9).

Remark 3. For the symmetric scalar product (x, y) in (5.26) equalities (5.29) and (5.30) coincide with the classical identities for the Riemann tensor

$$\langle R(u, v)x, y \rangle + \langle R(u, v)y, x \rangle = 0, \tag{5.31}$$

$$\langle R(u, v)x, y \rangle = \langle R(x, y)u, v \rangle. \tag{5.32}$$

Remark 4. We define in the space $T_x(M^n) + T_x(M^n)$ an algebraic submanifold W_H by the equation

$$W_H: Q_H(u, v) = \det(J_H(u, v) - 1) = 0. \tag{5.33}$$

Such a manifold is defined also in the space L^d of bi-vectors $\xi = u \wedge v$, having dimension $d = n(n-1)/2$. If the equalities (5.26) are valid for any u and v , then as in Lemma 1 we get that polynomial $Q_H(u, v)$ and the manifold W_H for any point $x \in M^n$ are invariant with respect to two reflections $u \rightarrow -u$ and $v \rightarrow -v$.

IV. Let us suppose that the Haantjes tensor H determines a structure of Lie algebra \mathcal{A}_x in the tangent space $T_x(M^n)$, and let G_x be the corresponding Lie group. In this case the polynomial $P_{\mathcal{A}}(u)$ and the manifold $V_{\mathcal{A}}$ are invariant with respect to all automorphisms of the Lie algebra \mathcal{A}_x , e.g. under the adjoint representation of its Lie group G_x . Thus the manifold $V_{\mathcal{A}}$ contains together with any point u all its orbit $\text{Ad}_{G_x}(u)$.

If $F: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a homomorphism of two Lie algebras, then $F(V_{\mathcal{A}_1}) \subset V_{\mathcal{A}_2}$.

If Lie algebra \mathcal{A}_x is semi-simple then operators $H_u = \text{ad}_u$ are skew-symmetric with respect to the Cartan-Killing form. Thus in view of Lemma 1 the corresponding polynomial $P_H(u)$ is even and the manifold V_H is invariant under the reflection $u \rightarrow -u$.

Remark 5. Let T be a linear representation of a Lie algebra \mathcal{A} in a linear space L . We define an algebraic manifold $V_T \subset \mathcal{A}$ by the equation

$$V_T: P_T(u) = \det(T(u) - 1) = 0. \tag{5.34}$$

It is easy to show that if two representations T_1 and T_2 are equivalent then the corresponding polynomials $P_{H_1}(u)$ and $P_{H_2}(u)$ and the algebraic manifolds V_{T_1} and V_{T_2} coincide. If a representation T is reducible, then the algebraic manifold V_T is reducible too.

6. Necessary Criteria for Existence of a Non-Degenerate Hamiltonian Structure

I. Let us consider a class of Hamiltonian systems (1.1) corresponding to the Poisson brackets

$$\{F_1(u), F_2(u)\} = \int_{-\infty}^{\infty} \frac{\delta F_1(u)}{\delta u^i(x)} I^{ij}(u) \frac{\delta F_2(u)}{\delta u^j(x)} dx, \tag{6.1}$$

where a skew-symmetric operator $I = (I^{ij})$ has the form

$$I^{ij}(u) = g^{ij}(u) \frac{d}{dx} + b_{\beta}^{ij} u_x^{\beta}. \tag{6.2}$$

A connection

$$\Gamma_{jk}^i(u) = -g_{j\alpha}(u) b_k^{\alpha i}(u) \tag{6.3}$$

naturally arises from (6.2). Skew-symmetry of the Poisson brackets (6.1) implies [7–9] the symmetry condition for the metric, $g^{ij}(u) = g^{ji}(u)$, and the compatibility condition of the connection (6.3) with the metric g_{ij} :

$$\frac{\partial g^{ij}}{\partial u^k} - b_k^{ij} - b_k^{ji} = \frac{\partial g^{ij}}{\partial u^k} + \Gamma_{\alpha k}^i g^{\alpha j} + \Gamma_{\alpha k}^j g^{i\alpha} = 0. \tag{6.4}$$

In [7–9] it is shown that the Poisson brackets (6.1)–(6.2) with a non-degenerate metric $g^{ij}(u)$ satisfy the Jacobi identity if and only if the torsion and the curvature of the connection (6.3) are zero. That means the metric $g_{ij}(u)$ is flat and therefore the operator $I^{ij}(u)$ in appropriate coordinates (u^1, \dots, u^n) takes the form

$$I^{ij}(u) = q^i \delta_j^i \frac{d}{dx} \tag{6.5}$$

with some constant coefficients q^i .

A Hamiltonian system (1.1) with a Hamiltonian $f(u^1, \dots, u^n)$ has the form

$$u_t^i = I^{i\alpha} \frac{\partial f}{\partial u^\alpha} = g^{i\alpha}(u) \frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} u_x^\beta + b_{\beta}^{i\alpha}(u) \frac{\partial f}{\partial u^\alpha} u_x^\beta. \tag{6.6}$$

In view of (6.5) any Hamiltonian system (6.6) can be transformed to the canonical form

$$u_t^i = q^i \frac{\partial^2 f}{\partial u^i \partial u^\alpha} u_x^\alpha. \tag{6.7}$$

II. Let us consider a general system (6.6) with symmetric connection (6.3)

$$\Gamma_{jk}^i = \Gamma_{kj}^i. \tag{6.8}$$

Metric g_{ij} (that is not necessarily flat and is not necessarily compatible with the connection (6.8)) defines the bilinear form

$$(v, w) = g_{\alpha\beta} v^\alpha w^\beta, \quad v, w \in T_x(M^n). \tag{6.9}$$

System (6.6) is connected with the operator tensor field

$$A^i_\beta(u) = g^{i\alpha}(u) \frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} + b^{i\alpha}_\beta(u) \frac{\partial f}{\partial u^\alpha}. \tag{6.10}$$

The symmetricity of the operator A

$$(Av, w) = (v, Aw) \tag{6.11}$$

follows from the condition (6.8). Indeed, (6.11) is equivalent to the equation

$$\sum_{\alpha, \beta, \gamma} g_{\alpha\gamma} A^\gamma_\beta (v^\alpha w^\beta - v^\beta w^\alpha) = 0. \tag{6.12}$$

The last one after substituting the formulae (6.10) and (6.3) takes the form

$$\sum_{\alpha, \beta, \gamma} \left(\frac{\partial^2 f}{\partial u_\alpha \partial u_\beta} - \Gamma^\gamma_{\alpha\beta} \frac{\partial f}{\partial u^\gamma} \right) (v^\alpha w^\beta - v^\beta w^\alpha) = 0, \tag{6.13}$$

and the statement follows. In particular we obtain that the operator A (6.10) for a Hamiltonian system (6.6) is symmetric with respect to the scalar product (6.9) [9].

III.

Theorem 1. *The operators H_u (2.5) for the Hamiltonian system (6.6) are skew-symmetric with respect to the bilinear form (6.9):*

$$(H_u v, w) + (v, H_u w) = 0. \tag{6.14}$$

Proof. The condition (6.14) is invariant. Thus it is enough to check (6.14) for the canonical form (6.7) of the Hamiltonian system. The system (6.7) is a special case of the system of conservation laws [2, 3]

$$u^i_t = f^i_x = \frac{\partial f^i}{\partial u^x} u^x. \tag{6.15}$$

We denote partial derivatives by low indices: $f^i_\alpha = \partial f^i / \partial u^\alpha$, $f^i_{\alpha\beta} = \partial^2 f / \partial u^\alpha \partial u^\beta$, etc.

System (6.15) is connected with the (1, 1) tensor field $A^i_\alpha(u) = \partial f^i / \partial u^\alpha$. The corresponding Nijenhuis tensor (2.2) has the following entries:

$$N^i_{jk} = f^i_{k\alpha} f^\alpha_j - f^i_{j\alpha} f^\alpha_k. \tag{6.16}$$

A direct calculation of the Haantjes tensor (2.4) for the system of conservation laws (6.15) gives the formula

$$\begin{aligned} H^i_{jk} &= f^i_\alpha f^\alpha_\beta f^\beta_{\gamma k} f^\gamma_j - f^i_\alpha f^\alpha_\beta f^\beta_{\gamma j} f^\gamma_k \\ &+ f^i_\alpha f^\alpha_{\beta j} f^\beta_\gamma f^\gamma_k - f^i_\alpha f^\alpha_{\beta k} f^\beta_\gamma f^\gamma_j \\ &+ f^i_{\alpha\beta} f^\alpha_k f^\beta_\gamma f^\gamma_j - f^i_{\alpha\beta} f^\alpha_j f^\beta_\gamma f^\gamma_k. \end{aligned} \tag{6.17}$$

The system (6.7) is embedded into (6.15) where $f^i = q^i \partial f / \partial u^i$. Therefore for the system (6.7) we get from (6.16)

$$N_{jk}^i = q^i q^\alpha (f_{ik\alpha} f_{\alpha j} - f_{ij\alpha} f_{\alpha k}) . \tag{6.18}$$

The Haantjes tensor for the system (6.7) in view of (6.17) is determined by the formulae

$$H_{jk}^i = q^i \tilde{H}_{jk}^i , \tag{6.19}$$

$$\tilde{H}_{jk}^i = \sum_{\sigma} \text{sign}(\sigma) \sum_{\alpha\beta\gamma} q^\alpha q^\beta q^\gamma f_{\sigma(i)\alpha} f_{\alpha\beta\sigma(j)} f_{\beta\gamma} f_{\gamma\sigma(k)} , \tag{6.20}$$

where summation is taken over all permutations σ of indices i, j, k . Coefficients \tilde{H}_{jk}^i are skew-symmetric with respect to the transpositions of any two indices i, j, k .

The condition of the skew-symmetry of the operators H_u (6.14) is equivalent to the equations

$$\sum_{\alpha, \beta, \gamma} g_{\alpha\gamma} H_{j\beta}^\gamma (v^\alpha w^\beta + v^\beta w^\alpha) = 0 \tag{6.21}$$

for all $j = 1, \dots, n$. Substituting here the formula (6.19) and $g_{\alpha\gamma} = (q^\alpha)^{-1} \delta_\alpha^\gamma$ we obtain the equations

$$\sum_{\alpha, \beta} \tilde{H}_{j\beta}^\alpha (v^\alpha w^\beta + v^\beta w^\alpha) = 0 .$$

These equations hold identically in view of

$$\tilde{H}_{j\beta}^\alpha = -\tilde{H}_{j\alpha}^\beta . \tag{6.22}$$

Theorem 1 is proven.

IV. For the Jacobi tensor J_{Hijk}^s we denote $J_{Hijks} = J_{Hijk}^\alpha g_{\alpha s}$.

Corollary 3. *The Jacobi tensor J_H for the Hamiltonian system (6.6) has the following properties:*

$$\begin{aligned} J_H(u, v)w &= -J_H(v, u)w , \\ J_H(u, v)w &= J_H(v, w)u = J_H(w, u)v , \\ (J_H(u, v)x, y) + (x, J_H(u, v)y) &= 0 , \\ (J_H(u, v)x, y) &= (J_H(x, y)u, v) , \end{aligned} \tag{6.23}$$

equivalent to the relations

$$J_{Hijks} = \text{sign}(\sigma) J_{H\sigma(i)\sigma(j)\sigma(k)\sigma(s)} \tag{6.24}$$

for all permutations σ of four indices i, j, k, s .

Proof. The equalities (6.23) follow from the general symmetries (3.8) and (3.9) of the Jacobi tensor and from Lemma 2 based on Theorem 1 for the Hamiltonian system (6.6). The equivalence of (6.23) and (6.24) is simple to verify.

The Haantjes tensor $H(u, v)$ for $n = 3$ for any $(1, 1)$ -tensor field $A_j^i(x)$ defines the structure of the Lie algebra in each tangent space $T_x(M^3)$, see Corollary 1. Thus in the whole we have the deformation of the structures of the Lie algebras in the tangent bundle $T(M^3)$. For the Hamiltonian system (6.6) this is obvious because of the vanishing of the Jacobi tensor (3.7) for $n = 3$ in view of (6.24).

Theorem 2. *For the Hamiltonian system (6.6) for $n = 3$ the following alternative is true:*

1) *The Haantjes tensor $H(u, v)$ is zero. Then the system (6.6) possesses the Riemann invariants and is integrable by the generalized hodograph transformation.*

2) *The Haantjes tensor $H(u, v)$ is non-zero. Then the arising Lie algebra for each tangent space $T_x(M^3)$ is simple, so it is isomorphic either to $so(3)$ or $sl(3)$. The Cartan-Killing form $(u, v)_H = \text{Tr}(H_u H_v)$ defines the non-degenerate metric h_{ij} on the manifold M^3 that has to be conformally flat and therefore has to satisfy the classical Weyl-Schouten equations:*

$$R_{ijk} = R_{ij,k} - R_{ik,j} - \frac{1}{4}(h_{ik}R_{,j} - h_{ij}R_{,k}) = 0. \quad (6.25)$$

Proof. 1) The existence of the Riemann invariants provided that the Haantjes tensor $H(u, v) = 0$ is proven in our paper [11]. The integrability by the generalized hodograph transformation follows from the Tsarev theorem [19].

2) The Haantjes tensor (6.19), (6.20) for the Hamiltonian system in the canonical form (6.7) has only the following non-zero components:

$$H_{12}^3 = -H_{21}^3 = q^3 c(x), \quad H_{23}^1 = -H_{32}^1 = q^1 c(x), \quad H_{31}^2 = -H_{13}^2 = q^2 c(x), \quad (6.26)$$

where $c(x) = \tilde{H}_{12}^3(x)$ (constants q^1, q^2, q^3 are nonzero because the metric g_{ij} is nondegenerate). Therefore the Lie algebra in $T_x(M^3)$ is simple if $c(x) \neq 0$ and is commutative if $c(x) = 0$.

From (6.26) we get for the metric $(u, v)_H$:

$$h_{ij}(x) = \sum_{\alpha, \beta}^3 H_{i\beta}^\alpha H_{j\alpha}^\beta = (-c(x)q^1 q^2 q^3)g_{ij}(x).$$

Thus the metric $h_{ij}(x)$ is conformal to the flat metric $g_{ij}(x)$. That follows also from the fact that the Lie algebra in $T_x(M^3)$ is simple and its operators $\text{ad}_u = H_u$ are skew-symmetric with respect to the two metrics $h_{ij}(x)$ and $g_{ij}(x)$.

Any conformally flat metric $h_{ij}(x)$ satisfies the Weyl-Schouten equations (6.25) [18] where R_{ij} is the Ricci tensor and R is the scalar curvature of the metric h_{ij} ; the $R_{ij,k}$ and $R_{,j}$ are their covariant derivatives.

Theorem 2 is proven.

Theorem 3. *The Haantjes tensor $H(u, v)$ for the Hamiltonian system (6.6) for $n = 4$ defines the structure of the Lie algebra $\mathcal{A}_4(x)$ in each tangent space $T_x(M^4)$. The Cartan-Killing form $h_{ij} = (e_i, e_j)_H$ is degenerate. If $h_{ij} = 0$ then the Haantjes tensor $H_{jk}^i = 0$. Thus the Lie algebra $\mathcal{A}_4(x)$ can be neither simple nor nilpotent.*

Proof. The Jacobi tensor J_H for a Hamiltonian system (6.6) for $n = 4$ has in view of (6.24) only the following nonzero components

$$J_{H\sigma(1)\sigma(2)\sigma(3)\sigma(4)} = \text{sign}(\sigma)J_{H1234}.$$

In the coordinates u^1, u^2, u^3, u^4 , where the system (6.6) has a canonical form (6.7), we have

$$J_{H123}^4 = \sum_{\alpha=1}^4 (H_{12}^\alpha H_{\alpha 3}^4 + H_{23}^\alpha H_{\alpha 1}^4 + H_{31}^\alpha H_{\alpha 2}^4). \quad (6.27)$$

After the substitution of the formulae (6.19), (6.20) we find that each summand in (6.27) is equal to zero because symbol \tilde{H}_{jk}^i (6.20) is equal to zero when any two of the indices i, j, k coincide. Therefore the Jacobi tensor J_H vanishes and hence the Haantjes tensor H in view of (3.7) defines the structure of the Lie algebra $\mathcal{A}_4(x)$ in each tangent space $T_x(M^4)$.

As known, the Cartan-Killing form for any 4-dimensional Lie algebra is degenerate. Thus rank r of the form $h_{ij} = (e_i, e_j)_H$ has to be $0 \leq r \leq 3$.

The Haantjes tensor H_{jk}^i in the canonical coordinates (6.7) has the form (6.19), (6.20). For $n = 4$ we denote

$$\tilde{H}_{23}^4 = c_1, \quad \tilde{H}_{34}^1 = c_2, \quad \tilde{H}_{41}^2 = c_3, \quad \tilde{H}_{12}^3 = c_4.$$

All entries H_{jk}^i are the linear combinations of c_1, c_2, c_3, c_4 . Substituting these expressions and (6.19) into the formula

$$h_{ij} = (e_i, e_j)_H = \sum_{\alpha, \beta} H_{i\alpha}^\beta H_{j\beta}^\alpha,$$

we obtain the relations

$$h_{ij} = -2 \frac{c_i c_j}{q_i q_j}, \quad h_{ii} = -\frac{2q_i}{q} (q_j c_j^2 + q_k c_k^2 + q_s c_s^2),$$

where

$$q = q_1 q_2 q_3 q_4, \quad i \neq j \neq k \neq s.$$

Hence it follows that if the metric $h_{ij} = 0$, then all $c_k = 0$, and therefore the Haantjes tensor $H_{jk}^i = 0$. Theorem 3 is proven.

Remark 6. If the Haantjes tensor $H(v, w)$ defines the structure of the simple Lie algebra in each tangent space $T_x(M^n)$ then the bilinear form (6.9) is proportional in view of (6.14) to the Cartan-Killing form $(v, w)_H = \text{Tr}(H_v H_w)$. Therefore the metric $h_{ij}(x)$ defined on the manifold M^n by the form $(v, w)_H$ must be conformally flat, or the Weyl tensor of the conformal curvature $C_{jke}^i(x)$ for the metric $h_{ij}(x)$ for $n > 3$ must be zero (for $n = 3$ this tensor is zero identically [18]).

V. The properties of a Hamiltonian system (6.6) for $n = 3, 4$ described by Theorems 2 and 3 do not depend on the metric g_{ij} and are the properties of the Haantjes tensor only. Therefore these properties are the definite necessary conditions for the general system (1.1) for $n = 3, 4$ to be Hamiltonian. In the following Theorem 4 we describe invariant necessary conditions for the existence of a Hamiltonian structure for a system (1.1) for all n .

Theorem 4. *If a system (1.1) possesses some Hamiltonian structure (6.6) then the following necessary conditions are fulfilled in any point $x \in M^n$ and for any vectors $u, v, w \in T_x(M^n)$:*

- 1) *The polynomial $P_H(u) = \det(H_u - 1)$ is even.*
- 2) *The algebraic manifold $V_H \subset T_x(M^n)$ is invariant under the involution $u \rightarrow -u$.*

3) The eigenvalues of an operator H_u are symmetric with respect to zero and the dimension of its image $H_u(T_x(M^n))$ is even in a general case.

4) The function $\omega(u, v, w) = \text{Tr}(H_u H_v H_w)$ is skew-symmetric with respect to the transpositions of u, v, w .

5) The differential 2-forms $\beta_{Hk}(u, v) = \text{Tr}(A^k[H_u, H_v]) = 0$.

6) The following transvections are equal to zero

$$J_{Hij\beta}^\alpha (A^m)_\alpha^\beta = \text{Tr}(J_{Hij} A^m) = 0 ,$$

$$\text{Tr}(H_u^k H_v^m) = 0 , \quad k + m = 2s + 1 .$$

Proof. The conditions 1), 2) and 3) follow from Lemma 1 in view of Theorem 1. Condition 4) is equivalent to the equality

$$\text{Tr}(H_u(H_v H_w + H_w H_v)) = 0 . \tag{6.28}$$

The proof of the equality (6.28) and the conditions 4), 5) and 6) is based on the following lemma known in the linear algebra.

Lemma 3. *If an operator A is symmetric and an operator B is skew-symmetric with respect to a non-degenerate bilinear form $(v, w) = g_{\alpha\beta} v^\alpha w^\beta$:*

$$(Av, w) = (v, Aw), \quad (Bv, w) = -(v, Bw) , \tag{6.29}$$

then $\text{Tr}(AB) = 0$.

Indeed, from (6.29) one gets

$$(ABv, w) = -(v, BAw) .$$

This equality implies

$$ABg = -g(BA)^t, \quad g^{-1}ABg = -(BA)^t .$$

Hence

$$\text{Tr}(AB) = \text{Tr}(g^{-1}ABg) = \text{Tr}(-(BA)^t) = -\text{Tr}(AB) ,$$

and the claim follows.

If a system (1.1) is Hamiltonian and so has the form (6.6) then the corresponding operator $A_j^i(x)$ is symmetric with respect to the bilinear form (6.9), see (6.11). Operators H_u are skew-symmetric in view of Theorem 1, see (6.14). Operators $J_H(u, v)$ are skew-symmetric in view of the Corollary 3, see (6.23). Therefore the conditions 4), 5), 6) and the equalities (6.28) follow from Lemma 3.

Theorem 4 is proven.

Remark 7. It is easy to derive from Lemma 3 the vanishing of many other transvections, for example

$$\begin{aligned} \text{Tr}(J_H^k(u, v)H_w^m) &= 0, \quad k + m = 2s + 1 , \\ \text{Tr}(H_u^k [H_v^l, H_w^m]) &= 0, \quad k + l + m = 2s , \\ \text{Tr}(A^k [H_u^l, H_v^m]) &= 0, \quad l + m = 2s , \\ \text{Tr}(H_u^k (H_v^l H_w^m + H_w^m H_v^l)) &= 0, \quad k + l + m = 2s + 1 . \end{aligned} \tag{6.30}$$

These equalities are also the necessary conditions for a system (1.1) to be Hamiltonian.

VI. Let us consider the bilinear tensor $M(u, v)$ (3.23).

Theorem 5. *Linear operators $M_u = N_{Au} - AN_u$ for a Hamiltonian system (6.6) are symmetric with respect to the bilinear form (6.9):*

$$(M_u v, w) = (v, M_u w). \tag{6.31}$$

Proof. The condition (6.31) is invariant. Therefore it is enough to prove it for the canonical form (6.7) of a Hamiltonian system (6.6). Tensor M_{jk}^i is connected with the Nijenhuis tensor N_{jk}^i in view of (3.23) by the formulae

$$M_{jk}^i = N_{\alpha k}^i A_j^\alpha - N_{jk}^\alpha A_\alpha^i. \tag{6.32}$$

This expression for a general system of conservation laws (6.15) in view of (6.16) takes the form

$$M_{jk}^i = f_{k\alpha}^i f_\beta^\alpha f_j^\beta - f_{\alpha\beta}^i f_j^\alpha f_k^\beta - f_{k\beta}^\alpha f_j^\beta f_\alpha^i + f_{j\beta}^\alpha f_k^\beta f_\alpha^i. \tag{6.33}$$

Hence we obtain after the substitution $f^i = q^i \partial f / \partial u^i$ for the canonical form (6.7):

$$M_{jk}^i = q^i \tilde{M}_{jk}^i, \tag{6.34}$$

$$\tilde{M}_{jk}^i = \sum_{\alpha, \beta} q^\alpha q^\beta (f_{ik\alpha} f_{\alpha\beta} f_{\beta j} - f_{i\alpha\beta} f_{\alpha j} f_{\beta k} - f_{k\alpha\beta} f_{\alpha j} f_{\beta i} + f_{j\alpha\beta} f_{\alpha i} f_{\beta k}). \tag{6.35}$$

Symmetricity condition (6.31) is equivalent to the equations

$$g_{\alpha\gamma} M_{j\beta}^\gamma = g_{\beta\gamma} M_{j\alpha}^\gamma, \quad g M_j = M_j^t g \tag{6.36}$$

for all α, β and $j = 1, \dots, n$. Substituting here the formulae (6.34) and $g_{\alpha\gamma} = (q^\alpha)^{-1} \delta_{\alpha, \beta}^\gamma$ we obtain the equations

$$\tilde{M}_{jk}^i = \tilde{M}_{ji}^k, \tag{6.37}$$

which hold obviously for the coefficients (6.35).

Theorem 5 is proven.

Remark 8. The operators H_u are the commutators $H_u = [M_u, A]$, see (2.6) and (3.23). Therefore the skew-symmetricity of the operators H_u (Theorem 1) follows from the symmetricity of the operators A and M_u (Theorem 5).

Corollary 4. *If metric g_{ij} for a Hamiltonian system (6.6) is positively (or negatively) defined then the scalar product*

$$(u, v)_M = \text{Tr}(M_u M_v) \tag{6.38}$$

is non-negatively defined and the scalar product

$$(u, v)_H = \text{Tr}(H_u H_v)$$

is non-positively defined.

Indeed, for the canonical form (6.7) we have in view of (6.34) and (6.37),

$$(u, u)_M = \sum_{\alpha\beta} q^\alpha q^\beta \tilde{M}_{u\beta}^\alpha \tilde{M}_{u\alpha}^\beta = \sum_{\alpha, \beta} q^\alpha q^\beta (\tilde{M}_{u\beta}^\alpha)^2 \geq 0,$$

because $q^\alpha q^\beta > 0$. Analogously in view of (6.19) and (6.22) we get

$$(u, u)_H = \sum_{\alpha, \beta} q^\alpha q^\beta \tilde{H}_{u\beta}^\alpha \tilde{H}_{u\alpha}^\beta = -\sum_{\alpha, \beta} q^\alpha q^\beta (\tilde{H}_{u\beta}^\alpha)^2 \leq 0.$$

Theorem 6. *If a system (1.1) possesses some Hamiltonian structure (6.6) then the following necessary conditions are satisfied in any point $x \in M^n$ and for any vectors u, v, w in the tangent space $T_x(M^n)$:*

1) *The polynomials*

$$\begin{aligned} P_k(u) &= \det([M_u, A^k] - 1), \\ Q(u, v) &= \det([M_u, M_v] - 1), \\ R(u, v) &= \det([H_u, H_v] - 1), \end{aligned} \tag{6.39}$$

and the corresponding algebraic manifolds are invariant with respect to two involutions $u \rightarrow -u$ and $v \rightarrow -v$.

2) *The eigenvalues of the operators*

$$[M_u, A^k], \quad [M_u, M_v], \quad \{H_u, M_v\} = H_u M_v + M_v H_u \tag{6.40}$$

are symmetric with respect to zero.

3) *The differential 2-forms vanish*

$$\gamma_k = \text{Tr}(A^k [M_u, M_v]) = 0. \tag{6.41}$$

The proof of these necessary conditions is based on the fact that operators (6.40) are skew-symmetric with respect to the bilinear form (6.9) and on Lemma 1.

A wide variety of necessary conditions for a system (1.1) to be Hamiltonian can be derived analogously to (6.30) from the fact that the operators A and M_u are symmetric and the operators H_u are skew-symmetric with respect to the bilinear form (6.9).

VII. The properties of symmetry of the Nijenhuis tensor with respect to the bilinear form (6.9) are described by the following theorem.

Theorem 7. *The Nijenhuis tensor N for a Hamiltonian system (6.6) satisfies the three identities*

$$(N(u, v), w) + (N(v, w), u) + (N(w, u), v) = 0, \tag{6.42}$$

$$(AN(u, v), w) + (AN(v, w), u) + (AN(w, u), v) = 0, \tag{6.43}$$

$$(A^2 N(u, v), w) + (A^2 N(v, w), u) + (A^2 N(w, u), v) = (H(u, v), w), \tag{6.44}$$

for any vectors $u, v, w \in T_x(M^n)$.

Proof. These identities are invariant. So it is enough to prove them in the coordinates where metric g_{ij} is diagonal and system (6.6) has the canonical form (6.7). The identities (6.42)–(6.44) are equivalent to the following tensor relations:

$$\begin{aligned} N_{ij}^\alpha g_{\alpha k} + N_{jk}^\alpha g_{\alpha i} + N_{ki}^\alpha g_{\alpha j} &= 0, \\ N_{ij}^\alpha A_\alpha^\beta g_{\beta k} + N_{jk}^\alpha A_\alpha^\beta g_{\beta i} + N_{ki}^\alpha A_\alpha^\beta g_{\beta j} &= 0, \\ N_{ij}^\alpha A_\alpha^\gamma A_\gamma^\beta g_{\beta k} + N_{jk}^\alpha A_\alpha^\gamma A_\gamma^\beta g_{\beta i} + N_{ki}^\alpha A_\alpha^\gamma A_\gamma^\beta g_{\beta j} &= H_{ij}^\alpha g_{\alpha k}. \end{aligned}$$

These relations are valid identically after the substitution of the formulae (6.18)–(6.20) and $A_\alpha^i = q^i f_{i\alpha}$ for the canonical form (6.7).

The identity (6.44) can also be derived in a pure algebraic way as a consequence of the two identities (6.42) and (6.43), the relation (2.4) connecting the Haantjes and the Nijenhuis tensors and the symmetricity relation (6.11) for the operator A . Theorem 7 is proven.

Remark 9. The property of the symmetricity of the operators $M_u = N_{Au} - AN_u$ (6.31) follows from the identities (6.42) and (6.43). Indeed, substituting into (6.42) Au instead of u and subtracting from the obtained expression the identity (6.43) we get in view of (6.11)

$$((N_{Au} - AN_u)v, w) - ((N_{Au} - AN_u)w, v) = 0,$$

that coincides with (6.31). This relation for the tensor $M(u, v)$ (3.19) takes the form

$$(M(u, v), w) = (M(u, w), v).$$

The Haantjes tensor for a Hamiltonian system (6.6) satisfies in view of the formulae (6.19), (6.20) the identities

$$(H(u, v), w) = (H(v, w), u) = (H(w, u), v).$$

These relations are also the immediate consequence of the identity (6.44) along with the skew-symmetricity of the operators H_u .

Remark 10. The three identities (6.42)–(6.44) are equivalent to the general identity

$$(P(A)N(u, v), w) + (P(A)N(v, w), u) + (P(A)N(w, u), v) = (H(u, v), w), \quad (6.45)$$

where $P(A)$ is the quadratic polynomial

$$P(A) = A^2 + pA + q$$

with arbitrary coefficients p and q .

If two vectors u and v belong to the kernel of the operator $P(A)$, or

$$P(A)u = 0, \quad P(A)v = 0, \quad (6.46)$$

then the identity (6.45) yields

$$H(u, v) = P(A)N(u, v). \quad (6.47)$$

The formula (6.47) under the conditions (6.46) generalizes the Haantjes formula (2.15) for the Hamiltonian system (6.6).

Remark 11. The Haantjes tensor $H(u, v)$ for the Hamiltonian system (6.6) satisfies the following identity

$$(T(u, v), w) + (T(v, w), u) + (T(w, u), v) = 0,$$

where the bilinear tensor $T(u, v)$ is

$$T(u, v) = H(u, P(A)v) + H(Q(A)u, v) - (P(A) + Q(A))H(u, v),$$

and $P(A)$ and $Q(A)$ are arbitrary polynomials of A (could be also arbitrary symmetric operators). This identity easily follows from the symmetricity of the operator A and skew-symmetricity of the operators H_u .

VIII. Let an operator tensor field $A(x)$ have a multiple eigenvalue $\lambda_k(x)$ with a multiplicity $n_k \geq 2$. Let $L_{kx} \subset T_x(M^n)$ be the field of the corresponding eigenspaces $(A - \lambda_k)L_k = 0$ and $e(x), g(x) \in L_{kx}$ be arbitrary eigenvector fields.

Theorem 8. *If the operator $A(x)$ corresponds to a Hamiltonian system (1.1) then*

1) *The relations hold*

$$N(L_k, L_k) \subset L_k, \quad M(L_k, L_k) = 0, \quad H(L_k, L_k) = 0, \quad (6.48)$$

$$(A - \lambda_k)^3[e(x), g(x)] = 0. \quad (6.49)$$

2) *If the Jordan normal form of the operator $A(x)$ is diagonal then the field of eigenspaces L_k is integrable and $N(L_k, L_k) = 0$.*

Proof. For the eigenvector fields $e(x), g(x) \in L_{kx}$ we have

$$M_e = N_{Ae} - AN_e = (\lambda_k - A)N_e. \quad (6.50)$$

Operators M_u for a Hamiltonian system (1.1) are symmetric with respect to the nondegenerate bilinear form (6.9). Thus from (6.11), (6.31) and (6.50) we obtain

$$(M_e g, v) = (g, M_e v) = (g, (\lambda_k - A)N_e v) = ((\lambda_k - A)g, N_e v) = 0, \quad (6.51)$$

where $v \in T_x(M^n)$ is an arbitrary vector. Equality (6.51) obviously implies

$$M_e g = (\lambda_k - A)N(e, g) = 0. \quad (6.52)$$

Hence the relations (6.48) follow.

The equality (6.49) follows from (6.52) after substituting the Nijenhuis formula (2.14) for $\lambda_k = \lambda_s$:

$$N(e(x), g(x)) = (A - \lambda_k)^2[e(x), g(x)]. \quad (6.53)$$

If the Jordan normal form of the operator $A(x)$ is diagonal then from (6.49) we obtain

$$[e(x), g(x)] \in L_{kx} \quad (6.54)$$

and therefore the field of eigenspaces L_{kx} is integrable. From (6.54) and (6.53) we obtain $N(e, g) = 0$. Theorem 8 is proven.

IX. A vector $w \in T_x(M^n)$ is called central vector for the Nijenhuis tensor if the operator $N_w = 0$, or $N(w, u) = 0$ for all u . Linear subspace of all central vectors is called the central subspace Z .

Proposition 5. *If the Nijenhuis tensor $N(u, v)$ for a Hamiltonian system (6.6) has a central subspace Z of a dimension k then all vectors $N(u, v)$ belong to a proper subspace L of the dimension $n - k$, the image $AN(u, v)$ belongs to the same subspace L .*

Proof. Let L be a subspace orthogonal to the central subspace Z . Obviously L has dimension $n - k$. For any central vector $w \in Z$ the identities (6.42) and (6.43) imply

$$(N(u, v), w) = 0, \quad (AN(u, v), w) = 0. \tag{6.55}$$

Hence we obtain

$$N(u, v) \in L, \quad AN(u, v) \in L. \tag{6.56}$$

7. Necessary Criteria for the Existence of Two Non-Degenerate Hamiltonian Structures

I. Let us suppose that system (1.1) has two Hamiltonian structures (6.6) with two non-degenerate metrics $(g_{ij})_1$ and $(g_{ij})_2$. In this case we have the $(1, 1)$ tensor field $B_i^j = g_{1i\alpha} g_2^{\alpha j}$ on the manifold M^n . Two Hamiltonian structures are in general position if all eigenvalues of the operator $B = g_1 g_2^{-1}$ are distinct (eigenvalues of the operator B can be complex).

Theorem 9. *If a system (1.1) has two linear independent non-degenerate Hamiltonian structures then*

1) *All operators A, M_u and H_u commute with the operator $B = g_1 g_2^{-1}$.*

2) *Tensors $M(u, v)$ and $H(u, v)$ are reducible, or all operators M_u and H_u have a non-trivial invariant subspace. The polynomials defined on the tangent space $T_x(M^n)$*

$$P_M(u) = \det(M_u - \lambda), \tag{7.1}$$

$$P_H(u) = \det(H_u - \lambda), \tag{7.2}$$

are reducible.

3) *If operator B has k distinct (complex) eigenvalues then polynomials (7.1) and (7.2) have at least k (complex) divisors.*

4) *If two Hamiltonian structures are in general position then the Haantjes tensor vanishes: $H(u, v) \equiv 0$ and all operators M_u and A reduce to the diagonal form simultaneously. The polynomial (7.1) decomposes into a product of n linear factors.*

Proof. 1) If a system (1.1) is Hamiltonian with respect to two Hamiltonian structures then in view of Theorem 1 and Theorem 5 operators A, M_u and H_u satisfy the following identities with respect to the two bilinear forms $(u, v)_1$ and $(u, v)_2$:

$$(Av, w)_1 = (v, Aw)_1, \quad (Av, w)_2 = (v, Aw)_2,$$

$$(M_u v, w)_1 = (v, M_u w)_1, \quad (M_u v, w)_2 = (v, M_u w)_2,$$

$$(H_u v, w)_1 = -(v, H_u w)_1, \quad (H_u v, w)_2 = -(v, H_u w)_2.$$

Hence we get for the corresponding matrices the equalities:

$$\begin{aligned} g_1^{-1}Ag_1 &= A^t, & g_2^{-1}Ag_2 &= A^t, \\ g_1^{-1}M_u g_1 &= M_u^t, & g_2^{-1}M_u g_2 &= M_u^t, \\ g_1^{-1}H_u g_1 &= -H_u^t, & g_2^{-1}H_u g_2 &= -H_u^t. \end{aligned} \tag{7.3}$$

These equalities lead to the commutativity relations with $B = g_1 g_2^{-1}$

$$AB = BA, \quad M_u B = B M_u, \quad H_u B = B H_u. \tag{7.4}$$

2) – 3) In view of (7.4) it is obvious that the reducibility of the tensors $M(u, v)$ and $H(u, v)$ follows from the classical Schur’s lemma [17]. Let $P_B(\lambda)$ be the characteristic polynomial of the operator B

$$P_B(\lambda) = \det(B - \lambda), \quad P_B(B) = 0. \tag{7.5}$$

Polynomial $P_B(\lambda)$ has the form

$$P_B(\lambda) = (-1)^n \prod_{i=1}^k (\lambda - \lambda_i)^{n_i}, \tag{7.6}$$

where λ_i are the eigenvalues of the operator B and n_i are their multiplicities. Let $L_i \subset R^n = T_x(M^n)$ be a maximal subspace annihilated by the polynomial

$$p_i(B) = (B - \lambda_i)^{n_i}, \quad p_i(B)(L_i) = 0. \tag{7.7}$$

Obviously we have $\dim L_i = n_i$ and

$$L_1 + L_2 + \dots + L_k = R^n, \quad n_1 + n_2 + \dots + n_k = n. \tag{7.8}$$

Operators A, M_u and H_u commute with operator $p_i(B)$ (7.7) and thus the equalities follow:

$$p_i(B)(A(L_i)) = 0, \quad p_i(B)(M_u(L_i)) = 0, \quad p_i(B)(H_u(L_i)) = 0. \tag{7.9}$$

Hence

$$A(L_i) \subset L_i, \quad M_u(L_i) \subset L_i, \quad H_u(L_i) \subset L_i. \tag{7.10}$$

Therefore all operators M_u, H_u and A have a block-diagonal structure in the decomposition (7.8). If operator B has $k \geq 2$ distinct eigenvalues then the operators M_u, H_u and A have k diagonal blocks in the decomposition (7.8). In this case the polynomials (7.1) and (7.2) are reducible and have k divisors.

Let $k = 1$ and thus

$$P_B(\lambda) = (-1)^n (\lambda - \lambda_1)^n,$$

then operator B must have nontrivial Jordan blocks otherwise the two Hamiltonian structures would be linear dependent. In this case operator $p_{11}(B) = B - \lambda_1$ annihilates subspace L_1 having dimension $\dim(L_1) = n_1 < n$. Thus all operators M_u, H_u and A also have invariant subspace L_1 . Therefore tensors $M(u, v)$ and $H(u, v)$ are reducible and the polynomials (7.1) and (7.2) are reducible too. We note that the polynomial (7.2) always has factor λ because $\det H_u \equiv 0$.

Obviously we have a filtration of invariant subspaces (which can coincide)

$$L_1 \subset L_2 \subset \dots \subset L_n = R^n, \tag{7.11}$$

where L_k is annihilated by the operator

$$p_{1k}(B) = (B - \lambda_1)^k, \quad p_{1k}(B)(L_k) = 0. \tag{7.12}$$

Filtration (7.12) is invariant also under the operators A , M_u and H_u and hence it leads to a more fine factorization of the polynomials (7.1) and (7.2).

4) Suppose the operator $B = g_1 g_2^{-1}$ has n distinct eigenvalues that means two Hamiltonian structures are in general position. Let e_1, \dots, e_n be the basis of its (complex) eigenvectors. All operators A, M_u and H_u commuting with operator B are diagonal in the basis on its eigenvectors. Thus all operators A, M_u, M_v commute one with another and hence $H_u = [M_u, A] = 0$. Therefore the Haantjes tensor vanishes: $H(u, v) \equiv 0$. The polynomial (7.1) is decomposed in this case into a product of n linear factors.

Theorem 9 is proven.

II.

Corollary 5. *If a system (1.1) has two Hamiltonian structures in general position and operator A has n real distinct eigenvalues then the system has n Riemann invariants and is integrable by the generalized hodograph transformation.*

Proof. The system (1.1) under these conditions has in view of the Theorem 9 zero Haantjes tensor. Therefore in view of Theorem 1 of [11] the system possesses n Riemann invariants and is transformed to the diagonal form after some change of coordinates. Thus in this case the integrability by the generalized hodograph transformation follows from the Tsarev Theorem [19].

Remark 12. Bi-Hamiltonian systems were studied in many papers, where the main method to prove their integrability was based on the Magri scheme [10]. The proof of Corollary 5 is independent on the Magri approach.

Theorem 10. *If a system (1.1) has two non-degenerate Hamiltonian structures and all eigenvalues λ_i of the operator $B = g_1 g_2^{-1}$ have multiplicities $m_i \leq 2$ then the Haantjes tensor $H_{jk}^i = 0$. If the multiplicities $m_i \leq 3$ then the Haantjes tensor defines the structure of the Lie algebra in each tangent space $T_x(M^n)$. This Lie algebra is a direct sum of the commutative one R^k and the 3-dimensional Lie subalgebras \mathcal{A}_3 with the commutators (3.14).*

Proof. The generalized eigenspaces L_i of the operator $B = g_1 g_2^{-1}$ belonging to the eigenvalues λ_i are invariant with respect to the operators A and H_u in view of (7.4). The Haantjes tensor $H(u, v)$ is zero on each L_i if $\dim L_i = 2$ in view of (6.47). If $\dim L_i = 3$ then the Haantjes tensor $H(u, v)$ defines the structure of the Lie algebra in the L_i with the commutators (3.14), in view of the formula (2.15) and $H_u = [M_u, A]$.

Theorem 10 is proven.

8. An application to the equations of gas dynamics

We consider the equations of one-dimensional gas dynamics

$$\begin{aligned} v_t &= -vv_x - \frac{1}{\rho} p_\rho \rho_x - \frac{1}{\rho} p_s s_x, \\ \rho_t &= -\rho v_x - v \rho_x, \\ s_t &= -vs_x, \end{aligned} \tag{8.1}$$

where $v(x, t)$ is the velocity of the gas, $\rho(x, t)$ is the density of mass and $s(x, t)$ is the density of entropy. Pressure p is determined by an equation of state $p = p(\rho, s)$.

The corresponding (1,1) tensor field A_j^i defined on the manifold M^3 with the coordinates v, ρ, s has the following eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and eigenvectors e_1, e_2, e_3 :

$$\lambda_1 = -v, \quad e_1 = p_s \frac{\partial}{\partial \rho} - p_\rho \frac{\partial}{\partial s}, \tag{8.2}$$

$$\lambda_2 = -v + \sqrt{p_\rho}, \quad e_2 = \sqrt{p_\rho} \frac{\partial}{\partial v} - \rho \frac{\partial}{\partial \rho},$$

$$\lambda_3 = -v - \sqrt{p_\rho}, \quad e_3 = \sqrt{p_\rho} \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho}.$$

In the hyperbolic case $p_\rho \geq 0$ it is easy to calculate the Haantjes tensor $H(u, v)$ in the basis of eigenvectors (8.2) using the formula (2.15). Thus we obtain:

$$H(e_1, e_2) = h(\rho, s)e_3, \quad H(e_1, e_3) = h(\rho, s)e_2, \quad H(e_2, e_3) = 0, \tag{8.3}$$

where

$$h(\rho, s) = p_\rho \left(p_\rho p_{\rho s} - p_s p_{\rho \rho} - \frac{2}{\rho} p_\rho p_s \right) = \rho^2 p_\rho^3 \left(\frac{p_s}{\rho^2 p_\rho} \right)_\rho. \tag{8.4}$$

Proposition 6. *The necessary and sufficient condition for the existence of Riemann invariants for the equations of gas dynamics (8.1) is*

$$p_\rho = 0 \quad \text{or} \quad \left(\frac{p_s}{\rho^2 p_\rho} \right)_\rho = 0. \tag{8.5}$$

This is also the necessary condition for the existence of a Hamiltonian structure of the type (6.6) with a nondegenerate metric g^j .

Proof. Condition (8.5) in view of (8.3), (8.4) is the condition of vanishing of the Haantjes tensor $H(u, v)$. Therefore the statement concerning the existence of Riemann invariants follows from Theorem 1 of [11].

If system (8.1) is Hamiltonian in the sense (6.6) then in view of Theorem 2 of this paper the Haantjes tensor $H(u, v)$ defines a deformation of structures of Lie algebras in the tangent bundle $T(M^3)$ which are simple or commutative. Lie algebras defined by the formulae (8.3) are solvable if $h(\rho, s) \neq 0$. Thus we get the necessary condition $h(\rho, s) = 0$, or (8.5) for the existence of a Hamiltonian structure (6.6). Proposition 6 is proven.

The operator H_u for a tangent vector $u = x_1 e_1 + x_2 e_2 + x_3 e_3$ has the form

$$H_u(e_1) = -hx_3e_2 - hx_2e_3, \quad H_u(e_2) = hx_1e_3, \quad H_u(e_3) = hx_1e_2. \quad (8.6)$$

The corresponding metric on the tangent space $T_x(M^3)$ is degenerate

$$(u, u)_H = \text{Tr}(H_u H_u) = 2h^2x_1^2. \quad (8.7)$$

The algebraic manifold $V_H \subset T_x(M^3)$ is defined by the formula

$$P_H(u) = \det(H_u - 1) = h^2x_1^2 - 1 = 0, \quad (8.8)$$

and thus consist of two parallel planes $x_1 = \pm h^{-1}$.

9. On Perturbations of the Benney Equations

I.The Benney equations [12] have the form

$$u_{it} = -u_i u_{ix} - \left(\sum_{j=1}^n \eta_{jx} \right), \quad (9.1)$$

$$\eta_{it} = -\eta_i u_{ix} - u_i \eta_{ix}.$$

On the corresponding manifold M^{2n} we consider the local coordinates $u_1, \dots, u_n, \eta_1, \dots, \eta_n$. In the tangent space $T_x(M^{2n})$ we have a basis of $2n$ vectors

$$e_i = \frac{\partial}{\partial u_i}, \quad g_i = \frac{\partial}{\partial \eta_i}, \quad i = 1, \dots, n. \quad (9.2)$$

The operator tensor field A_i^j defined by the Benney system (9.1) has the form

$$Ae_i = -u_i e_i - \eta_i g_i, \quad Ag_i = -(e_1 + \dots + e_n) - u_i g_i. \quad (9.3)$$

A direct calculation of the Nijenhuis tensor $N(v, w)$ (2.1) leads to the formulae

$$N(e_i, e_j) = 0, \quad N(g_i, g_j) = g_j - g_i,$$

$$N(e_i, g_j) = -N(g_j, e_i) = -e_i. \quad (9.4)$$

Hence for any two tangent vectors

$$v = \sum_{i=1}^n (x_i e_i + y_i g_i), \quad \alpha = \sum_{i=1}^n (\alpha_i e_i + \beta_i g_i), \quad (9.5)$$

we have

$$N_v e_i = N(v, e_i) = \omega(v) e_i, \quad N_v g_j = \omega(v) g_j - v, \quad (9.6)$$

$$N_v \alpha = N(v, \alpha) = \omega(v) \alpha - \omega(\alpha) v. \quad (9.7)$$

Here ω is the differential 1-form

$$\omega(v) = y_1 + \dots + y_n, \quad \omega(\alpha) = \beta_1 + \dots + \beta_n. \quad (9.8)$$

From (9.3) we get

$$Av = \sum_{i=1}^n (-x_i u_i - \omega(v)) e_i - \sum_{i=1}^n (x_i \eta_i + y_i u_i) g_i. \quad (9.9)$$

From (9.7) and (9.9) we obtain

$$M_v \alpha = (N_{Av} - AN_v) \alpha = -\omega(v)A\alpha - \tilde{\omega}(v)\alpha, \tag{9.10}$$

$$M_v = -\omega(v)A - \tilde{\omega}(v)1, \tag{9.11}$$

where

$$\tilde{\omega}(v) = \sum_{i=1}^n (x_i \eta_i - y_i u_i). \tag{9.12}$$

The Haantjes tensor $H(v, w)$ for the Benney system (9.1) is identically equal to zero, that follows from (9.11) and (2.6),

$$H_v = [M_v, A] = 0. \tag{9.13}$$

This gives in view of Theorem 1 of the work [11] another proof of the existence of the Riemann invariants for the Benney equations.

II. Let us show that the Nijenhuis tensor (9.7) for the Benney equations defines the structure of the Lie algebra in each tangent space $T_x(M^{2n})$. Indeed, let L^{2n-1} be the hyperplane of the tangent vectors, satisfying the condition $\omega(v) = 0$, and g be the tangent vector

$$g = \frac{1}{n}(g_1 + \dots + g_n). \tag{9.14}$$

Obviously $\omega(g) = 1$. In view of (9.7) for any two vectors $v, w \in L^{2n-1}$ we have

$$N(v, w) = 0, \quad N(g, v) = v. \tag{9.15}$$

The Jacobi identity for any three tangent vectors $u, v, w \in T_x(M^{2n})$

$$N(N(u, v), w) + N(N(v, w), u) + N(N(w, u), v) = 0 \tag{9.16}$$

follows easily from the formulae (9.15). Thus each tangent space $T_x(M^{2n})$ has the structure of the solvable Lie algebra \mathcal{A} , determined by the Nijenhuis tensor (9.7). The hyperplane L^{2n-1} is the maximal commutative ideal:

$$[L^{2n-1}, L^{2n-1}] = 0, \quad [\mathcal{A}, \mathcal{A}] = L^{2n-1}. \tag{9.17}$$

Obviously the structure of the Lie Algebra \mathcal{A} in $T_x(M^{2n})$ does not depend on the point $x \in M^{2n}$.

The invariant scalar product $(u, v)_N$ (2.8) coincides with the Cartan–Killing form

$$(u, v)_N = \text{Tr}(N_u N_v) = (2n - 1)\omega(u)\omega(v). \tag{9.18}$$

Thus we have

$$(L^{2n-1}, L^{2n-1})_N = 0, \quad (L^{2n-1}, g)_N = 0, \quad (g, g)_N = 2n - 1. \tag{9.19}$$

The algebraic manifold $V_N \in T_x(M^{2n})$ for the Benney system is determined by the equation

$$P_N(v) = \det(N_v - 1) = (\omega(v) - 1)^{2n-1} = 0. \tag{9.20}$$

Thus the manifold V_N is the hyperplane $\omega(v) = 1$, parallel to the plane L^{2n-1} .

Plane L^{2n-1} is the set Z_N of all N -nilpotent vectors, see (5.22)–(5.24).

III. We consider the following natural perturbations of the Benney equations:

$$\begin{aligned} u_{it} &= -u_i u_{ix} - \sum_{k=1}^n f_k(\eta_k) \eta_{kx}, \\ \eta_{it} &= -\eta_i u_{ix} - u_i \eta_{ix}, \end{aligned} \tag{9.21}$$

where $f_k(\eta_k)$ are arbitrary smooth functions. System (9.21) for $f_k(\eta_k) = \text{const}$ is equivalent to the Benney system (9.1).

The corresponding to (9.21) (1,1) tensor field A_j^i has the form

$$Ae_i = -u_i e_i - \eta_i g_i, \quad Ag_i = -f_i E - u_i g_i, \tag{9.22}$$

where

$$e_i = \frac{\partial}{\partial u_i}, \quad g_i = \frac{\partial}{\partial \eta_i}, \quad E = e_1 + \dots + e_n. \tag{9.23}$$

For any tangent vector v (9.5) we have

$$Av = -\sum_{k=1}^n x_k u_k e_k - \sum_{k=1}^n (x_k \eta_k + y_k u_k) g_k - \omega(v) E, \tag{9.24}$$

where $\omega(v)$ is the differential 1-form

$$\omega(v) = \sum_{k=1}^n f_k(\eta_k) y_k. \tag{9.25}$$

The Nijenhuis tensor for the system (9.21) has the form

$$\begin{aligned} N(e_i, e_j) &= 0, \quad N(g_i, g_j) = f_i g_j - f_j g_i, \\ N(g_j, e_i) &= -N(e_i, g_j) = f_j e_i - \eta_j \dot{f}_j \delta_j^i E, \quad \dot{f}_j = \frac{df_i(\eta_j)}{d\eta_j}. \end{aligned} \tag{9.26}$$

Hence for any two tangent vectors v, α (9.5) we obtain

$$N(v, \alpha) = N_0(v, \alpha) + \omega_0(v, \alpha) E, \tag{9.27}$$

where

$$N_0(v, \alpha) = \omega(v)\alpha - \omega(\alpha)v, \tag{9.28}$$

$$\omega_0(v, \alpha) = \sum_{k=1}^n (x_k \beta_k - y_k \alpha_k) \eta_k \dot{f}_k. \tag{9.29}$$

Tensor $N_0(v, \alpha)$ (9.28) is analogous to (9.7) and therefore satisfies the Jacobi identity (9.16). Tensor $N(v, \alpha)$ (9.27) satisfies the Jacobi identity only if $\omega_0(v, \alpha) \equiv 0$, or $f_k(\eta_k) = \text{const}$, i.e., only for the Benney system (9.1).

From (9.27) and (9.24) we get

$$\begin{aligned} M_v \alpha &= (N_{Av} - AN_v) \alpha \\ &= \omega_0(Av, \alpha) E - \omega_0(v, \alpha) AE - \omega(v) A\alpha - \tilde{\omega}(v) \alpha, \end{aligned} \tag{9.30}$$

where

$$\tilde{\omega}(v) = \sum_{k=1}^n f_k(x_k \eta_k - y_k u_k).$$

Hence the formula for the operators H_v follows

$$H_v \alpha = [M_v, A] \alpha = \omega_0(v, \alpha) A^2 E + \omega_1(v, \alpha) A E + \omega_2(v, \alpha) E, \quad (9.31)$$

where

$$\omega_1(u, \alpha) = \omega_0(A \alpha, u) + \omega_0(\alpha, A u), \quad (9.32)$$

$$\omega_2(u, \alpha) = \omega_0(A u, A \alpha). \quad (9.33)$$

Thus we get that operators H_v for the system (9.21) are not zero and transform a tangent space $T_x(M^{2n})$ into 3-dimensional subspace generated by vectors

$$E, A E, A^2 E. \quad (9.34)$$

Vectors (9.34) are linear independent and for a general vector v three 1-forms defined by the formulae (9.29), (9.32) and (9.33)

$$\omega_0(v, \alpha), \omega_1(v, \alpha), \omega_2(v, \alpha) \quad (9.35)$$

are linear independent too. Thus the dimension of the image $H_v(T_x(M^{2n}))$ for the general vector v is odd. Therefore in view of the Theorem 3 of this work (the necessary condition 3) is not fulfilled) and Theorem 1 of the work [11] we obtain the following result.

Proposition 7. *Any perturbation (9.21) of the Benney equations (9.1) with non-constant functions $f_k(\eta_k)$ has no Riemann invariants and has no Hamiltonian structure of the non-degenerate type (6.6).*

IV. Let $K \subset T_x(M^{2n})$ be the kernel of the operator H_v and h_1, \dots, h_{2n-3} be a basis in K . Operator H_v in the basis of $2n$ vectors

$$E, A E, A^2 E, h_1, \dots, h_{2n-3} \quad (9.36)$$

has nonzero entries only in the up-left 3×3 block. Therefore the polynomial

$$P_H(v) = \det(H_v - 1) \quad (9.37)$$

has the form

$$P_H(v) = -\det \begin{bmatrix} \omega_2(v, E) - 1 & \omega_2(v, A E) & \omega_2(v, A^2 E) \\ \omega_1(v, E) & \omega_1(v, A E) - 1 & \omega_1(v, A^2 E) \\ \omega_0(v, E) & \omega_0(v, A E) & \omega_0(v, A^2 E) - 1 \end{bmatrix}. \quad (9.38)$$

This polynomial has degree 3 and the corresponding algebraic manifold $V_H \subset T_x(M^{2n})$, determined by the equation $P_H(v) = 0$, is not invariant under the reflection $v \rightarrow -v$. Thus the necessary conditions 1) and 2) from Theorem 3 for the existence of a Hamiltonian structure are not fulfilled for the perturbation of the Benney equations (9.21).

10. An Application to the Certain Matrix Partial Differential Equations

I. We consider for an arbitrary Lie algebra \mathcal{A} the partial differential equation for a vector function $a(t, x) \in \mathcal{A}$ having the Lax form

$$a_t = [a, a_x]. \quad (10.1)$$

The Casimir functions $f(a)$, constant on the orbits of the adjoint representation of the corresponding Lie group G , are the pointwise first integrals of (10.1): $f(a(t, x))_t = 0$.

The manifold M^n corresponding to the system (10.1) coincides with the Lie algebra \mathcal{A} . The corresponding operator tensor field is $A = \text{ad}_a$. Let $u, v \in T_a(\mathcal{A})$ be two tangent vectors in a point $a \in A$. The operator tensor field A is skew-symmetric with respect to the Cartan–Killing form

$$(u, v) = \text{Tr}(\text{ad}_u \text{ad}_v), (Au, v) + (u, Av) = 0. \tag{10.2}$$

We consider the constant extensions \tilde{u}, \tilde{v} as the vector fields on the manifold $M^n = \mathcal{A}$. The Nijenhuis tensor is determined by the formula

$$N(u, v) = A^2[\tilde{u}, \tilde{v}]_g + [A\tilde{u}, A\tilde{v}]_g - A([A\tilde{u}, \tilde{v}]_g + [\tilde{u}, A\tilde{v}]_g), \tag{10.3}$$

where $[\alpha, \beta]_g$ is the geometric commutator of the vector fields α, β .

Let c^k_{ij} be structure constants of Lie algebra \mathcal{A} in the basis

$$e_1 = \frac{\partial}{\partial a^1}, \dots, e_n = \frac{\partial}{\partial a^n}, [e_i, e_j] = C^k_{ij} e_k. \tag{10.4}$$

Then we have

$$A\tilde{u} = \text{ad}_a \tilde{u} = \sum_{i,j,k} C^k_{ij} a^i u^j \frac{\partial}{\partial a^k}, A\tilde{v} = \sum_{i,j,s} C^s_{ij} a^i v^j \frac{\partial}{\partial a^s}, \tag{10.5}$$

and the formulae follow

$$[A\tilde{u}, A\tilde{v}]_g = [a, [u, v]], [\tilde{u}, \tilde{v}]_g = 0, \\ [A\tilde{u}, \tilde{v}]_g = [u, v], [\tilde{u}, A\tilde{v}]_g = [u, v].$$

substituting these formulae into (10.3), we obtain

$$N(u, v) = -[a, [u, v]], N_u = -\text{ad}_a \text{ad}_u. \tag{10.6}$$

Hence we obtain for the operators M_u :

$$M_u = N_{Au} - AN_u = N_{[a,u]} + \text{ad}_a^2 \text{ad}_u = \text{ad}_a \text{ad}_u \text{ad}_a. \tag{10.7}$$

Therefore for the Haantjes operators H_u we get the expression

$$H_u = [M_u, A] = \text{ad}_a \text{ad}_{[u,a]} \text{ad}_a. \tag{10.8}$$

From the formulae (10.2) (10.7) and (10.8) we obtain that all operators M_u, H_v and A are skew-symmetric with respect to the Cartan–Killing form (10.2).

Proposition 8. Equation (10.1) for any simple Lie algebra \mathcal{A} considered as a system of partial differential equations has no Riemann invariants and no Hamiltonian structure of the nondegenerate type (6.6).

Proof. If a system (10.1) for some Lie algebra \mathcal{A} is Hamiltonian in the sense of (6.6), then all operators M_u and A are symmetric with respect to the bilinear form (6.9) and all operators H_u are skew-symmetric. In this case analogously to Sect. 7 the non-degenerate operators B_a have to exist that anticommute with all operators M_u (10.7) and $A = \text{ad}_a$ and commute with all operators H_u (10.8). For a simple Lie

algebra \mathcal{A} such operators B_a do not exist in view of the explicit formulae (10.6)–(10.8). Therefore Eq. (10.1) for any simple Lie algebra \mathcal{A} is not Hamiltonian.

The nonexistence of the Riemann invariants follows from the nonvanishing of the Haantjes tensor $H(u, v)$ (10.8) and from Theorem 1 of [11]. Proposition 8 is proven.

II. Let us consider an equation for a matrix $a(t, x)$:

$$a_t = \lambda a a_x + \mu a_x a, \tag{10.9}$$

where λ and μ are some constants. Matrix entries a^{ij} are coordinates in the space M^N of all matrices, $N = n^2$. Tangent vectors $e_{ij} = \partial/\partial a^{ij}$ form a basis in a tangent space $T_a(M^N)$ which is naturally identified with the space of matrices M^N . Equation (10.9) in the coordinates a^{ij} has the form

$$a_t^{ij} = \lambda a^{i\alpha} a_x^{\alpha j} + \mu a_x^{i\alpha} a^{\alpha j}. \tag{10.10}$$

The corresponding operator A has the entries

$$A_{ks}^{ij} = \lambda a^{ik} \delta_s^j + \mu \delta_k^i a^{sj}, \quad Av = \lambda av + \mu va. \tag{10.11}$$

A direct calculation by the formula (2.2) leads to an expression for the Nijenhuis tensor

$$N(e_{ij}, e_{ks}) = \lambda \mu (\delta_i^s (a^{j\alpha} e_{k\alpha} - a^{\alpha k} e_{\alpha j}) + \delta_j^k (a^{\alpha i} e_{\alpha s} - a^{s\alpha} e_{i\alpha})).$$

Hence for any two tangent vectors $u, v \in T_a(M^N)$ we obtain

$$N(u, v) = \lambda \mu [a, [u, v]]. \tag{10.12}$$

That provides for $\lambda = 1$ and $\mu = -1$ the second proof of the formula (10.6).

Proposition 9. *Matrix equations*

$$a_t = P(a)a_x, \tag{10.13}$$

$$a_t = a_x P(a), \tag{10.14}$$

where $P(a)$ is an arbitrary analytic matrix function split into n noninteracting subsystems of partial differential equations in a neighbourhood of any matrix a with real and distinct eigenvalues.

Proof. We prove at first that the Nijenhuis tensor vanishes for Eqs. (10.13) and (10.14). Indeed, we get from (10.12) that equations

$$a_t = a a_x, \tag{10.15}$$

$$a_t = a_x a \tag{10.16}$$

have zero Nijenhuis tensor, because $\lambda \mu = 0$. If a matrix a has diagonal real Jordan normal form, then operators A (10.11), corresponding to Eqs. (10.15), (10.16) also have diagonal Jordan normal forms, because $Av = av$ or $Av = va$. Operator tensor fields, corresponding to Eqs. (10.13) and (10.14) are $P(A)$. Let e_1, \dots, e_N be eigenvectors of an operator A with diagonal Jordan normal form and $\lambda_1, \dots, \lambda_N$ be the corresponding eigenvalues. Operator $P(A)$ has the same eigenvectors e_1, \dots, e_N with eigenvalues $P(\lambda_1), \dots, P(\lambda_N)$. From the Nijenhuis formula

$$N(e_i, e_j) = (A - \lambda_i)(A - \lambda_j)[e_i, e_j] + (\lambda_i - \lambda_j)(e_i(\lambda_j)e_j + e_j(\lambda_i)e_i) \quad (10.17)$$

it follows that if the Nijenhuis tensor for the operator A is equal to zero then it is equal to zero for all operators $P(A)$ as well.

Eigenvalues of the operators $P(A)$ corresponding to Eqs. (10.13), (10.14) are equal to $P(\lambda_i)$ where λ_i form n eigenvalues of the matrix a and have multiplicities n . Therefore from the results of the work [16] in view of the vanishing of the Nijenhuis tensor we obtain that Eqs. (10.13), (10.14) split into n non-interacting subsystems of n equations (in each subsystem). These subsystems are tangent to n -dimensional eigenspaces of the operator $P(A)$ and have the form

$$\lambda_{it} = P(\lambda_i)\lambda_{ix}, \quad u_{ikt} = P(\lambda_i)u_{ikx}, \quad k = 1, \dots, n - 1. \quad (10.18)$$

The arising n^2 functions $\lambda_1, \dots, \lambda_n, u_{ik}, 1 \leq i \leq n, 1 \leq k \leq n - 1$ form another system of coordinates in the space of matrices M^N . Proposition 9 is proven.

III. We consider the matrix equation

$$a_t = aa_x a. \quad (10.19)$$

For the entries a^{ij} of the matrix a we get the equations

$$a_t^{ij} = a^{i\alpha} a_x^{\alpha\beta} a^{\beta j}. \quad (10.20)$$

The corresponding operator A has entries

$$A_{ks}^{ij} = a^{ik} a^{sj}, \quad Av = ava. \quad (10.21)$$

The operator A is symmetric with respect to the bilinear form

$$(u, v) = \text{Tr}(uv), \quad (Au, v) = (u, Av). \quad (10.22)$$

A direct calculation by the formula (2.2) leads to the following expression for the Nijenhuis tensor:

$$N(e_{ij}, e_{ks}) = (\delta_s^i a^{j\beta} a^{\beta\delta} a^{\gamma k} + \delta_k^j a^{\gamma\alpha} a^{\alpha i} a^{s\delta} - \delta_j^k a^{s\beta} a^{\beta\delta} a^{\gamma i} - \delta_i^s a^{\gamma\alpha} a^{\alpha k} a^{j\delta}) e_{\gamma\delta}. \quad (10.23)$$

Hence for any two tangent vectors $u, v \in T_a(M^N)$ we get

$$N(u, v) = a[a, [u, v]]a. \quad (10.24)$$

Thus we have

$$N_u = A\text{ad}_a \text{ad}_u. \quad (10.25)$$

For the operators M_u we get the formula

$$M_u = N_{A_u} - AN_u = A\text{ad}_a(\text{ad}_{aau} - A\text{ad}_u). \quad (10.26)$$

Therefore for the Haantjes tensor H_u we obtain

$$H_u = [M_u, A] = A\text{ad}_a[\text{ad}_{aau} - A\text{ad}_u, A] = A^2 \text{ad}_a \text{ad}_{[u, a]} \text{ad}_a. \quad (10.27)$$

Thus we get that the Haantjes tensor (10.27) is not equal to zero and so Eq. (10.19) as well as Eqs. (10.1) has no Riemann invariants.

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Note added in proof

11. Necessary Conditions for Existence of a Non-Local Non-Degenerate Hamiltonian Structure

I. Let us consider a (1,1) tensor

$$A_j^i(u) = g^{i\alpha} \nabla_\alpha \nabla_j f(u) + K f(u) \delta_j^i, \quad (11.1)$$

where g_{ij} is a non-degenerate metric of constant curvature K , $f(u)$ is a smooth function and ∇_j are operators of covariant differentiation with respect to the metric g_{ij} . The (1,1) tensors $A_j^i(u)$ (11.1) describe systems of partial differential equations (1.1) having non-local ($K \neq 0$) [20] or local ($K = 0$) [7–9] non-degenerate Hamiltonian structures.

Theorem 11. *If a system (1.1) has a non-local or local non-degenerate Hamiltonian structure then the Nijenhuis tensor $N_A(u, v)$ satisfies the algebraic identities (6.42)–(6.44) for the (1, 1) tensor A_j^i (11.1). The corresponding operators H_u (2.5) are skew-symmetric with respect to the bilinear form defined by the metric g_{ij} :*

$$(H_u v, w)_g = -(v, H_u w)_g, \quad (v, w)_g = g_{\alpha\beta} v^\alpha w^\beta. \tag{11.2}$$

The proof follows by a direct calculation using the classical formulae of differential geometry for the metric of constant curvature.

Corollary 6. *Theorems 3–10 hold true for the systems (1.1) possessing non-local and non-degenerate Hamiltonian structures.*

Indeed, Theorems 3-10 are direct consequences of the identity (6.14) and the algebraic identities (6.42)–(6.44). These identities hold true for the non-local Hamiltonian structures as well.

II. Let us assume that system (1.1) is hyperbolic. Then tensor $A_j^i(x)$ (11.1) has real and distinct eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ corresponding to the eigenvectors $e_1(x), \dots, e_n(x)$. Their commutator relations have the form

$$[e_i(x), e_j(x)] = \sum_{k=1}^n C_{ij}^k(x) e_k(x). \tag{11.3}$$

Theorem 12. *If a system (1.1) possesses a non-local or local non-degenerate Hamiltonian structure then the following necessary conditions are satisfied:*

1) *For any $p + 1$ distinct indices α, i, \dots, ℓ the following two products are equal*

$$C_{\alpha i}^j C_{\alpha j}^k \dots C_{\alpha m}^\ell C_{\alpha \ell}^i = (-1)^p C_{\alpha j}^i C_{\alpha k}^j \dots C_{\alpha \ell}^m C_{\alpha i}^\ell. \tag{11.4}$$

2) *For any p distinct indices i, j, \dots, ℓ and p arbitrary indices $\alpha, \beta, \dots, \sigma$ different from i, j, \dots, ℓ the following two products are equal*

$$D_{\alpha i}^j D_{\beta j}^k \dots D_{\gamma m}^\ell D_{\sigma \ell}^i = (-1)^p D_{\alpha j}^i D_{\beta k}^j \dots D_{\gamma \ell}^m D_{\sigma i}^\ell, \quad D_{\alpha i}^j = \frac{C_{\alpha i}^j}{(\lambda_\alpha - \lambda_i)^2}. \tag{11.5}$$

Proof. The metric g_{ij} is diagonal in the basis of eigenvalues

$$(e_i(x), e_j(x))_g = q_i(x) \delta_j^i, \tag{11.6}$$

because the (1,1) tensor $A_j^i(x)$ (11.1) is symmetric with respect to the metric $g_{ij}(x)$. The formulae (2.15), (11.2) and (11.3) imply

$$\frac{C_{\alpha i}^j}{(\lambda_\alpha - \lambda_i)^2} q_j = -\frac{C_{\alpha j}^i}{(\lambda_\alpha - \lambda_j)^2} q_i. \tag{11.7}$$

Multiplying the formulae (11.7) for p indices $\alpha, \beta, \dots, \delta$ and p pairs of indices $(i, j), (j, k), \dots, (\ell, i)$ we obtain the equalities (11.4) and (11.5).

III. Symmetricity of the (1,1) tensor $A_j^i(x)$ yields

$$(B(A)u, v)_g = (u, B(A)v)_g, \quad B(A, x) = \sum_{m=0}^k b_m(x) A^m(x) \tag{11.8}$$

for an arbitrary polynomial $B(A, x)$. We define a polynomial on the tangent bundle $T(M^n)$

$$P_B(u, \lambda) = \det(B(A)H_u - \lambda) \tag{11.9}$$

and a fibration of algebraic varieties $V_B \subset T(M^n)$ defined by the equation $P_B(u, 1) = 0$.

Theorem 13. *If a system (1.1) possesses a non-local or local non-degenerate Hamiltonian structure then the following necessary conditions are satisfied in any point $x \in M^n$ and for any tangent vector $u \in T_x(M^n)$ and for an arbitrary polynomial $B(A, x)$ (11.8) :*

- 1) *The polynomial $P_B(u, \lambda)$ (11.9) is even with respect to u and is even or odd with respect to λ whenever n is even or odd.*
- 2) *The fibration of the algebraic varieties $V_B \subset T(M^n)$ is invariant with respect to the involution $u \rightarrow -u$.*
- 3) *The set of eigenvalues of the operator $B(A)H_u$ is invariant with respect to the involution $\lambda \rightarrow -\lambda$.*

Proof of Theorem 13 follows from the identities (11.2) and (11.8) and the following two Lemmas.

Lemma 4. *The characteristic polynomials coincide for the operators BH and HB .*

Lemma 5. *If an operator B is symmetric with respect to a non-degenerate bilinear form (see(11.8)) and an operator H is skew-symmetric (11.2) then the set of eigenvalues of the operators BH and HB is invariant with respect to the involution $\lambda \rightarrow -\lambda$ along with their multiplicities.*

Proof. The relations (11.8) and (11.2) imply the matrix equations

$$Bg = gB^t, Hg = -gH^t, g^{-1}BHg = -(HB)^t. \tag{11.10}$$

Using the last formula we obtain for the characteristic polynomial

$$\begin{aligned} P_{BH}(\lambda) &= \det(BH - \lambda) = \det(g^{-1}BHg - \lambda) = \det(-(HB)^t - \lambda) \\ &= (-1)^n \det(HB + \lambda) = (-1)^n P_{HB}(-\lambda). \end{aligned} \tag{11.11}$$

Applying Lemma 4 we derive the equality

$$P_{BH}(\lambda) = (-1)^n P_{BH}(-\lambda), \tag{11.12}$$

that proves Lemma 5.

Theorem 14. *If two operators P and H are skew-symmetric with respect to a non-degenerate bilinear form (see (11.2)) then all non-zero eigenvalues of the operator PH have even multiplicities.*

Proof. We first assume that $P = BHB$ where B is a symmetric operator (see (11.8)). Then $PH = (BH)^2$. The set of eigenvalues of the operator BH is symmetric with respect to the involution $\lambda \rightarrow -\lambda$ along with their multiplicities (Lemma 5). Therefore all non-zero eigenvalues of the operator $(BH)^2$ have even multiplicities. Thus Theorem 14 is proven for the pairs BHB, H . For an arbitrary skew-symmetric pair P, H Theorem 14 follows by the continuity arguments because the set of pairs BHB, H is open and dense in the set of all skew-symmetric pairs P, H .

Corollary 7. *If a system (1.1) possesses a non-local or local non-degenerate Hamiltonian structure then non-zero eigenvalues of the operators $H_u^k H_v^m$ have even multiplicities when k and m are odd and are symmetric with respect to the involution $\lambda \rightarrow -\lambda$ when $k + m$ is odd.*

The methods of this paper lead to new geometric and algebraic constructions which will be published elsewhere.

