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Entropic Repulsion of the Lattice Free Field, II. The 0-Boundary Case

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Abstract: This paper is a continuation of [5]. We consider the Euclidean massless free field on a box V_N of volume N^d with 0-boundary condition; that is the centered Gaussian field with covariances given by the Green function of the simple random walk on $\mathbb{Z}^d, d \geqq 3$, killed as it exits V_N. We show that the probability, that all the spins are positive in the box V_N decays exponentially at a surface rate N^{d-1}. This is in contrast with the rate $N^{d-2}\log N$ for the infinite field of [5].

1. Introduction

The object of this paper is to analyze the asymptotical behavior of a Gibbsian Gaussian field, under the condition that the variables are positive in a large finite box. These asymptotics play an important role in the construction of droplets on a "hard surface", cf. [1, 6, 10], and in related questions dealing with quasi-locality, cf. [7], and entropic repulsion [7, 11].

More precisely, let $\Lambda = [-1,1]^d$ be the unit box in $\mathbb{R}^d$ and set $V_N = N\Lambda \cap \mathbb{Z}^d$. Next consider the Gaussian field P_N^0 on $\Omega_N = \mathbb{R}^{V_N}$ with density with respect to the Lebesgue measure $\lambda_N(dX) = \prod_{i\in V_N} dX(i)$ of the form

$$P_N^0(dX) = \frac{1}{Z_N}\exp\left(-\frac{1}{2}\sum_{\{i,j\}\cap V_N \neq \emptyset} Q_{\mathrm{d}}(i,j)(X(i)-X(j))^2\right)\lambda_N(dX)\,, \tag{1.1}$$

where Z_N is a normalizing constant, $Q_{\mathrm{d}}(i,j) = \frac{1}{2d}1_{|i-j|=1}$ is the transition matrix of the simple random walk on $\mathbb{Z}^d$, and we set $X(j) = 0$ for $j \notin V_N$. Thus the spins are "tied down" at the boundary of V_N. P_N^0 can be viewed as the finite Gibbs distribution on Ω_N to the nearest neighbor quadratic interaction

$$\mathscr{J} = \{J_{\{i,j\}}(X) = Q_{\mathrm{d}}(i,j)(X(i)-X(j))^2,\ \{i,j\} \subseteq \mathbb{Z}^d\}$$

with 0-boundary conditions on $V_N^{\complement}$. We will be working in the transient dimensions $d \geqq 3$; then P_N^0 converges weakly to P^0, the infinite Gibbs distribution, sometimes called (discrete) *Euclidean massless free field.* P^0 is the centered Gaussian field on

$\Omega = \mathbb{R}^{\mathbb{Z}^d}$ with covariance matrix G, the Green function of the simple random walk in $\mathbb{Z}^d$, cf. [8].

Let

$$\Omega_N^+ = \{X \in \Omega_N : X(k) \geqq 0,\ k \in V_N\} .$$

In a previous paper with E. Bolthausen and O. Zeitouni, we have shown,

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P^0(\Omega_N^+) = -2\mathbf{G}\mathbf{C}' , \tag{1.2}$$

where $\mathbf{G} = \lim_{N\to\infty} E_N^0[X(0)^2] = E^0[X(0)^2]$ and $\mathbf{C}' = \mathrm{cap}_{\mathbb{R}^d}(\Lambda)$ is the Newtonian capacity of Λ in $\mathbb{R}^d$, cf. [5]. The presence of the $\log N$ factor in the exponent, is best explained by the fact that, under the "hard wall" condition Ω_N^+, the spins are repelled to the height $\sqrt{4\mathbf{G}\log N}$ as $N \to \infty$, cf. Prop. 1.3 of [5] or (1.6) below.

In this paper we replace the infinite Gibbs measure P^0 by the *finite Gibbs measure* P_N^0 in (1.2). In particular we describe the effect of the 0-boundary condition on the entropic repulsion. We differentiate between two regimes, depending whether one looks

$$\text{inside the box, i.e. far from the boundary: } \Omega_{\delta N}^+ \text{ for some } \delta \in (0,1) , \tag{1.3}$$

or

$$\text{up to the boundary: } \Omega_N^+ . \tag{1.4}$$

In the first regime (1.3), we have a convergence very similar to (1.2):

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P_N^0(\Omega_{\delta N}^+) = -2\mathbf{G}\mathbf{C}(\delta) , \tag{1.5}$$

where $\mathbf{C}(\delta) = \mathrm{cap}_\Lambda(\delta\Lambda)$ is the Newtonian capacity of $\delta\Lambda = [-\delta,\delta]^d$ in Λ, cf. Theorem 2.2 below, and the same entropic repulsion as in [5]:

$$\begin{aligned} &\lim_{N\to\infty} \sup_{k\in V_{\delta N}} P_N^0(X(k) \leqq \sqrt{a\log N} \mid \Omega_N^+) \\ &= \lim_{N\to\infty} \sup_{k\in V_{\delta N}} P_N^0(X(k) \geqq \sqrt{b\log N} \mid \Omega_N^+) = 0 , \end{aligned} \tag{1.6}$$

for each $a < 4\mathbf{G} < b$ and $\delta \in (0,1)$.

Note that $\mathbf{C}(\delta) = O((1-\delta)^{-1})$ as $\delta \uparrow 1$, so that we expect a faster decay for $\delta = 1$. This is due to the 0-boundary condition, which makes it less likely for the variable to be positive. In fact, in the second regime (1.4), we have a surface order which can be interpreted as a purely boundary effect: let $\partial_L V_N = V_N \setminus V_{N-L} = \{k \in V_N : \mathrm{dist}(k, V_N^{\complement}) < L\}$, and set

$$\partial_L\Omega_N^+ \equiv \{X \in \Omega_N : X(i) \geqq 0,\ i \in \partial_L V_N\} ,$$

then we show in our main result, Theorem 4.1,

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\Omega_N^+) = \lim_{L\to\infty}\lim_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\partial_L\Omega_N^+) = -\sum_{i=-d}^{d} \kappa^0(e_i) , \tag{1.7}$$

where $\kappa^0(e_i)$ is a certain "surface tension" in the direction of the i^{th} unit vector e_i in $\mathbb{R}^d$.

The major tool in the derivation of (1.7) is the following interpolation in the "intermediate regime": let $\{L_N, N \in \mathbb{N}\}$ be a monotone increasing sequence with $2 \leqq L_N$ and $\lim_{N\to\infty} \frac{L_N}{N} = 0$, then

$$\begin{aligned} -\infty &< \liminf_{N\to\infty} \frac{L_N}{N^{d-1}\log L_N} \log P_N^0(\Omega^+_{(N-L_N)}) \\ &\leqq \limsup_{N\to\infty} \frac{L_N}{N^{d-1}\log L_N} \log P_N^0(\Omega^+_{(N-L_N)}) < 0\,, \end{aligned} \tag{1.8}$$

cf. Prop. 2.5 and 2.9. In fact, we can show that, under the condition Ω_N^+, we have at distance L_N from the boundary of V_N an entropic repulsion of the order $O(\sqrt{\log L_N})$.

The rest of the paper is divided into 4 sections. Section 2 gives a proof of (1.5) and (1.8). Our main tool is the random walk representation of the covariance of P_N^0 and a conditioning argument. In Sect. 3 we prove the entropic repulsion (1.6), here the argument is based on the FKG property of the conditional field $P_N^0(\,\cdot\,|\Omega_N^+)$. Section 4 deals with the convergence (1.7) in the boundary regime. Finally, the Appendix contains some useful estimates for the random walk.

Before concluding, let us state two important remarks. First it should be noted that the above results can be easily generalized to arbitrary *finite range interactions* Q and fixed *boundary conditions* $a \in \Omega$, cf. [5, 1]. That is, in the definition (1.1) of P_N^0, we can replace Q_{d} by the positive finite range matrix Q of an irreducible symmetric random walk on Z^d and set $X(k) = a(k)$, $k \notin V_N$. In particular, using monotonicity one can show that, for any log-tempered $a \in \Omega$, (1.5) and (1.6) hold with the same constants[1] $\mathbf{C}(\delta)$ and $\mathbf{G}$, cf. Remark 2.4. Also (1.7) is true for any constant boundary condition $a(k) \equiv a \in \mathbb{R}$, $k \in \mathbb{Z}^d$, with $\kappa^0(e_i)$ replaced by the corresponding $\kappa^a(e_i)$.

Second, much of what we have discussed above holds with some modifications for the recurrent dimension $d = 2$. The main difference here is the logarithmic divergence of the variance $G_N(0,0) = O(\log N)$ as $N \to \infty$. This of course implies that the infinite measure P^0 does not exist. We will treat this case in a separate paper with E. Bolthausen. In particular, we show that the boundary behavior (1.7) is the same as for $d \geqq 3$, however, in the interior of the box, we have a $(\log N)^2$-decay. More precisely, we show in [4], for each $\delta \in (0,1)$,

$$\begin{aligned} -2\mathbf{GC}(\delta) &\leqq \liminf_{N\to\infty} \frac{1}{(\log N)^2} \log P_N^0(\Omega^+_{\delta N}) \\ &\leqq \limsup_{N\to\infty} \frac{1}{(\log N)^2} \log P_N^0(\Omega^+_{\delta N}) \leqq -\frac{1}{2}\mathbf{GC}(\delta), \end{aligned}$$

where $\mathbf{C}(\delta) = \mathrm{cap}_\Lambda(\delta\Lambda)$ as above, and $\mathbf{G} = \lim_{N\to\infty} \frac{G_N(0,0)}{\log N}$.

2. The Behavior Inside the Box

In this section we give a proof of (1.5) and (1.8). Our main tool is the random walk representation of the covariance matrix of P_N^0: Let $\{\xi_n : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be the simple random walk in $\mathbb{Z}^d$ generated by Q_{d}. We denote by $\mathbb{P}_i$ and $\mathbb{E}_i$ the probability

[1] Of course, in case $Q \neq Q_{\mathrm{d}}$, the constant $\mathbf{C}(\delta)$ has to be adapted to the corresponding capacity, cf. [3]

and expectation for the walk with start at $i \in \mathbb{Z}^d$. Let $\tau_N = \inf\{n \in \mathbb{N}_0 : \xi_n \notin V_N\}$ be the first exit time of V_N, then the covariance of P_N^0 is given by

$$\mathrm{cov}_{P_N^0}(X(i),X(j)) = G_N(i,j) = \mathbb{E}_i\left[\sum_{n=0}^{\tau_N} 1_j(\xi_n)\right], \quad i,j \in V_N ,$$

cf. Appendix of [3]. Let $G(i,j) = \mathbb{E}_i[\sum_{n=0}^{\infty} 1_j(\xi_n)]$, $i,j \in \mathbb{Z}^d$, be the Green function of the discrete Laplacian, then, for each $\delta \in (0,1)$,

$$\lim_{N\to\infty} G_N(i,j) = G(i,j) \quad \text{uniformly on } V_{\delta N} . \tag{2.1}$$

This implies the weak convergence $\lim_{N\to\infty} P_N^0 = P^0$, where P^0, the infinite Gibbs state, is the centered Gaussian field with covariance G.

Let us fix some notation: $c_1, c_2, c_3, \ldots \in \mathbb{R}^+$ are generic constants which do not depend on N or L_N, but are not necessarily the same at different occurrences. Also for $\Lambda \subset \mathbb{Z}^d$ we write

$$\Omega^+(\Lambda) = \{X \in \Omega : X(k) \geqq 0,\ k \in \Lambda\} .$$

Our first result is the proof of

Theorem 2.2. *Let* $\delta \in (0,1)$, *then*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P_N^0(\Omega_{\delta N}^+) = -2\mathbf{G}\mathbf{C}(\delta) ,$$

where

$$\mathbf{C}(\delta) = cap_\Lambda(\delta\Lambda) = \inf\left\{\frac{1}{2d}\|\nabla h\|_{L^2(\Lambda)}^2 : h \in H_0^1(\Lambda),\ h \geqq 1_{\delta\Lambda}\right\}$$

is the Newtonian capacity[2] *of* $\delta\Lambda$ *in* Λ.

Proof. The proof follows exactly the argument of [5], so that we don't go into details and rather concentrate on the identification of the new constant $\mathbf{C}(\delta) = \mathrm{cap}_\Lambda(\delta\Lambda)$, which is alternatively given by

$$\mathbf{C}(\delta) = \sup\{2\langle\phi, 1_{\delta\Lambda}\rangle_\Lambda - \langle\phi 1_{\delta\Lambda}, \mathfrak{G}_\Lambda(1_{\delta\Lambda}\phi)\rangle_\Lambda : \phi \in C(\Lambda)\} ,$$

where $\langle\cdot,\cdot\rangle_\Lambda$ is the scalar product in $L^2(\Lambda)$ and $\mathfrak{G}_\Lambda\phi(x) = \int_\Lambda g_\Lambda(x,y)\phi(y)\,dy$, is the Green operator associated with the brownian motion, killed as it exits Λ, cf. [1].

We start with the *lower bound*

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P_N^0(\Omega_{\delta N}^+) \geqq -2\mathbf{G}\mathbf{C}(\delta) .$$

A glance at the proof of [5], shows that the only two new ingredients are, the uniform convergence (2.1), and the convergence of the relative entropy:

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \mathbf{H}_{V_{\delta N}}(P_N^{a(N)} \mid P_N^0) = \frac{a^2\mathbf{C}(\delta)}{2} , \tag{2.3}$$

[2] $H_0^1(\Lambda)$ is the usual Sobolev space

here $\mathbf{H}_{V_{\delta N}}(P_N^{a(N)} \mid P_N^0)$ is the relative entropy of $P_N^{a(N)}$ with respect to P_N^0 restricted to the box $V_{\delta N}$ and $P_N^{a(N)}$ is the Gaussian field on Ω_N with covariance G_N and constant mean $E_N^{a(N)}[X(k)] = a(N) = \sqrt{a \log N}$, $k \in V_N$, for some $a \in \mathbb{R}^+$. In order to prove (2.3), first note the identity

$$\mathbf{H}_{V_{\delta N}}(P_N^{a(N)} \mid P_N^0) = \frac{a(N)^2}{2} \langle 1_{V_{\delta N}}, G_{N,\delta}^{-1} 1_{V_{\delta N}} \rangle_{V_N} ,$$

where $G_{N,\delta}$ is the covariance matrix G_N restricted to $V_{\delta N}$, $G_{N,\delta}^{-1}$ the inverse of $G_{N,\delta}$, and $\langle \cdot, \cdot \rangle_{V_N}$ is the scalar product in $\ell^2(V_N)$, cf. [3]. Next, $\langle 1_{V_{\delta N}}, G_{N,\delta}^{-1} 1_{V_{\delta N}} \rangle_{V_N} = \mathrm{cap}_{V_N}(V_{\delta N})$ is the capacity of $V_{\delta N}$ in V_N with respect to the simple random walk, and using the same argument as in the proof of Lemma 2.2 of [3], one shows the convergence

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \mathrm{cap}_{V_N}(V_{\delta N}) = \mathrm{cap}_\Lambda(\delta\Lambda) ,$$

which yields (2.3). As far as the *upper bound* is concerned:

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P_N^0(\Omega_{\delta N}^+) \leqq -2\mathbf{GC}(\delta) ,$$

one again uses the convergence (2.1) and the fact that, for each $f \in C_b(\Lambda)$, with $f_N(k) = f(k/N)$, $k \in V_N$,

$$\lim_{N\to\infty} N^{-d-2} \langle f_N, G_N f_N \rangle_{V_N} = \langle f, \mathfrak{G}_\Lambda f \rangle_\Lambda ,$$

cf. [1]. Now the result follows from the equality

$$\mathbf{C}(\delta) = \sup\left\{ \frac{\langle 1_{\delta\Lambda}, h\rangle_\Lambda^2}{\langle h 1_{\delta\Lambda}, \mathfrak{G}_\Lambda(h 1_{\delta\Lambda})\rangle_\Lambda} : h \text{ piecewise constant on a uniform grid} \right\} . \qquad \square$$

Remark 2.4. Note that we do not use explicitly the geometry of V_N or $V_{\delta N}$ in the above argument. Also, using monotonicity, we could consider any log-tempered boundary condition $a \in \Omega_{\log} = \{a \in \Omega : \lim_{|k|\to\infty} \frac{|a(k)|^2}{\log|k|} = 0\}$. Thus, let Γ, Λ be two bounded open domains of $\mathbb{R}^d$ with piecewise smooth boundaries. Set

$$\Gamma_N = N\Gamma \cap \mathbb{Z}^d, \quad \Lambda_N = N\Lambda \cap \mathbb{Z}^d \quad \text{and} \quad P_{\Lambda_N}^a = P^0(\,\cdot \mid X(k) = a(k),\ k \notin \Lambda_N) .$$

Then if $\Gamma \subset \Lambda$ with $\mathrm{dist}(\Gamma, \Lambda^{\complement}) > 0$,

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P_{\Lambda_N}^a(\Omega^+(\Gamma_N)) = -2\mathbf{G}\mathrm{cap}_\Lambda(\Gamma) ,$$

where $\mathrm{cap}_\Lambda(\Gamma) = \inf\left\{ \frac{1}{2d} \|\nabla h\|_{L^2(\Lambda)}^2 : h \in H_0^1(\Lambda),\ h \geqq 1_\Gamma \right\}$ is the capacity of Γ in Λ.

Our next step is the upper bound in the intermediate regime.

Proposition 2.5. *Let* $\frac{L_N}{N} \searrow 0$ *with* $L_N \geqq 2$, *then*

$$\limsup_{N\to\infty} \frac{L_N}{N^{d-1}\log L_N} \log P_N^0(\Omega_{(N-L_N)}^+) < 0 .$$

Proof. Let $W_N = W_N(L_N) = \{k \in V_N : L_N \leqq \text{dist}(k, V_N^{\complement}) \leqq 2L_N\}$, denote by W_N^{o} the odd points of W_N and by W_N^{e} the even points in the interior of W_N. Let $\mathscr{F}_N^{\text{o}} = \sigma\{X(k) : k \in W_N^{\text{o}}\}$ be the sigma algebra generated by the odd points. By the Markov property of P_N^0, conditioned upon $\mathscr{F}_N^{\text{o}}$, the $\{X(k), k \in W_N^{\text{e}}\}$ are independent Gaussian with variance 1 and mean $\bar{X}(k) = \sum_j Q_{\text{d}}(k,j)X(j)$, cf. [3]. Thus

$$\begin{aligned} P_N^0(\Omega^+_{(N-L_N)}) &\leqq P_N^0(\Omega^+(W_N)) \leqq P_N^0(\Omega^+(W_N^{\text{e}}) \cap \Omega^+(W_N^{\text{o}})) \\ &\leqq E_N^0[P_N^0(\Omega^+(W_N^{\text{e}}) \mid \mathscr{F}_N^{\text{o}}); \Omega^+(W_N^{\text{o}})] \\ &= E_N^0\left[\prod_{k \in W_N^{\text{e}}} \{1 - \phi(\bar{X}(k))\}; \Omega^+(W_N^{\text{o}})\right], \end{aligned}$$

where

$$\frac{1}{2} e^{-x^2/2} \geqq \phi(x) \equiv (2\pi)^{-1/2} \int_x^\infty e^{-t^2/2}\, dt \geqq \frac{1}{2x} e^{-x^2/2}, \quad x \geqq 2\,. \tag{2.6}$$

For given $m = m(L_N) \geqq 2$, let $I_N \equiv \{k \in : \bar{X}(k) \leqq m\}$ and set $A_N = \{X : |I_N| \geqq \frac{1}{2}|W_N^{\text{e}}|\}$. Then

$$\begin{aligned} E_N^0\left[\prod_{k \in W_N^{\text{e}}} \{1 - \phi(\bar{X}(k))\}; \Omega^+(W_N^{\text{o}})\right] &= E_N^0\left[\prod_{k \in W_N^{\text{e}}} \{1 - \phi(\bar{X}(k))\}; \Omega^+(W_N^{\text{o}}) \cap A_N\right] \\ &\quad + E_N^0\left[\prod_{k \in W_N^{\text{e}}} \{1 - \phi(\bar{X}(k))\}; \Omega^+(W_N^{\text{o}}) \cap A_N^{\complement}\right] \\ &\leqq (1 - \phi(m))^{\frac{1}{2}|W_N^{\text{e}}|} + P_N^0(\Omega^+(W_N^{\text{o}}) \cap A_N^{\complement})\,. \end{aligned}$$

For the first term, we have, in view of (2.6), the a priori estimate

$$(1 - \phi(m))^{\frac{1}{2}|W_N^{\text{e}}|} \leqq \exp\left(-c_1 N^{d-1} L_N \frac{e^{-m^2/2}}{m}\right). \tag{2.7}$$

On $\Omega^+(W_N^{\text{o}}) \cap A_N^{\complement}$, we have

$$\bar{S}_N^{\text{e}} \equiv \frac{1}{|W_N^{\text{e}}|} \sum_{k \in W_N^{\text{e}}} \bar{X}(k) \geqq \frac{m}{2}, \quad \text{with } P_N^0\left(\bar{S}_N^{\text{e}} \geqq \frac{m}{2}\right) \leqq \exp\left(-\frac{m^2}{8\text{var}(\bar{S}_N^{\text{e}})}\right),$$

since $\bar{S}_N^{\text{e}}$ is centered Gaussian. Set $S_N^{\text{e}} = \frac{1}{|W_N^{\text{e}}|}\sum_{k \in W_N^{\text{e}}} X(k)$, then, since $\bar{X}(k)$ is the conditional expectation of $X(k)$, $\text{var}(\bar{S}_N^{\text{e}}) \leqq \text{var}(S_N^{\text{e}})$ with

$$\text{var}(S_N^{\text{e}}) = \frac{1}{|W_N^{\text{e}}|^2} \sum_{i,j \in W_N^{\text{e}}} G_N(i,j) \leqq c_2 \frac{L_N}{N^{d-1}}\,,$$

cf. (A.3) below. This yields

$$P_N^0(\Omega^+(W_N^{\text{o}}) \cap A_N^{\complement}) \leqq \exp\left(-\frac{c_3 N^{d-1} m^2}{L_N}\right). \tag{2.8}$$

In view of (2.7) and (2.8) we may choose $m(L_N) = \sqrt{a\log(L_N)}$ for some $0 < a < 2$ and conclude the proof. □

We now turn to the proof of the lower bound in the intermediate regime:

Proposition 2.9. *Let* $\frac{L_N}{N} \searrow 0$ *with* $L_N \geqq 2$, *then there exists a constant* $K < \infty$ *such that*

$$\liminf_{N\to\infty} \frac{L_N}{N^{d-1}\log L_N} \log P_N^0(\Omega^+_{(N-L_N)}) \geqq -K.$$

Proof. It will be enough to prove the existence of $K' < \infty$ such that

$$\liminf_{N\to\infty} \frac{L_N}{N^{d-1}\log L_N} \log P_N^0(\Omega^+(W_N(L_N))) \geqq -K'. \tag{2.10}$$

Namely, once (2.10) is proved, we can cover $V_{(N-L_N)}$ with $\{W_N(2^\ell L_N), \ell = 0,\dots,\ell_{\max}\}$, $\ell_{\max} \leqq -\log(L_N/N)/\log 2$. Then by FKG property, for large N,

$$\begin{aligned}
P_N^0(X(k) \geqq 0, k \in V_{(N-L_N)}) &\geqq \prod_{\ell=0}^{\ell_{\max}} P_N^0(X(k) \geqq 0,\ k \in W_N(2^\ell \delta_N))\\
&\geqq \exp\left(-2K' \sum_{\ell=0}^{\ell_{\max}} N^{d-1}\frac{\log(2^\ell L_N)}{2^\ell L_N}\right)\\
&\geqq \exp\left(-KN^{d-1}\frac{\log L_N}{L_N}\right),
\end{aligned}$$

for some $K < \infty$. In order to prove (2.10), we use a conditioning argument, which is quite different from the proof of the lower bound of [5]: For given L_N and $\varepsilon > 0$ let

$$\Lambda_N(\varepsilon) \equiv \left\{k : \frac{L_N}{2} \leqq \mathrm{dist}(k, V_N^{\mathrm{c}}) \leqq \frac{5L_N}{2}\right\} \cap [\varepsilon L_N^{2/d}]\mathbb{Z}^d\ ,$$

$\bar{W}_N = W_N(L_N) \setminus \Lambda_N(\varepsilon)$, and set

$$q_k^N(j) = \mathbb{P}_k(\xi_{\tau_\varepsilon} = j; \tau_\varepsilon < \tau_N), \quad j \in \Lambda_N(\varepsilon),\ k \in \bar{W}_N\ ,$$

where $\tau_\varepsilon = \inf\{n \in \mathbb{N}_0 : \xi_n \in \Lambda_N(\varepsilon)\}$. Note that

$$|\Lambda_N(\varepsilon)| \leqq \frac{c_1 N^{d-1}}{\varepsilon^d L_N}\ . \tag{2.11}$$

In the Appendix (Lemma A.9), we show that one can choose $\varepsilon > 0$ independently of N, such that

$$\inf_{k\in\bar{W}_N} \mathbb{P}_k(\tau_\varepsilon < \tau_N) = \inf_{k\in\bar{W}_N} \sum_{j\in\Lambda_N(\varepsilon)} q_k^N(j) \geqq \frac{1}{2}\ . \tag{2.12}$$

Let $\mathscr{F}_{\Lambda_N(\varepsilon)} = \sigma\{X(j),\ j \in \Lambda_N(\varepsilon)\}$. Then, conditioned upon $\mathscr{F}_{\Lambda_N(\varepsilon)}$, $\{X(k),\ k \in \bar{W}_N\}$ is a Gaussian field with positive covariance and conditional mean

$$\bar{X}(k) = E_N^0[X(k) \mid \mathscr{F}_{\Lambda_N(\varepsilon)}] = \sum_{j\in\Lambda_N(\varepsilon)} q_k^N(j)X(j), \quad k \in \bar{W}_N\ .$$

Thus for given $m = m(L_N) \geqq 2$,

$$P^0(\Omega^+(W_N)) = E_N^0[P_N^0(\Omega^+(W_N) \mid \mathscr{F}_{\Lambda_N(\varepsilon)})]$$
$$\geqq E_N^0[P(\Omega^+(\bar{W}_N) \mid \mathscr{F}_{\Lambda_N(\varepsilon)}); \; X(j) \geqq m, \; j \in \Lambda_N(\varepsilon)] .$$

In view of (2.11) and (2.12), we have on $\{X(j) \geqq m, j \in \Lambda_N(\varepsilon)\}$, by the FKG property and the fact that $\mathrm{var}(X(k) \mid \mathscr{F}_{\Lambda_N(\varepsilon)}) \leqq G_N(k,k) \leqq \mathbf{G}$, $k \in \bar{W}_N$,

$$P(\Omega^+(\bar{W}_N) \mid \mathscr{F}_{\Lambda_N(\varepsilon)}) \geqq \prod_{k \in \bar{W}_N} (1 - \phi(\bar{X}(k)/\sqrt{\mathbf{G}}))$$
$$\geqq (1 - \exp(-m^2/8\mathbf{G}))^{|\bar{W}_N|} \geqq \exp(-c_2 N^{d-1} L_N e^{-m^2/8\mathbf{G}}).$$

Also, again by FKG, (2.11) and $G_N(j,j) \geqq 1, j \in \Lambda_N(\varepsilon)$,

$$P_N^0(X(j) \geqq m, j \in \Lambda_N(\varepsilon)) \geqq \prod_{j \in \Lambda_N(\varepsilon)} P_N^0(X(j) \geqq m) \geqq \exp\left(-\frac{c_3 N^{d-1} m^2}{\varepsilon^d L_N}\right).$$

Thus

$$P_N^0(\Omega^+(W_N)) \geqq \exp(-c_2 N^{d-1} L_N e^{-m^2/8\mathbf{G}}) \exp\left(-\frac{c_3 N^{d-1} m^2}{\varepsilon^d L_N}\right),$$

and choosing $m(L_N) = \sqrt{b \log L_N}$ for some $b > 16\mathbf{G}$ yields (2.10). □

Remark 2.13. For $m > 0$ and $a \in \Omega$, consider the measure $P_N^{a,(m)}$ on Ω_N given by

$$P_N^{a,(m)}(dX) = \frac{1}{Z_N^{a,(m)}} \exp\left(-\frac{m}{2} \sum_{k \in V_N} X(k)^2\right) P_N^a(dX),$$

where $Z_N^{a,(m)}$ is a normalizing constant. Then, for each tempered

$$a \in \Omega' = \left\{X \in \Omega : \lim_{|k| \to \infty} |k|^{-\varepsilon} |a(k)| < \infty, \text{ for some } \varepsilon > 0\right\},$$

$P_N^{a,(m)}$ converges weakly as $N \to \infty$ to the centered Gaussian measure $P^{(m)}$ with covariance $G^{(m)} = (1+m)^{-1} \sum_{n=0}^{\infty} (1+m)^{-n} Q_{\mathrm{d}}^n$, called *Euclidean free field with positive mass m*. In this case the fixed boundary condition plays no role, in particular, one shows that, for each $\delta \in (0,1)$,

$$\lim_{N \to \infty} \frac{1}{N^d} \log P_N^{a,(m)}(\Omega_{\delta N}^+) = \lim_{N \to \infty} \frac{1}{N^d} \log P^{(m)}(\Omega_{\delta N}^+) = -\delta^d K(m)$$

for some $K(m) > 0$, cf. Sect. 3 of [5].

3. The Entropic Repulsion

The aim of this section is to prove the entropic repulsion (1.6). The crucial step in the proof will be the following FKG property of $P_N^0(\cdot \mid \Omega_N^+)$:

Lemma 3.1. *Let $\emptyset \neq V \subset W \subset V_N$, then, for all $k \in V$ and $a > 0$,*

$$P_N^0(X(k) \geqq a \mid \Omega^+(V)) \leqq P_N^0(X(k) \geqq a \mid \Omega^+(W)) . \tag{3.2}$$

Moreover

$$P_V^0(X(k) \geqq a \,|\, \Omega^+(V)) \leqq P_W^0(X(k) \geqq a \,|\, \Omega^+(\overline{V}))\,, \tag{3.3}$$

where $\overline{V} = \{k \in \mathbb{Z}^d : dist(k, V) \leqq 1\}$ *and* $P_W^0 = P_N^0(\,\cdot\,| X(k) = 0,\ k \in V_N \setminus W)$.

Proof. Note that $\Omega^+(W) = \Omega^+(V) \cap \Omega^+(W \setminus V)$. Thus, since (3.2) is equivalent with

$$P_N^0(\{X(k) \geqq a\} \cap \Omega^+(W \setminus V) \,|\, \Omega^+(V))$$
$$\geqq P_N^0(X(k) \geqq a \,|\, \Omega^+(V)) P_N^0(\Omega^+(W \setminus V) \,|\, \Omega^+(V))\,,$$

it suffices to show that $P_N^0(\,\cdot\,|\,\Omega^+(V))$ is positively correlated. We use a simple approximation argument: for $\beta > 0$, define

$$P_{V,N}^{0,\beta}(dX) = \frac{\exp(-\beta \sum_{k \in V} |X(k) \wedge 0|^2)}{Z_N(\beta)} P_N^0(dX)\,,$$

where $Z_N(\beta)$ is a normalizing constant. Then for each $\beta > 0$, by Theorem 1.3 of [9], cf. also Sect. 10.6 of this paper, we know that $P_{V,N}^{0,\beta}$ is positively correlated. Moreover, with respect to the weak convergence, we have

$$\lim_{\beta \to \infty} P_{V,N}^{0,\beta} = P_N^0(\,\cdot\,|\,\Omega^+(V))\,.$$

This implies (3.2). As for (3.3), let $\partial\overline{V} = \overline{V} \setminus V$ and note that, by continuity and the Markov property,

$$P_V^0(X(k) \geqq a \,|\, \Omega^+(V)) = P_W^0(X(k) \geqq a \,|\, \Omega^+(\overline{V}),\ X(j) \leqq 0,\ j \in \partial\overline{V})$$
$$= \lim_{\varepsilon \searrow 0} P_W^0(X(k) \geqq a \,|\, \Omega^+(\overline{V}),\ X(j) \leqq \varepsilon,\ j \in \partial\overline{V})\,.$$

Thus in order to prove (3.3), it suffices to show, for all $\varepsilon > 0$,

$$P_W^0(X(k) \geqq a \,|\, \Omega^+(\overline{V}), X(j) \leqq \varepsilon,\ j \in \partial\overline{V}) \leqq P_W^0(X(k) \geqq a \,|\, \Omega^+(\overline{V}))\,.$$

This is equivalent with

$$P_W^0(X(k) \geqq a, X(j) \geqq \varepsilon, j \in \partial\overline{V} \,|\, \Omega^+(\overline{V}))$$
$$\geqq P_W^0(X(k) \geqq a \,|\, \Omega^+(\overline{V})) P_W^0(X(j) \geqq \varepsilon,\ j \in \partial\overline{V} \,|\, \Omega^+(\overline{V}))\,,$$

and follows from the positive correlations of the measure $P_W^0(\,\cdot\,|\,\Omega^+(\overline{V}))$. □

Remark 3.4. Note that the FKG property of the conditional field $P_N^0(\,\cdot\,|\,\Omega_N^+)$, which was implicitly used in Sect. 4 of [5], does not follow immediately from the positive correlations of the original field P_N^0, since $\Omega^+(V)$ is not a cylinder set.

We now turn to the entropic repulsion, first inside the box:

Proposition 3.5. *Let* $\delta \in (0, 1)$ *and* $a < 4\mathbf{G} < b$, *then*

$$\lim_{N \to \infty} \sup_{k \in V_{\delta N}} P_N^0(X(k) \leqq \sqrt{a \log N} \,|\, \Omega_N^+)$$
$$= \lim_{N \to \infty} \sup_{k \in V_{\delta N}} P_N^0(X(k) \geqq \sqrt{b \log N} \,|\, \Omega_N^+) = 0\,.$$

Proof. Let $b > 4\mathbf{G}$, then, in view of (3.3),

$$\sup_{k\in V_{\delta N}} P_N^0(X(k) \geqq \sqrt{b\log N}\,|\,\Omega_N^+) \leqq \sup_{k\in V_{\delta N}} P^0(X(k) \geqq \sqrt{b\log N}\,|\,\Omega_{N+1}^+)\,,$$

where the RHS converges to 0 as $N \to \infty$ by Prop. 1.3 of [5]. Next, let $a < 4\mathbf{G}$ and $k \in V_{\delta N}$, then, by (3.2), for each $\delta < \delta' < 1$,

$$P_N^0(X(k) \leqq \sqrt{a\log N}\,|\,\Omega_N^+) \leqq P_N^0(X(k) \leqq \sqrt{a\log N}\,|\,\Omega_{\delta' N}^+)\,.$$

Now using Theorem 2.2 and precisely the same argument as in Sect. 4 of [5] (noticing in particular, that the height of the entropic repulsion in [5] depends on $\mathbf{G}$ only and not on the capacity $\mathbf{C}'$), one shows that

$$\lim_{N\to\infty} \sup_{k\in V_{\delta N}} P_N^0(X(k) \leqq \sqrt{a\log N}\,|\,\Omega_{\delta' N}^+) = 0. \qquad \square$$

Next for $\delta \in (0,1)$, let $W_{N,\delta}(L_N) = \bigcup_{i=1}^d W_{N,\delta}^i(L_N)$ be the "interior" of $W_N(L_N)$, where

$$W_{N,\delta}^i(L_N) = \{k \in W_N(L_N) : |k_j| \leqq \delta N,\ j \neq i\}\,. \tag{3.6}$$

Proposition 3.7. *Let* $\frac{L_N}{N} \searrow 0$ *and* $\lim_{N\to\infty} L_N = \infty$, *then there exist two constants* $0 < b \leqq B < \infty$ *such that, for all* $\delta \in (0,1)$,

$$\lim_{N\to\infty} \sup_{k\in W_{N,\delta}(L_N)} P_N^0(X(k) \leqq \sqrt{b\log L_N}\,|\,\Omega_N^+) = 0 \tag{3.8}$$

and

$$\limsup_{N\to\infty} \sup_{k\in W_N(L_N)} \frac{E_N^0[X(k)\,|\,\Omega_N^+]}{\sqrt{B\log L_N}} \leqq 1\,. \tag{3.9}$$

Our first step in the proof of (3.8) is the following

Lemma 3.10. *Let* $\mathbf{L}_{W_N} \equiv \frac{1}{|W_N|}\sum_{k\in W_N} \delta_{X(k)}$ *be the empirical measure of* $W_N = W_N(L_N)$, *then there exist* $b' > 0$, *such that, for all* $\varepsilon \in (0,1)$,

$$\lim_{N\to\infty} P_N^0(\mathbf{L}_{W_N}[0, \sqrt{b'\log L_N}] \geqq \varepsilon\,|\,\Omega_N^+) = 0\,.$$

Proof. First note that by (3.2),

$$P_N^0(\mathbf{L}_{W_N}[0,\sqrt{b'\log L_N}] \geqq \varepsilon\,|\,\Omega_N^+) \leqq P_N^0(\mathbf{L}_{W_N}[0,\sqrt{b'\log L_N}] \geqq \varepsilon\,|\,\Omega^+(W_N))\,. \tag{3.11}$$

We follow the argument of the proof of Prop. 4.1 of [5]: let W_N^{e} be the even elements of W_N, $\bar{X}(k) = \sum_j Q_{\mathrm{d}}(k,j)X(j)$ and

$$\mathbf{L}_{W_N^{\mathrm{e}}} = \frac{1}{|W_N^{\mathrm{e}}|}\sum_{k\in W_N^{\mathrm{e}}} \delta_{X(k)}, \qquad \bar{\mathbf{L}}_{W_N^{\mathrm{e}}} = \frac{1}{|W_N^{\mathrm{e}}|}\sum_{k\in W_N^{\mathrm{e}}} \delta_{\bar{X}(k)}$$

be the corresponding empirical measures. Then, in view of the proof of Theorem 2.2 above, we can choose $b' > 0$ such that

$$P_N^0(\bar{\mathbf{L}}_{W_N^e}[0, \sqrt{b' \log L_N}] \geqq \varepsilon;\ \Omega^+(W_N^o)) \leqq \exp(-c_1 \varepsilon N^{d-1} L_N^{c_2}),$$

for some $c_2 > 0$. Thus, by Prop. 2.9,

$$\lim_{N\to\infty} P_N^0(\bar{\mathbf{L}}_{W_N^e}[0, \sqrt{b' \log L_N}] \geqq \varepsilon \,|\, \Omega^+(W_N)) = 0 .$$

Next, we use the fact that, for each $\varepsilon' > 0$,

$$\limsup_{N\to\infty} \frac{1}{|W_N^e|} \log P_N^0 \left(\frac{1}{|W_N^e|} \sum_{k\in W_N^e} |X(k) - \bar{X}(k)| \geqq \varepsilon' \sqrt{\log L_N} \right) < 0 ,$$

in order to conclude

$$\lim_{N\to\infty} P_N^0(\mathbf{L}_{W_N^e}[0, \sqrt{b' \log L_N}] \geqq \varepsilon \,|\, \Omega^+(W_N)) = 0 , \tag{3.12}$$

cf. Proof of (4.2) in [5]. Now the result follows from (3.11), (3.12) and

$$\{\mathbf{L}_{W_N}[0, \sqrt{b' \log L_N}] \geqq \varepsilon\} \subseteq \{\mathbf{L}_{W_N^e}[0, \sqrt{b' \log L_N}] \geqq \varepsilon/2\}$$
$$\cup \{\mathbf{L}_{W_N^o}[0, \sqrt{b' \log L_N}] \geqq \varepsilon/2\} . \qquad \square$$

Proof of 3.8. It is enough to show, for each $i = 1, \ldots, d$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \sup_{k \in W_{N,\delta}^i} P_N^0(X(k) \geqq \sqrt{b \log L_N} \,|\, \Omega_N^+) = 0 .$$

Define $\Lambda_N^i = \{\ell \in \mathbb{Z}^d : |\ell_j| < \delta N/4, j \neq i, |\ell_i| < L_N/4\}$, $\tilde{V}_N = \bigcap_{\ell \in \Lambda_N^i} (V_N + \ell)$. Then, for each $k \in W_{N,\delta}^i(L_N)$, $\Lambda_N^i(k) = \Lambda_N^i + k \subset \tilde{V}_N$ with $\mathrm{dist}(\Lambda_N^i(k), \tilde{V}_N^{\mathrm{C}}) \geqq L_N/4$ for large N. A simple modification of the above lemma shows that there is $b > 0$, such that for each $\varepsilon > 0$,

$$\lim_{N\to\infty} \tilde{P}_N^0(\mathbf{L}_{\Lambda_N^i(k)}[0, \sqrt{b \log L_N}] \geqq \varepsilon \,|\, \tilde{\Omega}_N^+) = 0 , \tag{3.13}$$

where $\tilde{P}_N^0 = P_{\tilde{V}_N}^0$ and $\tilde{\Omega}_N^+ = \Omega^+(\tilde{V}_N)$. On the other hand, by (3.3) for each $\ell \in \Lambda_N^i$,

$$\begin{aligned} P_N^0(X(k) \leqq \sqrt{b \log L_N} \,|\, \Omega_N^+) &= P_{(V_N+\ell)}^0(X(k+\ell) \leqq \sqrt{b \log L_N} \,|\, \Omega^+(V_N+\ell)) \\ &\leqq \tilde{P}_N^0(X(k+\ell) \leqq \sqrt{b \log L_N} \,|\, \tilde{\Omega}_N^+) \\ &= \tilde{E}_N^0[1_{X(k+\ell) \leqq \sqrt{b \log L_N}} \,|\, \tilde{\Omega}_N^+] . \end{aligned}$$

Thus, taking the average over Λ_N^i yields

$$\begin{aligned} P_N^0(X(k) \leqq \sqrt{b \log L_N} \,|\, \Omega_N^+) &\leqq \tilde{E}_N^0[\mathbf{L}_{\Lambda_N^i(k)}[0, \sqrt{b \log L_N}] \,|\, \tilde{\Omega}_N^+] \\ &\leqq \varepsilon \tilde{P}_N^0(\mathbf{L}_{\Lambda_N^i(k)}[0, \sqrt{b \log L_N}] \leqq \varepsilon \,|\, \tilde{\Omega}_N^+) \\ &\quad + \tilde{P}_N^0(\mathbf{L}_{\Lambda_N^i(k)}[0, \sqrt{b \log L_N}] > \varepsilon \,|\, \tilde{\Omega}_N^+), \end{aligned}$$

for all $\varepsilon \in (0,1)$, and (3.8) follows from (3.13). $\square$

Proof of 3.9. For $k \in W_N(L_N)$, let $i \in \{1,\dots,d\}$ be such that $N - 2L_N \leqq |k_i| \leqq N - L_N$. Set $\bar{\Lambda}^i_N = \{\ell \in \mathbb{Z}^d : \ell_i = 0, |\ell_j| \leqq N/4, j \neq i\}$, $U_N = \bigcup_{\ell \in \Lambda_N}(V_N + \ell)$ and $\tilde{U}_N = \bigcup_{j:|j| \leqq L_N}(U_N + j)$. By (3.3) we have, for each $\ell \in \bar{\Lambda}^i_N$,

$$E^0_N[X(k) \,|\, \Omega^+_N] = E^0_{(V_N+\ell)}[X(k+\ell) \,|\, \Omega^+(V_N+\ell)] \leqq E^0_{U_N}[X(k+\ell) \,|\, \Omega^+(U_N)]$$
$$\leqq E^0_{\tilde{U}_N}[X(k+\ell) \,|\, \Omega^+(\tilde{U}_N)] = \tilde{E}^0_N[X(k+\ell) \,|\, \tilde{\Omega}^+_{(N-L_N)}]\,,$$

where $\tilde{P}^0_N \equiv P^0_{\tilde{U}_N}$ and $\tilde{\Omega}^+_{(N-L_N)} = \Omega^+(\tilde{U}_N)$. Thus

$$E^0_N[X(k) \,|\, \Omega^+_N] \leqq \tilde{E}^0_N[S_{\bar{\Lambda}^i_N(k)} \,|\, \tilde{\Omega}^+_{(N-L_N)}] \tag{3.14}$$

with $S_{\bar{\Lambda}^i_N(k)} = \frac{1}{|\bar{\Lambda}^i_N|}\sum_{\ell \in \bar{\Lambda}^i_N} X(k+\ell)$. For each $\alpha > 0$, we have the relative entropy bound

$$\alpha \tilde{E}^0_N[S_{\bar{\Lambda}^i_N(k)} \,|\, \tilde{\Omega}^+_{(N-L_N)}] \leqq -\log \tilde{P}^0_N(\tilde{\Omega}^+_{(N-L_N)}) + \frac{\alpha^2}{2}\tilde{E}^0_N[S^2_{\bar{\Lambda}^i_N(k)}]\,,$$

cf. Lemma 4.7 of [5]. That is, taking the optimal $\alpha > 0$,

$$(\tilde{E}^0_N[S_{\bar{\Lambda}^i_N(k)} \,|\, \tilde{\Omega}^+_N])^2 \leqq -2\log \tilde{P}^0_N(\tilde{\Omega}^+_{(N-L_N)})\tilde{E}^0_N[S^2_{\bar{\Lambda}^i_N(k)}]\,.$$

In the Appendix, we show that

$$\tilde{E}^0_N[S^2_{\bar{\Lambda}^i_N(k)}] = \frac{1}{|\bar{\Lambda}^i_N|^2}\sum_{\ell,j \in \bar{\Lambda}^i_N} \tilde{G}_N(\ell+k, j+k) \leqq c_1 \frac{L_N}{N^{d-1}}\,, \tag{3.15}$$

where $\tilde{G}_N$ is the covariance associated with $\tilde{P}^0_N$, cf. (A.3). Also, in view of Prop. 2.9 above, we can find $\tilde{K} < \infty$, such that

$$\liminf_{N\to\infty} \frac{L_N}{N^{d-1}\log L_N} \log \tilde{P}^0_N(\tilde{\Omega}^+_{(N-L_N)}) \geqq -\tilde{K}\,. \tag{3.16}$$

Now, (3.9) follows from (3.14), (3.15) and (3.16). □

4. Behavior at the Boundary

In this section we study the behavior of the conditional field $P^0_N(\,\cdot \,|\, \Omega^+_N)$ close to the boundary of V_N. In order to formulate the main result, it will be useful to move the boundary of V_N to the origin. Thus, let e_i denote the i^{th} unit vector in $\mathbb{R}^d$ and write $e_{-i} = -e_i$, $i = 1,\dots,d$. Next, let $\mathbb{Z}^d_i = \{k \in \mathbb{Z}^d : k \cdot e_i < 0\}$, $P^{i,0} = P^0(\,\cdot \,|\, X(k) = 0, k \notin \mathbb{Z}^d_i)$ and

$$\partial^i_L V_N = \{k \in \mathbb{Z}^d : -L-1 \leqq k \cdot e_i < 0, |k_j| \leqq N, j \neq i\}\,.$$

Theorem 4.1. *For each $i = -d,\dots,d$ and $L \in \mathbb{N}$ the following limits*[3]

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log P^{i,0}(\Omega^+(\partial^i_L V_N)) = -\kappa^0_L(e_i) \quad \textit{with } 0 < \kappa^0(e_i) \equiv \lim_{L\to\infty} \kappa^0_L(e_i) < \infty\,,$$

[3]Actually, $\kappa^o(e_i)$ does not depend on i, since Q_d is isotropic

exist. Moreover

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\log P_N^0(\Omega_N^+)=\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{N^{d-1}}\log P_N^0(\partial_L\Omega_N^+)=-\sum_{i=-d}^{d}\kappa^0(e_i)\,. \quad (4.2)$$

The proof of (4.2) requires some additional notation. Let $V_N^i=\{k\in\mathbb{Z}^d: -2N-1\leqq k\cdot e_i<0, |k_j|\leqq N, i\neq j\}$. Next, consider the centered Gaussian field $P_N^{i,0}=P^0(\,\cdot\,|X(k)=0, k\neq V_N^i)$ with covariances $G_N^i(k,j)=\mathbb{E}_k[\sum_{n=0}^{\tau_N^i}1_j(\xi_n)]$, where $\tau_N^i=\inf\{n\geqq 0:\xi_n\notin V_N^i\}$. Then $P_N^{i,0}$ converges weakly to $P^{i,0}$, the centered Gaussian field with covariance

$$G^i(k,j)=\mathbb{E}_k\left[\sum_{n=0}^{\tau^i}1_j(\xi_n)\right], \quad \text{where } \tau^i=\inf\{n\geqq 0:\xi_n\notin\mathbb{Z}_i^d\}\,.$$

The main step in the proof of (4.2), is to show that we can replace P_N^0 by $P^{i,0}$:

Lemma 4.3. *For each $L\in\mathbb{N}$, we have*

$$\liminf_{N\to\infty}\frac{1}{N^{d-1}}\log P_N^0(\partial_L\Omega_N^+)\geqq\sum_{i=-d}^{d}\liminf_{N\to\infty}\frac{1}{N^{d-1}}\log P^{0,i}(\Omega^+(\partial_L^i V_N))\,, \quad (4.4)$$

and

$$\limsup_{N\to\infty}\frac{1}{N^{d-1}}\log P_N^0(\partial_L\Omega_N^+)\leqq\sum_{i=-d}^{d}\limsup_{N\to\infty}\frac{1}{N^{d-1}}\log P^{0,i}(\Omega^+(\partial_L^i V_N))\,. \quad (4.5)$$

Proof. Set $\hat{\partial}_L^i V_N=\{k\in\partial_L V_N: N-L\leqq e_i\cdot k\leqq N\}$, then $\Omega^+(\partial_L V_N)=\bigcap_{i=-d}^d\Omega^+(\hat{\partial}_L^i V_N)$, and, by FKG and shift invariance, we have

$$P_N^0(\Omega^+(\partial_L V_N))\geqq\prod_{i=-d}^{d}P_N^0(\Omega^+(\hat{\partial}_L^i V_N))=\prod_{i=-d}^{d}P_N^{i,0}(\Omega^+(\partial_L^i V_N))\,,$$

and (4.4) will follow from

$$\liminf_{N\to\infty}\frac{1}{N^{d-1}}\log P_N^{i,0}(\Omega^+(\partial_L^i V_N))\geqq\liminf_{N\to\infty}\frac{1}{N^{d-1}}\log P^{i,0}(\Omega^+(\partial_L^i V_N))\,, \quad (4.6)$$

for each $i=-d,\ldots,d$. For fixed $\delta\in(0,1)$, we have by FKG,

$$\begin{aligned}P_N^{i,0}(\Omega^+(\partial_L^i\ V_N))&\geqq P_N^{i,0}(\Omega^+(\partial_L^i V_{\delta N}))P_N^{i,0}(X(k)\geqq 0, k\in\partial_L V_N^i\setminus\partial_L V_{\delta N}^i)\\&\geqq P_N^{i,0}(\Omega^+(\partial_L^i V_{\delta N}))\exp(-c_1(1-\delta)^{d-1}N^{d-1}L)\,. \quad (4.7)\end{aligned}$$

Thus, it suffices to show (4.6) with $\partial_L\Omega_N^{i,+}$ replaced by $\partial_L\Omega_{\delta N}^{i,+}$, for each fixed $\delta\in(0,1)$. Let $F_{N,L}^i(X)=\frac{P^{i,0}(dX)}{P_N^{i,0}(dX)}\Big|_{\partial_L V_{\delta N}^i}$, then, for each $p,q>1$ with $1/p+1/q=1$, we have by Hölder's inequality

$$P^{0,i}(\Omega^+(\partial_L^i V_{\delta N}))\leqq P_N^{0,i}(\Omega^+(\partial_L^i V_{\delta N}))^{1/p}\|F_{N,L}^i\|_{L^q(P_N^{0,i})}\,.$$

In order to verify (4.6), it is enough to prove that, for each $q > 1$,

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \|F^i_{N,L}\|_{L^q(P_N^{0,i})} = 0 . \tag{4.8}$$

Let $G^i_{N,\delta}$ and G^i_δ denote the covariance matrices of $P_N^{i,0}$ and $P^{i,0}$ restricted to the box $\partial_L V^i_{\delta N}$, then

$$\|F^i_{N,L}\|_{L^q(P_N^{0,i})} = \det(I+R_N)^{1/2}\det(I+qR_N)^{-1/2q} ,$$

where I is the identity on the box $\partial_L V^i_{\delta N}$ and $R_N = G^i_{N,\delta}(G^i_\delta)^{-1} - I$. Set $\|R_N\| = \sup_k \sum_\ell |R_N(k,\ell)|$. Then we will show in the Appendix (Lemma A.6) that, for each fixed $L \in \mathbb{R}^+$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \|G^i_{N,\delta} - G^i_\delta\| = 0 \quad \text{and} \quad \|(G^i_{N,\delta})^{-1}\| \leqq 2, \quad \|(G^i_\delta)^{-1}\| \leqq 2 . \tag{4.9}$$

Therefore $\lim_{N\to\infty} \|R_N\| = 0$. Thus, if $\{\lambda_j(N),\ j = 1,\dots,k_{\max}\} \subset \mathbb{R}$, with $k_{\max} = |\partial_L V^i_{\delta N}|$ denote the eigenvalues of the matrix R_N, then as $\max_i |\lambda_i(N)| \leqq \|R_N\|$, we see from the above, for N large enough with $\|R_N\| < 1/q$, that

$$\begin{aligned}\frac{1}{N^{d-1}} \log \|F^i_{N,L}\|_{L^q(P_N^{0,i})} &= \frac{1}{2N^{d-1}} \sum_{j=1}^{k_{\max}} \log(1+\lambda_j(N)) - \frac{1}{2qN^{d-1}} \sum_{j=1}^{k_{\max}} \log(1+q\lambda_j(N)) \\ &\leqq \frac{k_{\max}}{2N^{d-1}} \log(1+\|R_N\|) - \frac{k_{\max}}{2qN^{d-1}} \log(1-q\|R_N\|) ,\end{aligned}$$

which yields (4.8).

We now turn to the upper bound (4.5), which will follow from

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\partial_L \Omega^+_{\delta N}) \leqq \sum_{i=-d}^{d} \limsup_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^{0,i}(\Omega^+(\partial^i_L V_{\delta N})) , \tag{4.10}$$

for each fixed $\delta \in (0,1)$: Once (4.10) is proved, we can proceed as above using (4.7) and (4.8), interchanging the roles of $P_N^{i,0}$ and $P^{i,0}$. Note that for $i \neq j$ and large $N \geqq 2L$,

$$\mathrm{dist}(\hat\partial^i_L V_{\delta N}, \hat\partial^j_L V_{\delta N}) \geqq c_2 \delta N . \tag{4.11}$$

Also using the estimate (A.2) in the Appendix one shows the existence of a constant $c_3 < \infty$ such that for all $R \geqq 2L$,

$$\sup_{k\in\partial_L V_N} \sum_{j\in\partial_L V_N, |j-k|\geqq R} G_N(k,j) \leqq c_3 \frac{L^3}{R} . \tag{4.12}$$

In view of (4.9), (4.11) and (4.12), we can then apply the hypercontractive estimate derived in the proof of Prop. A.18 of [5] and get

$$P_N^0(\partial_L \Omega_N^+) \leqq P_N^0\left(\bigcap_{i=-d}^{d} \Omega^+(\hat\partial^i_L V_{\delta N})\right) \leqq \prod_{i=1}^{d} P_N^{i,0}(\Omega^+(\hat\partial^i_L V_{\delta N}))^{\frac{2}{p(c_2\delta N)}} ,$$

where $\lim_{N\to\infty} p(c_2\delta N) = 1$. This implies (4.10) and concludes the proof. □

Lemma 4.13. *For each $L \in \mathbb{R}^+$ and $i = -d,\dots,d$, we have*

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log P^{0,i}(\Omega^+(\partial_L^i V_N)) = \limsup_{N\to\infty} \frac{1}{N^{d-1}} \log P^{0,i}(\Omega^+(\partial_L^i V_N)) \equiv -\kappa_L^0(e_i)\,.$$

Proof. We use a simple subadditive argument: Take, for example $i = d$. For $\mathbf{M} = (M_1,\dots,M_{d-1}) \in \mathbb{N}^{d-1}$ let

$$\partial_L^d V_{\mathbf{M}} = \{k \in \mathbb{Z}^d : -L-1 \leqq k_d < 0,\ 0 \leqq k_j \leqq M_j,\ j = 1,\dots,d-1\}\,.$$

Then by the FKG property and stationarity of $P^{d,0}$, we have

$$P^{d,0}(\Omega^+(\partial_L^d V_{\mathbf{M}+\mathbf{M}'})) \geqq P^{d,0}(\Omega^+(\partial_L^d V_{\mathbf{M}}))P^{d,0}(\Omega^+(\partial_L^d V_{\mathbf{M}'}))\,,$$

where $\mathbf{M}, \mathbf{M}'$ and $\mathbf{M}+\mathbf{M}'$ satisfy $(M+M')_j = M_j + M_j'$ for some $1 \leqq j \leqq d-1$ and $M_i = M_i' = (M+M')_i$ for $i \neq j$. That is, $\mathbf{M} \to -\log P^{d,0}(\Omega^+(\partial_L^d V_{\mathbf{M}}))$ is subadditive in each coordinate M_j, $j = 1,\dots,d-1$, and therefore the limit

$$\lim_{N\to\infty} -\frac{1}{N^{d-1}} \log P^{d,0}(\Omega^+(\partial_L^d V_N)) = \inf_{N\in\mathbb{N}} -\frac{1}{N^{d-1}} \log P^{d,0}(\Omega^+(\partial_L^d V_N)) = -\kappa_L^0(e_d)$$

exists. □

Proof of (4.2). By Lemmas 4.3 and 4.13, we know that for each fixed $L \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\partial_L \Omega_N^+) = \sum_{i=-d}^{d} \lim_{N\to\infty} \frac{1}{N^{d-1}} \log P^{0,i}(\Omega^+(\partial_L^i V_N)) = -\sum_{i=-d}^{d} \kappa_L^0(e_i)\,.$$

Also $\kappa_L^0(e_i)$ is increasing in L with $0 < \kappa_1^0(e_i) \leqq \kappa_L^0(e_i) \leqq K < \infty$, cf. Prop. 2.9. This implies $\lim_{L\to\infty} \kappa_L^0(e_i) = \kappa^0(e_i) \in (0,\infty)$. Trivially we have

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\Omega_N^+) \leqq \lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\partial_L \Omega_N^+) = -\sum_{i=-d}^{d} \kappa^0(e_i)\,.$$

On the other hand by FKG,

$$P_N^0(\Omega_N^+) \geqq P_N^0(\Omega_{(N-L)}^+) P_N^0(\partial_L \Omega_N^+)\,,$$

where, by Prop. 2.9,

$$\lim_{L\to\infty} \liminf_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\Omega_{(N-L)}^+) = 0\,.$$

This shows

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\Omega_N^+) \geqq -\lim_{L\to\infty} \sum_{i=-d}^{d} \kappa_L^0(e_i) = -\sum_{i=-d}^{d} \kappa^0(e_i)\,,$$

and concludes the proof. □

Remark 4.14. Note that we can view $\hat{P}^{i,0}$ as a Gaussian field on $\Omega_i \equiv (\mathbb{R}^{\mathbb{Z}^-})^{\mathbb{Z}^{d-1}}$, invariant under the shift on $\mathbb{Z}^{d-1}$. Set $P_{N,L}^{i,+} \equiv P_N^{i,0}(\,\cdot \mid \Omega^+(\partial_L^i V_N))$, Then, in view of

(3.2) and (3.9) we have

$$\lim_{L\to\infty}\limsup_{N\to\infty} E_N^{i,0}[X(k)\,|\,\partial_L\Omega_N^{i,+}] \leqq \sqrt{c\log|k_i|}, \quad k\in\mathbb{Z}_i^d\,,$$

for some constant $c<\infty$ and thus $\{P_{N,L}^{i,+} : N \geqq L\in\mathbb{N}\}$ is tight. Using the monotonicity (3.3), or a Gibbsian characterization, we can then show the weak convergence on Ω_i

$$\lim_{L\to\infty}\lim_{N\to\infty} P_{N,L}^{i,+} = P^{i,+}\,, \tag{4.15}$$

for some $P^{i,+}$, stationary with respect to the shift on $\mathbb{Z}^{d-1}$.

Remark 4.16. Let $\omega\in S^{d-1}$ be a unit vector and set $\mathbb{Z}_\omega^d=\{k\in\mathbb{Z}^d : \omega\cdot k<0\}$,

$$\partial_L^\omega V_N=\{k\in\mathbb{Z}^d : -L-1\leqq k\cdot\omega<0,\ |k\cdot\tilde e_j|\ \leqq N,\ j=1,\dots,d-1\},$$

where $(\omega,\tilde e_1,\dots,\tilde e_{d-1})$ is an orthonormal basis of $\mathbb{R}^d$. Next, for fixed $\alpha\in\mathbb{R}$, let $P^{\omega,\alpha}=P^\alpha(\,\cdot\,|X(j)=\alpha, j\notin\mathbb{Z}_\omega^d)$, then using the same arguments as above, one shows that the limits

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\log P^{\omega,\alpha}(\Omega^+(\partial_L^\omega V_N))=-\kappa_L^\alpha(\omega), \qquad \kappa^\alpha(\omega)\equiv\lim_{L\to\infty}\kappa_L^\alpha(\omega)$$

exist and do not depend on the choice of the basis $\{\tilde e_1,\dots,\tilde e_{d-1}\}$. Moreover

$$0<\inf_{\omega\in S^{d-1}}\kappa^\alpha(\omega)<\sup_{\omega\in S^{d-1}}\kappa^\alpha(\omega)<\infty\,.$$

Next consider a bounded open domain Λ in $\mathbb{R}^d$ with polygonal boundary $\partial\Lambda=\bigcup_{i=1}^{\Delta}\partial_i\Lambda$ and set $\Lambda_N=N\Lambda\cap\mathbb{Z}^d$. Then a simple modification of the proof of Theorem 4.1 yields

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N^{d-1}}\log P_{\Lambda_N}^\alpha(\Omega^+(\Lambda_N)) &= \lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{N^{d-1}}\log P_{\Lambda_N}^\alpha(\Omega^+(\partial_L\Lambda_N))\\ &= -\sum_{i=1}^{\Delta}\kappa^\alpha(n_i)\,|\,\partial_i\Lambda|\,,\end{aligned}$$

where n_i is the unit normal of $\partial_i\Lambda$ and $|\partial_i\Lambda|$ is the area.

Note that in the derivation of the above limit, one uses explicitly the fact that the pieces of boundary $\partial_i\Lambda$ are flat in the above argument. In particular a generalization as suggested by D. Ioffe:

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\log P_N^\alpha(\Omega^+(\Lambda_N))=-\int_{\partial\Lambda}\kappa^\alpha(n_x)\,dx\,,$$

where Λ is an open domain of $\mathbb{R}^d$ with piecewise smooth boundary is not immediate.

5. Appendix

The object of this Appendix is to derive some useful estimates for the covariance matrices, based on the random walk representation. The basic estimate is the following

Lemma A.1. *There exists a constant* $c_1 < \infty$, *such that for all* $k, j \in \partial_L V_N$,

$$G_N(k,j) \leqq \begin{cases} c_1|k-j|^{2-d} & 1 \leqq |k-j| < 2L\,, \\ c_1 L^2 |k-j|^{-d} & |k-j| \geqq 2L\,. \end{cases} \tag{A.2}$$

In particular,

$$\sup_{k\in\partial_L V_N} \sum_{j\in\partial_L V_N} G_N(k,j) \leqq c_2 L^2\,. \tag{A.3}$$

Proof. First note that $G_N(k,j) \leqq G(k,j)$, where

$$\lim_{|k-j|\to\infty} G(k,j)|k-j|^{d-2} = \frac{\Gamma(d/2)d^{(3-d)/2}}{(d-2)\pi^{d/2}}\,,$$

cf. Sect. 26 of [12]. This shows (A.2) for $|k-j| < 2L$. Next take k, j with $|k-j| \geqq 2L$. As above, let us move the boundary of V_N to the origin. Thus suppose that $k, j \in \partial_L^d V_N$ with $-L-1 \leqq j_d \leqq k_d < 0$. Then, by the reflection principle,

$$G_N^d(k,j) \leqq G^d(k,j) \leqq G^d(k',j) = G(k',j) - G(k',\hat{j})\,, \tag{A.4}$$

where $k' = (k_1,\dots,k_{d-1},j_d)$, $\hat{j} = (j_1,\dots,j_{d-1},-j_d)$. We claim that

$$\limsup_{|k'-j|\to\infty} |k'-j|^d G^d(k',j) \leqq c_3|j_d|^2\,. \tag{A.5}$$

This will imply the second inequality in (A.2). In order to prove (A.5), we use harmonic analysis as in Sect. 7 of [12]: let $\hat{Q}_{\mathrm{d}}(\theta) = \sum_{k\in\mathbb{Z}^d} Q_{\mathrm{d}}(0,k)e^{ik\cdot\theta} = \frac{1}{d}\sum_{j=1}^d \cos(\theta_j)$, then

$$\begin{aligned} G^d(k',j) &= \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} (e^{-i(k'-j)\cdot\theta} - e^{-i(k'-\hat{j})\cdot\theta})\psi(\theta)\,d\theta \\ &= \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} e^{-i(k'-j)\cdot\theta}(1-\cos(2j_d\theta_d))\psi(\theta)\,d\theta \\ &= \frac{\Delta^{-d}}{(2\pi)^d} \int_{(-\pi\Delta,\pi\Delta]^d} e^{-i\frac{(k'-j)}{\Delta}\cdot\theta}\left(1-\cos\left(2\frac{j_d}{\Delta}\theta_d\right)\right)\psi\left(\frac{\theta}{\Delta}\right)d\theta\,, \end{aligned}$$

where $\psi(\theta) = \frac{1}{1-\hat{Q}_{\mathrm{d}}(\theta)}$ and $\Delta = |k'-j|$. Thus, by (A.4) and rotation invariance

$$\limsup_{\Delta\to\infty} \Delta^d G_N^d(k',j) \leqq \frac{4d|j_d|^2}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\theta_1}\frac{|\theta_d|^2}{|\theta|^2}\,d\theta < \infty\,.$$

Finally, (A.3) follows from (A.2). □

Next, recall the definitions of $G^i_{N,\delta}$ and G^i_δ: the covariance matrices of $P^{i,0}_N$ and $P^{i,0}$ restricted to the box $\partial^i_L V_{\delta N}$, cf. Proof of Lemma 4.3.

Lemma A.6. *For each fixed $L > 0$ and $\delta \in (0,1)$ we have*

$$\lim_{N\to\infty} \|G^i_{N,\delta} - G^i_\delta\| = 0\,. \tag{A.7}$$

Moreover

$$\|(G^i_{N,\delta})^{-1}\| \leqq 2, \qquad \|(G^i_\delta)^{-1}\| \leqq 2\,. \tag{A.8}$$

Proof. Using the random walk representation, we have

$$\sum_{j\in\partial^i_L V_{\delta N}} |\,G^i(k,j) - G^i_N(k,j)\,| = \mathbb{E}_k\left[\sum_{n=\tau^i_N+1}^{\tau^i} 1_{\partial^i_L V_{\delta N}}(\xi_n); \tau^i_N < \tau^i\right].$$

Thus, by the strong Markov property of the random walk and (A.3),

$$\sum_{j\in\partial^i_L V_{\delta N}} |\,G^i_N(k,j) - G^i(k,j)\,| = \mathbb{E}_k\left[\mathbb{E}_{\xi_{\tau^i_N}}\left[\sum_{n=+1}^{\tau^i} 1_{\partial_L V^i_{\delta N}}(\xi_n)\right]; \xi_{\tau^i_N} \notin \mathbb{Z}^d_i\right]$$

$$\leqq c_1 L^2 \mathbb{P}_k(\xi_{\tau^i_N} \notin \mathbb{Z}^d_i) = c_1 L^2 \mathbb{P}_k(\tau^i_N < \tau^i)\,.$$

This implies (A.7), since

$$\lim_{N\to\infty} \sup_{k\in\partial^i_L V_{\delta N}} \mathbb{P}_k(\tau_N < \tau^i_N) = 0\,.$$

Next note that $\langle f, (G^i_{N,\delta})^{-1} f\rangle_{\partial^i_L V_{\delta N}}$ and $\langle f, (G^i_\delta)^{-1} f\rangle_{\partial^i_L V_{\delta N}}, f \in \ell^2(\partial^i_L V_{\delta N})$, are the Dirichlet forms associated with the simple random walk embedded in the box $\partial^i_L V_{\delta N}$ and killed as it exits V^i_N and $\mathbb{Z}^d_i$ respectively. That is, if $\tilde\tau = \inf\{n \geqq 1 : \xi_n \in \partial^i_L V_{\delta N}\}$, then $(G^i_{N,\delta})^{-1}(k,k) = (G^i_\delta)^{-1}(k,k) = 1$ and, for $j \neq k$,

$$(G^i_{N,\delta})^{-1}(k,j) = -\mathbb{P}_k(\xi_{\tilde\tau} = j; \tilde\tau < \tau^i_N),\ (G^i_\delta)^{-1}(k,j) = -\mathbb{P}_k(\xi_{\tilde\tau} = j; \tilde\tau < \tau^i)\,.$$

Therefore

$$\sum_{j\in\partial^i_L V_{\delta N}} |\,(G^i_{N,\delta})^{-1}(k,j)\,| \leqq 1 + \mathbb{P}_k(\tilde\tau < \tau^i_N) \leqq 2,$$

$$\sum_{j\in\partial^i_L V_{\delta N}} |\,(G^i_\delta)^{-1}(k,j)| \leqq 1 + \mathbb{P}_k(\tilde\tau < \tau^i) \leqq 2\,,$$

which shows (A.8). □

Finally, let us prove that a random walk starting at distance L_N from the boundary of the box V_N, is more likely to get trapped in a sub-lattice of mesh $L_N^{2/d}$, before it exits V_N. More precisely

Lemma A.9. *Let $\Delta_N = [L_N^{2/d}]$, and set*

$$\Lambda_N(\varepsilon) \equiv [\varepsilon\Delta_N]\mathbb{Z}^d \cap \left\{k : \frac{L_N}{2} \leqq \mathrm{dist}(k, V_N^\complement) \leqq \frac{5L_N}{2}\right\}\,.$$

Let $\tau_\varepsilon = \inf\{n \in \mathbb{N}_0 : \xi_n \in \Lambda_N(\varepsilon)\}$, then we can choose $\varepsilon > 0$ independently of N, such that

$$\inf_{k\in W_N(L_N)} \mathbb{P}_k(\tau_\varepsilon < \tau_N) \geqq \frac{1}{2}\,. \tag{A.10}$$

Proof. Let $\tilde{\tau}_N = \inf\{n \geqq 0 : dist(\xi_n, V_N^{\complement}) \notin [\frac{L_N}{2}, \frac{5L_N}{2}]\}$, then

$$\mathbb{P}_k(\tau_N \leqq \tau_\varepsilon) \leqq \mathbb{P}_k(\tilde{\tau}_N \leqq \tau([\varepsilon\Delta_N])) \leqq \mathbb{P}_k(\tilde{\tau}_N \leqq \varepsilon L_N^2) + \mathbb{P}_k(\tau([\varepsilon\Delta_N]) \geqq \varepsilon L_N^2) .$$

Note that $\sup_{k\in W_N(L_N)} \mathbb{P}_k(\tilde{\tau}_N \leqq \varepsilon L_N^2)$ converges to 0 as $\varepsilon \searrow 0$, uniformly in N. On the other hand by (A.8) of [3],

$$\sup_{k\in W_N(L_N)} \mathbb{P}_k(\tau([\varepsilon\Delta_N]) \geqq \varepsilon L_N^2) \leqq \exp\left(-c_1 \frac{\varepsilon L_N^2}{(\varepsilon L_N^{2/d})^d}\right) = \exp(-c_1\varepsilon^{1-d}) . \qquad \square$$

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